# Optimality in a financial economy with outside money and restricted participation

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**Abstract**

We analyze an economy with inside financial assets and outside money. Households have different restricted access on both types of assets and according with a well known approach they use money to pay taxes. Since competitive equilibria are generically inefficient, we perform a Pareto Improvability analysis through a monetary intervention. It results that if government modifies the amount of money endowments just for one consumer in period one, he can Pareto Improve upon the market equilibrium hence money is effective.

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## 1 Introduction

In a standard general equilibrium framework with incomplete markets, consumers face the same opportunities to transfer wealth across spot markets. In real life, we can find many cases where the participation constraints on financial markets varies from a class of consumers to another. Think for example of collateral securities in American real estate market or of a credit line which is secured by financial assets.

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In recent literature we can find several models (see Balasko, Cass and Siconolfi (1990), Cass, Siconolfi and Villanacci (1992), Polemarchakis and Siconolfi (1997), Siconolfi (1988)) which present a wide range of restrictions on financial market participation. These kind of general equilibrium models are called "restricted participation model" and they can be seen as generalization of the incomplete market case. While Cass, Siconolfi and Villanacci (1992) propose a model where the individual participation constraint is described by a differentiable strictly quasi-concave function of consumer's assets $a_h$, our model in enriched by the presence of the outside money[1] whose exchange is restricted too. We assume $a_h$ is a function of both consumer's assets and outside money demands, i.e. $a_h(b_h, m_h^0) \geq 0$. From now on, unless it is otherwise specified, money means outside money.

The role of money and how money takes place in a general equilibrium framework has been interested economic theorists since long time and we can find a large amount of papers in economic literature that "justify" the existence of money ( for a survey on recent contributions, the interested reader may see Starr (1989) and Magill and Quinzii (1996).

In an incomplete market framework or in a restricted participation framework, people are not able to freely transfer wealth from a state to another. Due to this, money and other assets perform the important function of a store of value. Unfortunately, in a finite-horizon model, (unless there is an additional element exogenously preventing the price of money from going to zero) zero may be the only equilibrium price of money and that is a serious problem since a nil price of money effectively demonetizes the model (Hahn (1965), Cass and Shell (1980) ).

In recent literature, we can find several papers which try to overcome the well-known hot potato problem (see Cass and Shell (1980), Dubey and Geanakoplos (1992), Grandmont and Younes (1975), Magill and Quinzii (1992), Starr (1974 and 1989)) and here, we follow a well known approach in terms of needs of money to pay taxes (e.g. Lerner (1947) , Starr (1974), Villanacci (1991 and 1993)) according with money is used to pay taxes at the end of the second period; taxes are linear function of households' wealths. Households are obliged to pay taxes and no default is possible. We do not try to explain why money exists, but we are interested in analyze the efficiency of monetary competitive equilibria and the effectiveness of monetary policy. Consistently with the result in the incomplete market case, there exists an

[1]By the term *Outside money* we refer to money which is a direct debt of the public sector, e.g. circulating currency, or is based on such debt, e.g. commercial bank deposits matched by bank holdings of public sector debt. Examples are fiat money, gold and foreign exchange reserves. On the other hand, *Inside money* is a form of money which is based on private sector debt.

open and full measure subset of economies, whose associated equilibrium allocations are not Pareto Optimal and that leads to the following question: can a "limited" government intervention Pareto improve upon the market equilibrium? Answers of this question in an incomplete market framework can be found in Geanakoplos Polemarchakis (1986) and Citanna Kajii Villanacci (1998) which perform Pareto Improvability analysis by allowing asset redistribution (see also Cass (1995)). In our case the presence of money suggests a monetary intervention and we prove that if government can modify the amount of money endowments just for one consumer then Pareto Improvement are possible; that implies money is not neutral.

The paper is organized as follows: section two is devoted to features of the model while section three is about existence, regularity and efficiency of competitive equilibria. It is worth noticing that we omit the proof of the results in section three since they are "more or less" variations or extensions of others that the reader can find in the literature. However a complete proof of existence theorem can be found in Carosi (1999a) while regularity and Pareto non Optimality are in Carosi (1999b). Finally the Pareto Improvability analysis and effectiveness of money are dealt in section four.

# 2 Set up of the model

We consider an exchange economy with two periods; today and tomorrow where $S$ states of world are possible. Using a standard notation we index today as $s = 0$, and the $S$ states of tomorrow as $s = 1, ..., S$. The timing is the following: in state 0, households receive endowments of goods and money, they exchange goods and assets and consume the goods they acquired. Households are not allowed to buy and sell assets freely, but they must take into account their own participation constraints. Tomorrow uncertainty is resolved, one of the $S$ states occurs and households receive their endowments of goods and money. They exchange goods and fulfill the obligations underwritten in state 0. Finally households consume the goods they acquired and they use money to pay taxes. We will use the following notations:

- Households are labelled by $h \in \{1, 2, ..., H\}$, goods in each state by $c \in \{1, ..., C\}$ while assets are denoted by $i \in \{1, ..., I\}$. The total number of goods is $G = C(S + 1)$.
- $e_h^{sc}$ and $x_h^{sc}$ are respectively, the endowment and demand of good $c$ in state $s$, of household $h$.
- $e_h^s \equiv (e_h^{sc})_{c=1}^C$ , $e_h = (e_h^s)_{s=0}^S$ , $e = (e_h)_{h=1}^H$

- $x_h^s \equiv (x_h^{sc})_{c=1}^C$ , $x_h = (x_h^s)_{s=0}^S$ , $x = (x_h)_{h=1}^H$
- $e_h^{sm}$ is the endowment of money in state $s$, owned by household $h$
- $e_h^m = (e_h^{sm})_{s=0}^S$ , $e^m = (e_h^m)_{h=1}^H$
- $p^{sc}$ is the price of good $c$ in state $s$, $p^s \equiv (p^{sc})_{c=1}^C$ , $p = (p^s)_{s=0}^S$
- $b_h^i$ is the demand of asset $i$, of household $h$, $b_h \equiv (b_h^i)_{i=1}^I$
- $q^i$ is the price of asset $i$ in state 0, $q = (q^i)_{i=1}^I$
- $q^{sm}$ is the price of money in state $s$, $q^m = (q^{sm})_{s=0}^S$
- $m_h^s$ is the demand of money in state $s$ of household $h$. $m_h = (m_h^s)_{s=0}^S$ , $m = (m_h)_{h=1}^H$

Households' utility functions satisfies standard smoothness assumption, that is:

**Assumption 1** *i) $u_h : \mathbb{R}_{++}^G \to \mathbb{R}$ , is a smooth function, i.e., a $\mathcal{C}^\infty$ function.*
*ii) $u_h$ is differentiably strictly increasing, i.e., $Du_h(x_h) \gg 0$.*
*iii) the Hessian matrix $D^2 u_h$ is negative definite*
*iv) For any $\underline{u} \in \mathbb{R}$, $Cl\left\{x \in \mathbb{R}_{++}^G : u_h(x) \geq \underline{u}\right\} \subseteq \mathbb{R}_{++}^G$.*

We assume consumers cannot issue outside money.

**Assumption 2** *$m_h^s \geq 0$ for all $s$ and all $h$.*

Prices of goods, money and assets are expressed in units of account. We assume that prices of goods and money are strictly positive.

**Assumption 3** *$p^{sc} \in \mathbb{R}_{++}$ for all $s$ and all $c$, $q^m \in \mathbb{R}_{++}^{S+1}$.*

An asset $i$ costs $q^i$ in state 0 and it gives the right to receive $y^{si}$ if tomorrow state $s$ occurs. We denote the matrix of assets yields by $Y = \begin{bmatrix} y^{11} & .. & y^{1I} \\ & & \\ y^{S1} & & y^{SI} \end{bmatrix}$; $Y$ is a $S \times I$ matrix . Moreover $Y^M = \begin{bmatrix} Y & \mathbf{1} \end{bmatrix}$ is a $S \times (I+1)$, matrix where $[\mathbf{1}]$ it the vector of yields associate with money

It greatly simplifies our analysis to assume that

**Assumption 4** *$S > I + 1$, $RankY = I$ and $rankY^M = I + 1$.*

**Remark 5** The previous Assumption means there are no redundant assets in the economy. As Cass, Siconolfi and Villanacci (1992) say, "..In this context, Assumption 4 is not at all innocuous. When their portfolio holdings are constrained, households may very well benefit from the opportunity afforded by the availability of additional bonds whose yields are not linearly independent".

Households deal with two different kinds of constraints in the assets market. On one hand they must take into account the incompleteness of the asset market (i.e. $rankY = I < S$) and on the other hand, they must consider their own participation constraint. The latter is expressed by the following function :

$$\begin{aligned} & a_h : \mathbb{R}^I \times \mathbb{R} \rightarrow \mathbb{R}^{\#J_h} \\ & a_h^j : \left(b_h, m_h^0\right) \mapsto a_h^j \left(b_h, m_h^0\right) \quad j = 1, ..., \#J_h \end{aligned} \tag{1}$$

where $a_h = \left[a_h^j \left(b_h, m_h^0\right)\right]_{j=1}^{\#J_h}$ , $J_h$ is a set of indexes such that $J_h \subseteq I$. Each household faces to the constraint $a_h \left(b_h, m_h^0\right) \geq 0$ and $a_h^j$ verifies the following Assumption.

**Assumption 6** *$a_h^j$ is a $C^2$, differentially strictly quasi-concave function, i.e. for every $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ and every $\Delta \in \mathbb{R}^{I+1}$*

$$Da_h^j \left(b_h, m_h^0\right) \Delta = 0 \Rightarrow \Delta^T D^2 a_h^j \left(b_h, m_h^0\right) \Delta < 0.$$

Moreover function $a_h$ verifies the following Assumption.

**Assumption 7** *i)* $a_h (0, 0) \gg 0$.

*ii) For every $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ such that $\left\{a_h^j \left(b_h, m_h^0\right) = 0\right\}_{j \in J'_h}$,*

*$rank \left(Da_h^{J'_h} \left(b_h, m_h^0\right)\right) = \#J'_h$, for every index subset $J'_h \subseteq J_h$.*

*iii) For every asset $i$, there exists at least one consumer $h'$ such that for every $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ the following condition holds: $D_{b_{h'}^i} a_{h'} \left(b_{h'}, m_{h'}^0\right) = 0$*

*iv) there exists at least one consumer $h'$ such that: $D_{m_{h'}^0} a_{h'} \left(b_{h'}, m_{h'}^0\right) = 0$.*

**Remark 8** Assumption 7 has important economic meanings.

i) people are not obliged to operate in the assets and/or money markets. Moreover, every consumer can freely operate when both his assets and money demand are "very low".

iii) for every asset there exists at least one household who is unrestricted on that asset market.

iv) there exists at least one consumer who can arbitrary vary his money demand.

Assumptions 7 i),...,iv) are used in order to prove the existence of the competitive equilibrium and their generic regularity.

We impose the following Assumption on the relationship between the number of consumers and the number of assets which strengthens Assumption 4, $S > I$.

**Assumption 9** $S \geq H > I + 1$.

**Remark 10** Assumption 9 will be used in order to prove the non neutrality of policy intervention. In fact $H > I+1$ allows to simplify the computations (see Case 2 page. 13 ) and $S \geq H$ guarantees that the number of independent policy instruments is greater than the number of households.

Households are not able to create wealth by acting on the assets and money markets. Hence we obtain the following no arbitrage condition that allows us to define the set of no arbitrage assets and money prices

**Definition 11** Let us define the no-arbitrage asset and money price set as: $\widehat{Q}_h = \{\hat{q} = (q, q^m) \in \mathbb{R}^I \times \mathbb{R}^{S+1}_{++} : \nexists \, (b_h, m_h^0)$, such that $a_h \left(b_{h,} m_h^0\right) \geq 0$ and $\begin{bmatrix} -q & -q^{m0} \\ Y & q^{m\mathbf{1}} \end{bmatrix} \begin{bmatrix} b_h \\ m_h^0 \end{bmatrix} > 0 \}$

where $q^{m\mathbf{1}}$ is the vector $q^{m\mathbf{1}} = \left(q^{ms}\right)_{s=1}^S = \left(q^{m1}, .., q^{mS}\right)$ of dimension $S \times 1$.

$\widehat{Q} = \underset{h \in H}{\cap} \widehat{Q}_h$ is the set of no arbitrage.

It is straightforward to check that $\widehat{Q}$ is non empty since it is larger than the standard set of no-arbitrage asset prices whose non-emptiness is a well known result.

In period 1, Mr. $h$ pays taxes using money; taxes are proportional to the value of his endowments.

- $\tau_h^{sc} \in [0,1] \subseteq \mathbb{R}$ is the percentage of taxes that Mr. $h$ has to pay for good $c$, in state $s$. $\tau_h^s = \left(\tau_h^{sc}\right)_{c=1}^C$ $\tau_h = \left(\tau_h^s\right)_{s=1}^S$ ,$\tau = \left(\tau_h\right)_{h=1}^H$

An economy is described by a vector $\omega = (e, e^m, \tau)$ of endowments of goods and money and tax parameters.

**Assumption 12** $\omega \in \Omega = \mathbb{R}^{GH}_{++} \times X^m \times T$ *where*

*i)* $X^m \equiv \left\{ e^m \in \mathbb{R}^{(S+1)H} : \sum\limits_{h=1}^H e_h^{m0} > 0 \text{ and for } s \geq 1,\ \sum\limits_{h=1}^H \left(e_h^{m0} + e_h^{ms}\right) > 0 \right\}$

$$ii)\ T \equiv \left\{ \begin{array}{l} \tau \in [0,1]^{SCH} : a) \forall s \geq 1, \\ \exists h \ and \ \exists c : \tau_h^{sc} > 0 \\ b) \exists h^* \ such \ that \ \forall s \geq 1, \exists c \ : \tau_{h^*}^{sc} \neq 1 \end{array} \right\}$$

**Remark 13** As Villanacci (1993) observes, condition i) implies that in each states of the world there exists a positive amount of money; moreover part a) of ii) means that taxes are a nontrivial function of wealth, while part b) says there exists at least one consumer, who, in every states, does not use all his wealth to pay taxes.

# 3 Competitive Equilibria

Each household maximizes his utility function subject to his budget constraints which depends on his endowments and taxes and on participation constraints in both assets and money market. Note that in every state of period 1 households use money only to pay taxes, no one wants to hold an amount of money greater than the one required to meet his tax obligations.

For $(\omega, \widehat{p}, \widehat{q}, \widehat{q}^m) \in \Omega \times \mathbb{R}_{++}^G \times \hat{Q}$,we have :

$$\begin{array}{lll} (\mathbf{P1}) & \max\limits_{(x_h, b_h, m_h)} u(x_h) & s.t. \\ & \widehat{p}^0 x_h^0 + \widehat{q}^{0m} m_h^0 + \widehat{q} b_h & \leq \widehat{p}^0 e_h^0 + \widehat{q}^{0m} e_h^{m0} \\ & m_h^0 & \geq 0 \\ (s = 1, ..., S) & \widehat{p}^s x_h^s + \sum\limits_{c=1}^{C} \tau_h^{sc} \widehat{p}^{sc} e_h^{sc} & \leq \sum\limits_{i=1}^{I} \widehat{q}^{sm} y^{si} b_h^i \widehat{q}^{sm} + \widehat{q}^{sm} \left(e_h^{ms} + m_h^0\right) \\ & a_h \left(b_h, m_h^0\right) & \geq 0 \end{array} \tag{2}$$

**Remark 14** We can easily check that the solution $(x_h, b_h, m_h)$ of $(P1)$ does not change if we multiply by a positive number the vector prices $(p, q)$. Then we can normalize prices of goods and money. In order to eliminate technical complication, we normalize prices using the price of good $C$ in state 0, while in the other states we normalize prices using the price of money. From now on we will always refer to the normalized prices. We have:

$$p^0 \equiv \frac{\widehat{p}^0}{p^{01}},\ q^{0m} \equiv \frac{\widehat{q}^{0m}}{p^{01}},\ \ q \equiv \frac{\widehat{q}}{p^{01}} \text{ and } p^s = \frac{\widehat{p}s}{q^{sm}} \text{ for } s > 0$$

Denote

$$Q = \left\{ \exists \left(\hat{q}, \hat{q}^m\right) \in \hat{Q} \text{ such that } q \equiv \tfrac{\hat{q}}{p^{01}},\ q^{0m} \equiv \tfrac{\hat{q}^{0m}}{p^{01}},\ q^{ms} = 1, \text{ for } s = 1..S \right\}$$

Define the demand map of household $h$. Given $\omega \in \Omega$, it associates with every vector prices, a vector of demand of goods, money and assets.

$$\left(x_h, b_h, m_h^0\right) : \mathbb{R}_{++}^G \times Q \rightarrow \mathbb{R}_{++}^G \times \mathbb{R}^I \times \mathbb{R}_+ \tag{3}$$
$$\left(x_h, b_h, m_h^0\right) : (p, q, q^m) \mapsto \arg\max (P1) . \tag{4}$$

It can be proved (see Carosi (1999a)) that the demand function of Mr. $h$ is continuous and it is $C^1$ in open and full measure set of $\mathbb{R}_{++}^G \times Q \times \Omega$.

From now on the maximizing behavior of households will be described by the following first order conditions that can be easily derived from Kuhn-Tucker necessary and sufficient conditions of the maximization problem.

$$(Foc_h) \left( \begin{array}{rr} D_{x_h} u_h (x_h) - \lambda_h \Phi & = 0 \\ -\Phi (x_h - e_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi (\tau_h, p) e_h + R b_h & = 0 \\ \lambda_h R + \mu_h D_{b_h} a_h (b_h, m_h^0) & = 0 \\ (\forall j \in J_h) \qquad \min \left[\mu_h^j, a_h^j (b_h, m_h^0)\right] & = 0 \\ \lambda_h q^m + \mu_h D_m a_h (b_h, m_h^0) + \gamma_h & = 0 \\ \min \left[\gamma_h, m_h^0\right] & = 0 \end{array} \right) \tag{5}$$

where $(\lambda_h, \mu_h, \gamma_h)$ are the Lagrange multipliers

- $\Phi$ is a $\widehat{S} \times G$ matrix where $\widehat{S} = S + 1$,

$$\Phi \equiv \left[ \begin{array}{ccc} p^0 & & \\ & ... & \\ & & p^S \end{array} \right]$$

  with $p^0, p^1, ..., p^S \in \mathbb{R}_{++}^C$,

- $q^m$ is an $\widehat{S}$ vector, $q^m \equiv \left[ \begin{array}{c} -q^{om} \\ \mathbf{1} \end{array} \right]$, and $\mathbf{1} = (1, ..., 1)^T$.

- $\hat{U}$ is an $\widehat{S} \times \widehat{S}$ diagonal matrix,

$$\hat{U} \equiv \left[ \begin{array}{cc} q^{0m} & \\ & -I_{S \times S} \end{array} \right]$$

  where $I_{S \times S}$ is the identity matrix whose dimension is $S$

- $R$ is an $\widehat{S} \times I$ matrix, $R = \left[ \begin{array}{c} -q \\ Y \end{array} \right]$.

- $\Psi(\tau_h, p)$ is an $\widehat{S} \times G$ matrix,

$$\Psi(\tau_h, p) \equiv \begin{bmatrix} 0 & & & \\ & \tau_h^{11} p^{11} ... \tau_h^{1C} p^{1C} & & \\ & & .. & \\ & & & \tau_h^{S1} p^{S1} ... \tau_h^{SC} p^{SC} \end{bmatrix}$$

We are ready to define the equilibrium for any given economy $\omega \in \Omega$.

**Definition 15 (Equilibrium)** Given an economy $\omega = (e, e^m, \tau) \in \Omega$, the vector $(p, q, q^m)$ is an equilibrium prices system if and only if there exists $(x_h, b_{h,} m_h^0)_{h=1}^H$ such that

1) $(x_h, b_{h,} m_h^0)$ solves consumer's maximization problem (for every $h$ );

2) $\sum_{h=1}^{H} (x_h^{sc} - e_h^{sc}) = 0$ for every $s, c$ i.e. markets for goods clear;

3) $\sum_{h=1}^{H} b_h^i = 0$ for every $i$ i.e. markets for assets clear;

4) money markets clear i.e.

$$\begin{cases} \sum_{h=1}^{H} (m_h^0 - e_h^{m0}) = 0 \\ \sum_{h=1}^{H} \left( -m_h^0 - e_h^{ms} + \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc} \right) = 0 \quad (s = 1..S). \end{cases} \tag{6}$$

From now on we use the following notations :

$$x^{\setminus} \equiv \left(x_h^{\setminus}\right)_{h=1}^{H} \equiv \left(\left(x_h^{s\setminus}\right)_{s=0}^{S}\right)_{h=1}^{H} \equiv \left(\left((x_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$

$$e^{\setminus} \equiv \left(e_h^{\setminus}\right)_{h=1}^{H} \equiv \left(\left(e_h^{s\setminus}\right)_{s=0}^{S}\right)_{h=1}^{H} \equiv \left(\left((e_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$

We define the function $F : \Xi \times \Omega \to \mathbb{R}^{\dim \Xi}$ with
$\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{H\widehat{S}} \times \mathbb{R}^{HI} \times \mathbb{R}^{\sum \# J_h} \times \mathbb{R}^{H} \times \mathbb{R}^{H} \times \mathbb{R}_{++}^{G-1} \times Q$

$$F:(\xi,\omega)\mapsto$$

$$\left(\begin{array}{lr} \text{Left Hand Side of equations (5)} & \\ (M1) & \sum_{h=1}^{H}\left(x_h^{\backslash}-e_h^{\backslash}\right) \\ (M2) & \sum_{h=1}^{H} b_h \\ (M3) & \sum_{h=1}^{H}\left(m_h^0-e_h^{m0}\right) \\ (M4) & \sum_{h=1}^{H}\left(-m_h^0-e_h^{ms}+\sum_{c=1}^{C}\tau_h^{sc}p^{sc}e_h^{sc}\right)(s>0) \end{array}\right) \tag{7}$$

with $\xi\equiv\left(\left(x_h,\lambda_h,b_h,\mu_h,m_h^0,\gamma_h\right)_{h=1}^{H},p^{\backslash 01},q,q^{0m}\right)$.

**Definition 16** The set of equilibria associated with the economy $\overline{\omega}=(\overline{e},\overline{e^m},\overline{\tau})\in\Omega$, is given by $EQ_{\overline{\omega}}\equiv F_{\overline{\omega}}^{-1}(0)$ where $F_{\overline{\omega}}$ is the restriction of the function $F$ to $\overline{\omega}$, i.e. $F_{\overline{\omega}}:\xi\mapsto F(\xi,\overline{\omega})$

We now recall the basic properties of the model which are going to used in the next section. First of all the existence result can be proven by means of a Degree Argument (Carosi 1999a).

**Theorem 17 (Existence )** *For every economy $\omega\in\Omega$, $EQ_{(\omega)}\neq\emptyset$.*

Using a similar approach given by Cass, Siconolfi and Villanacci (1991) we get the generic uniqueness and regularity of equilibria, that is:

**Theorem 18 (Regularity)** *There exists an open and full measure set $\widetilde{\Omega}$ such that*
*i) $\sharp F^{-1}(0)$ is finite*
*ii) for every $(\xi,\omega)\in F^{-1}(0)$, there exists an open set $U\subseteq\Xi\times\Omega$ of $(\xi,\omega)$, such that the restriction of the function $\pi$ on $U\cap(F^{-1}(0))$ is a diffeomorphism*

Finally as a trivial variation of inefficiency result for incomplete markets (see for example Citanna Kajii and Villanacci (1998)) we obtain the following:

**Theorem 19 (Inefficiency)** *There exists a full measure and open subset $\Omega^{IN}\subseteq\Omega$, such that the equilibrium allocations associated with every $\omega\in\Omega^{IN}$ are not Pareto Optima.*

The interested reader can find complete proofs of the previous theorems in Carosi (1999b).

## 4 Monetary policy and Pareto Improvability.

We suppose that the policy maker can modify only the amount of money endowments of one consumer, say consumer $H$, in period 1. Let $T_H$ be the set of independent instruments of monetary policy whose generic element is $t = (t_H^s)_{s=1}^S$. The planner does not have to respect any constraint and so the space of policy instruments collapses with the space of independent instruments.

We perform a quadratic perturbation the function $u_h$. So we construct the finite dimensional subset $\mathcal{A}_u \equiv \prod \mathcal{A}_{u_h}$ of $\mathcal{U}$. With this regard, the reader can see Citanna, Kajii and Villanacci (1998).

The monetary intervention modifies the maximization problem of household $H$, that is, he solves his optimization problem if and only if the following $Foc_H$ are satisified

$$Foc_H \left( \begin{array}{rr} & D_{x_H} u_H(x_1) - \lambda_H \Phi = 0 \\ & -\Phi(x_H - e_H) + q^m m_H^0 + \hat{U}\left(e_H^m + (0, t_H)^T\right) - \Psi(\tau_H, p)\, e_H + R b_H = 0 \\ & \lambda_H R + \mu_H D_{b_H} a_H(b_H, m_H^0) = 0 \\ (\forall j \in J_H) & \min\left[\mu_H^j, a_H^j(b_H, m_H^0)\right] = 0 \\ & \lambda_H q^m + \mu_H D_m a_H(b_H, m_H^0) + \gamma_H = 0 \\ & \min\left[\gamma_H, m_H^0\right] = 0 \end{array} \right) \tag{8}$$

Taking into account that planner's intervention also modifies market clearing conditions, a new equilibrium is defined as follows.

**Definition 20** Given an economy $(\omega, u) \in \Omega \times \mathcal{U}$ $\xi$ is a vector of equilibrium endogenous variables with respect to "an economy with planner intervention $t$" if and only if $F_{pl}(\xi, t, \omega, u) = 0$ where $F_{pl}$ is defined as follows:

$$F_{pl} : \Xi \times T \times \Omega \times \mathcal{U} \to \mathbb{R}^n \qquad F_{pl} : (\xi, t, \omega, u) \mapsto$$

$$\left( \begin{array}{lr} (Foc_h)_{h\neq H} = (\text{left hand side of equations 5})_{h\neq H} & \\ (Foc_H) = (\text{left hand side of equations 8}) & \\ (M1) & \sum_{h=1}^{H}\left(x_h^{\setminus} - e_h^{\setminus}\right) \\ (M2) & \sum_{h=1}^{H} b_h \\ (M3) & \sum_{h=1}^{H}\left(m_h^0 - e_h^{m0}\right) \\ (M4)\,(s>0) & \sum_{h=1}^{H}\left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C}\tau_h^{sc} p^{sc} e_h^{sc} + t_1^s\right) \end{array} \right)$$

Note that $F_{eq}(\xi,\omega,u) = 0 \Leftrightarrow F^{pl}(\xi,0,\omega,u) = 0$. That means $F_{pl}$ is a variation of $F$.

From the regularity result we have that there exists an open and dense set $\mathcal{E}_r \subseteq \Omega \times \mathcal{U}$ that verifies the following properties :

for every $(\omega,u) \in \mathcal{E}_r$, $F_{eq}(\xi,\omega,u) = 0 \Rightarrow rank D_\xi \tilde{F}(\xi,\omega,u) = n.$

Finally the properness of the projection function $\pi^{eq} : \Xi \times \Omega \times \mathcal{U} \to \Omega \times \mathcal{U}$, $\pi^{eq} : (\xi,\omega,u) \mapsto (\omega,u)$ can be easily checked.

We recall a very well known result about Pareto Improvability. The reader can find the proof in Citanna, Kajii and Villanacci (1998) and Cass (1995).

**Proposition 21** *There exists an open and dense set $\mathcal{E}_I$ in $\Omega \times \mathcal{U}$ such that for any $(\omega,u) \in \mathcal{E}_I$, every associated equilibrium $\xi$ is Pareto Improvable if one of the following conditions holds:*

*i) The following system has no solution*

$$\begin{cases} F(\xi,\omega,u) & = 0 & (1) \\ \left[D_{\xi,t}\left(F_{pl}(\xi,0,\omega,u), U(\xi,0,\omega,u)\right)\right]^T \kappa & = 0 & (2) \\ \frac{1}{2}\kappa^T\kappa - 1 & = 0 & (3) \end{cases} \tag{9}$$

*ii) There exists a subset $D^*$ which is dense in $\Omega \times \mathcal{U}$ and such that for every $(\omega,t) \in D^*$, the matrix $D\widetilde{F}_{\mathcal{A}_u}$*

$$\begin{array}{l} \\ F\left(\xi,\rho,(u(.;A_h))_{h=1}^H\right) \\ \left[D_{\xi,t}(F_{pl},U)\right]\kappa \\ 1\backslash 2\kappa^T\kappa - 1 \end{array} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\xi} & \multicolumn{1}{c}{\kappa} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{\omega} \\ \hline D_\xi F & 0 & 0 & * \\ \hline * & \left[D_{\xi,t}(F_{pl},U)\right]^T & N(\kappa_x) & * \\ \hline 0 & \kappa & 0 & 0 \\ \hline \end{array} \tag{10}$$

*has full row rank.*

*iii) There exists a subset $D^*$ which is dense in $\Omega \times \mathcal{U}$ and such that for every $(\omega,t) \in D^*$, the matrix*

$$M\left(\xi,\omega,(u(.;A_h))_{h=1}^H\right) \equiv \begin{bmatrix} \left[D_{\xi,t}(F_{pl},U)\right]^T & N(\kappa_x) \\ \kappa & 0 \end{bmatrix} \tag{11}$$

*has full rank*

The following result allows us to state that there exists an open and dense set of economies such that equilibria are Pareto Improvable. That has a remarkable consequence: even a limited monetary policy has real effects, i.e. money is not neutral.

**Theorem 22 (Pareto Improvability)** *Suppose that the policy maker can modify only the amount of money endowments of one consumer, in period*

1. *Then there exists an open and dense set $\mathcal{E}_I$ in $\Omega \times \mathcal{U}$ such that for any $(\omega, u) \in \mathcal{E}_I$, every associated equilibrium $\xi$ is Pareto Improvable.*

**Proof.** We are going to show one of the conditions of the previous theorem is verified. The proof is quite long and not so easy to read. Hence we split it in several different parts. We describe the strategy of the proof and we give details in the following lemmas. Note that we are dealing even with endogenous variables and that enforced us to distinguish several cases. The submatrix $N\kappa_x$ depends on $\kappa_{x_h}$ $\forall h$ (see also Citanna, Kajii and Villanacci (1998)) and according to this, we consider the following:

i) CASE 1. $\kappa_{x_h} \neq 0$ for every $h$, that is $N\kappa_{x_h}$ has full row rank for every $h$. In this case, we have the desired result by showing matrix $D\widetilde{F}_{\mathcal{A}_u}$ (see 10) has full rank. This is proved in Lemma 23.

ii) CASE 2. $\kappa_{x_h} = 0$ for every $h$, that is $N\kappa_{x_h}$ does not have full row rank for every $h$. In Lemma 24 we show that system 9 has no solution and then we get the desired result

iii)CASE 3. There exists at least an $h$ such that $\kappa_{x_h} = 0$, that is $N\kappa_{x_h}$ does not have full rank. If there exists at least a consumer such that $N\kappa_{x_h}$ has full rank and $\kappa_{U_h} \neq 0$ we can follow the same procedure we have seen for CASE 1.

Otherwise the result can be obtained by combining the two strategies we have already presented in CASE 1 and CASE 2. More precisely, as in Lemma 24 we eliminate redundant equations and we consider a reduced system where the number of equations is greater than the number of variables. Then as in Lemma 23, by a Transversality argument on the reduced system we get the desired result. We omit the details of the proof since they are similar to the ones presented in previous cases. The interested reader can see a complete proof in Carosi (1999b). ■

**Lemma 23** *If $\kappa_{x_h} \neq 0$ for every $h$, then matrix 10 has full row rank*

**Proof.** We consider the submatrix

| | $\xi$ | $\kappa$ | $A$ | $\tau, e^{m1}$ |
|---|---|---|---|---|
| $F\left(\xi, \rho, (u(.;A_h))_{h=1}^H\right)$ | $*$ | $0$ | $0$ | $*$ |
| $[D_{\xi,t}(F_{pl}, U)]\kappa$ | $*$ | $[D_{\xi,t}(F_{pl}, U)]^T$ | $N(\kappa_x)$ | $*$ |
| $1\backslash 2\kappa^T\kappa - 1$ | $0$ | $\kappa$ | $0$ | $0$ |

(12)

We know that there exists at least a $\kappa_{U_h} \neq 0$, without any loss of generality we assume $\kappa_{U_H} \neq 0$ (this will allow us to find an easy perturbation of the last row). It is easy to check that if the matrix (12) has full row rank, then the matrix (10) has full row rank. We write the matrix (12) extensively:

| | $n$... | | $G^{\backslash}$ | $I$ | 1 | $S$.. | .1 | $G'$.. | .$\widehat{G}$.. | .$S$.. | .$S$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\xi$... | $\kappa_{x_H}$...$\kappa_{\gamma_H}$ | $\kappa_{p^{\backslash}}$ | $\kappa_q$ | $\kappa_{q^{m0}}$ | $\kappa_{p'}$.. | .$\kappa_{U_H}$ | $A_H$.. | .$\widehat{\tau}_H$. | .$\tau_H^{\bullet 1}$. | .$e_H^{\mathbf{1}m}$ |
| $(1_1^F)$ | $*$ | | | | | | | | | | |
| ..... | | | | | | | | | | | |
| $(1_H^F)$ | $*$ | | | | | | | $0^*$ | | | |
| $(2_H^F)$ | $*$ | | | | | | | | $\widehat{\Pi}_H$ | $\Pi_H^{\bullet 1}$ | 0<br>$I$ |
| ...... | | | | | | | | | | | |
| $(10^F)$ | $*$ | | | | | | | | | | |
| $(11^F)$ | $*$ | | | | | | | | $\widetilde{\Pi}_H$ | $\widetilde{\Pi}_H^{\bullet 1}$ | -$I$ |
| $(1_H)$ | $*$ | $*$ | $[0I]^T$ | | | | $D_H^T$ | $N_{(\bullet)}$ | | | |
| $(2_H)$ | $*$ | $*$ | | | | | | | | | |
| $(3_H)$ | $*$ | $*$ | | $I$ | | | | | | | |
| $(4_H)$ | $*$ | $*$ | | | 1 | -$1^T$ | | | | | |
| $(5_H)$ | $*$ | $*$ | | | | | | | | | |
| $(6_H)$ | $*$ | $*$ | | | | | | | | | |
| $(7_H)$ | $*$ | $*$ | | | | | | | | | |
| | | | | | | | | | | | |
| (8) | $*$ | $\left[\Lambda_H^{\backslash}\right]^T \widetilde{Z}_H^{\backslash}$ | | | | $\mathbb{T}^{\backslash}$ | | | $\widehat{O}_H$ | | |
| (9) | $*$ | $[b_H, \mathbf{0}]$-$\lambda_H^0 I_I$ | | | | | | | | | |
| (10) | $*$ | $[z_H^{m0}, \mathbf{0}]$-$\lambda_H^0$ | | | | | | | | | |
| (11) | $*$ | $[\Lambda_H^1]^T \widetilde{Z}_H^1$ | | | | $\mathbb{T}^1$ | | | | $O_H^{\bullet 1}$ | |
| (12) | | 0 $I$ | | | | $-I$ | | | | | |
| (13) | | $\kappa_{x_H}$...$\kappa_{\lambda_H}$ | $\kappa_{p^{\backslash}}$ | $\kappa_q$ | $\kappa_{q^{0m}}$ | $\kappa_{p'}$ | $\kappa_{U_h}$ | | | | |

(13)

$G' = \frac{G(G+1)}{2}$, $D_H^T = (D_x u_H)^T$, $N_{(\bullet)} = N(\kappa_{x_H})$, $\Pi_h^{\bullet 1}$ and $\widetilde{\Pi}_h^{\bullet 1}$ are respectively

| | $\tau_h^{11}$ | ... | $\tau_h^{S1}$ |
|---|---|---|---|
| 0 | 0 | | 0 |
| 1 | $-p^{11}e_h^{11}$ | ... | |
| ... | | $\ddots$ | |
| $S$ | | | $-p^{S1}e_h^{S1}$ |

and

| | $\tau_h^{11}$ | ... | $\tau_h^{S1}$ |
|---|---|---|---|
| 1 | $p^{11}e_h^{11}$ | ... | |
| ... | | $\ddots$ | |
| $S$ | | | $p^{S1}e_h^{S1}$ |

$p' = (p^{s1})_{s=1}^{S}$, $\left(\Lambda_h^{\backslash}\right)^T$ and $(\Lambda_h^1)^T$ are a $G^{\backslash} \times G$ and $S \times G$ matrix respectively such that

$$\left(\Lambda_h^{\backslash}\right)^T = \begin{array}{|c|c|c|c|c|}\hline 0 & \lambda_h^0 I_{C-1} & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \lambda_h^S I_{C-1} \\ \hline \end{array}, \quad \left(\Lambda_h^1\right)^T = \begin{array}{|c|c|c|c|}\hline 0 & & & \\ \hline & \ddots & & \\ \hline & & \lambda_h^S & \\ \hline & & & 0 \\ \hline \end{array}$$

$\widetilde{Z}_h^{\backslash}$ is a $G^{\backslash} \times S+1$ matrix and $\widetilde{Z}_h^1$ is a $S \times S+1$ matrix such that

$$\widetilde{Z}_h^{\backslash} = \begin{array}{|c|c|c|}\hline \widetilde{z}_h^{02} & & \\ \hline & & \vdots \\ \hline & & \widetilde{z}_h^{SC} \\ \hline \end{array} \text{ and } \widetilde{Z}_h^1 = \begin{array}{|c|c|c|c|}\hline 0 & & & \\ \hline \vdots & \widetilde{z}_h^{11} & & \\ \hline & & \ddots & \\ \hline & & & \widetilde{z}_h^{S1} \\ \hline \end{array}$$

where $\widetilde{z}_h^{sc} = -x_h^{sc} + \left(1 - \tau_h^{sc}\right) e_h^{sc}$ for $s > 0$ and $\widetilde{z}_h^{0c} = -x_h^{sc} + e_h^{sc}$
$\mathbb{T}^{\backslash}$ is a $G^{\backslash} \times S$ matrix such that

| | 1 | .. | $S$ |
|---|---|---|---|
| $p^{02}$ | 0 | | 0 |
| ... | | | |
| $p^{0C}$ | 0 | | 0 |
| $p^{12}$ | $\sum_{h \in H} \tau_h^{12} e_h^{12}$ | | |
| ... | | $\ddots$ | |
| $p^{SC}$ | | | $\sum_{h \in H} \tau_h^{SC} e_h^{SC}$ |

$\mathbb{T}^1$ is a $S \times S$ matrix such that

| | | | |
|---|---|---|---|
| $p^{11}$ | $\sum_{h \in H} \tau_h^{11} e_h^{11}$ | | |
| ... | | | |
| $p^{S1}$ | | | $\sum_{h \in H} \tau_h^{S1} e_h^{S1}$ |

$\widehat{O}_h$ is a $\widehat{G} \times \widehat{G}$ matrix such that

| | $\tau_h^{12}$ | ... | $\tau_h^{SC}$ |
|---|---|---|---|
| $\left(p^{0c}\right)_{c=2}^{C}$ | 0 | | 0 |
| $p^{12}$ | $\left(\kappa_{p'}^1 - \kappa_{\lambda_h}^1\right) e_h^{12}$ | | |
| $\vdots$ | | $\ddots$ | |
| $p^{SC}$ | | | $\left(\kappa_{p'}^S - \kappa_{\lambda_h}^S\right) e_h^{SC}$ |

$O_h^{\bullet 1}$ is a $S \times S$ matrix such that

| | $\tau_h^{11}$ | ... | $\tau_h^{S1}$ |
|---|---|---|---|
| $p^{11}$ | $\left(\kappa_{p'}^1 - \kappa_{\lambda_h}^1\right) e_h^{11}$ | | |
| .... | | $\ddots$ | |
| $p^{S1}$ | | | $\left(\kappa_{p'}^S - \kappa_{\lambda_h}^S\right) e_h^{S1}$ |

Observe that the submatrix obtained by erasing rows (12),(13) has full row rank in a dense and open set of economies. That follows from regularity result. Then we are left to perturb the last two superrows.

**Perturbation of row** (12). We consider two different cases

**CASE A: there exists at least a consumer** $h$ **such that** $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) \neq 0$ for every $s$.

Previous condition implies that the block matrices $\left[\widehat{O}_h\right]_{h\in H}$ and $[O_h^{\bullet 1}]_{h\in H}$ has full rank and that allows us to use these submatrix in order to perturb superrows (8) and (11) respectively. Without any loss of generality we can assume that $\left(\kappa_{p'}^s - \kappa_{\lambda_H}^s\right) \neq 0$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (12) | $\leftarrow$ | $(\Delta\kappa_{p'})$ | $\rightsquigarrow$ | (11) | $\leftarrow$ | $(\Delta\tau_H^{\bullet 1})$ | $\rightsquigarrow$ | $\binom{2_H^F}{11^F}$ | |
| | | | $\rightsquigarrow$ | (8) | $\leftarrow$ | $(\Delta\widehat{\tau}_H)$ | $\rightsquigarrow$ | $\binom{2_H^F}{11^F}$ | |
| | | | $\rightsquigarrow$ | $(4_h)_{h=1}^H$ | $\leftarrow$ | $(\Delta\kappa_{q^{0m}})$ | | | |
| $\binom{2_H^F}{11^F}$ | $\leftarrow$ | $(\Delta e_H^{m1})$ | | | | | | | |

**CASE B: There exists at least an** $s$ $(s>0)$ **such that** $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) = 0$ **for every** $h$ **.** We come back to system (9). If $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) = 0$ for every $h$ then we substitute rows $(8^s)$, $(11^s)$ and $(12^s)$ with following:

| | |
|---|---|
| | $\sum_{h=1}^{H}\left(\lambda_h^s\kappa_{x_h}^s + \left[-x_h^{s1} + \left(1-\tau_h^{s1}\right)e_h^{s1}\right]\kappa_{\lambda_h}^s + \tau_h^{s1}e_h^{s1}\kappa_{p'}^s\right) =$ |
| | $= \sum_{h=1}^{H}\left(\lambda_h^s\kappa_{x_h}^{s1} + \left(e_h^{s1} - x_h^{s1}\right)\kappa_{\lambda_h}^s - \tau_h^{s1}e_h^{s1}\left(\kappa_{\lambda_h}^s - \kappa_{p'}^s\right)\right) =$ |
| $(11^s)$ | $= \sum_{h=1}^{H}\left(\lambda_h^s\kappa_{x_h}^{s1} + \left(e_h^{s1} - x_h^{s1}\right)\kappa_{x_h}^s\right) = 0$ |
| $(8^s)$ | $\sum_{h=1}^{H}\left(\lambda_h^s\kappa_{x_h}^{sc} + \left(e_h^{sc} - x_h^{sc}\right)\kappa_{\lambda_h}^s\right) = 0$ (for every $c>1$) |
| $(12^s)$ | $\kappa_{\lambda_h}^s - \kappa_{p'}^s = 0$ |

We study the rank of the matrix which corresponds to the new system. Note that the columns of $\mathbb{T}^1$ and $\mathbb{T}^{\setminus}$ corresponding to the state $s$ are zero. That means the perturbation of row $(12^s)$ does not alter rows (8),(11).

**Perturbation of row** (13)

| | | | | | | |
|---|---|---|---|---|---|---|
| (13) | $\leftarrow$ | $\Delta\kappa_{U_H}$ | $\rightsquigarrow$ | $(1_H)$ | $\leftarrow$ | $\Delta A_H$ |

. ■

**Lemma 24** *If CASE 2 occurs, then System (9) has no solution.*

**Proof.** System (9) can be written as follows

$$
\begin{array}{lll}
(1_h) & (D_h^2)^T \kappa_{x_h} - \Phi \kappa_{\lambda_h} + [0I]^T \kappa_{p\backslash} + \kappa_{U_h} D_x U_h & = 0 \\
(2_h) & -\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} & = 0 \\
(3_h) & R^T \kappa_{\lambda_h} + (\beta_{b_h})^T \kappa_{b_h} + \left[\beta_{b_h m_h^0}\right]^T \kappa_{m_h^0} + \left[D_{b_h} a_h^{\mathbf{1}\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & \begin{array}{l} (q^m)^T \kappa_{\lambda_h} + \beta_{b_h m_h^0} \kappa_{b_h} + \beta_{m_h^0} \kappa_{m_h^0} + \left[D_{m_h^0} a_h^{\mathbf{1}\backslash}\right]^T \zeta_h^1 + \\ + \chi_{h\left[m_h^0=0\right]} \kappa_{\gamma_h} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} \end{array} & = 0 \\
(5_h) & \left[D_{b_h} a_h^{\mathbf{1}\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{\mathbf{1}\backslash}\right] \kappa_{m_h^0} & = 0 \\
(6_h) & \left[D_{b_h} a_h^{\mathbf{3}\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{\mathbf{3}\backslash}\right] \kappa_{m_h^0} + I\kappa_{\zeta_h^3} & = 0 \\
(7_h) & \kappa_{\kappa_{m_h^0}} + \chi_{h\left[\gamma_h^0=0\right]} \kappa_{\gamma_h} & = 0 \\
(8) & \sum\limits_{h=1}^{H} \left( \left[\Lambda_h^{\backslash}\right]^T \kappa_{x_h} + \widetilde{Z}_h^{\backslash} \kappa_{\lambda_h} \right) + \mathbb{T}^{\backslash} \kappa_{p'} & = 0 \\
(9) & \sum\limits_{h=1}^{H} \left( \left( \begin{array}{cc} b_h & \mathbf{0} \end{array} \right) \kappa_{\lambda_h} - \lambda_0 I_I \kappa_{b_h} \right) & = 0 \\
(10) & \sum\limits_{h=1}^{H} \left( \left( \begin{array}{cc} z_h^{m0} & \mathbf{0} \end{array} \right) \kappa_{\lambda_h} - \lambda_0 \kappa_{m_h^0} \right) & = 0 \\
(11) & \sum\limits_{h=1}^{H} \left( [\Lambda_h^1]^T \kappa_{x_h} + \widetilde{Z}_h^1 \kappa_{\lambda_h} \right) + \mathbb{T}^1 \kappa_{p'} & = 0 \\
(12) & \left( \begin{array}{cc} 0 & I \end{array} \right) \kappa_{\lambda_H} + I\kappa_{p'} & = 0 \\
(13) & \begin{array}{l} \sum\limits_{h=1}^{H} \kappa_{x_h}^T \kappa_{x_h} + \ldots + \sum\limits_{h=1}^{H} \kappa_{\gamma_h}^T \kappa_{\gamma_h} + \kappa_{p\backslash}^T \kappa_{p\backslash} + \kappa_q^T \kappa_q + \\ + \kappa_{q^{0m}}^T \kappa_{q^{0m}} + \kappa_{p'}^T \kappa_{p'} + \kappa_U^T \kappa_U + 1 \end{array} & = 0
\end{array}
\qquad (14)
$$

**Step 1.** $\kappa_{\lambda_h}^s = \kappa_{U_h} \lambda_h^s$ for every $s$. $\kappa_{p\backslash} = 0$.

Since $\kappa_{x_h} = 0$ for every $h$, from $(1_h)$ we have $-p^{s1} \kappa_{\lambda_h}^s + \kappa_{U_h} D_{x_h^{s1}} = 0$.

From First Order Condition we have $-\lambda_h^s p^{s1} + D_{x_h^{s1}} = 0$ and so $p^{s1} = \frac{D_{x_h^{s1}}}{\lambda_h^s}$, hence $\frac{D_{x_h^{s1}}}{\lambda_h^s} \kappa_{\lambda_h}^s = \kappa_{U_h} D_{x_h^{s1}}$. We get $\kappa_{\lambda_h}^s = \kappa_{U_h} \lambda_h^s$.

Since $\kappa_{\lambda_h}^s = \kappa_{U_h} \lambda_h^s$, from $(1_h)$ we have $\kappa_{p\backslash} = 0$.

**Step 2.** $\kappa_{b_h} = 0$ . $\kappa_{m_h^0} = 0$.

Taking into account $\kappa_{x_h} = 0$ and the rank condition on $(R, q^m)$ from $(2_h)$ we get $\kappa_{b_h} = 0$ . $\kappa_{m_h^0} = 0$..

**Step 3.** $\kappa_{p'}^s = -\lambda_H^s \kappa_{U_H}$ .

From (12) we have $\kappa_{\lambda_H}^s - \kappa_{p'}^s = 0$ for every $s > 0$. Hence from step 1 we get $\kappa_{p'}^s = -\lambda_H^s \kappa_{U_H}$.

**Step 4.** $\kappa_{\zeta_h^3} = 0$, $\kappa_{\gamma_h} = 0$.

It follows respectively from $(6_h)$ and $(7_h)$ and from step 2.

Due to step 1-4, in order to get the desired result we can study the following reduced system:

$$\begin{array}{lll} (3_h) & R^T\left(\kappa_{U_h}\lambda_h\right)+\left[D_{b_h}a_h^{\mathbf{1}\backslash}\right]^T\kappa_{\zeta_h^1}+I\kappa_q & =0\\ (11) & \sum\limits_{h=1}^{H}\widetilde{Z}_h^1\left(\lambda_h\kappa_{U_h}\right)+\mathbb{T}^1\left(\kappa_{U_H}\lambda_H\right) & =0 \end{array} \tag{15}$$

If we can prove that (15) has only the solution $\left(\left(\kappa_{\zeta_h^1},\kappa_{U_h}\right)_{h=1}^{H},\kappa_q\right)=0$, from equation $(4_h)$ of system (14) it follows $\kappa_{q^{0m}}=0$ and then system (14) has no solution. Using a Transversality argument, we can claim there exists an open and dense set of economies such that the matrix

$\left[\left[\begin{array}{cc}\left(\widetilde{Z}_h^1\lambda_h\right)_{h\neq H} & \widetilde{Z}_H^1\lambda_H\end{array}\right]-\left[\begin{array}{cc}0 & \mathbb{T}^1\end{array}\right]\right]$ has full rank. Then (11) implies $\kappa_{U_h}=0$ for every $h$ in a dense and open set of economies. Consequently we deal with the following system:

$$\begin{array}{llll} (3_1) & I & \left[D_{b_1}a_1^{\mathbf{1}\backslash}\right]^T\kappa_{\zeta_1^1}+\kappa_q I & =0\\ & & \ldots\ldots & \\ (3_H) & I & \left[D_{b_H}a_H^{\mathbf{1}\backslash}\right]^T\kappa_{\zeta_H^1}+I\kappa_q & =0 \end{array}$$

Hence we are left to study the rank of the following matrix

$$\begin{array}{cc|c|c|c|c|} & & \sharp J_1^{1\backslash} & & \sharp J_h^{1\backslash} & I\\ & & \kappa_{\zeta_1^1} & \ldots & \kappa_{\zeta_H^1} & \kappa_q\\ \hline I & (3_1) & \left[D_{b_1}a_1^{\mathbf{1}\backslash}\right]^T & & & I\\ \hline & \ldots & & \ddots & & \\ \hline I & (3_H) & & & \left[D_{b_H}a_H^{\mathbf{1}\backslash}\right]^T & I\\ \hline \end{array} \tag{16}$$

By Assumption on Participation Constraints we know that for every asset $i$, there exists at least one consumer $h'$ such that $D_{b_{h'}^i}a_{h'}\left(b_{h'},m_{h'}^0\right)=0$; consequently, at least $I$ rows of the following submatrix are zero. By using these rows we perform some elementary row operations to obtain a submatrix (which can be the matrix itself) of matrix (16) such that:

| | $\sharp J_1^{1\backslash}$ | | $\sharp J_H^{1\backslash}$ | $I$ |
|---|---|---|---|---|
| | $\kappa_{\zeta_1^1}$ | | $\kappa_{\zeta_H^1}$ | $\kappa_q$ |
| | $\left[D_{b_1} a_1^{\mathbf{1}\backslash}\right]^T$ | | | $0$ |
| $\sum_{h=1}^{H} \sharp J_h^{1\backslash}$ | | $\ddots$ | | $\vdots$ |
| | | | $\left[D_{b_H} a_H^{\mathbf{1}\backslash}\right]^T$ | $0$ |
| $I$ | | | | $I$ |

It is easy to show that this matrix has full rank (It follows from Assumption 7.iii)). Hence $\left(\kappa_{\zeta_h^1}\right)_{h=1}^{H} = 0$ and $\kappa_q = 0$. Then system 14 has no solution. ■

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