ISOMETRIES OF CARNOT GROUPS AND SUB-FINSLER HOMOGENEOUS MANIFOLDS

ENRICO LE DONNE AND ALESSANDRO OTTAZZI

Abstract. We show that isometries between open sets of Carnot groups are affine. This result generalizes a result of Hamenstädt. Our proof does not rely on her proof. We show that each isometry of a sub-Riemannian manifold is determined by the horizontal differential at one point. We then extend the result to sub-Finsler homogeneous manifolds. We discuss the regularity of isometries of homogeneous manifolds equipped with homogeneous distances that induce the manifold topology.

1. Introduction

A fundamental problem in geometry is the study of spaces that are isometrically homogeneous, i.e., metric spaces on which the group of isometries acts transitively. Such spaces have particular differentiable structures when in addition they are finite dimensional, locally compact, and their distance is geodesic. Indeed, one can characterize these spaces as particular sub-Finsler manifolds by using the theory of locally compact groups and methods from Lipschitz analysis on metric spaces, [17, 8, 27, 6, 7]. Despite the fact that the group of global isometries of these manifolds is a Lie group and acts smoothly and by smooth maps, the local isometries are still not completely understood. By 'local isometry' we mean isometry between open subsets. In this paper we give a complete description of the space of local isometries for those homogeneous spaces that also admit dilations. These spaces, called Carnot groups, are particular nilpotent groups equipped with left-invariant geodesic distances.

The study of isometries of distinguished Riemannian manifolds, such as homogeneous spaces, symmetric spaces, and Lie groups, has been a flourishing subject. References for the regularity of isometries are the seminal papers [29, 30], see also [10, 34]. Regarding the Finsler category, we mention the work [15]. The space of isometries is well studied in Banach spaces, see [16, 4]. A number of authors have tried to understand isometries of sub-Riemannian manifolds, see [32, 33, 18, 22, 20]. For sub-Finsler homogeneous spaces, we refer to [5, 6], see also [13]. Sub-Finsler geometry is needed to deal with first-order differential operators over vector bundles on a manifold, see [14].

Regarding Carnot groups, U. Hamenstädt [18] showed that isometries are affine, in the case that the isometry is globally defined and that the distance is sub-Riemannian. We say that a homeomorphism of a group (equipped with a left-invariant distance inducing the manifold topology) is *affine* if it is the composition of a left translation with a group isomorphism.

We generalize Hamenstädt's result to the setting of a sub-Finsler distance and isometries defined only on some open set. We need to point out, that to obtain such a local result, one cannot use the same argument as in [18] to deduce smoothness of the map. Actually, the issue of smoothness was a

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subtle point, which was clarified only later by I. Kishimoto in [22] for global isometries. Moreover, in Hamenstädt's strategy, one needs to consider a blow down of the isometry, which requires the map to be globally defined. Hence, we shall provide a new method of proof. In fact, our technique also shows that, as in Riemannian geometry, isometries of homogeneous spaces are uniquely determined by their 'first-order' expansion at one point. Throughout the paper, Lie groups and manifolds are always finite dimensional.

1.1. **Statement of the main results.** Let G be a Lie group and H be a compact subgroup. Let $M = G/H$ be the homogeneous manifold of left cosets. Hence, the group G acts transitively on M on the left. Let Δ be a G-invariant subbundle of the tangent bundle TM. We assume that Δ is bracket-generating and call it the *horizontal bundle*. Fix a norm on Δ_p, for every $p \in M$, and assume that it is G-invariant. Then the Carnot-Carathéodory distance between two points of the manifold is the infimum of the lengths of curves tangent to Δ and connecting the two points. Since the length is measured using the norm, the distance is also called *sub-Finsler*. Unlike the general case, we shall only consider symmetric norms, i.e., each $X \in \Delta_p$ has the same norm as $-X$. We do so because we are interested in a metric-space viewpoint. Theorem 2.6 will clarify that we are in fact considering very general homogeneous metric spaces.

When H is trivial, i.e., $M = G$, we set $V_1 := \Delta_e$ and $V_{j+1} := [V_1, V_j]$. If

$$\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_s,$$

then the space is called a *sub-Finsler Carnot group*. See Section 2 for more details.

Our first theorem characterizes local isometries between sub-Finsler Carnot groups as affine maps.

Theorem 1.1. *Let G_1, G_2 be sub-Finsler Carnot groups and for $i = 1, 2$ consider $\Omega_i \subset G_i$ open sets. If $F : \Omega_1 \to \Omega_2$ is an isometry, then there exists a left translation τ on G_2 and a group isomorphism ϕ between G_1 and G_2, such that F is the restriction to Ω_1 of $\tau \circ \phi$, which is a global isometry.*

The fact that G_1 is isomorphic to G_2 is a consequence of Pansu's Differentiability Theorem [31]. Note that in the statement above we require the domain Ω_1 to be open. In Section 5, we shall see that this assumption is necessary, unlike in the Euclidean case. However, connectedness is not required.

In particular, Theorem 1.1 states that *global* isometries of Carnot groups are affine maps. This fact, already present in [18, 22], is of crucial importance in the paper [12], where L. Capogna and the first-named author prove the smoothness of isometries of sub-Riemannian manifolds. One of the steps in their argument is the invariance of Popp measures, which rests on the global version of Theorem 1.1.

Later in the paper, with the tools of [12], we show that isometries of a sub-Riemannian manifold are characterized by their value at one point and the differential at this point (see Corollary 4.1 and also Remark 4.3). In [32, 33], similar conclusions were obtained for particular sub-Riemannian manifolds. In the case that for us is of particular interest, i.e., the class of sub-Finsler homogeneous manifolds, the result reads as follows.

Theorem 1.2. *For $i = 1, 2$, let $M_i = (G_i/H_i, d_i)$ be a connected homogeneous manifold equipped with a G_i-invariant sub-Finsler distance inducing the manifold topology. Let Δ be the horizontal bundle of M_1. Let $\Omega_i \subset M_i$ be open sets. Let $F_1 : \Omega_1 \to \Omega_2$ be an isometry. Then F_1 is C^∞. Moreover, if $F_2 : \Omega_1 \to \Omega_2$ is another isometry with the properties that $F_1(p) = F_2(p)$ and $(dF_1)_p|_{\Delta_p} = (dF_2)_p|_{\Delta_p}$, for some $p \in M_1$, then $F_1 = F_2$.*

The differentiability of the maps in the above theorems is a subtle point. Indeed, the fact that isometries of Carnot groups are smooth is for us a consequence of a result by Capogna and M. Cowling

[11]. Regarding smoothness of isometries of sub-Finsler homogeneous manifolds, we use a well-known trick that uses John's ellipsoids to bring the problem back to the sub-Riemannian case, which is solved in [12]. However, when the isometries are globally defined in the whole homogeneous manifold, we prove that they are analytic maps. This result holds true on any homogeneous space G/H equipped with a G-invariant homogeneous distance which induces the manifold topology but is not necessarily sub-Finsler.

Even under the additional assumption that the manifold in Theorem 1.2 is actually a Lie group, one cannot have the same conclusions as in Theorem 1.1. Namely, for general Lie groups, it is not true that isometries are necessarily affine maps, as, for example, some rotations in the group of Euclidean motions. Moreover, both in Riemannian geometry and sub-Riemannian geometry, there are cases of isometric Lie groups that are not isomorphic. Also, local isometries of homogeneous spaces do not always extend to global isometries. For all these last remarks see Section 5.

Theorem 1.2 can be rephrased saying that every isometry F is determined by $F(p)$ and by its blow up at p. Namely, every quasiconformal map between two Carnot-Carathéodory spaces admits a blow up map at almost every point that is an isomorphism between two Carnot groups, see [24]. If F is an isometry, as in Theorem 1.2, then F is smooth and $dF(p)|_{\Delta_p}$ coincides with the differential of the blow up at p restricted to Δ_p.

Theorem 1.3. *Let G/H be a homogeneous space of a Lie group G modulo a compact subgroup H. Assume that d is a G-invariant distance that induces the manifold topology. If $F : (G/H, d) \to (G/H, d)$ is an isometry, then F is analytic.*

The above theorem is a consequence of Montgomery-Zippin's solution of Hilbert's fifth problem, from which it is immediate that the group of isometries $\mathrm{Iso}(M, d)$ of the manifold $M = G/H$ is a Lie group acting transitively on M. Hence, M admits an analytic structure for which $\mathrm{Iso}(M, d)$ acts by analytic maps. Our result states that this analytic structure coincides with the initial one.

1.2. **Structure of the paper.** The paper is organized as follows. In Section 2 we fix the notation and we show some results that will be used in the proof of our main results. Section 3 is devoted to the proof of Theorem 1.1, whereas in Section 4 we prove Theorem 1.2 and Theorem 1.3. We conclude the paper with Section 5, where we collect a number of remarks and counterexamples.

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2. Preliminaries

2.1. **General notation.** Let G be a Lie group. Denote by $\mathrm{Lie}(G)$ the Lie algebra of G whose elements are tangent vectors at the identity e of G.

Let H be a closed subgroup of G. Hence, the space G/H of left cosets gH, with $g \in G$, has a natural structure of an analytic manifold, see [19, page 123]. We may assume that H does not contain any non-trivial normal subgroup of G, for otherwise we may factor it out. Given our later assumptions, there is no loss of generality in assuming H to be compact; see Remark 2.4. The group G is a Lie transformation group of $M = G/H$. Namely, every element $g \in G$ acts by left translations on M, i.e., induces the diffeomorphism

$$L_g : g'H \mapsto g \cdot (g'H) := gg'H. \tag{2.1}$$

We fix a G-invariant subbundle Δ of the tangent bundle TM of M. The choice of Δ can be seen in the following way. There is a one-to-one correspondence between H-invariant subspaces Δ_H in $T_H(M)$ and Ad_H-invariant subspaces V in $\mathrm{Lie}(G)$ that contain $\mathrm{Lie}(H)$. We choose such a subspace Δ_H in $T_H(G/H)$, and therefore, a corresponding $V \subseteq \mathrm{Lie}(G)$. Then, for all $gH \in G/H$, the subbundle Δ is defined as

$$\Delta_{gH} := (dL_g)_H \Delta_H,$$

The subbundle is well defined, i.e., the definition does not depend on the representative in gH, because Δ_H is H-invariant.

If the subspace $V \subset \mathrm{Lie}(G)$ associated to Δ_H has the property that $\mathrm{Lie}(G)$ is the smallest Lie subalgebra of $\mathrm{Lie}(G)$ containing V, then V (or, equivalently, Δ) is said to be *bracket-generating*.

We shall fix a norm on Δ that is G-invariant. These norms exist when H is compact; for example, they can be obtained by averaging any norm on Δ by H. The choice of such a norm can be seen in the following way. Fix a seminorm on V that is Ad_H-invariant and for which the kernel is $\mathrm{Lie}(H)$. The projection from G to M gives an H-invariant norm $\|\cdot\|$ on Δ_H. Hence, we have an induced G-invariant norm on Δ by

$$\|v\| := \left\|(dL_{g^{-1}})_{gH} v\right\|, \qquad \forall v \in \Delta_{gH}.$$

Since the initial norm is Ad_H-invariant, it follows that the above equation is independent of the choice of the representative in gH.

An absolutely continuous curve $\gamma : [0,1] \to M$ is said to be *horizontal* (with respect to Δ) if the derivative $\dot\gamma(t)$ belongs to Δ, for almost every $t \in [0,1]$. Each horizontal curve γ has an associated length defined as

$$L(\gamma) := \int_0^1 \|\dot\gamma(t)\| \, \mathrm{d}t.$$

Definition 2.2 (sub-Finsler homogeneous manifolds)**.** Let $M = G/H$ be a homogeneous space formed by a Lie group G modulo a compact subgroup H. We are given a bracket-generating G-invariant subbundle $\Delta \subseteq TM$ and a G-invariant norm $\|\cdot\|$ on Δ. We may choose the norm by fixing an Ad_H-invariant seminorm on a bracket-generating subspace $V \subseteq \mathrm{Lie}(G)$ whose kernel is $\mathrm{Lie}(H)$. The *sub-Finsler distance* between two points $p, q \in M$ is defined as

$$d(p,q) := \inf\{L(\gamma) \mid \gamma \text{ horizontal and } \gamma(0) = p, \gamma(1) = q\}. \tag{2.3}$$

We call the pair (M, d) a *sub-Finsler homogeneous manifold*.

By Chow's Theorem [28, Chapter 2], the topology induced by d coincides with the manifold topology. Notice that, by construction, the above sub-Finsler distance is left-invariant, i.e., every left translation (2.1) is an isometry of (M, d).

Remark 2.4. We remark that if a homogeneous space $M = G/H$ with H closed admits a G-invariant distance and the action of G on G/H is effective, i.e., kernel free, then H is compact. Indeed, H is compact by the Ascoli-Arzelà theorem, as it is the stabilizer of the origin in M. Since the isometry group of a homogeneous manifold is a Lie group, we conclude that any sub-Finsler homogeneous manifold M is of the form G/H with G a Lie group and H a compact subgroup.

Definition 2.5 (sub-Finsler Carnot groups)**.** Given a subspace V_1 of the Lie algebra of a Lie group G, define the subspaces V_j as

$$V_j := [V_1, V_{j-1}], \qquad \forall j > 1.$$

If

$$\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_k \oplus \cdots,$$

then G is said to be a (nilpotent) *stratified group* and V_1 is called the *first stratum* (of the stratification $\{V_j\}$). If d is the sub-Finsler distance associated to $(G, V_1, \|\cdot\|)$, then the pair (G, d) is called a *sub-Finsler Carnot group*, or simply a Carnot group.

Sub-Finsler Carnot groups are special cases of sub-Finsler homogeneous manifolds that admits metric dilations, coming from the stratification. Indeed, one considers the family of linear operators $\delta_t : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ for $t > 0$, which act by scalar multiplication by t^i on V_i. These operators are Lie algebra automorphisms which preserve V_1, and define correspondingly Lie group automorphisms on G, for which $d(\delta_t(p), \delta_t(q)) = td(p, q)$.

By the work of V. N. Berestovskiĭ, we know that sub-Finsler homogeneous manifolds are the only metric spaces that are isometrically homogeneous, are locally compact, have finite topological dimension, and whose distance is intrinsic (see, e.g., [9]). This result is based on Montgomery-Zippin's characterization of Lie groups, see Theorem 4.4.

Theorem 2.6 (Consequence of [27], [7], and [26], see also [23])**.** *Let X be a locally compact and finite-dimensional topological space. Assume that X is equipped with an intrinsic distance d such that its isometry group $\mathrm{Iso}(X, d)$ acts transitively on X. Then (X, d) is isometric to a sub-Finsler homogeneous manifold.*

If, moreover, the space (X, d) admits a nontrivial dilation, i.e., there exists $\lambda > 1$ such that $(X, \lambda d)$ is isometric to (X, d), then (X, d) is a sub-Finsler Carnot group.

If the norm in Definition 2.2 comes from a scalar product, then the associated distance is called *sub-Riemannian*. If this is the case for a sub-Finsler Carnot group, then we call it a *sub-Riemannian Carnot group*. More generally, a *sub-Riemannian manifold* is a triplet (M, Δ, ρ) where M is a connected smooth manifold, Δ is a subbundle of the tangent bundle TM that bracket-generates TM, and ρ is a Riemannian metric restricted to vectors in Δ, see [28]. Given a sub-Riemannian manifold (M, Δ, ρ), we iteratively set $\Delta^1 := \Delta$, and $\Delta^{k+1} := \Delta^k + [\Delta^k, \Delta]$, for $k \in \mathbb{N}$. Namely, for every $p \in M$,

$$\Delta_p^k = \mathrm{span}\{[Y_1, [Y_2, [\dots [Y_{l-1}, Y_l]]]]_p \mid l \leq k, Y_j \in \Gamma(\Delta), \forall j = 1, \dots, l\}.$$

The bracket-generating condition is expressed by the existence of $s \in \mathbb{N}$ such that, for all $p \in M$, one has

$$\Delta_p^s = T_p M.$$

A sub-Riemannian manifold (M, Δ, ρ) is *equiregular* if, for all $i \in \mathbb{N}$, the dimension of Δ_p^k is constant in $p \in M$. In other words,

$$\Delta^1 \subseteq \Delta^2 \subseteq \dots \subseteq \Delta^s = TM$$

is a flag of subbundles.

Sub-Riemannian manifolds have an associated length structure on horizontal curves and a distance defined as in (2.3). One can show [28] that a curve in a sub-Riemannian manifold has finite length if and only if it is a horizontal curve, up to reparametrization. Consequently, if an isometry F of a sub-Finsler homogeneous manifold is C^1, then it is a *contact map*, i.e., its differential preserves the subbundle. Namely,

$$dF_p(\Delta_p) \subseteq \Delta_{F(p)}, \qquad \forall p \in G.$$

Moreover, F being an isometry, $dF_p|_{\Delta_p}$ is an isometry between the two normed vector spaces Δ_p and $\Delta_{F(p)}$.

2.2. **Riemannian extensions and the horizontal differential.** In this section we show that every smooth isometry of a sub-Riemannian manifold is determined by its value at one point and that of its differential at that point, restricted to the subbundle Δ.

Given a sub-Riemannian metric (M, Δ, ρ) we say that a Riemannian metric $\hat{\rho}$ is *a Riemannian extension* of ρ if ρ equals $\hat{\rho}$ restricted to Δ. For a metric space $M = (M, d)$ we denote by $\mathrm{Iso}(M)$ the group of isometries of M. Moreover, for all $p \in M$, we set

$$\mathrm{Iso}_p(M) = \{F \in \mathrm{Iso}(M) \mid F(p) = p\}.$$

Lemma 2.7. *Let M be a sub-Riemannian manifold and $p \in M$. Assume that $\mathrm{Iso}_p(M) \subset C^\infty(M)$. Then there exists a Riemannian extension $\hat{\rho}$ on M such that $\mathrm{Iso}_p(M) \subset \mathrm{Iso}(M, \hat{\rho})$.*

Proof. Let $H := \mathrm{Iso}_p(M)$. By Ascoli-Arzelà's theorem, H is compact. Actually by a result of Montgomery and Zippin [27, Theorem 2 at page 208 and Theorem at page 212], H is a compact Lie group acting smoothly on M. In particular, the map $F \mapsto dF$ with $F \in H$ is continuous. Let μ_H be a probability Haar measure on H. Fix an auxiliary Riemannian extension $\tilde{\rho}$ on M and define

$$\hat{\rho} = \int_H F_* \tilde{\rho} \, \mathrm{d}\mu_H(F).$$

Here, $F_*\tilde{\rho}(v, w) := \tilde{\rho}(dF(v), dF(w))$. Since by assumption H acts on M smoothly, the set $\mathcal{G} := \{F_*\tilde{\rho} : F \in H\}$ is a compact set of Riemannian tensors extending the sub-Riemannian metric on M. In particular, in local coordinates, all $\rho' \in \mathcal{G}$ can be represented with a matrix with uniformly bounded eigenvalues. Hence, $\hat{\rho}$ is a Riemannian tensor, which is H-invariant by the linearity of integrals. $\square$

Proposition 2.8. *Let (M, Δ, ρ) be a connected equiregular sub-Riemannian manifold. Let $\tilde{\rho}$ be a Riemannian extension of ρ. Let $F : M \to M$ be a C^∞ map. Assume that F is an isometry for both ρ and $\tilde{\rho}$. If there exists $p \in M$ such that $F(p) = p$ and $dF|_{\Delta_p} = \mathrm{id}_{\Delta_p}$ then $dF|_{T_pM} = \mathrm{id}_{T_pM}$ and hence $F = \mathrm{id}_M$.*

Proof. Since (M, Δ, ρ) is equiregular, in a neighborhood of p we can find a frame $X_1, \ldots, X_n$ of TM such that, if $r_k = \dim \Delta^k_p$, then $X_1, \ldots, X_{r_k}$ is a frame of Δ^k, for every $k = 1, \ldots, s$. We shall prove that $dF|_{\Delta^k_p} = Id_{\Delta^k_p}$ by induction on k. The case $k = 1$ holds by assumption. Assume now that $dF|_{\Delta^l_p} = Id_{\Delta^l_p}$ for every $l < k$. Let $X \in \Gamma(\Delta)$, $Y \in \Gamma(\Delta^{k-1})$. Since dF preserves the horizontal bundle, $dF(\Gamma(\Delta^j)) = \Gamma(\Delta^j)$ for every j. Therefore

$$dF(X) = X + \sum_{j=1}^{r_1} a^j X_j \quad \text{and} \quad dF(Y) = Y + \sum_{l=1}^{r_{k-1}} b^l X_l.$$

Moreover, $a^j(p) = b^l(p) = 0$ for every j, l by the inductive hypothesis. Hence

$$\begin{aligned}
dF[X, Y] &= [dF(X), dF(Y)] \\
&= [X + \sum_{j=1}^{r_1} a^j X_j, Y + \sum_{l=1}^{r_{k-1}} b^l X_l] \\
&= [X, Y] + \sum_l X b^l X_l + \sum_j a^j [X_j, Y] + \sum_l b^l [X, X_l] \\
&\quad - \sum_j Y a^j X_j + \sum_{j,l} a^j b^l [X_j, X_l] + \sum_{j,l} a^j X_j b^l X_l - \sum_{j,l} b^l X_l a^j X_j.
\end{aligned}$$

At p,

$$dF[X,Y]_p = [X,Y]_p + \sum_l (Xb^l)_p (X_l)_p - \sum_j (Ya^j)_p (X_j)_p. \tag{2.9}$$

Note that $\sum_l (Xb^l)_p (X_l)_p - \sum_j (Ya^j)_p (X_j)_p$ is in Δ_p^{k-1}. Hence, on the one hand the linear map dF_p acts as a unipotent matrix on Δ_p^k. On the other hand, since F is a Riemannian isometry, the linear map dF_p is an orthogonal matrix. Since the only unipotent and orthogonal matrix is the identity, we conclude that $dF|_{\Delta_p^k} = Id_{\Delta_p^k}$. The classic argument based on the Riemannian geodesics gives that $F = \mathrm{id}_M$. □

Corollary 2.10. *Let M and N be two connected, equiregular, sub-Riemannian manifolds. Assume that $\mathrm{Iso}_p(M) \subset C^\infty(M)$. Let $p \in M$ and let Δ be the horizontal bundle of M. Let $F_1, F_2 : M \to N$ be two C^∞ isometries. If $F_1(p) = F_2(p)$ and $dF_1|_{\Delta_p} = dF_2|_{\Delta_p}$, then $F_1 = F_2$.*

Proof. This follows from Lemma 2.7 and Proposition 2.8 applied to $F_1 \circ F_2^{-1}$. □

2.3. **Disconnected domains.**

The proposition proved in this section will be instrumental to show that our result on isometries of Carnot groups (Theorem 1.1) holds with no connectedness assumptions on the domain of the isometries.

Regarding the next theorem, notice that every homogeneous sub-Riemannian manifold is analytic, i.e., both the manifold and the subbundle are analytic.

Theorem 2.11 ([1, Theorem 1]). *Let M be an analytic sub-Riemannian manifold and set $q_0 \in M$. Then there exists an open and dense subset $\Sigma_{q_0} \subseteq M$ such that for any $q \in \Sigma_{q_0}$ there exists a unique length minimizing curve γ connecting q_0 to q. Moreover, the curve γ is analytic.*

The following result holds for general sub-Riemannian manifolds. We show that an isometry is completely determined by its behavior on an open set.

Proposition 2.12. *Let M be an analytic sub-Riemannian manifold. Let $F : \Omega_1 \to \Omega_2$ be an isometry of two open sets in M. Assume that F is the identity on an open subset Ω of Ω_1. Then F is the identity.*

Proof. Pick $q \in \Omega_1$. According to the notation in Theorem 2.11, consider $\Sigma = \Sigma_q \cap \Sigma_{F(q)}$. Fix $p \in \Sigma \cap \Omega$. Since $p \in \Sigma_q$, Theorem 2.11 implies that there exists a unique and analytic length minimizing curve γ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since Ω is open, one can choose $s_0 \in (0,1)$ such that $p' := \gamma(s_0) \neq p$ and $\gamma(s_0) \in \Omega$. Denote by σ a length minimizing curve such that $\sigma(0) = p'$ and $\sigma(1) = F(q)$. Let $\tilde\gamma$ be the curve formed by joining $\gamma|_{[0,s_0]}$ with σ. We claim that $\tilde\gamma$ minimizes the length between p and $F(q)$. Indeed, since $F(p) = p$, $F(p') = p'$, and F is an isometry,

$$\begin{aligned} d(p, F(q)) &= d(p,q) = d(p,p') + d(p',q) \\ &= d(p,p') + d(p', F(q)). \end{aligned}$$

So $\tilde\gamma$ attains the distance from p to $F(q)$. Since $p \in \Sigma_{F(q)}$, it follows that $\tilde\gamma$ is analytic. Since γ and $\tilde\gamma$ coincide on an interval, they are both analytic and have the same length, we conclude that they coincide everywhere. In particular, $q = F(q)$. □

2.4. **Sub-Finsler structures.** There are several natural definitions for the notion of sub-Finsler manifold. One considers triplets (M, Δ, N) where M is a connected smooth manifold, $\Delta \subseteq TM$ a bracket-generating subbundle, and, for each $p \in M$, $N_p := N(p)$ is a norm on Δ_p. The various definitions differ depending on the requested regularity for N. The requirement that N is the restriction to Δ of a Finsler structure is an assumption that for us is too strong. In fact, we need to allow any arbitrary norm on vector spaces, which may not be smooth away from the origin. Low-regularity assumptions are that the function N is merely continuous, or Lipschitz in smooth coordinates. In our case the norm changes smoothly with respect to the point of the manifold in a suitable sense, which can be formalized following the notion of partially smooth Finsler metrics presented in [25]; see [3]. The case of interest for us are those norms that are of *constant type*. Namely, norms for which there exists a smooth frame, such that the norm of all linear combinations of its elements is independent on the point. Sub-Finsler homogeneous manifolds are examples. Hence, we prove the following lemma only for homogeneous manifolds and discuss in Remark 2.16 the general case.

Lemma 2.13. *Let G/H be a homogeneous manifold equipped with a G-invariant sub-Finsler distance. Then there exists a G-invariant sub-Riemannian distance d_{SR} with same horizontal bundle as d_{SF}, such that any isometry among open subsets of G/H with respect to d_{SF} is an isometry with respect to d_{SR}.*

Proof. The proof uses a well-known trick based on John's Ellipsoid Theorem, see [21], together with Margulis-Mostow Differentiability Theorem, [24].

Let Δ_p be the horizontal bundle at $p \in M := G/H$. Denote $K_p = \{v \in \Delta_p \mid \|v\| \leq 1\}$, where $\|\cdot\|$ is the norm defining d_{SF}. John's Ellipsoid Theorem states that there exists a unique ellipsoid E_p contained in K_p with maximal volume and, moreover, there exists $c \geq 1$ depending only on the topological dimension of M, such that

$$E_p \subseteq K_p \subseteq c \cdot E_p. \tag{2.14}$$

First, let F be a C^1 d_{SF}-isometry. We claim that for all p in the domain of F,

$$df_p(E_p) = E_{F(p)}. \tag{2.15}$$

Indeed, dF_p restricts to a linear isometry between $(\Delta_p, \|\cdot\|)$ and $(\Delta_{F(p)}, \|\cdot\|)$. In particular, $dF_p(K_p) = K_{F(p)}$ and $dF_p(E_p)$ is an ellipsoid contained in $K_{F(p)}$. Since $E_{F(p)}$ is the maximal-volume ellipsoid, it follows that $\mathrm{vol}(dF_p(E_p)) \leq \mathrm{vol}(E_{F(p)})$. Since dF_p^{-1} also restricts to a linear isometry, we obtain the reverse inequality and, by uniqueness, (2.15) follows. In particular, taking $F = L_g$ the left translation by $g \in G$, we conclude that $E_{gH} = (dL_g)_H E_H$. Recall that g can be chosen smoothly as gH varies, see [19]. Therefore, $\{E_p\}_{p \in M}$ defines a G-invariant scalar product $\langle\cdot,\cdot\rangle$ on Δ_p which in turn gives a G-invariant sub-Riemannian distance d_{SR} on M. Also, by (2.14), d_{SF} and d_{SR} are c-bi-Lipschitz.

Let F be a map between open subsets of M that is an isometry with respect to d_{SF}, but for which we ask no a priori smoothness assumptions. The two distances being bi-Lipschitz equivalent, the map F is bi-Lipschitz with respect to d_{SR}. By Margulis-Mostow Theorem, the Pansu differential of F exists almost everywhere. Namely, for almost every $p \in M$ the blow up of F at p is a group isomorphism $(\mathrm{P}F)_p : G_p \to G_{F(p)}$. In particular, for these p, the derivative XF exists for all $X \in \Delta_p$ and the horizontal differential is a linear map that uniquely determines $(\mathrm{P}F)_p$. Being the blow up of an isometry, $(\mathrm{P}F)_p$ is an isometry between $(G_p, \|\cdot\|_p)$ and $(G_{F(p)}, \|\cdot\|_{F(p)})$ and then $(\mathrm{P}F)_p(G_p, \langle\cdot,\cdot\rangle_p) \to (G_{F(p)}, \langle\cdot,\cdot\rangle_{F(p)})$ is an isometry as well. In other words, almost every tangent of F is a sub-Riemannian isometry.

We plan to show that F is a sub-Riemannian isometry, by showing that it preserves the sub-Riemannian length of curves. Take any smooth horizontal curve γ, which we may suppose to be locally a straight segment, in some coordinates. Let $\Omega \subseteq M$ be the full-measure set on which F is Pansu-differentiable. By Fubini's theorem, there exists curves of the form $\gamma_n := g_n \cdot \gamma$ with $g_n \in G$, g_n arbitrarily close to the identity in G such that $\gamma_n \cap \Omega$ has full $\mathcal{H}^1$-measure. In other words, $\sigma_n := F \circ \gamma_n$ is differentiable almost everywhere and $\langle \dot\sigma_n, \dot\sigma_n \rangle = \langle \dot\gamma_n, \dot\gamma_n \rangle$. Hence, the sub-Riemannian length of σ_n equals the sub-Riemannian length of γ_n. Passing to the limit, the sub-Riemannian length of $F \circ \gamma$ equals the sub-Riemannian length of γ. Therefore F is an isometry with respect to d_{SR}. □

Remark 2.16. We note that the above proof extends to sub-Finsler structures with norm of constant type and to those that admit a Finsler extension. However, it does not immediately generalize to the case of partially smooth sub-Finsler manifolds. The issue is that the scalar product defined using the John Ellipsoid Theorem may not depend smoothly on the base point, see for example [25, Example 3.2. page 7]. However, in [25], the authors consider a different Riemannian metric associated to a Finsler one and prove that if the structure is partially smooth then the Riemannian metric is indeed smooth. One can use a similar construction to associate a sub-Riemannian metric to a partially smooth sub-Finsler metric, for which the analogue of Lemma 2.13 holds.

3. Isometries of Carnot groups

In this section we prove Theorem 1.1. First, we consider the particular case when the distance is sub-Riemannian and the domain of the map is connected. Namely, in Section 3.1 we show that every isometry defined between two connected and open subsets of a Carnot group G endowed with a sub-Riemannian metric is affine. Next, in Section 3.2 we use Lemma 2.13 and Proposition 2.12 to deduce the general case.

3.1. Sub-Riemannian isometries on connected sets.

Theorem 3.1. *Let (G, d) be a sub-Riemannian Carnot group. Let $\Omega_1, \Omega_2 \subset G$ be two* connected *open sets. Let $F : \Omega_1 \to \Omega_2$ be an isometry. If $F(e) = e$, then F is the restriction to Ω_1 of a group isomorphism of G.*

Proof. By [11, Theorem 1.1], F is smooth. Let ϕ be the blow up of F at e, i.e., the Pansu differential at the identity. We plan to show that $F = \phi|_{\Omega_1}$.

By [31], the map ϕ is a group isomorphism and, moreover, it is an isometry, being the limit of isometries. Hence, the maps ϕ and F are C^∞ isometries with the property that $F(e) = \phi(e) = e$ and $d\phi_e|_{V_1} = dF_e|_{V_1}$, see [35], where V_1 is the first stratum of the stratification of G. By [11, Theorem 1.1] we know that $\mathrm{Iso}_e(\Omega_1, d) \subset C^\infty(\Omega_1, \Omega_1)$, we can apply Corollary 2.10 and conclude that $F = \phi|_{\Omega_1}$. □

3.2. General case: Proof of Theorem 1.1. We recall that by Pansu's Differentiability Theorem [31], we may assume that $G_1 = G_2$. Let (G, d_{SF}) be a sub-Finsler Carnot group. Let $\Omega_1, \Omega_2 \subseteq G$ be two open sets and $F : \Omega_1 \to \Omega_2$ a sub-Finsler isometry, which a priori is not smooth. Let d_{SR} be the sub-Riemannian distance on G associated to d_{SF} by Lemma 2.13. Thus, by Lemma 2.13, the map F is also an isometry with respect to d_{SR}. Up to composing with a translation, we may assume $F(e) = e$. By Theorem 3.1, the map F is a group isomorphism ϕ on the connected component Ω of Ω_1 containing e, . Then the map $\phi^{-1} \circ F$ is an isometry that is the identity on Ω. By Proposition 2.12, we get that $\phi^{-1} \circ F$ is the identity on Ω_1. □

4. Isometries of manifolds

In this section we consider the regularity of isometries of sub-Riemannian manifolds. First we discuss the case of general manifolds, then the case of global isometries of homogeneous manifolds.

4.1. Isometries of manifolds and the horizontal differential.

We show here Theorem 1.2, which is a a consequences of our preliminary results together with the regularity theorem obtained by Capogna and the first-named author in [12].

Corollary 4.1. *Let M and N be connected equiregular sub-Riemannian manifolds. Let $p \in M$ and let Δ be the horizontal bundle of M. Let $F_1, F_2 : M \to N$ be two isometries. If $F_1(p) = F_2(p)$ and $dF_1|_{\Delta_p} = dF_2|_{\Delta_p}$, then $F_1 = F_2$.*

Proof. From the regularity result in [12, Corollary 1.5], F_1, F_2 are C^∞ and $\mathrm{Iso}_p(M) \subset C^\infty(M)$. Corollary 2.10 concludes. □

Remark 4.2. Using Proposition 2.12, one has the following more general consequence. With M, N, Δ, and p as in the corollary above, let $U \subseteq M, V \subset N$ be two open subsets, not necessarily connected. Let $F_1, F_2 : U \to V$ be isometries. If $F_1(p) = F_2(p)$ and $dF_1|_{\Delta_p} = dF_2|_{\Delta_p}$, then $F_1 = F_2$.

Proof of Theorem 1.2. Let $F_1, F_2 : \Omega_1 \to \Omega_2$ be two isometries for the sub-Finsler distance. By Lemma 2.13, they are also isometries with respect to some sub-Riemannian distance. By [12, Corollary 1.5], F_1, F_2 are C^∞. By Corollary 4.1, we conclude. □

Remark 4.3. It is possible to generalize both the C^∞ regularity and Corollary 4.1 to isometries of (partially smooth) sub-Finsler manifolds. Indeed, one can use the analogue of Lemma 2.13 as explained in Remark 2.16.

4.2. Analytic regularity for global isometries of sub-Finsler homogeneous spaces.

Theorem 4.4 (Gleason-Montgomery-Zippin)**.** *If a second countable and locally compact group H acts by isometries, continuously, effectively, and transitively on a locally compact, locally connected, and finite-dimensional metric space X, then H is a Lie group and X is a differentiable manifold.*

Proposition 4.5 (Consequence of Theorem 4.4)**.** *Let $M = G/H$ be a homogeneous manifold equipped with a G-invariant distance d, inducing the manifold topology. Then the isometry group* $\mathrm{Iso}(M)$ *is a Lie group, the action*

$$\begin{array}{rcl} \mathrm{Iso}(M) \times M & \to & M \\ (F, p) & \mapsto & F(p) \end{array} \tag{4.6}$$

is analytic, and, for all $p \in M$, the space $\mathrm{Iso}_p(M)$ *is a compact Lie group.*

Proof. By the Ascoli-Arzelà Theorem, $\mathrm{Iso}(M)$ is locally compact and $\mathrm{Iso}_p(M)$ is compact (both equipped with the compact open topology). Obviously they both are groups with the composition as multiplication. Furthermore, since G acts transitively on M, so does $\mathrm{Iso}(M)$. Therefore, by Theorem 4.4 it follows that $\mathrm{Iso}(M)$ is a Lie group. Being a compact subgroup, $\mathrm{Iso}_p(M)$ is a Lie group as well.

For the proof that the action of $\mathrm{Iso}(M)$ on M is analytic, we make explicit the analytic structures considered. The group G and the manifold M are given with their analytic structures, which we denote ω_G and ω_M, respectively. Hence, by assumption, the action

$$(G, \omega_G) \times (M, \omega_M) \longrightarrow (M, \omega_M), \tag{4.7}$$

given by (2.1), is analytic. The group $\mathrm{Iso}(M)$ has an analytic structure ω_I of Lie group and, since it is acting transitively (and continuously) on M, there exists an analytic structure $\tilde{\omega}_M$ on M such that the map

$$(\mathrm{Iso}(M), \omega_I) \times (M, \tilde{\omega}_M) \longrightarrow (M, \tilde{\omega}_M), \tag{4.8}$$

given by (4.6), is analytic, see [19, page 123]. Every element of G induces an isometry. Hence, we have a map

$$\iota : (G, \omega_G) \longrightarrow (\mathrm{Iso}(M), \omega_I), \tag{4.9}$$

induced by (2.1). The map ι is a continuous homomorphism. By [19, Theorem 2.6], ι is, in fact, analytic. By composition of (4.8) and (4.9),

$$(G, \omega_G) \times (M, \tilde{\omega}_M) \longrightarrow (M, \tilde{\omega}_M), \tag{4.10}$$

again given by (2.1), is analytic. By [19, Theorem 4.2] there is a unique analytic structure on M for which the action given by (2.1) is analytic. Therefore, we conclude that $\omega_M = \tilde{\omega}_M$. Hence, the map (4.6) is analytic when M is equipped with ω_M. □

5. Remarks and examples

We conclude the article with some comments. In particular, we present examples in order to show that: in Theorem 1.1 the hypothesis that the set Ω is open cannot be dropped; isometries of the blow up spaces of G/H do not necessarily come from isometries in G/H; isometries are not always restrictions of global isometries, unless we are in the setting of Carnot groups; isometries in a general Lie group are not necessarily compositions of translations and isomorphisms.

Unlike in the Euclidean space, Theorem 1.1 cannot be generalized to arbitrary subsets. Here we present a counterexample. We take the sub-Riemannian Heisenberg group $(\mathbb{H}, d_{SR})$ and we define exponential coordinates (x, y, z) with respect to the basis of its Lie algebra given by vectors X, Y and Z, whose only nontrivial bracket is $[X, Y] = Z$. Consider the set given by the xy plane together with the third axis, namely,

$$E := \exp(\mathbb{R}X \oplus \mathbb{R}Y) \cup \exp(\mathbb{R}Z).$$

Then the map $(x, y, z) \mapsto (x, y, -z)$ is an isometry of E into itself. However, this map is not the restriction of a group isomorphism. In discussion with J. Tyson, we observed that the above map does not even extend to a quasiconformal map, in particular, to a bi-Lipschitz map. Indeed, suppose such a map F exists. By Pansu's theorem, a quasi-conformal map admits blow ups that are group isomorphisms. Now, every group isomorphism of the Heisenberg group is topological orientation preserving. On the other hand, one can show that F is topological orientation reversing and hence the blow ups of F are not orientation preserving. We have reached a contradiction.

Given an isometry F of a sub-Finsler homogeneous space G/H, the differential of the blow up at a point p equals the differential at p, when they are both restricted to Δ_p. Therefore Theorem 1.2 claims that $\mathrm{Iso}_p(G/H)$ injects into $\mathrm{Iso}_p((G/H)_p)$, where $(G/H)_p$ denotes the Gromov tangent cone of G/H at p, which is a Carnot group. However, it is not true that isometries of $(G/H)_p$ are always blow ups of isometries of G/H. We can find counterexamples already in Riemannian Lie groups. Take, for instance, the three dimensional Heisenberg group, endowed with a Riemannian distance. We denote it by $(\mathbb{H}, d_R)$. Then its tangent cone at every point is the Euclidean 3-space, which contains all the rotations among its isometries. However, rotations with respect to horizontal lines are not isometries for $(\mathbb{H}, d_R)$. This follows from the observation that $\mathrm{Iso}(\mathbb{H}, d_R) = \mathrm{Iso}(\mathbb{H}, d_{SR})$. The identification of the two isometry groups rests upon the study of length minimizing curves: for both metric models of $\mathbb{H}$, the only infinite geodesics are the 1-parameter groups corresponding to horizontal vectors. Since isometries must preserve infinite geodesics, it follows that the horizontal space is preserved by the differential of every isometry.

For general Lie groups, isometries between open sets may not be restrictions of global isometries. For example, every point in the flat cylinder $\mathbb{R} \times \mathbb{S}^1$ has a neighborhood isometric to a disk in $\mathbb{R}^2$ and hence, all rotations are isometries of this neighborhood. Of course, not all of them extend to global isometries.

Not even in Euclidean space it is true that all group isomorphisms are isometries. However, an automorphism of a sub-Finsler Carnot group is an isometry if and only if its differential preserves the first stratum (ad hence all strata) and restricted to the first stratum preserves the norm defining the sub-Finsler distance. Hence we have a complete description of local isometries of sub-Finsler Carnot groups. We notice that the statements of Theorem 1.1 and Theorem 1.2 become equivalent if $G/H = G$ is a Carnot group. If this is not the case, we cannot conclude that an isometry in a sub-Finsler Lie group G is affine. As counterexample, we take the universal covering group $\tilde{G}$ of the group $G = E(2)$ of Euclidean motions of the plane. One can see that there exists a Riemannian distance on $\tilde{G}$ that makes it isometric to the Euclidean space $\mathbb{R}^3$. In particular, they have the same isometry group. However, a straightforward calculation of the automorphisms shows that not all isometries fixing the identity are group isomorphisms of $\tilde{G}$. Indeed, let $\{X, Y, T\}$ be a basis of $\mathrm{Lie}(G)$ and let $[X, Y] = T$, $[Y, T] = X$ be the nonzero brackets. It turns out that only the rotations in the plane $\mathbb{R}X + \mathbb{R}T$ are automorphisms. Notice that the latter discussion also implies that $\mathbb{R}^3$ and $\tilde{G}$ are not isomorphic. Examples of isometric Lie groups that are not isomorphic can be found also in the strict sub-Riemannian context. A. Agrachev e D. Barilari [2] showed that $SL(2,\mathbb{R})$ and $A^+(\mathbb{R}) \oplus \mathbb{R}$ are isometric with respect to suitable sub-Riemannian structures. Here $A^+(\mathbb{R})$ is the group of orientation preserving affine maps on $\mathbb{R}$. A direct computation shows that the isometric isomorphisms fixing the identity in $SL(2,\mathbb{R})$ form a one-dimensional space, whereas the identity map is the only isometric isomorphism fixing the identity in $A^+(\mathbb{R}) \times \mathbb{R}$. The following question arises naturally. Let (G, d_{SR}) be a sub-Riemannian Lie group for which not all isometries are affine. Does there exist another sub-Riemannian Lie group (G', d'_{SR}) isometric to (G, d_{SR}) for which all isometries are affine?

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