A Note on Optimal Debt Reduction Policies*

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Abstract

We analyze the optimal debt reduction problem in an uncertainty context. The social planner has a finite horizon and seeks to minimize the social costs associated with debt repayment by taking into account not only the short run costs of the policy but also the long run costs associated with the outstanding level of debt. We characterize the optimal policy and the dynamics of the debt to GDP ratio, showing that it will decrease over time if economic policy is effective enough. We characterize how the evolution of the debt to GDP ratio depends on the main parameters and we present a simple calibration based on Greek data to illustrate the implications of our analysis in real world setups.

Keywords: Debt Reduction, Optimal Policy, Uncertainty
JEL Classification: C61, E60, H63

1 Introduction

The recent global financial crisis has strongly affected several European economies by generating long lasting recessions characterized by sharp falls in per capita income and increases in unemployment rates. In order to mitigate the negative effects of the crisis most countries have implemented, either automatic or discretionary, countercyclical fiscal policies, accompanied also by fiscal-costly measures to support banking stability (Lane, 2012). This has put strong pressure on central governments to rely more than ever on the international market to finance their rising deficits, resulting in a surge in the debt to GDP ratio in several countries. The most famous example is represented by Greece, whose debt to GDP ratio increased from 109.4% in 2008 to 172.1% in 2011, but similar even if milder figures apply to other countries as well, including France, Italy, Portugal, and the UK. Figure 1 represents the dynamics of debt (left panel) and per capita income (right panel) in these countries over the last twenty years, showing that in some countries a first dramatic increase in the debt to GDP ratio occurs in 2007 (in an early phase of the crisis) while in others in 2008 (in a later and more severe phase). This rapid increase in the debt to GDP ratio has raised growing concerns for the sustainablity of the public debt in several countries which are required to act now in order to avoid more severe consequences later. For example, in their attempt to manage the Greek debt crisis, the EU and the IMF have demanded Greece to reduce its debt to GDP ratio by 40 percentage points in a few decades, forcing it to implement severe austerity measures which might undermine the development prospects of the country for long time. Greece, as other countries as well, face thus the problem of how to pay off its public debt in order to minimize the associated social costs. The nature of such a problem is all but trivial and the scope of this paper consists of developing a simple analytical framework to give a preliminary answer to

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such a delicate problem. Several works analyze the relation between economic growth and public debt in order to assess the sustainability of debt dynamics (see, among others, Futagami et al., 2008; Greiner and Fincke, 2009; Greiner, 2011, 2012; Maebayashi et al., 2017), but to the best of our knowledge, the problem of how to optimally repay public debt has never been considered thus far. Since no other study has looked at this issue in a way comparable to ours, as a starting point for the analysis we propose a simple framework to account for the key features of the problem to support policymakers in their decision process.

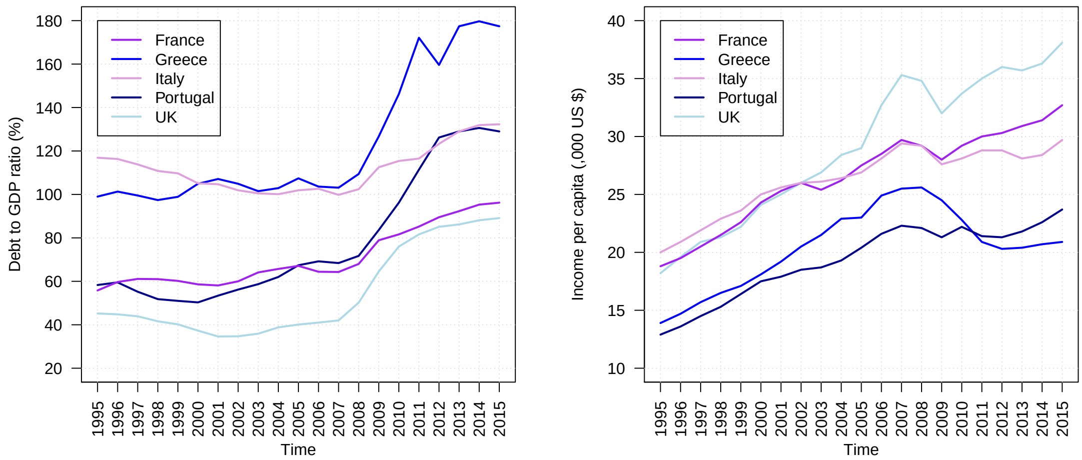

Figure 1: Evolution of the debt to GDP ratio (left) and income per capital (right) in selected EU countries.

This brief paper proceeds as follows. Section 2 presents our model, which consists of a finite horizon cost minimization problem in which the debt to GDP ratio is subject to random shocks and the planner cares also for the level of debt at the end of the planning horizon. In Section 3 we explicitly solve the stochastic optimization problem and we characterize the optimal policy and the optimal dynamics of debt to GDP ratio, showing how the debt to GDP ratio is critically affected by specific factors. In Section 4 we present a calibration of our model based on Greek data to illustrate the implications of our analysis in real world circumstances. In Section 5 we discuss a simple extension of our baseline model showing that our results apply even in a more general setting. Section 6 contains concluding remarks and highlights directions for future research. All mathematical technicalities are included in the appendix A.

2 The Model

We consider a model of debt reduction over a finite horizon in which the evolution of public debt is subject to random shocks. We focus on a small open economy in which the rate of interest on international borrowing is exogenously given, and we abstract from population growth such that aggregate and per capita variables coincide. The macroeconomic framework is extremely simple: households consume completely their disposable income: $C_t = (1 - \tau_t)Y_t$, where C_t denotes consumption, Y_t income and $\tau_t \in (0, 1)$ is the tax rate. Government spending is assumed to be null, such that the tax revenue $\tau_t Y_t$ represents also the government's budget surplus, which is entirely devoted to debt repayment. Income grows at a constant rate $g \in \mathbb{R}$, and it is subject to random shocks driven by a geometric Brownian motion: $dY_t = gY_t dt + \sigma Y_t dW_t$, where $\sigma \geq 0$ is the standard deviation of income and dW_t the increment of a Wiener process. We assume the economic growth rate, g, to take real values in order to distinguish between situations of expansions ($g > 0$) and recessions ($g < 0$), understanding their implications for the debt dynamics. The level of public debt (which consists entirely of external debt due to the absence of domestic funding sources) , B_t, accumulates according to the difference between its interest payments, rB_t, where $r > 0$ is the constant interest rate, and the government's budget surplus, $\tau_t Y_t$: $\dot{B}_t = rB_t - \tau_t Y_t$. From the dynamic equations of income and debt, straightforward algebra allows us to derive the evolution of the debt to GDP ratio, $x_t = \frac{B_t}{Y_t}$, which is thus given by the following equation: $dx_t = [(r-g)x_t - \tau_t]dt + \sigma x_t dW_t$. Note that (randomness apart), in absence of any debt repayment, that is $\tau_t = 0$, the trend of the debt to GDP ratio (what we shall refer to as the *"debt ratio"* for expositional simplicity) is intuitively determined by the difference between the interest rate and the economic growth rate, suggesting that during expansions the increase in the debt ratio might be small or even negative while during recessions the debt ratio will tend to increase fast. This implies that the tax rate, τ_t, required in order to effectively reduce the debt ratio changes with the pace of economic growth. In the following we will restrict our analysis to the case in which $r > g$ such that in absence of specific debt reduction policies the debt ratio will on average tend to rise over time, consistently with the recent after-crisis experience of several European economies.

Similar to Barro (1979), the social planner wishes to minimize the social costs of debt reduction by choosing the optimal level of the tax rate, τ_t, which determines the amount of resources allocated to debt repayment. The social cost function, $\mathcal{C}$, is the weighted sum of two different terms: the expected discounted ($\rho > 0$ is the rate of time preference) sum of instantaneous losses generated by the debt reduction policy, and the discounted damage associated with the remaining level of debt at the end of the planning horizon, T. The instantaneous loss function, $c(\tau_t)$, is assumed to be increasing and convex and for the sake of simplicity to be quadratic: $c(\tau_t) = \frac{\tau_t^2}{2}$. Similarly, the damage function, $d(x_T)$, is assumed to be increasing and convex and to take a quadratic form:[1] $d(x_T) = \frac{x_T^2}{2}$. Note that the quadratic formulation of the loss and damage functions are due to tax smoothing motives (Barro, 1979). The social planner needs thus to choose the tax rate τ_t in order to minimize the expected social cost function, by taking into account the evolution of the debt ratio and its given initial condition. The planner's problem can be summarized as follows:

$$\min_{\tau_t} \quad \mathcal{C} = \mathbb{E}\left[\int_0^T \frac{\tau_t^2}{2} e^{-\rho t} dt + \phi \frac{x_T^2}{2} e^{-\rho T}\right] \tag{1}$$

$$s.t. \quad dx_t = [(r-g)x_t - \tau_t]dt + \sigma x_t dW_t \tag{2}$$

$$x_0 > 0 \text{ given,} \tag{3}$$

where $\phi \geq 0$ represents the relative weight of the end-of-planning horizon damage function. If $\phi = 0$ then only the sum of instantaneous losses will matter (and in this case the cost minimizing tax rate will clearly be null), while if $\phi \to \infty$ then only the end-of-planning-horizon damage will matter. For any positive and finite value of ϕ both the short-term loss and end-of-planning-horizon damage functions will be relevant to determine the optimal debt reduction policy. The parameter ϕ captures thus the concern for long run sustainabliity of the public debt, and we shall refer to this as the "degree of sustainability concern" (La Torre et al., 2017).

3 The Optimal Policy and Dynamics

Solving the above stochastic problem requires to find an explicit expression for the value function solving the Hamilton-Jacobi-Belllman equation associated with the problem (1), (2) and (3). After some algebra it is possible to claim the following (the proofs of all the following propositions are postponed in the Appendix A).

[1] In Section 5 we shall generalize our model to allow the loss and damage functions to be not necessarily quadratic, showing that our qualitative results extend even in more general contexts. It seems thus convenient to present the model in the simplest possible form first.

Proposition 1. *The value function associated with the problem (1), (2) and (3) is given by:*

$$\mathcal{J}(t, x_t) = \frac{1}{2}\frac{\lambda\phi e^{-(\lambda+\rho)t}}{e^{-\lambda T}(\lambda-\phi)+\phi e^{-\lambda t}}x_t^2, \tag{4}$$

where $\lambda \equiv 2(r-g)+\sigma^2-\rho$. The optimal rule for the tax rate, τ_t^* and the optimal dynamic path of the debt ratio are respectively given by:

$$\tau_t^* = \frac{\lambda\phi e^{-\lambda t}}{(\lambda-\phi)e^{-\lambda T}+\phi e^{-\lambda t}}x_t^* \tag{5}$$

$$x_t^* = x_0\left[\frac{(\lambda-\phi)e^{-\lambda T}+\phi e^{-\lambda t}}{(\lambda-\phi)e^{-\lambda T}+\phi}\right]e^{\left(r-g-\frac{1}{2}\sigma^2\right)t+\sigma W_t}. \tag{6}$$

First of all, note that since $r > g$, the most realistic case for our analysis is represented by $\lambda > 0$, which is naturally verified unless the rate of time preference is particularly large; in what follows we shall restrict our analysis to this case.[2] Proposition 1 clearly states that both the optimal tax rate and the optimal debt ratio are not constant but time-varying; moreover, the optimal tax rate is proportional to the debt to GDP ratio, which since affected by random shocks requires the optimal policy to respond to the shock as well. Understanding thus the dynamics of both the tax rate and the debt ratio is not possible, but we can explicitly analyze their expected behavior. Indeed, the expected value of (5) and (6) is given by the following expressions, respectively:

$$\mathbb{E}[\tau_t^*] = \frac{\lambda\phi e^{-\lambda t}}{(\lambda-\phi)e^{-\lambda T}+\phi e^{-\lambda t}}\mathbb{E}[x_t^*] \tag{7}$$

$$\mathbb{E}[x_t^*] = x_0\left[\frac{(\lambda-\phi)e^{-\lambda T}+\phi e^{-\lambda t}}{(\lambda-\phi)e^{-\lambda T}+\phi}\right]e^{(r-g)t}. \tag{8}$$

Note that the degree of sustainability concern plays a crucial role in determining the expected dynamics of the tax rate and the debt ratio. If $\phi = 0$, the optimal solution dictates no public intervention (i.e., $\mathbb{E}[\tau_t^*] = 0, \forall t \in [0, T]$) and thus the expected value of the debt ratio will grow over time at the exogenous rate determined by the gap between the interest rate and the economic growth rate (i.e., $\mathbb{E}[x_t^*] = x_0 e^{(r-g)t}$). If $\phi > 0$, the optimal solution prescribes a strictly positive level of public intervention to reduce the expected growth rate of the debt ratio. However, also a strictly positive level of public intervention does not necessarily reduce the debt ratio, since it could simply slow down its positive growth rate. In fact, in order for public policy to be effective in its goal of reducing the debt ratio the degree of sustainability concern needs to fall in a specific range as characterized in the following proposition.

Proposition 2. *Public policy is effective if the debt ratio decreases over time, namely whenever $r-g < \phi < 1$.*

Proposition 2 states that the debt ratio will effectively decrease if the degree of sustainability concern is large enough ($\phi > r-g$) but not too large ($\phi < 1$). While the former condition is needed in order to ensure a reduction in the debt ratio, the latter is required to ensure the optimal tax rate to well defined, that is $\mathbb{E}[\tau_t^*] \in [0, 1]$. Whenever these two conditions are verified public policy can effectively achieve its goal of reducing the debt ratio. Intuitively, on the one hand, if the degree of sustainability concern is not large enough, the sum of instantaneous losses generated by public policy dominates the damage associated with the outstanding level of debt and thus the optimal tax rate would be small and not enough to turn the growth rate of the debt ratio from positive to negative; on the other hand, if the degree of sustainability concern is too large, the debt-induced damage dominates and thus the optimal tax rate would be excessively large requiring to devote more than the available GDP to reduce the debt ratio. In what follows we shall

[2] It is straightforward to extend the analysis to the case in which $\lambda < 0$. Intuitively, in such a case within the relevant parameter range most of our conclusions will simply need to be reversed.

assume that the condition in Proposition 2 is met such that public policy is effective, in order to characterize how the optimal debt ratio is affected by the main parameters. The result is summarized in the following proposition.

Proposition 3. *The expected optimal debt ratio decreases with the degree of sustainability concern (ϕ), and, provided that the time horizon is not too short (i.e., $T \geq \frac{1}{\lambda-\phi}$), it increases with uncertainty (σ) and decreases with the rate of time preference (ρ).*

The results in Proposition 3 are intuitive by considering how the main parameters affect the optimal tax rate, and in turn the debt ratio. The higher the degree of sustainability concern the stronger the debt-induced damage and thus the stronger the incentive to increase the tax rate, resulting in a lower debt ratio at any moment in time. The other parameters determine in a similar way the debt reduction motive through their effects on the sum of instantaneous losses and end-of-planning-horizon damage, but it is generally not possible to unambiguously quantify the short run and long run costs. Only whenever the time horizon is large enough it is possible to show that intuitively the debt ratio monotonically rises with the degree of uncertainty and falls with the rate of time preference. The effects of the interest rate and the economic growth rate are generally ambiguous since they affect in opposite directions the incentives to debt reduction and the trend of the debt ratio in absence of public policy.

4 A Calibration on the Greek Case

We now illustrate our model's implications by presenting a simple calibration based on Greek data. This case study is interesting because of the extensive coverage that the Greek debt crisis has received in the media, and also because it fits well our model's setup since public debt in Greece consists to a large extent of external debt. Based on historical data over the last twenty years (1995–2015), per capita GDP in Greece has shown a yearly growth rate of 2% with standard deviation of 5%; the debt ratio in 2015 is 177.4% implying an average cost of borrowing of 8.5% (i.e., average interest rate on 10-years government bonds during 2016). By setting the end of planning horizon at 2040, the Greek government's goal consists of reducing the debt ratio by determining the tax rate to devote to debt repayment. Note that given these parameter values, a null debt repayment (i.e., $\tau_t = 0, \forall t$) will lead the debt ratio to grow at a strictly positive rate given by $r - g = 6.5\%$, thus a strictly positive tax rate is needed to effectively reduce the debt ratio. The pure rate of time preference is set at 4% as in traditional macroeconomic settings (Barro and Sala–i–Martin, 2004), while the degree of sustainability concern has been calibrated to 7% in order to bring the debt ratio below 120% by 2040, as required by the European Commission. The results of our calibration exercise are shown in Figure 2 where the solid curves represent the (expected) paths of the tax rate (left panel) and the debt ratio (right panel).

We can observe that the tax rate monotonically falls over time and thus initially overshoots its final value; this allows the debt ratio to fall more rapidly in earlier times than at the end of the planning horizon. Note that such an initial overshooting followed by a gradual reduction in the tax rate is consistent with tax-smoothing motives and, despite it generates large early time losses, it is effectively needed in order to ensure the required reduction of the debt ratio, due to the large gap between the interest and growth rates. Moreover, note that the optimal tax rate, which coincides with the budget surplus as a percentage of GDP, at any moment in time exceeds by far the 3.5% level established by the European Commission in its negotiations with Greece; this suggests that the current plan of the EU for reducing Greece's public debt is likely not to achieve its target in the proposed time frame. The effects of an increase in the economic growth rate (or a reduction of the interest rate, since the quantitative effects are exactly the same) by half and one percentage point are represented by the dashed and dotted curves, respectively. A higher economic growth rate (a lower interest rate), by reducing the gap between the interest and growth rates, requires a smaller initial overshooting and thus lower early time losses, leading thus to a decrease in the optimal tax rate at

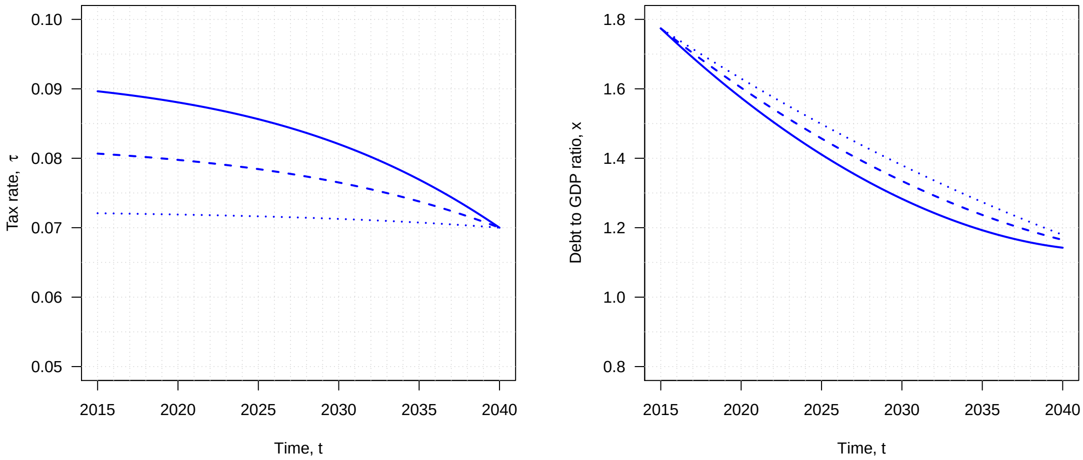

Figure 2: Optimal tax rate, τ_t (left), and debt to GDP ratio, x_t (right), in the case of Greece. Parameter values: $\sigma = 0.05$, $r = 0.085$, $\rho = 0.04$, $x_0 = 1.774$, $T = 25$, $\phi = 0.07$, with either $g = 0.02$ (solid curves) or $g = 0.025$ (dashed curves) or $g = 0.03$ (dotted curves).

any moment in time which in turn slows down the reduction in the debt ratio. This suggests that promoting economic growth may be an effective strategy to achieve the required debt reduction at lower social costs.

The results of our above calibration need to be taken with some grain of salt. Since our model is extremely stylized abstracting from the analysis of the mutual links between fiscal policy and economic growth and from the consideration of important factors that have influenced Greek debt dynamics (such as extensions of debt maturity obligations and rising concerns of international markets about Greece's solvency capabilities), it does not pretend to capture all the complex issues underlying the Greek debt crisis (see Papageorgiou, 2012, for a more involved discussion of some of these issues). Despite the simplicity of our framework, we believe that our calibration can provide useful (but not exhaustive) insights to understand the implications of alternative debt reduction policies on the Greek debt dynamics from a macroscopic point of view, clearly illustrating the applicability of our model in real world circumstances.

5 A Model Extension

We now present a simple extension of our baseline model in order to show that the results that we have earlier discussed apply in a more general context. We focus on a generalization of the quadratic cost and damage functions introduced in the objective function (1). Rather than quadratic functionals, we now consider the following problem:

$$\min_{\tau_t} \quad \mathcal{C} = \mathbb{E}\left[\int_0^T \frac{\tau_t^\gamma}{\gamma} e^{-\rho t} dt + \phi \frac{x_T^\gamma}{\gamma} e^{-\rho T}\right] \tag{9}$$

$$s.t. \quad dx_t = [(r-g)x_t - \tau_t]dt + \sigma x_t dW_t \tag{10}$$

$$x_0 > 0 \text{ given,} \tag{11}$$

where $\gamma > 1$ in order to ensure that the cost and damage functions are increasing and convex. Clearly this formulation collapses to our baseline model whenever $\gamma = 2$. By repeating the same calculations as in our baseline model, it is possible to show that the value function solving the Hamilton-Jacobi-Bellman equation associated with the above problem takes the following form:

$$\mathcal{J}(t,x_t)=\frac{1}{\gamma}\left\{\frac{\lambda\phi^{\frac{1}{\gamma-1}}e^{-\lambda t}}{\left[\lambda-\phi^{\frac{1}{\gamma-1}}(\gamma-1)\right]e^{-\lambda T}+\phi^{\frac{1}{\gamma-1}}(\gamma-1)e^{-\lambda t}}\right\}^{\gamma-1}e^{-\rho t}x_t^{\gamma} \tag{12}$$

where $\lambda\equiv\gamma(r-g)+\frac{\gamma(\gamma-1)}{2}\sigma^2-\rho$, while the optimal tax rate and debt ratio are given by the following expressions:

$$\tau_t^* = \frac{\lambda\phi^{\frac{1}{\gamma-1}}e^{-\lambda t}}{\left[\lambda-\phi^{\frac{1}{\gamma-1}}(\gamma-1)\right]e^{-\lambda T}+\phi^{\frac{1}{\gamma-1}}(\gamma-1)e^{-\lambda t}}\,x_t^*. \tag{13}$$

$$x_t^* = x_0\left\{\frac{\left[\lambda-\phi^{\frac{1}{\gamma-1}}(\gamma-1)\right]e^{-\lambda T}+\phi^{\frac{1}{\gamma-1}}(\gamma-1)e^{-\lambda t}}{\left[\lambda-\phi^{\frac{1}{\gamma-1}}(\gamma-1)\right]e^{-\lambda T}+\phi^{\frac{1}{\gamma-1}}}\right\}e^{\left(r-g-\frac{1}{2}\sigma^2\right)t+\sigma W_t} \tag{14}$$

A quick comparison between the above expression for τ_t^* and x_t^* and those in (5) and (6) shows that, qualitatively speaking, nothing changes with respect to our baseline model apart from the introduction of an additional parameter γ. This allows us to conclude that, without any loss of generality, our previous results extend also in this more general framework, and thus our baseline model is a good representative case of this more general setup.

6 Conclusions

Following the recent global financial crisis, several countries are now faced with the problem of how to reduce their level of outstanding debt in order to ensure the sustainability of their public finance. Our paper develops a simple debt control problem in which the social planner by determining the tax rate effectively determines the budget surplus to devote to debt repayment. The planner needs to take into account not only the short run costs associated with economic policy but also the long run costs induced by the level of debt remaining at the end of the planning horizon. We characterize the optimal tax rate and the optimal debt ratio dynamics, showing that if public policy is effective enough the debt ratio will decrease over time; we also show how the debt ratio is affected by the main parameters. We then present a calibration based on the case of Greece to illustrate the implications of our model and the extent to which the current EU plan to reduce Greece's public debt differs from the optimal strategy prescribed by our stylized model. We also show that our results derived from a quadratic cost minimization problem extend to a more general non-quadratic setting.

To the best of our knowledge, this is the first paper analyzing the optimal debt repayment issue thus the setup has been kept as simple as possible in order to understand the underlying mechanisms in the most intuitive way. However, the analysis could be extended in order to enrich the model's structure by introducing capital accumulation or government spending such that fiscal policy determines not only debt repayments but also its accumulation. These further tasks are left for future research.

A Technical Appendix

By denoting with $\mathcal{J}(t,x_t)$ the value function associated with our stochastic problem (1), (2) and (3), the Hamilton-Jacobi-Bellman equation reads as:

$$-\frac{\partial\mathcal{J}}{\partial t} = \min_{\tau_t}\left\{\frac{1}{2}\tau_t^2e^{-\rho t}+[(r-g)x_t-\tau]\frac{\partial\mathcal{J}}{\partial x_t}+\frac{1}{2}\sigma^2x_t^2\frac{\partial^2\mathcal{J}}{\partial x_t^2}\right\} \tag{15}$$

while the corresponding terminal condition:

$$\mathcal{J}(T, x_T) = \frac{\phi}{2} x_T^2 e^{-\rho T} \tag{16}$$

The first order necessary and sufficient (see below) condition for τ_t yields:

$$\tau_t = e^{\rho t} \frac{\partial \mathcal{J}}{\partial x_t} \tag{17}$$

We proceed by guessing the form of the value function and verifying that our guess is correct. Our guess is:

$$\mathcal{J}(t, x_t) = \frac{1}{2} x_t^2 A_t e^{-\rho t} \tag{18}$$

where A_t is a variable to be determined. By computing its derivatives: $\frac{\partial \mathcal{J}}{\partial t}$, $\frac{\partial \mathcal{J}}{\partial x_t}$ and $\frac{\partial^2 \mathcal{J}}{\partial x_t^2}$, and plugging these into (17) and (15), we obtain respectively:

$$\tau_t = x_t A_t \tag{19}$$

$$\dot{A}_t = A_t^2 - A_t[2(r-g) + \sigma^2 - \rho] \tag{20}$$

At the terminal time, by comparing (16) and (18) evaluated at T, it follows that $A_T = \phi \geq 0$. Note that (20) is a Bernoulli differential equation with a terminal condition $A_T = \phi \geq 0$, and its exact solution is given by:

$$A_t = \frac{\lambda \phi e^{-\lambda t}}{e^{-\lambda T}(\lambda - \phi) + \phi e^{-\lambda t}}, \tag{21}$$

where $\lambda \equiv 2(r-g) + \sigma^2 - \rho$. For later reference, note that since $\dot{A}_t = \frac{-\lambda^2 \phi e^{-\lambda t} e^{-\lambda T}(\lambda - \phi)}{[e^{-\lambda T}(\lambda - \phi) + \phi e^{-\lambda t}]^2}$, A_t increases over time if $\lambda - \phi < 0$, while it decreases if $\lambda - \phi > 0$. By plugging (21) into (19) we get the optimal dynamics of the policy given by (5), and by plugging (5) in (2), it is possible to determine the time evolution of the debt ratio given in (6).

In order to prove sufficiency, it is straightforward to show that the minimized Hamiltonian, given by the following expression:

$$\hat{\mathcal{H}} = \frac{1}{2} A_t^2 x_t^2 e^{-\rho t} + [(r-g) - A] x_t^2 A_t e^{-\rho t} + \frac{1}{2} \sigma^2 x_t^2 A_t e^{-\rho t}, \tag{22}$$

is strictly convex in x_t for all t, ensuring that the pair (τ_t^*, x_t^*) represents the optimal solution for our minimization problem. This happens whenever the following condition holds true: $2(r-g) + \sigma^2 > A_t$. Note that, due to the above discussion of the sign of $\dot{A}_t$ it follows that $A_t \leq \min\{\lambda, \phi\}$, which implies that the above condition is automatically verified whenever $\lambda \geq \phi$, while it requires that $2(r-g) + \sigma^2 \geq \phi$ whenever $\lambda < \phi$.

The temporal evolution of $\mathbb{E}[x_t^*]$ is given by the following derivative, whose sign is unambiguously determined (provided that $\lambda > 0$) whenever $\phi > r - g$:

$$\frac{\partial \mathbb{E}[x_t^*]}{\partial t} = -\frac{x_0 e^{(r-g)t} e^{-\lambda t} \left[[\lambda - \phi)(r-g) e^{-\lambda(T-t)} + \phi(r - g - \lambda) \right]}{(\lambda - \phi) e^{-\lambda T} + \phi}. \tag{23}$$

Moreover, for $\mathbb{E}[\tau_t^*]$ to be lower than unity we need that $\phi < 1$, provided that $\mathbb{E}[x_t^*]$ is not too large. These two conditions jointly determine the result in Proposition 2.

Straightforward but tedious algebra leads to the derivatives of $\mathbb{E}[x_t^*]$ with respect to the main parameters, which are given by the following expressions:

$$\frac{\partial \mathbb{E}[x_t^*]}{\partial \phi} = -\frac{\lambda x_0 e^{(r-g)t} e^{-\lambda T} \left(1 - e^{-\lambda t}\right)}{[(\lambda - \phi) e^{-\lambda T} + \phi]^2} \tag{24}$$

$$\frac{\partial \mathbb{E}[x_t^*]}{\partial \lambda} = \frac{x_0 e^{(r-g)t} \left\{ \left(1 - e^{-\lambda t}\right) \phi e^{-\lambda T} [1 - T(\lambda - \phi)] - t \phi e^{-\lambda t} [(\lambda - \phi) e^{-\lambda T} + \phi] \right\}}{[(\lambda - \phi) e^{-\lambda T} + \phi]^2} \tag{25}$$

$$\frac{\partial \mathbb{E}[x_t^*]}{\partial (r-g)} = \frac{x_0 e^{(r-g)t} \left\{ \left(1 - e^{-\lambda t}\right) \phi e^{-\lambda T} [1 - T(\lambda - \phi)] + t(\lambda - \phi) e^{-\lambda T} [(\lambda - \phi) e^{-\lambda T} + \phi] \right\}}{[(\lambda - \phi) e^{-\lambda T} + \phi]^2} \tag{26}$$

Note that $\frac{\partial \mathbb{E}[x_t^*]}{\partial \phi}$ is positive provided that $\lambda > 0$. Note instead that $\frac{\partial \mathbb{E}[x_t^*]}{\partial \lambda}$ is unambiguously determined only whenever $T > \frac{1}{\lambda - \phi}$ and in this case the derivative is negative; from this we can directly determine the signs of $\frac{\partial \mathbb{E}[x_t^*]}{\partial \sigma}$ and $\frac{\partial \mathbb{E}[x_t^*]}{\partial \rho}$, which are positive and negative, respectively. The sign of $\frac{\partial \mathbb{E}[x_t^*]}{\partial (r-g)}$ is instead undetermined, and thus we cannot conclude anything a priori about the signs of $\frac{\partial \mathbb{E}[x_t^*]}{\partial r}$ and $\frac{\partial \mathbb{E}[x_t^*]}{\partial g}$.

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