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MINIMAL CLUSTERS OF FOUR PLANAR REGIONS WITH THE SAME AREA *

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Abstract. We prove that the optimal way to enclose and separate four planar regions with equal area using the less possible perimeter requires all regions to be connected. Moreover, the topology of such optimal clusters is uniquely determined.

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1. Introduction

We consider the problem of enclosing and separating N regions of $\mathbb{R}^2$ with prescribed area and with the minimal possible interface length.

The case $N = 1$ corresponds to the celebrated isoperimetric problem whose solution, the circle, was known since antiquity.

For $N \geq 1$ first existence and partial regularity in $\mathbb{R}^n$ was given by Almgren [1] while Taylor [21] describes the singularities for minimizers in $\mathbb{R}^3$. Existence and regularity of minimizers in $\mathbb{R}^2$ was proved by Morgan [14] (see also [12]): the regions of a minimizer in $\mathbb{R}^2$ are delimited by a finite number of circular arcs which meet in triples at their end-points (see Thm. 2.3).

Foisy *et al.* [8] proved that for $N = 2$ in $\mathbb{R}^2$ the two regions of any minimizer are delimited by three circular arcs joining in two points (standard double bubble) and are uniquely determined by their enclosed areas. Wichiramala [23] proved that for $N = 3$ in $\mathbb{R}^2$ the three regions of any minimizer are delimited by six circular arcs joining in four points. Such configuration (standard triple bubble) is uniquely determined by the given enclosed areas, as shown by Montesinos [13]. The case $N = 4$ has been considered in [11] where some partial information on minimal clusters is obtained.

The minimization problem can be stated also for $N = \infty$ regions with equal areas (the *honeycomb conjecture*, see [15]): Hales [9] proved that the hexagonal grid is indeed the solution.

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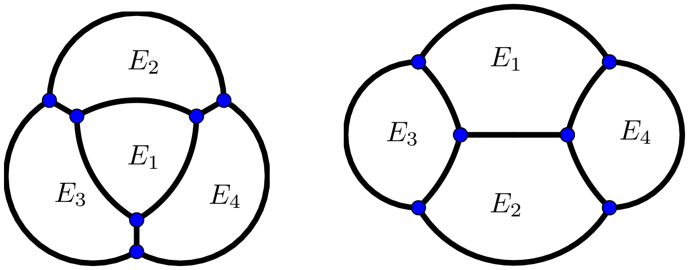


FIGURE 1. The *flower* (*left hand side*) and *sandwich* (*right hand side*) topologies.

In obtaining the results with $N = 2$ or $N = 3$ planar regions, the main difficulty is to prove that each region of the minimizer is connected. In fact, in general, this is an open question (soap bubble conjecture, Conjecture 2.11, see Morgan and Sullivan [16]).

To investigate such a conjecture, in this paper (which originates from the Ph.D. thesis [20] of the second author) we consider the case of $N = 4$ regions in the plane. In Theorem 6.5 we prove that if the four planar regions have equal areas then the conjecture is true: the minimizing clusters must be connected. However, in this case, connectedness and stationarity is not enough to uniquely determine the topology of minimizers. In fact there are two nontrivial possible topologies: we call them the *flower* and the *sandwich* topologies (see Fig. 1). We then exclude the flower topology, to conclude that minimizers have the sandwich type (Thm. 7.3).

We conjecture that the minimizer with equal areas is symmetric *i.e.*: the regions E_1 and E_3 are congruent to the regions E_2 and E_4 respectively (see Conjecture 7.4 for more details).

The problem of dividing the sphere in regions of equal areas has also been considered. See for example [7] where it is proven that the minimizer for four equal areas in the sphere is given by a geodesic tetrahedron.

The plan of the paper is as follows. In Section 2 we set up the notation and collect the known results that we need in the rest of the paper. In Section 3 we present some tools which apply to general planar clusters. In particular notice that Proposition 3.3 gives an estimate by below on the measure of each connected component of a minimal cluster. This estimate can be used to obtain an upper bound on the total number of connected components of a cluster as in Theorem 3.4.

In Section 4 we start the analysis of planar clusters with four equal areas. In particular we find a precise estimate on the length of the minimizers (Prop. 4.1), we prove that possible components of a disconnected region cannot be too small (Prop. 4.2) and cannot be too big (Prop. 4.3). This estimates enable us to prove that a minimizer can have at most six connected components (Prop. 4.9). In Section 5 we exclude the clusters with six components. In Section 6 we exclude the clusters with five components and obtain the connectedness result Theorem 6.5. In Section 7 we consider all connected clusters (four components) and exclude the flower topology (Prop. 7.2, Thm. 7.3).

2. NOTATION AND PRELIMINARY RESULTS

Let us denote with $\boldsymbol{E} = (E_1, \dots, E_N)$ an N-uple of measurable subsets of $\mathbb{R}^2$. We will say that $\boldsymbol{E}$ is an N-*cluster* if $m(E_i \cap E_j) = 0$ for all $i \neq j$ ($m(\cdot)$ is the Lebesgue measure). The *external region* E_0 is defined as

$$E_0 = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} E_i.$$

The sets $E_0, E_1, \dots, E_N$ will be called the *regions* of the cluster $\boldsymbol{E}$.

We define the *measure* and the *perimeter* of a cluster by:

$$\boldsymbol{m}(\boldsymbol{E}) := (m(E_1), \dots, m(E_N)), \qquad P(\boldsymbol{E}) := \frac{1}{2}\sum_{i=0}^{N} P(E_i)$$

where $P(E_i)$ is the *perimeter* of the measurable set E_i. For regular sets E_i one has $P(E_i) = \mathcal{H}^1(\partial E_i)$ which is the length of the boundary of E_i.

Given a measurable set E we say that C with $m(C) > 0$ is a *component* of E if

$$m(E) = m(C) + m(E \setminus C) \quad \text{and} \quad P(E) = P(C) + P(E \setminus C)$$

(*i.e.* the decomposition $E = C \cup (E \setminus C)$ does not add any boundary). We say that E is *connected* if it has no component C with $0 < m(C) < m(E)$ ($C = E$ is a trivial component). Notice that in our definitions a *component* does not need to be connected: in general a component can be a union of connected components. We say that a cluster $\boldsymbol{E}$ is *connected* if each region E_i, for $i = 1, \dots, N$, is connected. We say that a cluster is *disconnected* if it is not connected (*i.e.* at least one region is not connected).

A component C of a region E_i of the cluster $\boldsymbol{E}$ (with $i \neq 0$) is said to be *external* if is adjacent to the external region E_0 (formally $P(C \cup E_0) < P(C) + P(E_0)$) otherwise it is said to be *internal*.

Given a vector of positive numbers $\boldsymbol{a} \in \mathbb{R}^N_+$, $\boldsymbol{a} = (a_1, \dots, a_N)$, $a_i > 0$ we will define the family of *competitors* as the clusters with measure $\boldsymbol{a}$:

$$\mathcal{C}(\boldsymbol{a}) = \{\boldsymbol{E} \colon \boldsymbol{m}(\boldsymbol{E}) = \boldsymbol{a}\}$$

among these we will consider the following optimization problem:

$$p(\boldsymbol{a}) = \inf\{P(\boldsymbol{E}) \colon \boldsymbol{E} \in \mathcal{C}(\boldsymbol{a})\}$$

and the corresponding minimizers:

$$\mathcal{M}(\boldsymbol{a}) = \{\boldsymbol{E} \in \mathcal{C}(\boldsymbol{a}) \colon P(\boldsymbol{E}) = p(\boldsymbol{a})\}.$$

We will also consider the *weak* variants of this minimization problem:

$$\begin{aligned}
\mathcal{C}^*(\boldsymbol{a}) &= \{\boldsymbol{E} \colon \boldsymbol{m}(\boldsymbol{E}) \geq \boldsymbol{a}\} \\
p^*(\boldsymbol{a}) &= \inf\{P(\boldsymbol{E}) \colon \boldsymbol{E} \in \mathcal{C}^*(\boldsymbol{a})\} \\
\mathcal{M}^*(\boldsymbol{a}) &= \{\boldsymbol{E} \in \mathcal{C}^*(\boldsymbol{a}) \colon P(\boldsymbol{E}) = p^*(\boldsymbol{a})\}.
\end{aligned}$$

(the comparison between vectors of $\mathbb{R}^N$ is understood componentwise).

Definition 2.1 (Regular cluster)**.** We say that a planar N-cluster $\boldsymbol{E}$ is *regular* when:

(1) each region (including the external region E_0) is (up to a negligible set) a closed set which is equal to the closure of its interior points (and in the following we will assume that the Lebesgue representant of the regions E_i is always a closed set);
(2) each region, but the external one E_0, is bounded;
(3) the boundary of the cluster, defined by

$$\partial \boldsymbol{E} = \bigcup_{k=1}^{N} \partial E_k$$

is the continuous embedding of a finite planar graph (*i.e.* there are a finite number of simple continuous curves which we will call *edges* which can only meet in their end-points which we will call *vertices* and the *faces* of the graph correspond to the connected components of the regions);

(4) each *vertex* has order at least three (*i.e.* it coincides with at least three end-points of the edges).

Notice that the perimeter of a region E_i of a regular cluster $\boldsymbol{E}$ is the sum of the length of the edges of E_i. Moreover since each edge belongs to the boundary of exactly two regions, we have

$$P(\boldsymbol{E}) = \frac{1}{2}\sum_{k=0}^{N} P(E_k) = \sum_{\sigma \text{ edge of } \boldsymbol{E}} \ell(\sigma) = \mathcal{H}^1(\partial \boldsymbol{E}).$$

Definition 2.2 (Stationary cluster)**.** We say that a regular planar cluster $\boldsymbol{E} = (E_1, \dots, E_N)$ is *stationary* if it satisfies the following conditions:

(1) every edge is either a circular arc or a straight segment (which, in the following, we will identify with an arc of zero curvature);
(2) in every *vertex* exactly three arcs meet, defining three equal angles of 120 degrees;
(3) it is possible to associate a real number p_i (which we will call *pressure*) to each region E_i of the cluster, so that $p_0 = 0$ and such that any arc between the regions E_i and E_j has curvature $|p_i - p_j|$ (it is a straight segment if $p_i = p_j$) and the region with higher pressure is towards the side where the the arc is convex.

In particular it follows that the sum of the signed curvatures of the three arcs meeting in a vertex is always zero.

Theorem 2.3 (Existence and regularity)**.** [12,14] *Given $\boldsymbol{a} \in \mathbb{R}^N_+$ the family of clusters $\mathcal{M}(\boldsymbol{a})$ is not empty and every minimal cluster $\boldsymbol{E} \in \mathcal{M}(\boldsymbol{a})$ is regular and stationary.*

Theorem 2.4 (Existence and regularity, weak case)**.** [14] *Given $\boldsymbol{a} \in \mathbb{R}^N_+$ the family of clusters $\mathcal{M}^*(\boldsymbol{a})$ is not empty and every minimal cluster $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ is regular and stationary.*

Weak minimizers have some additional properties which makes them a better ambient space for our investigation.

Proposition 2.5 (Properties of weak minimizers)**.** [23] *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$, $\boldsymbol{a} \in \mathbb{R}^N_+$. Then*:

(1) *the external region E_0 is connected*;
(2) *all the pressures p_i are nonnegative*;
(3) *if $m(E_i) > a_i$ then $p_i = 0$.*

Theorem 2.6 (Pressure formula)**.** [6] *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ with $\boldsymbol{a} \in \mathbb{R}^N_+$. Then*

$$P(\boldsymbol{E}) = 2\sum_{i=1}^{N} p_i m(E_i).$$

Lemma 2.7 (Turning angle)**.** [23] *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$, $\boldsymbol{a} \in \mathbb{R}^N_+$ and let C be a connected component of some region E_i of $\boldsymbol{E}$. Let n be the number of edges of C and let L_j be the total length of the edges of C in common with the region E_j ($L_j = 0$ if C and E_j have not edges in common). Then, if $i \neq 0$, it holds*

$$\frac{(6-n)\pi}{3} = \sum_{j=0}^{N} (p_i - p_j) L_j$$

where p_j is the pressure of the region E_j. For $i = 0$ we have instead

$$\frac{(6+n)\pi}{3} = \sum_{j=1}^{N} (p_j - p_0) L_j.$$

Proposition 2.8 ([4,5,20])**.** *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ with $\boldsymbol{a} \in \mathbb{R}^N_+$. Let M be the total number of bounded connected components of the regions of $\boldsymbol{E}$.*

(1) *Every bounded connected component is simply connected.*
(2) *Two connected components of $\boldsymbol{E}$ cannot share more than a single edge.*
(3) *If $N > 2$ then each connected component C of $\boldsymbol{E}$ has at least three edges.*
(4) *Each connected component of a region with k connected components has at most $M+1-k$ edges, and if it is internal it has at most $M-k$ edges.*
(5) *The total number of edges is $3(M-1)$ and the total number of vertices is $2(M-1)$.*
(6) *If $M \le 6$ then $\boldsymbol{E} \in \mathcal{M}(\boldsymbol{a})$ (i.e. $\boldsymbol{E}$ is a strong minimizer).*

Theorem 2.9 (Removal of triangle components)**.** [23] *Let $\boldsymbol{E} \in \mathcal{C}(\boldsymbol{a})$ be a stationary regular cluster and suppose that a connected component C of some region E_i has three edges. Consider the three arcs which arrive at the three vertices of C but are not edges of C. The circles containing these three arcs meet in a point P inside the component C.*

Moreover the cluster $\boldsymbol{E}'$ obtained from $\boldsymbol{E}$ by removing the component C and prolonging the three edges, is itself a stationary regular cluster $\boldsymbol{E}' \in \mathcal{C}(\boldsymbol{a}')$ with $a'_i = a_i - m(C)$ (and the region E_i disappears if C was the only component of E_i) and $a'_j \ge a_j$ for all $j \ne i$. Also the pressures p'_j of the regions of $\boldsymbol{E}'$ are equal to the pressure p_j of the corresponding regions of $\boldsymbol{E}$ (if E_i disappears because C was the only component of E_i, the regions must be relabeled but again the pressures of the corresponding regions remain the same).

Theorem 2.10 (Double bubble monotonicity)**.** [8] *Given $r_1 > 0$ and $r_2 > 0$, up to isometries, there exists a unique double bubble $\boldsymbol{E}$ such that the external radii of the two regions E_1 and E_2 are r_1 and r_2 respectively. If we increase one radius (say) r_1 then the area of the corresponding region E_1 increases while the area of the other region E_2 decreases. As a consequence there is a unique double bubble with prescribed areas.*

Conjecture 2.11 (Soap bubble conjecture)**.** [16] *For all $\boldsymbol{a} \in \mathbb{R}^N_+$ each $\boldsymbol{E} \in \mathcal{M}(\boldsymbol{a})$ is connected.*

The main aim of this paper is to prove that the conjecture holds in the case $\boldsymbol{a} = (1,1,1,1)$.

3. Estimates on general clusters

Lemma 3.1 (Isoperimetric inequality for clusters)**.** *Given $\boldsymbol{E} \in \mathcal{C}^*(\boldsymbol{a})$ one has*

$$P(\boldsymbol{E}) \ge \sqrt{\pi}\left(\sqrt{\sum_{k=1}^N a_k} + \sum_{k=1}^N \sqrt{a_k}\right).$$

Proof. Given any $\boldsymbol{E} \in \mathcal{C}^*(\boldsymbol{a})$, by applying the isoperimetric inequality

$$P(E) \ge 2\sqrt{\pi}\sqrt{\min\{m(E), m(\boldsymbol{R}^2 \setminus E)\}}$$

one has:

$$P(\boldsymbol{E}) = \frac{1}{2}\sum_{k=0}^N P(E_k) \ge \sqrt{\pi}\left(\sqrt{\sum_{k=1}^N m(E_k)} + \sum_{k=1}^N \sqrt{m(E_k)}\right). \qquad \square$$

Proposition 3.2 (Variation I)**.** *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ and suppose that C_i is a component of the region E_i. Let ℓ be the sum of the lengths of the edges of C_i in common with the region $E_k \ne E_i$ ($k=0$ is also admitted). Then*

$$\ell \le 2\sqrt{\pi}\sqrt{m(C_i)}.$$

Proof. Let B be any ball disjoint from $\boldsymbol{E}$ with the same area as C_i, so that $P(B) = 2\sqrt{\pi}\sqrt{m(C_i)}$. Consider the cluster $\boldsymbol{E}'$ obtained by $\boldsymbol{E}$ by means of the following variations on the regions E_i and E_j:

$$E'_i = (E_i \setminus C_i) \cup B, \qquad E'_j = E_j \cup C_i.$$

Clearly we have $m(E'_i) = m(E_i)$ and $m(E'_j) > m(E_j)$. Hence $\boldsymbol{E}' \in \mathcal{C}^*(\boldsymbol{a})$. Moreover, since the edges of length ℓ has been removed and the ball B has been added, by the minimality of $\boldsymbol{E}$ we have:

$$0 \le P(\boldsymbol{E}') - P(\boldsymbol{E}) = P(B) - \ell = 2\sqrt{\pi}\sqrt{m(C_i)} - \ell. \qquad \square$$

Proposition 3.3 (Variation II)**.** *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ and suppose that C_i is a component of the region E_i with $0 < m(C_i) < m(E_i)$. Let ℓ_k be the sum of the lengths of the edges of C_i in common with the region $E_k \neq E_i$ ($k = 0$ is also admitted). Then*

$$\ell_k \le \frac{m(C_i)}{|2a_i - m(C_i)|} P(\boldsymbol{E}). \tag{3.1}$$

Moreover, if we denote by $r \le N$ the number of regions which have an edge in common with C_i, for all $\lambda \ge P(\boldsymbol{E})$ one has:

$$m(C_i) \ge \frac{16\pi a_i^2}{r^2\lambda^2}\left(1 - \frac{16\pi a_i}{r^2\lambda^2}\right). \tag{3.2}$$

Proof. Let

$$t = \sqrt{\frac{m(E_i)}{m(E_i) - m(C_i)}} = \sqrt{1 + \frac{m(C_i)}{m(E_i) - m(C_i)}} \le 1 + \frac{1}{2}\frac{m(C_i)}{m(E_i) - m(C_i)}$$

and consider a new cluster $\boldsymbol{E}'$ whose regions are defined by $E'_i = t(E_i \setminus C_i)$, $E'_k = t(E_k \cup C_i)$ and $E'_j = tE_j$ when $j \notin \{i, k\}$. Simply speaking, the cluster $\boldsymbol{E}'$ has been obtained from $\boldsymbol{E}$ by giving C_i to E_k and then rescaling of a factor $t > 1$.

Notice that t was defined so that

$$m(E'_i) = t^2(m(E_i) - m(C_i)) = m(E_i)$$

and clearly every other region does not decrease its measure since $t > 1$. So $\boldsymbol{E}' \in \mathcal{C}^*(\boldsymbol{a})$ is a weak competitor to $\boldsymbol{E}$. On the other hand since in the cluster $\boldsymbol{E}'$ all edges in common between the component tC_i and the region tE_k have been removed (and these edges have a total length of $t\ell_k$) we have

$$P(\boldsymbol{E}') = t(P(\boldsymbol{E}) - \ell_k).$$

Since $P(\boldsymbol{E}) \le P(\boldsymbol{E}')$ one obtains:

$$\begin{aligned} P(\boldsymbol{E}) &\le t(P(\boldsymbol{E}) - \ell_k) \le \left(1 + \frac{m(C_i)}{2(m(E_i) - m(C_i))}\right)(P(\boldsymbol{E}) - \ell_k) \\ &= P(\boldsymbol{E}) + \frac{m(C_i)}{2(m(E_i) - m(C_i))}P(\boldsymbol{E}) - \frac{2m(E_i) - m(C_i)}{2(m(E_i) - m(C_i))}\ell_k \end{aligned}$$

which is equivalent to

$$\ell_k \le \frac{m(C_i)}{2m(E_i) - m(C_i)} P(\boldsymbol{E}).$$

Using $0 \le a_i \le m(E_i)$ and $m(C_i) \le m(E_i)$ one can easily check that

$$2m(E_i) - m(C_i) \ge |2a_i - m(C_i)|$$

so that (3.1) is proven.

Now if the component C_i has edges in common with at least r other regions, there is k such that $\ell_k \geq P(C_i)/r$. By also applying the isoperimetric inequality $P(C_i) \geq 2\sqrt{\pi}\sqrt{m(C_i)}$ we obtain:

$$2\sqrt{\pi}\sqrt{m(C_i)} \leq r\ell_k \leq \frac{rm(C_i)}{|2a_i - m(C_i)|}P(\boldsymbol{E}) \leq \frac{r\lambda m(C_i)}{|2a_i - m(C_i)|}$$

if $P(E) \leq \lambda$ as in the statement of the Theorem being proved. Whence, by squaring and then dividing by $m(C_i)$, we obtain

$$4\pi \leq \frac{r^2\lambda^2 m(C_i)}{(2a_i - m(C_i))^2} = \frac{r^2\lambda^2 m(C_i)}{4a_i^2 - 4a_i m(C_i) + m^2(C_i)}$$

which is equivalent to the following quadratic inequality in $m(C_i)$:

$$m^2(C_i) - \left(4a_i + \frac{r^2\lambda^2}{4\pi}\right) m(C_i) + 4a_i^2 \leq 0.$$

The corresponding equation has two positive solutions, and $m(C_i)$ is larger than the smaller of the two. So we obtain:

$$m(C_i) \geq 2a_i + \frac{r^2\lambda^2}{8\pi} - \sqrt{\left(2a_i + \frac{r^2\lambda^2}{8\pi}\right)^2 - 4a_i^2} = 2a_i - \frac{r^2\lambda^2}{8\pi}\left(\sqrt{1 + \frac{32\pi a_i}{r^2\lambda^2}} - 1\right)\cdot \tag{3.3}$$

By using the inequality:

$$\sqrt{1+x} \leq 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

with $x = \frac{32\pi a_i}{r^2\lambda^2}$, after some straightforward simplifications, we obtain (3.2). $\square$

The following result is not used in the rest of the paper, but might be interesting by itself.

Theorem 3.4. *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ be an N-cluster with $N \geq 3$ and suppose that C_i is a component of the region E_i with $0 < m(C_i) < m(E_i)$ and suppose that r is the number of regions which are adjacent to C_i.*

Let

$$\|\boldsymbol{a}\|_{\frac{1}{2}} = \left(\sum_{j=1}^N \sqrt{a_j}\right)^2, \qquad \|\boldsymbol{a}\|_{-1} = \left(\sum_{j=1}^N (a_j)^{-1}\right)^{-1}.$$

Then

$$m(C_i) \geq \frac{20}{9}\frac{a_i^2}{r^2\,\|\boldsymbol{a}\|_{\frac{1}{2}}} \geq \frac{20}{9}\frac{a_i^2}{N^2\,\|\boldsymbol{a}\|_{\frac{1}{2}}}\cdot \tag{3.4}$$

In particular, the number M_i of connected components of E_i has the following bound

$$M_i \leq \frac{9}{20}N^2\frac{\|\boldsymbol{a}\|_{\frac{1}{2}}}{a_i}$$

and hence the total number M of connected components of $\boldsymbol{E}$ is bounded by

$$M \leq \frac{9}{20}N^2\frac{\|\boldsymbol{a}\|_{\frac{1}{2}}}{\|\boldsymbol{a}\|_{-1}}\cdot$$

Proof. Consider, as a competitor, a cluster $\boldsymbol{E}'$ whose regions E_i' are disjoint balls with area a_i and let

$$\lambda = P(\boldsymbol{E}') = 2\sqrt{\pi}\sum_{j=1}^N \sqrt{a_j} = 2\sqrt{\pi}\sqrt{\|\boldsymbol{a}\|_{\frac{1}{2}}}$$

Since $\boldsymbol{E}' \in \mathcal{C}^*(\boldsymbol{a})$ we have $P(\boldsymbol{E}) \le \lambda$. Notice that $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ implies that $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a}^*)$ with $\boldsymbol{a}^* = \boldsymbol{m}(\boldsymbol{E})$, we can apply Proposition 3.3 with $\boldsymbol{a}^*$ in place of $\boldsymbol{a}$ and with λ defined as above. So (3.2) holds with this value of λ and $\boldsymbol{a}^*$ in place of $\boldsymbol{a}$.

Notice also that $\lambda = P(\boldsymbol{E}') \ge P(\boldsymbol{E}) \ge P(E_i) \ge 2\sqrt{\pi}\sqrt{m(E_i)} = 2\sqrt{\pi}\sqrt{a_i^*}$. Moreover $r \ge 3$ since, by Proposition 2.8, we know that for $N \ge 3$ every component has at least three edges. Hence we know that

$$1 - \frac{16\pi a^* i}{r^2\lambda^2} \ge 1 - \frac{16\pi a_i^*}{9 \cdot 4\pi a_i^*} = \frac{5}{9}\cdot$$

So (3.2) becomes (notice that $r \le N$)

$$m(C_i) \ge \frac{16\pi (a_i^*)^2}{4\pi r^2 \|\boldsymbol{a}\|_{\frac{1}{2}}} \cdot \frac{5}{9} = \frac{20}{9} \cdot \frac{(a_i^*)^2}{r^2 \|\boldsymbol{a}\|_{\frac{1}{2}}} \ge \frac{20}{9} \cdot \frac{(a_i^*)^2}{N^2 \|\boldsymbol{a}\|_{\frac{1}{2}}}$$

and, noting that $a_i^* = m(E_i) \ge a_i$, (3.4) is proved.

Now suppose that C_i be the component of E_i with smaller area. Then $a_i^* = m(E_i) \ge M_i \cdot m(C_i)$ and we have

$$M_i \le \frac{a_i^*}{m(C_i)} \le \frac{a_i^*}{\frac{20}{9}\frac{(a_i^*)^2}{N^2\|\boldsymbol{a}\|_{\frac{1}{2}}}} = \frac{9}{20} \cdot \frac{N^2\|\boldsymbol{a}\|_{\frac{1}{2}}}{a_i^*} \le \frac{9}{20} \cdot \frac{N^2\|\boldsymbol{a}\|_{\frac{1}{2}}}{a_i}$$

and summing up for $i = 1, \dots, N$ we obtain:

$$M = \sum_{i=1}^N M_i \le \frac{9}{20} N^2 \|\boldsymbol{a}\|_{\frac{1}{2}} \sum_{i=1}^N \frac{1}{a_i} = \frac{9}{20} N^2 \frac{\|\boldsymbol{a}\|_{\frac{1}{2}}}{\|\boldsymbol{a}\|_{-1}}\cdot \qquad \square$$

Proposition 3.5. *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ and let C be a connected component of some region E_i. Let n be the number of edges of C. Then we have the following estimate on the pressure of the region E_i:*

$$p_i \ge \frac{(6-n)\pi}{3P(C)} + \left(1 - \frac{\ell}{P(C)}\right) p_{\min} \ge \frac{(6-n)\pi}{3P(C)}$$

where ℓ is the length of the external edge of C ($\ell = 0$ if C is internal) and $p_{\min}$ is the lowest pressure of the bounded regions which are adjacent to C.

Proof. By Lemma 2.7 we have

$$\begin{aligned}\frac{(6-n)\pi}{3} &= \sum_j (p_i - p_j) L_j = p_i \sum_j L_j - \sum_{j\ne 0} p_j L_j \\ &\le p_i \sum_j L_j - p_{\min} \sum_{j \ne 0} L_j = p_i P(C) - p_{\min}(P(C) - \ell)\end{aligned}$$

where the sum in j is extended to the regions E_j which are adjacent to C. The first estimate of the statement follows.

To get the second estimate recall that $p_{\min} \ge 0$ in view of Proposition 2.5. $\square$

Proposition 3.6 (Variation III)**.** *Let $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ be a cluster and let B and C be two different components of the same bounded region E_i of $\boldsymbol{E}$. Let p_i be the pressure of E_i. Suppose that B is external and let L be the length of the external arc of B and n be the number of different regions which are adjacent to C. Then*

$$p_i \ge \frac{P(C)}{n\, m(C)} - \frac{2}{L} \ge \frac{2\sqrt{\pi}}{n\sqrt{m(C)}} - \frac{2}{L}\cdot$$

Proof. Suppose $i = 1$ and consider all the regions which are adjacent to C. Suppose that E_2 is the region whose edges in common with C have largest total length. Let ℓ be such total length in common between C and E_2: we have that $n\ell \geq P(C)$.

Let γ be the external edge of B and let v and w be its vertices. The arc γ has radius $R = 1/p_1$, length L and spans an angle $\theta = L/R$. Given $h > 0$ we are going to modify B by increasing the radius R up to $R + h$. Just consider the two radii in v and w: extend them of a length h and join them with a parallel arc of radius $R + h$. Let D be the strip between these two parallel arcs. We have $m(D) = ((R+h)^2 - R^2)\theta/2 = Lh + Lh^2/(2R) \geq Lh$. It is easy to see that $D \subseteq E_0$ (since all the external arcs are convex and meet at angles of 120 degrees). Fix $h = m(C)/L$ and consider the following variation:

$$E'_1 = (E_1 \setminus C) \cup D, \qquad E'_2 = E_2 \cup C.$$

If we let $\boldsymbol{E}' = (E'_1, E'_2, E_3, \dots, E_N)$ we notice that $m(E'_1) \geq m(E_1)$ (since $m(D) \geq Lh = m(C)$) so $\boldsymbol{E}' \in \mathcal{C}^*(\boldsymbol{a})$. Moreover, in computing the perimeter of $\boldsymbol{E}'$ the edges in common between C and E_2 have been removed so we gain ℓ while the arc of length L has increased to length $2h + L(R + h)/R$ and so we have, by the minimality of $\boldsymbol{E}$:

$$0 \leq P(\boldsymbol{E}') - P(\boldsymbol{E}) \leq -\ell + 2h + L\frac{R+h}{R} - L = m(C)\left(\frac{1}{R} + \frac{2}{L}\right) - \ell$$

To obtain the statement just remember that $1/R = p_1$ and remember that $\ell \geq P(C)/n$. □

Lemma 3.7. *Let $\boldsymbol{E} \in \mathcal{C}(a_1, a_2)$ be a connected stationary cluster (a double bubble) with $a_1 \geq a_2$. Then the pressures p_1, p_2 satisfy the following relations*

$$\frac{k_8}{\sqrt{a_1}} \leq p_1 \leq p_2 \leq \frac{k_8}{\sqrt{a_2}}$$

with

$$k_8 := \sqrt{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}, \qquad 1.5897 < k_8 < 1.5898.$$

Proof. By Theorem 2.10 we know that the external radii r_1, r_2 and areas a_1, a_2 of a double bubble are in one-to-one correspondence. Moreover we know that when $r_1 = r_2$ we have $a_1 = a_2$ because the resulting double bubble is symmetric and a direct computation gives $a_1 = a_2 = k_8^2 r^2$ (with $r = r_1 = r_2$). Hence, by the monotonicity proven in Theorem 2.10, since we have $a_1 \geq a_2$ by assumption, we know that $r_1 \geq r_2$ and hence $p_1 \leq p_2$ (remember that $p_i = 1/r_i$). Hence monotonicity gives also:

$$a_1 \geq k_8^2 r_1^2, \qquad a_2 \leq k_8^2 r_2^2$$

whence

$$p_1 = \frac{1}{r_1} \geq \frac{k_8}{\sqrt{a_1}}, \qquad p_2 = \frac{1}{r_2} \leq \frac{k_8}{\sqrt{a_2}}.$$

□

Lemma 3.8 (Reduction to double-bubble)**.** *Let $\boldsymbol{E} = (E_1, \dots, E_N)$ be a stationary cluster which is reducible to a double bubble (E'_i, E'_j) by subsequent removal of triangular components where $E'_i \supseteq E_i$, $E'_j \supseteq E_j$, $E'_i \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_j)$ and $E'_j \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_i)$. Let $\boldsymbol{a} = \boldsymbol{m}(\boldsymbol{E})$ and $a = \sum_{k=1}^N a_k$.*

Then

$$\frac{k_8}{\sqrt{\max\{a - a_i, a - a_j\}}} \leq \min\{p_i, p_j\} \leq \max\{p_i, p_j\} \leq \frac{k_8}{\sqrt{\min\{a_i, a_j\}}}.$$

Proof. By Theorem 2.9 we know that the pressures of the double bubble are equal to the corresponding pressures of the cluster $\boldsymbol{E}$. Also notice that, for $k = i, j$ one has $m(E'_k) \geq m(E_k) = a_k$ $(k = i, j)$, while $m(E'_i) \leq m(\mathbb{R}^2 \setminus (E_0 \cup E_j)) = a - a_j$ and $m(E'_j) \leq m(\mathbb{R}^2 \setminus (E_0 \cup E_i)) = a - a_i$ so, by Theorem 2.10 we obtain the desired result. □

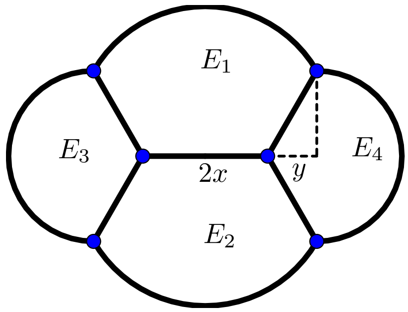


FIGURE 2. The competitor cluster defined in Proposition 4.1.

Lemma 3.9 (Perimeter of triple bubble)**.** *One has*

$$p^*(1,1,1) = 6\sqrt{\frac{\pi}{2} + \frac{1}{\sqrt{3}}} \geq k_{10} := 8.7939$$

Proof. From [23] we know that each $\boldsymbol{E} \in \mathcal{M}^*(1,1,1) = \mathcal{M}(1,1,1)$ is a standard triple bubble where each region E_i is composed by the union of an half-circle and an isosceles triangle with two angles of 30 degrees. If r is the radius of the half circles, the area of each region turns out to be

$$1 = \left(\frac{\pi}{2} + \frac{1}{\sqrt{3}}\right) r^2$$

while the perimeter is given by

$$P(\boldsymbol{E}) = (3\pi + 2\sqrt{3})r = 6\left(\frac{\pi}{2} + \frac{1}{\sqrt{3}}\right) r = 6\sqrt{\frac{\pi}{2} + \frac{1}{\sqrt{3}}} \qquad \square$$

4. ESTIMATES ON $\mathcal{M}^*(1,1,1,1)$

Proposition 4.1 (The competitor)**.** *We have $p^*(1,1,1,1) \leq k_0 := 11.1962$.*

Proof. Let $x := 0.2707$, $y := 0.394$ and $R = 2(x+y)/\sqrt{3}$. Consider the cluster represented in Figure 2. The area of the regions with four edges is given by:

$$m(E_1) = m(E_2) = (2x+y)y\sqrt{3} + \frac{\pi}{3}R^2 - \frac{\sqrt{3}}{4}R^2 > 1$$

while the area of the regions with three edges is:

$$m(E_3) = m(E_4) = \sqrt{3}y^2 + \frac{\pi}{2}(y\sqrt{3})^2 > 1.$$

So $\boldsymbol{E} \in \mathcal{C}^*(1,1,1,1)$. And we have

$$P(\boldsymbol{E}) = 2\frac{2\pi}{3}R + 2\pi\sqrt{3}y + 2x + 8y \geq k_0. \qquad \square$$

Proposition 4.2. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ and suppose that C is a component of some region. Then*:

$$m(C) \geq k_2 := 0.0244.$$

Moreover, if the number of regions which have an edge in common with C is not larger than 3 one has

$$m(C) \geq k_6 := 0.0425.$$

Proof. We can apply Proposition 3.3 with $a_i = 1$, $r \leq 4$, $P(\boldsymbol{E}) \leq k_0$ so $rP(\boldsymbol{E}) \leq \lambda := 4k_0$. We obtain:

$$m(C) \geq \frac{\pi}{k_0^2}\left(1 - \frac{\pi}{k_0^2}\right) \geq k_2.$$

And with $r \leq 3$ we would have

$$m(C) \geq \frac{16\pi}{9k_0^2}\left(1 - \frac{16\pi}{9k_0^2}\right) \geq k_6. \qquad \square$$

Proposition 4.3. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be such that the region E_1 can be decomposed in two parts $E_1 = E_1' \cup C_1$ with*

$$m(E_1) = m(E_1') + m(C_1), \quad m(E_1') \geq m(C_1), \quad P(E_1) = P(E_1') + P(C_1)$$

then

$$m(C_1) \leq k_1 := 0.1605, \qquad P(C_1) \leq k_7 := 1.4199$$

Proof. Let $m = m(C_1)$. By Lemma 3.1, one has

$$P(\boldsymbol{E}) \geq \sqrt{\pi}\left(\sqrt{4} + \sqrt{m} + \sqrt{m(E_1')} + 3\sqrt{1}\right) = \sqrt{\pi}\left(\sqrt{m} + \sqrt{m(E_1')} + 5\right)$$

whence

$$\sqrt{m} + \sqrt{m(E_1')} \leq \frac{P(\boldsymbol{E})}{\sqrt{\pi}} - 5 \leq \frac{k_0}{\sqrt{\pi}} - 5 \leq c_1 := 1.3168$$

On one hand we have assumed that $m(E_1') \geq m(C_1) = m$, so $2\sqrt{m} \leq c_1 < \sqrt{2}$ which gives $m \leq 1/2$. On the other hand we know that $m(E_1') = m(E_1) - m \geq 1 - m$, whence

$$\sqrt{m} + \sqrt{1-m} \leq c_1.$$

Now let $f(x) = \sqrt{x} + \sqrt{1-x}$. By computing the sign of $f'(x)$ we easily notice that $f(x)$ is increasing for $x \in [0, 1/2]$. By direct computation one checks that $f(k_1) > c_1$ (in fact k_1, which is defined in the statement of the theorem being proved, has been choosen to satisfy this relation). Since we know that $f(m) \leq c_1$ and $m \leq 1/2$ we conclude that $m = m(C_1) < k_1$. To get the estimate on the perimeter, we use again the isoperimetric inequality:

$$\begin{aligned} P(C_1) &= 2P(\boldsymbol{E}) - \left(P(E_1') + P(E_0) + \sum_{i=2}^{4} P(E_i)\right) \leq 2k_0 - 2\sqrt{\pi}(\sqrt{1 - m(C_1)} + \sqrt{4} + 3\sqrt{1}) \\ &\leq 2k_0 - 2\sqrt{\pi}(\sqrt{1-k_1} + 5) \leq k_7. \end{aligned} \qquad \square$$

Definition 4.4 (Big/small components)**.** Let $\boldsymbol{E}$ be a regular N-cluster. We say that a component C of a region E_i is *small* if $m(C) \leq m(E_i)/2$. Otherwise we say that C is *big*. Notice that at most one connected component of each region can be big.

Corollary 4.5. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then each region E_i has exactly one big connected component E_i'. Furthermore $m(E_i') \geq 1 - k_1$, where k_1 is the constant introduced in Proposition 4.3.*

Proof. Let $E_i^1, \dots, E_i^M$ be the connected components of the region E_i. Suppose by contradiction that all E_i^j are small: $m(E_i^j) \le m(E_i)/2$, $j = 1, \dots, M$. Let K be the smallest index such that

$$\sum_{j=1}^{K} m(E_i^j) > k_1. \tag{4.1}$$

We claim that

$$\sum_{j=1}^{K} m(E_i^j) < m(E_i) - k_1. \tag{4.2}$$

Otherwise we would have (notice that $k_1 < 1/4$)

$$\begin{aligned}\sum_{j=1}^{K-1} m(E_i^j) &= \sum_{j=1}^{K} m(E_i^j) - m(E_i^K) \ge m(E_i) - k_1 - m(E_i^K) \ge m(E_i) - k_1 - \frac{m(E_i)}{2} \ge \frac{m(E_i)}{2} - k_1 \\ &\ge \frac{1}{2} - k_1 > k_1\end{aligned}$$

which is a contradiction since K was the minimal index satisfying the inequality (4.1). So, if we define

$$E_i' = \bigcup_{j=1}^{K} E_i^j, \qquad E_i'' = E_i \setminus E_i'$$

we have (by (4.1) and (4.2))

$$m(E_i') > k_1, \qquad m(E_i'') > k_1.$$

This is now a contradiction with Proposition 4.3, since the smaller of the two components E_i', E_i'' should have a measure smaller than k_1.

Finally if E_i' is the big connected component of the region E_i, applying Proposition 4.3 with $C_i = E_i \setminus E_i'$, we find $m(E_i) \ge 1 - k_1$. □

Corollary 4.6. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then at most one of the big components is internal.*

Proof. Suppose by contradictions that two big components E_i^1 and E_j^1 are internal. Then by the isoperimetric inequality:

$$\begin{aligned}P(\boldsymbol{E}) &\ge P(E_i^1 \cup E_i^2) + P(E_0) \\ &\ge 2\sqrt{\pi}\left(\sqrt{m(E_i^1) + m(E_i^2)} + \sqrt{m(E_1) + m(E_2) + m(E_3) + m(E_4)}\right) \\ &\ge 2\sqrt{\pi}\left(\sqrt{2(1-k_1)} + \sqrt{4}\right) \ge 11.6831 > k_0 \ge p^*(1,1,1,1).\end{aligned}$$

Which is a contradiction. □

Proposition 4.7. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be such that both regions E_i and E_j are disconnected ($i \ne j$). Then every small component C of either E_i or E_j satisfies*:

$$m(C) \le k_3 := 0.0408, \qquad P(C) \le k_9 := 0.7154.$$

Proof. Without loss of generality we might suppose that $i = 1$, $j = 2$. Let E_1' be the larger small component of E_1 and let E_2' be the larger small component of E_2. Suppose moreover that $m := m(E_1') \geq m(E_2')$. Then we have

$$m(E_1 \setminus E_1') \geq 1 - m, \qquad m(E_1') = m,$$
$$m(E_2 \setminus E_2') \geq 1 - m, \qquad m(E_2') \geq k_2.$$

So, from the isoperimetric inequality:

$$\frac{P(\boldsymbol{E})}{\sqrt{\pi}} \geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sum_{i=1}^{2} \sqrt{m(E_i \setminus E_i')} + \sum_{i=1}^{2} \sqrt{m(E_i')} + \sum_{j=3}^{4} \sqrt{m(E_j)}$$

we obtain:

$$\frac{P(\boldsymbol{E})}{\sqrt{\pi}} \geq \sqrt{4} + \sqrt{1-m} + \sqrt{m} + \sqrt{1-m} + \sqrt{k_2} + 2\sqrt{1} = 4 + 2\sqrt{1-m} + \sqrt{m} + \sqrt{k_2}.$$

If we set $f(x) = 2\sqrt{1-x} + \sqrt{x}$ and remember that $P(\boldsymbol{E}) \leq k_0$ (Prop 4.1) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 4 - \sqrt{k_2} \leq c_2 := 2.1606$$

We have:

$$f'(x) = -(1-x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{1}{2}(1-x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that $f'(k_1) > 0.1565 > 0$ and since $f'' < 0$ we know that f is strictly increasing on $[0, k_1]$. By direct computation one checks k_3 was choosen so that $f(k_3) > c_2$. If, by contradiction, $m > k_3$ since $m \in [k_2, k_1]$ (by Prop. 4.2 and Prop. 4.3) we would have $f(m) > f(k_3) > c_2$ against (4). So $m < k_3$.

Since m was the measure of the largest small component we obtain the first estimate: $m(C) \leq m \leq k_3$.

To prove the estimate on the perimeter $P(C)$ suppose now that $C = E_1'$ (not it will not matter if E_1' is larger or smaller than E_2'). Recall that (Prop. 4.2)

$$m(E_1') \geq k_2, \qquad m(E_2') \geq k_2$$

and the previous estimate gives:

$$m(E_1 \setminus E_1') \geq 1 - k_3, \qquad m(E_2 \setminus E_2') \geq 1 - k_3.$$

Hence, using the isoperimetric inequality we have

$$\begin{aligned} 2P(\boldsymbol{E}) &= P(E_1') + P(E_2') + \sum_{i=1}^{2} P(E_i \setminus E_i') + \sum_{i=3}^{4} P(E_i) + P(\mathbb{R}^2 \setminus E_0) \\ &\geq P(E_1') + 2\sqrt{\pi}\left(\sqrt{k_2} + 2\sqrt{1-k_3} + 2\sqrt{1} + \sqrt{4}\right) \end{aligned}$$

whence, recalling also that $P(\boldsymbol{E}) \leq k_0$:

$$P(E_1') \leq 2k_0 - 2\sqrt{\pi}(\sqrt{k_2} + 2\sqrt{1-k_3} + 4) \leq k_9. \qquad \square$$

Proposition 4.8. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be such that the region E_i has at least three components. Then every small component C of E_i satisfies:*

$$m(C) \leq k_4 := 0.0411.$$

Proof. Without loss of generality we might suppose that $i = 1$. Notice that, by Corollary 4.5, there are at least two small components of E_1. Let E_1' be the larger small component of E_1 and E_1'' be another small component of E_1. Let $m := m(E_1') \geq m(E_1'')$. Then we have

$$m(E_1 \setminus (E_1' \cup E_1'')) \geq 1 - m - m, \qquad m(E_1') = m, \qquad m(E_1'') \geq k_2.$$

So, from the isoperimetric inequality:

$$\frac{P(\boldsymbol{E})}{\sqrt{\pi}} \geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sqrt{m(E_i \setminus (E_i' \cup E_i''))} + \sqrt{m(E_i')} + \sqrt{m(E_i'')} + \sum_{j=2}^{4} \sqrt{m(E_j)}$$

we obtain:

$$\frac{P(\boldsymbol{E})}{\sqrt{\pi}} \geq \sqrt{4} + \sqrt{1-2m} + \sqrt{m} + \sqrt{k_2} + 3\sqrt{1} = 5 + \sqrt{1-2m} + \sqrt{m} + \sqrt{k_2}. \tag{4.3}$$

If we set $f(x) = \sqrt{1-2x} + \sqrt{x}$ and remember that $P(\boldsymbol{E}) \leq k_0$ (Prop. 4.1) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 5 - \sqrt{k_2} \leq c_3 := 1.1606$$

We have:

$$f'(x) = -(1-2x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -(1-2x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that $f'(k_1) > 0.0344 > 0$ and since $f'' < 0$ we know that f is strictly increasing on $[0, k_1]$. By direct computation one checks that k_4 has been choosen so that $f(k_4) > c_3$. If, by contradiction, $m > k_4$ since $m \in [k_2, k_1]$ (by Props. 4.2 and 4.3) we would have $f(m) > f(k_4) > c_3$ against (4.3). So $m \leq k_4$. ☐

Proposition 4.9. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then the total number of small components is not larger than two.*

Proof. Suppose by contradiction that the cluster $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ has at least three small components C_1, C_2, C_3. Suppose $m := m(C_1) \geq m(C_2) \geq m(C_3)$. Let $C = C_1 \cup C_2 \cup C_3$ and let $E_i' = E_i \setminus C$ for $i = 1, \dots, 4$.

From the isoperimetric inequality:

$$\frac{P(\boldsymbol{E})}{\sqrt{\pi}} \geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sum_{i=1}^{4} \sqrt{m(E_i')} + \sum_{i=1}^{3} \sqrt{m(C_i)}.$$

Now consider the quantity

$$A = \sum_{i=1}^{4} \sqrt{m(E_i')}$$

to get an estimate of A from below we use the estimates $k_2 \leq m(C_i) \leq m$ but we have to distinguish three different cases:

(1) if the small components all belong to the same region we have $A \geq \sqrt{1-3m} + 3\sqrt{1}$;
(2) if only two of the small components belong to the same region: $A \geq \sqrt{1-2m} + \sqrt{1-m} + 2\sqrt{1}$;
(3) if the three small components belong to three different regions: $A \geq 3\sqrt{1-m} + \sqrt{1}$.

With a straightforward algebraic manipulation one can check that for all $x \in [0, 1/3]$ one has

$$3\sqrt{1-x} + 1 \geq \sqrt{1-2x} + \sqrt{1-x} + 2 \geq \sqrt{1-3x} + 3$$

so that in every case it holds $A \geq \sqrt{1-3m}+3$.

Hence

$$\frac{P(\boldsymbol{E}')}{\sqrt{\pi}} \geq \sqrt{4}+\sqrt{1-3m}+3+\sqrt{m}+2\sqrt{k_2} = \sqrt{1-3m}+\sqrt{m}+5+2\sqrt{k_2} \tag{4.4}$$

If we set $f(x)=\sqrt{1-3x}+\sqrt{x}$ and remember that $P(\boldsymbol{E}')=P(\boldsymbol{E})\leq k_0$ (Prop. 4.1) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 5 - 2\sqrt{k_2} \leq k_5 := 1.0044$$

We have:

$$f'(x) = -\frac{3}{2}(1-3x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{9}{4}(1-x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that $f(k_1) > 1.1206 > k_5$ and $f(k_2) > 1.1189 > k_5$. And since $f'' < 0$ we know that f is concave and hence $f(x) > k_5$ if $x \in [k_2, k_1]$. Since $f(m) \leq k_5$ and we already know that $m \geq k_2$ (Prop. 4.2) we conclude that $m > k_1$, which is a contradiction. $\square$

Corollary 4.10. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then there are at most six bounded connected components. Four connected components are big and at most two are small (see Def. 4.4).*

If the small components are exactly two, they have measure between k_2 and k_4, they are external, and they have edges in common with all the other regions. If the two small components belong to the same region they both have four edges, while if they belong to different regions they might have four or five edges.

If there is only one small component it has measure not larger than k_1.

Proof. By Proposition 4.9 there are at most two small components, so the total number of bounded connected components is at most six.

If we have two small components they can either belong to the same region, and then by Proposition 4.8 each small component has measure not larger than k_4. Or, the two components belong to different regions and then by Proposition 4.7 each small component has measure not larger than $k_3 < k_4$. Every small component which is adjacent only to three other regions would have measure larger than k_6 by Proposition 4.2 and since $k_6 > k_4$ this is impossible. So every small component must have edges in common with all the other four regions, included the external one: so they have at least four edges and are external. If the two components belong to two different regions they can have four or five edges (the two small component might have an edge in common). If the two components belong to the same region, each other region is connected and hence they cannot have more than four edges (each edge is adjacent to a different component).

If there is only one small component we can only apply Proposition 4.3 to get the estimate with the constant k_1. $\square$

5. Clusters with six components

In this section we will consider possible minimizers $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ with exactly six bounded components and we will exclude that they exist.

The following Corollary assures that we have $m(E_i)=1$ for $i=1,\dots,4$. This will be used in the following without further notice.

Corollary 5.1. $\mathcal{M}^*(1,1,1,1) = \mathcal{M}(1,1,1,1)$.

Proof. Given any $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ by Corollary 4.10 we know that $\boldsymbol{E}$ has no more than six bounded components. By Proposition 2.8 we conclude that $\boldsymbol{E} \in \mathcal{M}(1,1,1,1)$, hence $\mathcal{M}^*(1,1,1,1) \subseteq \mathcal{M}(1,1,1,1)$. Since $\mathcal{M}^*(1,1,1,1)$ is not empty (Thm. 2.4) we obtain $p^*(1,1,1,1) = P(\boldsymbol{E}) = p(1,1,1,1)$.

On the other hand, given $\boldsymbol{E}' \in \mathcal{M}(1,1,1,1)$ we have $\boldsymbol{E}' \in \mathcal{C}^*(1,1,1,1)$ and since $P(\boldsymbol{E}') = p(1,1,1,1) = p^*(1,1,1,1)$ we conclude that $\boldsymbol{E}' \in \mathcal{M}^*(1,1,1,1)$. $\square$

Corollary 5.2. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then we exclude that one region E_i can have three components.*

Proof. Suppose by contradiction that the region E_1 is composed by three components: one big and two small (recall that, by Corollary 4.5, each region has one big component). By Proposition 2.8 we know that every component has at least three edges. By Corollary 4.10, a small component has four edges, so, the two small components have exactly four vertices and the region E_1 has at least $3+4+4 = 11$ vertices. But the total number of bounded connected components is $M = 6$ and by Proposition 2.8 the number of vertices should be $v = 2(M-1) = 10$. This is a contradiction. □

Proposition 5.3. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then we exclude that two different regions are disconnected.*

Proof. By contradiction suppose that C_1 and C_2 are small components of E_1 and E_2 respectively and let $E_1' = E_1 \setminus C_1$ and $E_2' = E_2 \setminus C_2$ be the two big components.

Recall that, by Corollary 4.10, the small components C_1 and C_2 have four or five edges.

If the component C_i $(i = 1,2)$ has five edges, by Proposition 3.5 and Proposition 4.7, one finds that

$$p_i \geq \frac{\pi}{3P(C)} \geq \frac{\pi}{3k_9} > 1.4637 > \frac{k_0}{8} \tag{5.1}$$

On the other hand if C_i has only four edges, one finds:

$$p_i \geq \frac{2\pi}{3P(C)} \geq \frac{2\pi}{3k_9} > \frac{k_0}{4}\cdot$$

Remember that, by Theorem 2.6 and Proposition 4.1, we have

$$p_1 + p_2 + p_3 + p_4 = \frac{P(\boldsymbol{E})}{2} \leq \frac{k_0}{2}\cdot$$

Without loss of generality we might and shall suppose that $p_1 \geq p_2$.

Notice that p_1 and p_2 are both larger than the average and, in particular, p_2 is not the lowest pressure: $p_2 > \min\{p_3, p_4\}$. If both regions C_1 and C_2 had four edges, we would find $p_1 + p_2 > k_0/2$ which is a contradiction. Hence we know that C_1 has four or five edges and C_2 has five edges (if C_i has four edges p_i is the higher pressure).

Step 1. we claim that at most one component is internal. By Corollary 4.10 we know that the small components are external and by Corollary 4.6 we know that at most one big component is internal. The claim follows.

Step 2. we claim that E_2' is external and has three or four edges.

Notice that since at most one component is internal, and we have a total of 6 bounded components, the external region E_0 has either 5 or 6 vertices. On the other hand the big component E_2' has at least 3 vertices and the small component C_2 has 5 vertices. Two of the vertices of C_2 are in common with the vertices of E_0 and, if E_2' were internal, all its vertices would be distinct from the vertices of E_0 and, of course, from the vertices of C_2. So we find at least $3+3+5 = 11$ distinct vertices of the cluster $\boldsymbol{E}$ while we know (Prop. 2.8) that $\boldsymbol{E}$ has exactly 10 vertices.

The same contradiction holds in the case that E_2' has more than four vertices since also in this case at least three of them would be internal.

Step 3. we claim that E_1' and E_2' are adjacent. Let ℓ_1 and ℓ_2 be the lengths of the external edges of E_1' and E_2' respectively ($\ell_i = 0$ if E_i' is internal). Suppose by contradiction that E_1' and E_2' have no common edge. Then

$$k_0 \geq P(\boldsymbol{E}) \geq P(E_1') + P(E_2') + P(E_0) - (\ell_1 + \ell_2)$$

and by applying the isoperimetric inequality and the estimates $m(E_i') \geq 1 - k_3$ we obtain:

$$k_0 \geq 2\sqrt{\pi}(2\sqrt{1-k_3} + \sqrt{4}) - (\ell_1 + \ell_2)$$

whence

$$\frac{\ell_1 + \ell_2}{2} \geq 2\sqrt{\pi}(\sqrt{1-k_3} + 1) - \frac{k_0}{2} > c_4 := 1.4186.$$

If we let ℓ_i be the largest between ℓ_1 and ℓ_2 we have $\ell_i > c_4$ and from Proposition 3.6 we obtain the following estimate on the pressure of the corresponding region E_i (remember that every component of $\boldsymbol{E}$ is adjacent to at most four different regions):

$$p_i \geq \frac{\sqrt{\pi}}{2\sqrt{m(C_i)}} - \frac{2}{\ell_i} \geq \frac{\sqrt{\pi}}{2\sqrt{k_3}} - \frac{2}{c_4} > 2.9776 > \frac{k_0}{4}. \tag{5.2}$$

Remember that $p_1 + p_2 + p_3 + p_4 \leq k_0/2$ so p_i is the highest pressure (actually $i = 1$ since we decided that $p_1 \geq p_2$). Then let $n \geq 3$ be the number of edges of E_i' and let $L_{i,j}$ be the total length of the edges in common between E_i' and E_j (so that $L_{i,0} = \ell_i$):

$$\pi \geq \frac{(6-n)\pi}{3} = \sum_j (p_i - p_j) L_{i,j} \geq p_i \ell_i$$

whence:

$$p_i \leq \frac{\pi}{\ell_i} \leq \frac{\pi}{c_4} < 2.2146$$

which is in contradiction with with (5.2).

Step 4. if a connected region E_i $(i = 3, 4)$ is internal, it is adjacent to both E_1' and E_2'.

The proof is the same as in the previous Step. Just take E_i in place of E_2' and E_1' or E_2' in place of E_1'. Notice that $\ell_2 = 0$ so that $\ell_i = \ell_1$ and the proof completes in exactly the same way (the estimates are actually stronger).

Step 5. we claim that if one of E_3 or E_4 is internal and the other one is external with only three edges, then E_3 and E_4 must be adjacent. We proceed in a similar way as the step before. Suppose by contradiction that E_3 is internal and not adjacent to E_4.

So E_3 is only adjacent to the components of E_1 and E_2 and it has at most four edges, so, by Lemma 2.7, we have

$$0 < \frac{(6-4)\pi}{3} \leq \sum_{i=1}^{2} (p_3 - p_i) L_{3,i}.$$

We deduce that $p_3 \geq p_2$ since otherwise (being $p_1 \geq p_2$) the right hand side of the previous equation would be negative. So $p_3 \geq p_2 \geq k_0/8$.

Now, let ℓ_i be the length of the external edges of E_i' (recall that only one component can be internal hence E_i' is external and $\ell_i > 0$). We have

$$k_0 \geq P(\boldsymbol{E}) \geq P(E_1' \cup E_2' \cup E_3) + P(E_0) - (\ell_1 + \ell_2)$$

whence, by applying the isoperimetric inequality,

$$\frac{\ell_1 + \ell_2}{2} \geq \sqrt{\pi}(\sqrt{2(1-k_3)+1} + \sqrt{4}) - \frac{k_0}{2} > c_5 := 0.9747.$$

Now if ℓ_i is the maximum between ℓ_1 and ℓ_2 we know that $\ell_i > c_5$. By Proposition 3.6 (since any component can be adjacent to at most 4 different regions), we have

$$p_i \geq \frac{2\sqrt{\pi}}{4\sqrt{m(C_i)}} - \frac{2}{\ell_i} \geq \frac{\sqrt{\pi}}{2\sqrt{k_3}} - \frac{2}{c_5} > 2.3355 > \frac{3}{16}k_0.$$

So $p_1 > 3k_0/16$ (since p_1 has been choosen to be the maximum between p_1 and p_2).

Now we work on E_4 which is external with $m = 3$ edges. Remember that p_2 cannot be the lowest pressure and since $p_1 \geq p_2$ and $p_3 \geq p_2$ we deduce that p_4 is the lowest pressure. Hence, by Lemma 2.7

$$\pi = \frac{(6-m)\pi}{3} = \sum_j (p_4 - p_j)L_{4,j} \leq p_4 L_{4,0}$$

and by Proposition 3.2

$$p_4 \geq \frac{\pi}{L_{4,0}} \geq \frac{\pi}{2\sqrt{\pi}\sqrt{m(E_4)}} = \frac{\sqrt{\pi}}{2} > 0.8862 > \frac{k_0}{16}$$

So, we have found that

$$P(\boldsymbol{E}) = 2(p_1 + p_2 + p_3 + p_4) > 2\left(\frac{3k_0}{16} + \frac{k_0}{8} + \frac{k_0}{8} + \frac{k_0}{16}\right) = k_0$$

which contradicts the minimality of $\boldsymbol{E}$. The claim is proved.

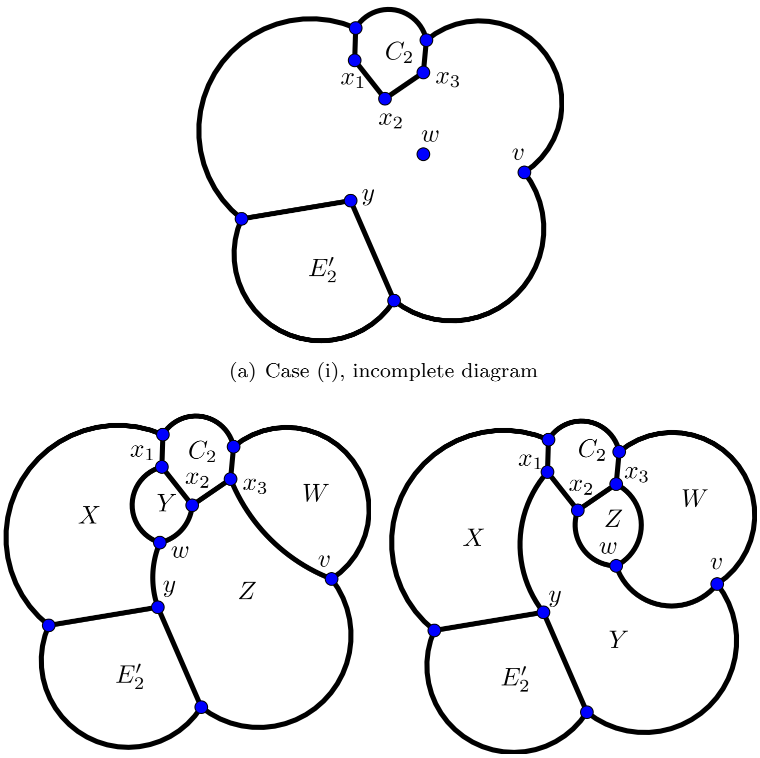
Step 6. we claim that E_0 has not five edges. Suppose by contradiction that E_0 has exactly five edges and consider two possible cases: E_2' has either (i) three or (ii) four edges.

If E_2' has three edges the region $E_2 = C_2 \cup E_2'$ has 8 distinct vertices (since C_2 has five vertices). Three vertices of C_2 (let us call them x_1, x_2 and x_3) are not vertices of E_0, and one vertex of E_2' (let us call it y) is not a vertex of E_0. On the other hand E_0 has five vertices, and four of them are shared by C_2 and E_2'. We denote by v the remaining vertex. Up to now we have considered 9 vertices in total, since the cluster $\boldsymbol{E}$ has exactly 10 vertices, there is an additional vertex w belonging to neither E_0 nor E_2. The situation is depicted in Figure 3a. We see that 11 edges have been already identified, so 4 edges are missing.

Consider the three edges which meet in the vertex w. At least two of them should connect w to the vertices x_k of C_2. In fact if only one edges connects w to C_2 the other two edges of w should go to v and y and hence the two remaining vertices of C_2 should be joined together which is not admitted (we would obtain a two sided component). Not all three edges of w can join the three free vertices of C_2 because otherwise we would obtain two three-sided internal components. But we know that at most one component can be internal. So, exactly two edges join w with two vertices of C_2. The two vertices of C_2 must be consecutive, otherwise the third vertex x_2 could not be connected to anything (the edge would be closed in the loop: w, x_3, x_2 x_1). We have two possibilities: the two vertices are either x_1 and x_2 or x_2 and x_3 (the order of the vertices is given by the Figure, where x_1 is "closer" to the component E_2').

In the first case (x_1 and x_2 are joined to w) the third edge in w cannot go to x_3 (already excluded) and cannot go to v because otherwise the edge from x_3 to y would cross the already defined edges. So the diagram is completed by an edge joining w with y and an edge joining v with x_3. The resulting diagram is depicted in Figure 3b. We know that C_1 is external and has four or five edges: the only possibility is $X = C_1$. Then E_1' must be adjacent to E_2' so it must be $Z = E_1'$: however E_1' cannot be adjacent to C_1 and we get a contradiction.

In the second case (x_2 and x_3 are joined to w) we can complete the diagram in a unique way, by adding an edge from w to v and an edge from y to x_1 as represented in Figure 3c. In this case we have $X = E_1'$ since E_1' must be adjacent to E_2' but cannot have six edges. So $W = C_1$ because C_1 is external and not adjacent to E_1'. So Y and Z are the two connected regions E_3 and E_4. However in *Step 4* we proved that the connected region,



(a) Case (i), incomplete diagram

(b) (c)

FIGURE 3. Diagrams used in the proof of Proposition 5.3, Step 6, case (i).

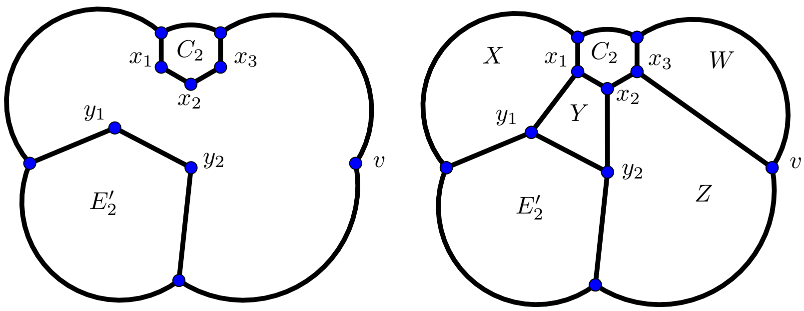


FIGURE 4. Diagram used in the proof of Proposition 5.3, Step 6, case (ii).

if internal, must be adjacent to both E_1' and E_2' which is not the case for the component Z. So this configuration must be excluded, too.

So, the case when E_2' has only three edges has been completed and excluded. Suppose now (ii) that E_2' has four edges. In this case no additional vertex must be added, and we are in the situation depicted in Figure 4. Let x_1, x_2 and x_3 be the free vertices of C_2 and v be the free vertex of E_0, as before. Let y_1 and y_2 be the two free vertices of E_2'. There are three edges missing in the diagram and there is only one possibility (since the edges from C_2 cannot go back to C_2 and they cannot cross each other): x_1 is joined to y_1, x_2 to y_2 and x_3 to v. The component C_1 is external with four or five edges, hence C_1 is either X or Z. The component E_1' is adjacent

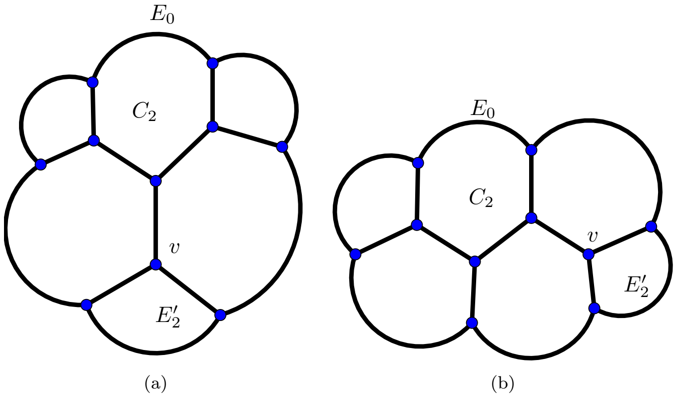


FIGURE 5. Diagrams used in the proof of Proposition 5.3 Step 7.

to E'_2 but cannot be adjacent to C_1 hence E'_1 is either X or Z. So Y and W are the two connected regions E_3 and E_4: say $Y = E_3$ and $W = E_4$.

But now we notice that E_3 is internal and E_4 is external with only three edges, hence by Step 5 they should be adjacent, which is not the case.

Step 7. Conclusion. We know now that E_0 has six edges. Recall that C_2 is external and has five vertices, two of which are shared with E_0 while E'_2 has at least three vertices (all distinct from C_2) two of which are shared with the vertices of E_0. So we have identified 6 vertices of E_0 and at least $3+1=4$ internal vertices of $E_2 = C_2 \cup E'_2$. We know that the cluster has 10 vertices in total, so we have identified all of them. In particular we conclude that E'_2 has three vertices. Let x_1, x_2 and x_3 be the three internal vertices of C_2 and let v be the internal vertex of E'_2.

If we look at the edges, we have already identified the six edges of E_0, other four are the internal edges of C_2 and other two are the internal edges of E'_2. To reach the total of 15 edges, we need to place other three edges. No edge can join two points of C_2 (otherwise a two sided component would rise). So the three missing edges start from the three internal points of C_2. One of them goes to the internal vertex of E'_2 and the other two go to the two free vertices of E_0.

There are now two possibilities: either (a) the vertex v is connected to the middle of the three internal vertices of C_2 or (b) it is connected to one lateral vertex (see Fig. 5)

We can easily exclude case (a) because the component C_1 must be one of the two five-sided components (C_1 has either four or five edges and there are no components with four edges) while E'_1 must be adjacent to E'_2 and hence must be the other component with five edges. But this is a contradiction since C_1 cannot be adjacent to E'_1.

So we remain with the configuration of case (b). The region with three edges adjacent to C_2 is not C_1 (because C_1 has four or five edges) and it cannot be E'_1 because E'_1 must be adjacent to E'_2. Hence we conclude that it is one of E_3 and E_4. Let us say it is E_3. Then E_4 must be the region with five edges, because otherwise C_1 and E'_1 would be adjacent to each other. So C_1 has four edges and hence $p_1 \geq k_0/4$ is the region with higher pressure and $p_2 \geq k_0/8$ is the second higher pressure while $p_3 + p_4 \leq k_0/8$.

We know that E_3 has three edges, E_4 has five edges and both E_3 and E_4 are external. Let $L_{j,k}$ be the total length of the edges between E_j and E_k. Applying Proposition 3.2 we obtain, for $j = 3, 4$:

$$L_{j,0} \leq 2\sqrt{\pi}\sqrt{m(E_j)} = 2\sqrt{\pi} \tag{5.3}$$

Since p_1 and p_2 are the largest pressures and E_3 is not adjacent to E_4 we have, for $j = 3, 4$

$$L_{j,0}\, p_j \geq \sum_{k=1}^{4} L_{j,k}(p_j - p_k)$$

hence, by Lemma 2.7

$$L_{3,0}\, p_3 \geq \pi, \qquad L_{4,0}\, p_4 \geq \frac{\pi}{3} \tag{5.4}$$

and putting together with (5.3) we obtain

$$p_3 \geq \frac{\pi}{L_{3,0}} \geq \frac{\sqrt{\pi}}{2}, \qquad p_4 \geq \frac{\pi}{3L_{4,0}} \geq \frac{\sqrt{\pi}}{6}.$$

Now we are going to improve the estimates on p_1 and p_2. First notice that if we denote by ℓ_i the length of the external edge of C_i we have, by Proposition 3.3 (notice that $m(C_i) < k_3 < 1$),

$$\ell_i \leq \frac{m(C_i)}{|2 - m(C_i)|} P(\boldsymbol{E}) \leq \frac{m(C_i)}{2 - k_3} k_0$$

while, by the isoperimetric inequality, we have

$$P(C_i) \geq 2\sqrt{\pi}\sqrt{m(C_i)}.$$

Now, applying Proposition 3.5 to the component C_i with $i = 1, 2$, which has $n_i = i + 3$ edges, we have

$$\begin{aligned} p_i &\geq \frac{(6 - n_i)\pi}{3P(C_i)} + p_{\min}\left(1 - \frac{\ell_i}{P(C_i)}\right) \geq \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6}\left(1 - \frac{m(C_i)k_0}{(2 - k_3)2\sqrt{\pi}\sqrt{m(C_i)}}\right) \\ &= \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6}\left(1 - \frac{\sqrt{m(C_i)}k_0}{2\sqrt{\pi}(2 - k_3)}\right) \geq \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6}\left(1 - \frac{\sqrt{k_3}k_0}{2\sqrt{\pi}(2 - k_3)}\right) \geq \frac{(3 - i)\pi}{3k_9} + c_7 \end{aligned}$$

with $c_7 := 0.1992$. By using (5.4)

$$P(\boldsymbol{E}) = 2(p_1 + p_2 + p_3 + p_4) \geq 2\left(\frac{2\pi}{3k_9} + c_7 + \frac{\pi}{3k_9} + c_7 + \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{6}\right) = \frac{2\pi}{k_9} + 4c_7 + \frac{4}{3}\sqrt{\pi} \geq 11.9428 > k_0$$

which is a contradiction. □

6. Clusters with five components

In this section we consider a weak minimizer $\boldsymbol{E} \in \mathcal{M}^*(1, 1, 1, 1)$ with five bounded components. Only one region is disconnected: we will assume the region is E_1 and we denote with E_1' and C_1 respectively, its big and small connected components.

Proposition 6.1. *Let $\boldsymbol{E} \in \mathcal{M}^*(1, 1, 1, 1)$ be a cluster with 5 connected components. Then, up to a relabeling of the components, the topology of $\boldsymbol{E}$ is one of the cases represented in Figure 6.*

Proof. Suppose that E_1 is the only disconnected region and let E_1' and C_1 respectively be the big and small connected components of E_1. By Proposition 2.8 we know that $\partial\boldsymbol{E}$ is composed by 12 edges and 8 vertices moreover both E_1' and C_1 may have at most 4 edges if they are external and 3 edges if they are internal.

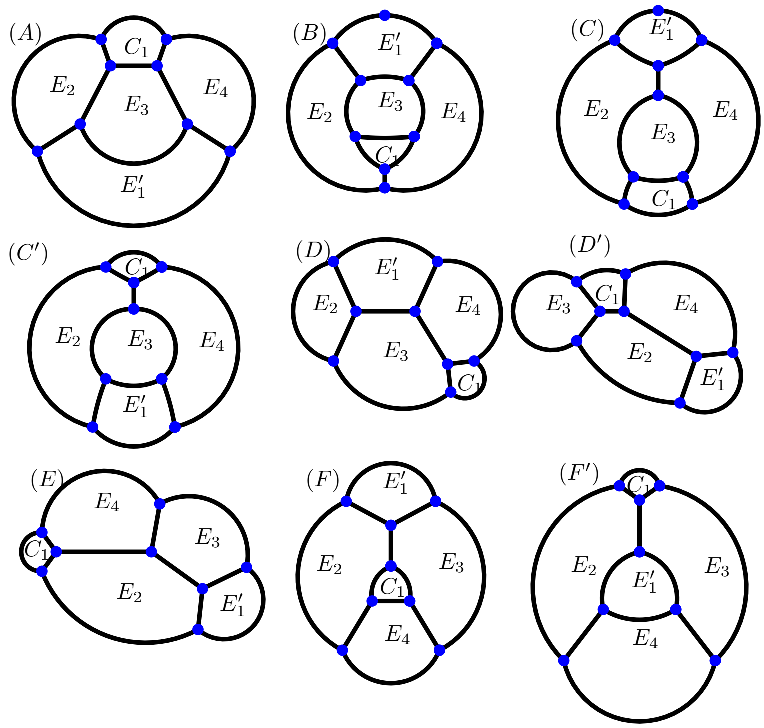


FIGURE 6. Classification of clusters with five components, Proposition 6.1.

Step 1. Suppose that both E'_1 and C_1 have four edges (and hence they are external). All the 8 vertices of the cluster are vertices of either E'_1 or C'_1 and both E'_1 and C_1 have an external edge with two external vertices. The external region E_0 has four edges.

The remaining two internal vertices of E'_1 must be connected with the two internal vertices of C_1 (remember that we cannot have two edges with the same end points, because two-sided components are not allowed). Hence the cluster is of type (A) in Figure 6.

Step 2. Suppose that E'_1 has 4 edges (hence it is external) and suppose C_1 is external with 3 edges. In this case we need to add an additional vertex v.

If v is external then the external region E_0 has five edges. The vertex v must be connected to an internal vertex of E'_1 while the other internal vertex of E'_1 must be connected to the internal vertex of C_1. The resulting topology is (D).

If, instead, the additional vertex v is internal, it must be connected to the two internal vertices of E'_1 and to the internal vertex of C_1. Hence we are in case (C').

Step 3. Suppose E'_1 has 4 edges (hence it is external) and suppose C_1 is internal with 3 edges. Since the external region must have at least three edges, there is an additional external vertex v and E_0 has three edges. One of the three vertices of C_1 must be connected to the vertex v while the other two vertices of C_1 must be connected to the two internal vertices of E'_1. The resulting topology is (B).

Step 4. Suppose E'_1 has 3 edges and is external while C_1 has four edges (and hence is external). We repeat the same reasoning of Step 2 with E'_1 and C_1 exchanged and we obtain cases (D') and (C).

Step 5. Suppose E_1' has 3 edges and is internal while C_1 has four edges (and hence is external). We repeat the same reasoning of Step 3 and obtain case (B) with E_1' and C_1 exchanged. But in this case we would have two big internal components: E_1' and E_3 and this is impossible in view of Corollary 4.6.

Step 6. Suppose that both E_1' and C_1 have three edges and are external. There are two additional vertices v, w which are not vertices of E_1' or C_1. Since the external region E_0 has at most 5 edges (there are only 5 bounded components) one of the two vertices, say v, is internal. The other vertex w cannot be internal, because otherwise v and w need to be joined by two different edges, which is not possible. The internal vertex v must be connected to w and to the two internal vertices of E_1' and C_1. The resulting topology is (E).

Step 7. Suppose that both E_1' and C_1 have three edges and suppose E_1' is external and C_1 is internal. We need to place two additional vertices v and w. Certainly one among v and w is external, since E_0 has at least three edges. In case both v and w are external E_0 has four edges.

If two of the three vertices of C_1 are connected to the same vertex, we would obtain an additional three sided component (say it is E_2). Hence we notice we have three components with three edges: E_1', C_1 and E_2. Let n_0, n_3 and n_4 be the number of edges of E_0, E_3 and E_4. By Euler's formula we have $24 = 3 \times 3 + n_0 + n_3 + n_4 \leq 9 + 4 + n_3 + n_4$, which means that $\max\{n_3, n_4\} \geq \frac{11}{2}$, i.e $\max\{n_3, n_4\} \geq 6$ (notice that n_3 and n_4 are integers), which is impossible by Proposition 2.8 (each component can only have one edge in common with each other component).

So the three vertices of C_1 are connected to v, w and to the internal vertex of E_1'. Necessarily v and w are also connected to the external vertices of E_1' hence they are both external and E_0 has 4 edges. The resulting cluster is of type (F).

Step 8. Suppose that both E_1' and C_1 have three edges and suppose that E_1' is internal and C_1 is external. We obtain the same classification of Step 7 but with E_1' and C_1 exchanged. We obtain case (F').

Step 9. Suppose that both E_1' and C_1' have three edges and are both internal. This is impossible because the external region would only have two edges, which is excluded. ☐

Proposition 6.2. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$. Then $\boldsymbol{E}$ cannot have the topologies (B), (C), (C′), (D), (D′), (E), (F) of Figure 6.*

Proof. Notice that in each case it is possible (by subsequently removing triangular components) to reduce the cluster $\boldsymbol{E}$ to a double bubble (E_1'', E_2'') where $E_1'' \supseteq E_1'$ and $E_2'' \supseteq E_2$.

So, by applying Lemma 3.8 we obtain at once

$$p_1 \leq \frac{k_8}{\sqrt{\min\{m(E_1'), m(E_2)\}}} = \frac{k_8}{\sqrt{1 - m(C_1)}} \leq \frac{k_8}{\sqrt{1 - k_1}} \leq 1.7352. \tag{6.1}$$

In the case when C_1 has only three edges (*i.e.* cases (B), (C'), (D), (E), and (F)) we can apply Proposition 3.5 and then Proposition 4.3 to obtain

$$p_1 \geq \frac{(6-3)\pi}{3P(C_1)} \geq \frac{\pi}{k_7} \geq 2.2125$$

and this is in contradiction with (6.1).

In both cases (C) and (D') we can reduce the triangular components to find a double bubble (E_2'', E_4'') with $E_2'' \supseteq E_2$ and $E_4'' \supseteq E_4$. Moreover $E_2'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_4)$ and $E_4'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$ so that $m(E_2'') \leq 3$ and $m(E_4'') \leq 3$. So, by using Lemma 3.8 we obtain

$$\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{\min\{4 - m(E_4), 4 - m(E_2)\}}} = \frac{k_8}{\sqrt{3}}.$$

In case (D') we can find another reduction to a double bubble (E_2'', E_3'') and, as before, we find

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}$$

so that, in this case, $\min\{p_2, p_3, p_4\} \geq k_8/\sqrt{3}$.

In case (C) we apply Proposition 3.5 to the component C_1 to obtain:

$$p_1 \geq \frac{(6-4)\pi}{3P(C_1)} \geq \frac{2\pi}{3k_7} \geq 1.4750 > 0.9179 \geq \frac{k_8}{\sqrt{3}}$$

and then we apply the same Proposition 3.5 to E_3 to obtain (notice that we consider $\ell = 0$ since E_3 is internal):

$$p_3 \geq \frac{(6-3)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} \geq \min\{p_1, p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

So, in both cases C and D', we obtain

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now we need to estimate the length ℓ of the external edge of C_1. By Proposition 3.3 we have (notice that $m(C_1) < k_1 < 1$),

$$\ell \leq \frac{m(C_1) \cdot P(\boldsymbol{E})}{|2 - m(C_1)|} \leq \frac{m(C_1)k_0}{2 - k_1}$$

while, by Proposition 4.3, we have

$$P(C_1) \leq k_7.$$

By applying Proposition 3.5, and using the previous estimates, we get

$$\begin{aligned} p_1 &\geq \frac{(6-4)\pi}{3P(C_1)} + \min\{p_2, p_3, p_4\}\left(1 - \ell \cdot \frac{1}{P(C_1)}\right) \\ &\geq \frac{2\pi}{3k_7} + \frac{k_8}{\sqrt{3}}\left(1 - \frac{m(C_1)k_0}{2 - k_1} \cdot \frac{1}{2\sqrt{\pi}\sqrt{m(C_1)}}\right) \\ &\geq \frac{2\pi}{3k_7} + \frac{k_8}{\sqrt{3}}\left(1 - \frac{\sqrt{k_1}k_0}{2\sqrt{\pi}(2 - k_1)}\right) \geq 1.7615 \end{aligned}$$

which, again, is in contradiction with (6.1). ☐

Proposition 6.3. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ has 5 components. Then we exclude that $\boldsymbol{E}$ has the topology (F') of Figure 6.*

Proof. By removing the triangular components we are able to reduce the cluster $\boldsymbol{E}$ to a double bubble (E_2'', E_3'') with $E_2'' \supseteq E_2$ and $E_3'' \supseteq E_3$. Notice that $E_2'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_3)$ and $E_3'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$ so that $m(E_2'') \leq 3$ and $m(E_3'') \leq 3$. So, by Lemma 3.8, we obtain

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}.$$

We repeat the same argument with E_4 in place of E_3 to obtain $\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}$ so that

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now we estimate the length ℓ of the external edge of C_1 by using Proposition 3.3:

$$\ell \leq \frac{m(C_1)}{|2-m(C_1)|} \cdot P(\boldsymbol{E})$$

i.e. (notice that $m(C_1) < k_1 < 1$)

$$\frac{\ell}{P(C_1)} \leq \frac{\ell}{2\sqrt{\pi}\sqrt{m(C_1)}} \leq \frac{\sqrt{m(C_1)}P(\boldsymbol{E})}{2\sqrt{\pi}(2-m(C_1))} \leq \frac{\sqrt{k_1}k_0}{2\sqrt{\pi}(2-k_1)}$$

and we apply Proposition 3.5 to obtain

$$\begin{aligned} p_1 &\geq \frac{(6-3)\pi}{3P(C_1)} + \min\{p_2,p_3\}\left(1-\frac{\ell}{P(C_1)}\right) \\ &\geq \frac{\pi}{k_7} + \frac{k_8}{\sqrt{3}}\left(1-\frac{\sqrt{k_1}k_0}{(2-k_1)2\sqrt{\pi}}\right) \geq c_8 := 2.4990. \end{aligned}$$

By Lemma 2.7 applied to the component E_1' we have

$$\begin{aligned} \pi = \frac{(6-3)\pi}{3} = \sum_{j=0}^{4}(p_1-p_j)L_j &\geq (p_1 - \max\{p_0,p_2,p_3,p_4\})P(E_1') \\ &= (p_1-\max\{p_2,p_3,p_4\})2\sqrt{\pi}\sqrt{1-k_1} \end{aligned}$$

so that

$$\max\{p_2,p_3,p_4\} \geq p_1 - \frac{\sqrt{\pi}}{2\sqrt{1-k_1}}$$

Hence

$$\begin{aligned} P(\boldsymbol{E}) &= 2(p_1+p_2+p_3+p_4) \geq 2(p_1+\max\{p_2,p_3,p_4\}+2\min\{p_2,p_3,p_4\}) \\ &\geq 4c_8 - 2\cdot\frac{\sqrt{\pi}}{2\sqrt{1-k_1}} + 4\cdot\frac{k_8}{\sqrt{3}} \geq 11.5561 \geq k_0 \end{aligned}$$

which is a contradiction. □

Proposition 6.4. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be a cluster with 5 components. Then we exclude that $\boldsymbol{E}$ has the topology (A) depicted in Figure 6.*

Proof. First of all notice that

$$\begin{aligned} 2k_0 \geq 2P(\boldsymbol{E}) &= P(E_1') + P(C_1) + P(E_2) + P(E_4) + P(E_0) + P(E_3) \\ &\geq 2\sqrt{\pi}\left(\sqrt{1-k_1}+\sqrt{k_2}+2\sqrt{1}+\sqrt{4}\right) + P(E_3) \end{aligned}$$

so that

$$P(E_3) \leq 2k_0 - 2\sqrt{\pi}\left(\sqrt{1-k_1}+\sqrt{k_2}+4\right) \leq c_9 := 4.4111.$$

Now let ℓ_j be the total length of the external edges of the region E_j $(j=1,2,4)$. If we remove E_1 from $\boldsymbol{E}$ we obtain a 3-cluster $\boldsymbol{E}' = (E_2,E_3,E_4)$ with $\boldsymbol{E}' \in \mathcal{C}^*(1,1,1)$. Hence, by Lemma 3.9 we have $P(\boldsymbol{E}') \geq k_{10}$. Moreover

$$\ell_1 = P(\boldsymbol{E}) - P(\boldsymbol{E}') \leq k_0 - k_{10}.$$

We can repeat the same argument for ℓ_2 and ℓ_4 to obtain

$$\max\{\ell_1, \ell_2, \ell_4\} \leq k_0 - k_{10}. \tag{6.2}$$

By Proposition 3.5 we have (notice that we let $\ell = 0$ since E_3 is internal)

$$p_3 \geq \frac{(6-4)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} > \min\{p_1, p_2, p_4\}. \tag{6.3}$$

The same proposition applied to the component C_1 gives

$$p_1 \geq \frac{(6-4)\pi}{3P(C_1)} \geq \frac{2\pi}{3k_7} \geq 1.4750.$$

Since

$$k_0 \geq P(\boldsymbol{E}) = 2(p_1 + p_2 + p_3 + p_4) \geq 2p_1 + 6\min\{p_2, p_3, p_4\} \geq \frac{4\pi}{3k_7} + 6\min\{p_2, p_3, p_4\},$$

we obtain

$$\min\{p_2, p_3, p_4\} \leq \frac{k_0}{6} - \frac{2\pi}{9k_7} \leq 1.3744,$$

so that

$$p_1 > \min\{p_2, p_3, p_4\}. \tag{6.4}$$

Putting together (6.3) and (6.4) we can say that the minimum among p_1, p_2, p_3, p_4 is either p_2 or p_4. Without loss of generality we can assume that such a minimum is p_2.

Hence, applying Lemma 2.7 to the region E_2 we obtain

$$\frac{(6-4)\pi}{3} = \sum_{i=0}^{4}(p_2 - p_i)L_i \leq p_2\ell_2$$

where L_i is the total length of the edges between E_2 and E_i (so that $L_0 = \ell_2$) and we used the estimate $p_2 - p_i \leq 0$ for $i \neq 0$. So, using (6.2)

$$\min\{p_1, p_2, p_3, p_4\} = p_2 \geq \frac{2\pi}{3\ell_2} \geq \frac{2\pi}{3(k_0 - k_{10})}.$$

Now, use again Proposition 3.5 on the region E_3 to obtain

$$p_3 \geq \frac{(6-4)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} \geq \frac{2\pi}{3c_9} + \frac{2\pi}{3(k_0 - k_{10})} \geq c_{10} := 1.3466. \tag{6.5}$$

Finally we apply Lemma 2.7 to the region E_0 to obtain

$$\frac{(6+4)\pi}{3} = p_1\ell_1 + p_2\ell_2 + p_4\ell_4 \leq \max\{\ell_1, \ell_2, \ell_4\}(p_1 + p_2 + p_4)$$

hence, using also (6.2)

$$p_1 + p_2 + p_4 \geq \frac{10\pi}{3(k_0 - k_{10})}.$$

So, using also (6.5), we have

$$P(\boldsymbol{E}) = 2p_3 + 2(p_1 + p_2 + p_4) \geq 2c_{10} + \frac{20\pi}{3(k_0 - k_{10})} \geq 11.4116 > k_0$$

which is a contradiction. $\square$

Theorem 6.5. *Let $\boldsymbol{E} \in \mathcal{M}(1,1,1,1)$. Then $\boldsymbol{E}$ is connected.*

Proof. By Corollary 5.1 we know that $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$.

By Corollary 4.5 and by Proposition 4.9 we know that each region E_i has exactly one big component and the total number of small components is not larger than two.

If the cluster has exactly two small components, with Corollary 5.2, we exclude that they belong to the same region and with Proposition 5.3 we exclude that they belong to two different regions.

Finally, from Proposition 6.1 and Propositions 6.2, 6.3 and 6.4 we exclude that the cluster has exactly one small connected component (which means five connected components in total). □

7. Connected clusters (four components)

Proposition 7.1. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be a connected cluster. Then $\boldsymbol{E}$ has one of the two topologies depicted in Figure 1.*

Proof. Since every region is connected, by Proposition 2.8 every region (comprising E_0) has three or four edges and the cluster has a total of nine edges and six vertices. Let x be the number of regions (bounded or unbounded) with four edges and let y the number of regions (bounded or unbounded) with three edges. We have one unbounded region E_0 and four bounded regions, hence: $x + y = 5$. Moreover summing up all the edges of all the regions we would count each edge twice, hence we have: $4x + 3y = 18$. Solving the system of two equations gives $x = 3$, $y = 2$ hence we have three regions with four edges and two regions with four edges.

If the unbounded region E_0 has three edges (note that there is a total of six vertices), there is one internal region and three external regions. The internal region can only have three edges (because it is not adjacent to E_0) and we are in the first case of the statement.

If the unbounded region E_0 has four edges, all the bounded regions are external: two of them have three edges and two have four edges. The regions with four edges are adjacent to all other regions hence the regions with three edges don't touch each other. We are in the second case of the statement. □

Proposition 7.2. *Let $\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$ be a connected cluster. Then $\boldsymbol{E}$ has not the flower topology.*

Proof. Suppose by contradiction that $\boldsymbol{E}$ has the flower topology and let E_1 be the internal three sided region.

First of all notice that

$$k_0 \geq P(\boldsymbol{E}) \geq P(E_0) + P(E_1) \geq 2\sqrt{\pi}\sqrt{4} + P(E_1)$$

so that

$$P(E_1) \leq k_0 - 4\sqrt{\pi} \leq c_{11} := 4.1064.$$

Now let ℓ_2 be the length of the external edge of the region E_2. If we remove E_2 from $\boldsymbol{E}$ we obtain a 3-cluster $\boldsymbol{E}' = (E_1, E_3, E_4)$ with $\boldsymbol{E}' \in \mathcal{C}^*(1,1,1)$. Hence, by Lemma 3.9 we have $P(\boldsymbol{E}') \geq k_{10}$. Moreover

$$\ell_2 = P(\boldsymbol{E}) - P(\boldsymbol{E}') \leq k_0 - k_{10}.$$

We can repeat the same argument for the lengths ℓ_3 and ℓ_4 of the external edges of E_3 and E_4, to obtain

$$\max\{\ell_2, \ell_3, \ell_4\} \leq k_0 - k_{10}. \tag{7.1}$$

By removing the triangular components we are able to reduce the cluster $\boldsymbol{E}$ to a double bubble (E_2'', E_3'') with $E_2'' \supseteq E_2$ and $E_3'' \supseteq E_3$. Notice that $E_2'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_3)$ and $E_3'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$ so that $m(E_2'') \leq 3$ and $m(E_3'') \leq 3$. So, by Lemma 3.8, we obtain

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}.$$

We repeat the same argument with E_4 in place of E_3 to obtain $\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}$ so that

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now, use again Proposition 3.5 on the region E_1 to obtain (notice that we let $\ell = 0$ since E_1 is internal)

$$p_1 \geq \frac{(6-3)\pi}{3P(E_1)} + \min\{p_2, p_3, p_4\} \geq \frac{\pi}{c_{11}} + \frac{k_8}{\sqrt{3}} \geq c_{12} := 1.6829. \tag{7.2}$$

Finally we apply Lemma 2.7 to the region E_0 to obtain

$$\frac{(6+3)\pi}{3} = p_2\ell_2 + p_3\ell_3 + p_4\ell_4 \leq \max\{\ell_1, \ell_2, \ell_4\} \cdot (p_2 + p_3 + p_4)$$

hence, using also (7.1)

$$p_2 + p_3 + p_4 \geq \frac{3\pi}{k_0 - k_{10}}.$$

So, using also (7.2), we have

$$P(\boldsymbol{E}) = 2p_1 + 2(p_2 + p_3 + p_4) \geq 2c_{12} + \frac{6\pi}{k_0 - k_{10}} \geq 11.2124 > k_0$$

which is a contradiction. □

Theorem 7.3. *Let $\boldsymbol{E} \in \mathcal{M}(1,1,1,1)$. Then $\boldsymbol{E}$ has the* sandwich *topology as in Figure 1.*

Proof. By Theorem 6.5 we know that $\boldsymbol{E}$ is connected so by Proposition 7.1 we know that $\boldsymbol{E}$ can either have the flower or the sandwich topology. With Proposition 7.2 we exclude the flower topology and the result follows. □

Conjecture 7.4. *Up to isometries there is a unique stationary cluster $\boldsymbol{E} \in \mathcal{C}(1,1,1,1)$ with the* sandwich *topology. In such a cluster the regions E_1 and E_3 are isometric to E_2 and E_4 (respectively).*

A stationary sandwich cluster with given external radii r_1, r_2, r_3, r_4 can be uniquely constructed by taking the double bubble with external radii r_1, r_2 and then "growing" a triangular region in each vertex so that it reaches the prescribed external radius In fact both the area and the radii of a growing triangular camera are strictly increasing, see [13, 23]

Not only that, since when the triangular region grows, the area of the quadrangular regions decreases, if r_1 and r_2 are fixed it is possible to find $r_3 = r_4$ so that the area of the triangular regions become equal to the area of the smaller quadrangular region. If $r_1 = r_2$ we then find that all four regions have equal area and with a rescaling we obtain a stationary symmetric sandwich cluster in $\mathcal{C}(1,1,1,1)$.

We believe that when $r_1 > r_2$ then the previous construction would yield a cluster where $m(E_1) > m(E_2) = m(E_3) = m(E_4)$. This would exclude that asymmetric sandwich cluster can have all equal areas.

We have some numerical computations [18] which confirm this.

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