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MARKOVIANITY AND ERGODICITY FOR A SURFACE GROWTH PDE

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The paper analyzes a model in surface growth where the uniqueness of weak solutions seems to be out of reach. We prove existence of a weak martingale solution satisfying energy inequalities and having the Markov property. Furthermore, under nondegeneracy conditions on the noise, we establish that any such solution is strong Feller and has a unique invariant measure.

1. Introduction. This paper deals with a model arising in the theory of growth of surfaces, where an amorphous material is deposited in high vacuum on an initially flat surface. Details on this model can be found in Raible, Linz and Hänggi [27] and Raible et al. [28] or Siegert and Plischke [30]. After rescaling the equation reads

$$\dot{h} = -h_{xxxx} - h_{xx} + ({h_x}^2)_{xx} + \eta \tag{1.1}$$

with periodic boundary conditions on the interval $[0, L]$ (i.e., h and all its derivatives are L-periodic), where the noise η is white in space and time. One could also think of h being defined on a circle of circumference L.

Periodic boundary conditions are the standard condition in these models. The boundary is not considered to be important and L is very large. Sometimes the model is considered also on the whole real line, even though we do not examine this case. We remark that from a mathematical point of view Neumann or Dirichlet boundary conditions are quite similar for the problem studied here. The key point ensured by any of these boundary conditions is that there is a suitable cancellation in the nonlinearity, namely

$$\int_0^L h({h_x}^2)_{xx}\, dx = 0,$$

which is the main (and only) ingredient to derive useful a priori estimates.

The main terms are the dominant linear operator and the quadratic nonlinearity. The linear instability $-h_{xx}$, which leads to the formation of hills, is sometimes neglected (as we shall do in the analysis of the long-time behavior in Section 5).

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For general surveys on surface growth processes and molecular beam epitaxy see Barabási and Stanley [1] or Halpin-Healy and Zhang [21]. Recently the equation has also become a model for ion-sputtering, where a surface is eroded by an ion-beam; see Cuerno and Barabási [9], Castro et al. [7].

Sometimes one adds to the model an additional nonlinear term ${-h_x}^2$ of Kuramoto–Sivashinsky type, but in the present form the equation is mass conserving [i.e., $\int_0^L h(t,x)\,dx = 0$]. This comes from the fact that (1.1) is considered to be subject to a moving frame, which is a time-dependent coordinate system, that takes into account the average growth of the surface due to the deposition process.

Known results on the model. Before stating the main results of the paper, we give a short account of the previously known results concerning both the deterministic and the stochastic version of the model.

- If $\eta = 0$, then the equation has an absorbing set in L^2, although the solution may not be unique (Stein and Winkler [31]).
- There exists a unique local solution in $L^p([0,\tau), H^1) \cap C^0((0,\tau), H^1)$ for initial conditions in H^γ with $\gamma > 1 - \frac{1}{p}$ and $p > 8$ (see Blömker and Gugg [3]).
- There are stationary solutions that can be constructed as limit points of stationary solutions of Galerkin approximations (see Blömker and Hairer [6]).
- There are weak martingale solutions by means of the Galerkin approximation (see Blömker and Gugg [4], Blömker, Gugg and Raible [5]).

The main problem of the model, which is shared by both the deterministic and the stochastic approach, is the lack of uniqueness for weak solutions. This is very similar to the celebrated Navier–Stokes equation. With this problem in mind, a possible approach to analyze the model is to look for solutions with *special* properties, possibly with a physical meaning, such as the balance of energy—we shall often refer to it as *energy inequality*—or the Markov property.

Main results. Here we use the method developed by Flandoli and Romito [18–20] in order to establish the existence of weak solutions having the Markov property. For the precise formulation of the concept of solution see Definitions 2.2 and 2.5.

The method is essentially based on showing a multi-valued version of the Markov property for sets of solutions and then applying a clever selection principle (Theorem 3.1). The original idea can be found in Krylov [24] (see also Stroock and Varadhan [32], Chapter 12).

A key point in this analysis is the definition of weak martingale solutions. The above-described procedure needs to handle solutions which incorporate all the necessary bounds on the size of the process (solution to the SPDE) in different norms. These bounds must be compatible with the underlying Markov structure. This justifies the extensive study of the energy inequality in Section 2.

Once the existence of at least one Markov family of solutions is ensured, the analysis of such solutions goes further. Indeed, the selection principle provides a family of solutions whose dependence with respect to the initial conditions is just measurability. By slightly restricting the set of initial conditions, this dependence can be improved to continuity in the total variation norm (or *strong Feller* in terms of the corresponding transition semigroup). In a few words, we show that the smaller space $\mathcal{H}^1$ (see the next section for its precise definition) is the natural framework for the stochastic model.

Our last main result concerns the long-time behavior of the model. We are able to show that any Markov solution has a unique invariant measure whose support covers the whole state space. In principle the existence of stationary states has been already proved by Blömker and Hairer [6]. Their results are not useful in this framework, as we have a transition semigroup that depends on the generic selection under analysis, which is in general not obtained by a suitable limit of Galerkin approximations. In this way, our results are more powerful, as they apply to *every* Markov solution. The price to pay is that the proof of existence of an invariant measure is painfully long and technical (see Section 5).

We finally remark that, even though our results show that every Markov solution is strong Feller and converges to its own invariant measure, well-posedness is still an open problem for this model and these results essentially do not improve our knowledge on the problem. Even the invariant measures are different, as they depend on different Markov semigroups.

A comparison with previous results on the Markov property. There are several mathematical interests in this model, in comparison with the theory developed in Flandoli and Romito [18–20] for the Navier–Stokes equations. Essentially, in this model we have been able to find the *natural space* for the Markov dynamics, thus showing the existence of the (unique) invariant measure. It is still an open problem for the Navier–Stokes equations, in the framework of Markov selections, to find a space that allows for both strong Feller and the existence of an invariant measure.

Another challenge of this model concerns the analysis of the energy inequality. Here the physics of the model requires a noise white in time and space, while the analysis developed in the above-cited papers has been based on a trace-class noise with quite regular trajectories.

Finally, we remark that there is a different approach to handle the existence of solutions with the Markov property, based on spectral Galerkin methods, which has been developed by Da Prato and Debussche [10] (see also Debussche and Odasso [14]) for the Navier–Stokes equations (no result with these techniques is known for the model analyzed in this paper). Their methods are similar to [4–6].

Layout of the paper. The paper is organized as follows: In Section 2 we state the martingale problem and define weak and energy solutions. We also give a few restatements of the energy balance. We next show in Section 3 that there is at least one family of energy solutions with the Markov property. In Section 4 we show that the transition semigroup associated with any such solution has the strong Feller property. Existence and uniqueness of the invariant measure is then shown in Section 5.

Finally, Sections 6 and 7 contain a few technical results that are used throughout the paper. They have been confined to the last part of the paper to allow the reader to focus on the main topics rather than on such details.

2. The martingale problem.

2.1. *Notation and assumptions.* Let $\mathcal{D}^\infty$ be the space of infinitely differentiable L-periodic functions on $\mathbf{R}$ with zero mean in $[0, L]$. We work with periodic boundary conditions on $[0, L]$ and mean zero and we define for $p \in [1, \infty]$

$$\mathcal{L}^p = \left\{ h \in L^p(0, L) : \int_0^L h(x)\,dx = 0 \right\}$$

with the standard L^p-norms. For instance, $|f|^2_{\mathcal{L}^2} = \int_0^L f^2(x)\,dx$ and the scalar product $\langle f, g\rangle_{\mathcal{L}^2} = \int_0^L f(x)g(x)\,dx$.

Let Δ be the operator ∂_x^2 on $\mathcal{L}^2$ subject to periodic boundary conditions. The leading linear operator in (1.1) is $A = -\Delta^2$. Let $(e_k)_{k\in\mathbf{N}}$ be the orthonormal basis of $\mathcal{L}^2$ given by the trigonometric functions $\sin(2m\pi x/L)$ and $\cos(2m\pi x/L)$ with $m \in \mathbf{N}$, and let λ_k be the eigenvalues of A such that

$$Ae_k = \lambda_k e_k.$$

Notice that $\lambda_k \sim -k^4$.

Let $\mathcal{Q} : \mathcal{L}^2 \to \mathcal{L}^2$ be a bounded linear operator such that

$$\mathcal{Q}e_k = \alpha_k^2 e_k, \qquad k \in \mathbf{N},$$

so that $\mathcal{Q}$ is a nonnegative self-adjoint operator. This is sufficient to model all kinds of spatially homogeneous Gaussian noise η such that

$$\mathbb{E}\eta(t,x) = 0 \quad \text{and} \quad \mathbb{E}\eta(t,x)\eta(s,y) = \delta(t-s)q(x-y),$$

where q is the spatial correlation function (or distribution). Now $\mathcal{Q} = q\star$, which is the convolution operator with q. For details see Blömker [2] and the references therein.

In a formal way we can rewrite (1.1) as an abstract stochastic evolution equation

$$dh = \big(Ah - \Delta h + B(h,h)\big)\,dt + dW,$$

where W is a suitable $\mathcal{Q}$-Wiener process [for details see (2.2)], and $B(u,v) = -\Delta(\partial_x u \cdot \partial_x v)$.

Let us finally comment on the spaces we are using. The Sobolev spaces $\mathcal{H}^\gamma$ for $\gamma \in \mathbf{R}$ are defined as the domains of fractional powers of $1 - A$, which are equivalent to standard Sobolev spaces. See, for example, Henry [22], Pazy [26] or Lunardi [25]. Here we use the explicit expansions of norms in terms of Fourier series

$$\mathcal{H}^\gamma = \left\{ u = \sum_{k\in\mathbf{N}} \alpha_k e_k : |u|^2_{\mathcal{H}^\gamma} = \sum_{k\in\mathbf{N}} \alpha_k^2 (1-\lambda_k)^{\gamma/2} < \infty \right\}.$$

This is equivalent to saying that $\mathcal{H}^\gamma$ consists of all functions h in $H^\gamma_{\mathrm{loc}}(\mathbf{R})$ which are L-periodic and satisfy $\int_0^L h(x)\,dx = 0$. Furthermore, the standard $H^\gamma([0, L])$-norm, which is defined by fractional powers of the Laplacian, is an equivalent norm on $\mathcal{H}^\gamma$. We also use the space

$$\mathcal{W}^{1,4} := \{u \in \mathcal{H}^1 : |u|_{\mathcal{W}^{1,4}} = |u|_{\mathcal{L}^4} + |\partial_x u|_{\mathcal{L}^4} < \infty\}.$$

Note that all usual Sobolev embeddings, such as $\mathcal{H}^1 \subset C^0([0, L])$ or $\mathcal{H}^2 \subset \mathcal{W}^{1,4}$, still hold. For a more detailed presentation we refer to Blömker, Gugg and Raible [5], Section 2.

2.1.1. *The underlying probability structure.* Let $\Omega = C([0,\infty); \mathcal{H}^{-4})$ and let $\mathcal{B}$ be the σ-algebra of Borel subsets of Ω. Let $\xi : [0,\infty) \times \Omega \to \mathcal{H}^{-4}$ be the *canonical process* on Ω, defined as $\xi(t,\omega) = \omega(t)$.

For each $t \geq 0$, let $\mathcal{B}_t = \sigma[\xi(s) : 0 \leq s \leq t]$ be the σ-field of events up to time t and $\mathcal{B}^t = \sigma[\xi(s) : s \geq t]$ be the σ-field of events after time t. The σ-field $\mathcal{B}_t$ can be seen as the Borel σ-field of $\Omega_t = C([0,t]; \mathcal{H}^{-4})$ and, similarly, $\mathcal{B}^t$ as the Borel σ-field of $\Omega^t = C([t,\infty]; \mathcal{H}^{-4})$. Notice that both Ω_t and Ω^t can be seen as Borel subsets of Ω (by restriction to corresponding sub-intervals). Define finally the *forward shift* $\Phi_t : \Omega \to \Omega^t$ as

$$\Phi_t(\omega)(s) = \omega(s-t), \qquad s \geq t. \tag{2.1}$$

Given a probability measure P on $(\Omega, \mathcal{B})$ and $t > 0$, we shall denote by $\omega \mapsto P|^{\omega}_{\mathcal{B}_t} : \Omega \to \Omega^t$ a regular conditional probability distribution of P given $\mathcal{B}_t$. Notice that Ω is a Polish space and $\mathcal{B}_t$ is countably generated, so a regular conditional probability distribution does exist and is unique, up to P-null sets.

In particular, $P|^{\omega}_{\mathcal{B}_t}[\omega' : \xi(t,\omega') = \omega(t)] = 1$ and, if $A \in \mathcal{B}_t$ and $B \in \mathcal{B}^t$, then

$$P[A \cap B] = \int_A P|^{\omega}_{\mathcal{B}_t}[B] P(d\omega).$$

One can see the probability measures $(P|^{\omega}_{\mathcal{B}_t})_{\omega\in\Omega}$ as measures on Ω such that $P|^{\omega}_{\mathcal{B}_t}[\omega' \in \Omega : \omega'(s) = \omega(s), \text{ for all } s \in [0,t]] = 1$ for all ω in a $\mathcal{B}_t$-measurable P-full set. We finally define the *reconstruction* of probability measures (details on this can be found in Stroock and Varadhan [32], Chapter 6).

DEFINITION 2.1. Given a probability measure P on $(\Omega, \mathcal{B})$, $t > 0$ and a $\mathcal{B}_t$-measurable map $Q : \Omega \to \Pr(\Omega^t)$ such that $Q_\omega[\xi_t = \omega(t)] = 1$ for all $\omega \in \Omega$, $P \otimes_t Q$ is the unique probability measure on $(\Omega, \mathcal{B})$ such that:

1. $P \otimes_t Q$ agrees with P on $\mathcal{B}_t$.
2. $(Q_\omega)_{\omega\in\Omega}$ is a regular conditional probability distribution of $P \otimes_t Q$, given $\mathcal{B}_t$.

2.2. *Solutions to the martingale problem.*

DEFINITION 2.2 (Weak martingale solution). Given $\mu_0 \in \mathrm{Pr}(\mathcal{L}^2)$, a probability measure P on $(\Omega, \mathcal{B})$ is a solution, starting at μ_0, to the martingale problem associated to (1.1) if:

[W1] $P[L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^1)] = 1$,
[W2] for every $\varphi \in \mathcal{D}^\infty$, the process $(M_t^\varphi, \mathcal{B}_t, P)_{t\geq 0}$, defined P-a.s. on $(\Omega, \mathcal{B})$ as

$$M_t^\varphi = \langle \xi(t) - \xi(0), \varphi\rangle_{\mathcal{L}^2} + \int_0^t \langle \xi(s), \varphi_{xxxx} + \varphi_{xx}\rangle_{\mathcal{L}^2}\, ds + \\ - \int_0^t \langle (\xi_x(s))^2, \varphi_{xx}\rangle_{\mathcal{L}^2}\, ds$$

is a Brownian motion with variance $t|\mathcal{Q}^{1/2}\varphi|^2_{\mathcal{L}^2}$,
[W3] the marginal at time 0 of P is μ_0.

REMARK 2.3. It is not difficult to prove that the definition of a weak martingale solution given above coincides with the usual definition given in terms of the existence of an underlying probability space and a Wiener process. This equivalence is proved in Flandoli [16] for the Navier–Stokes equations and one can proceed similarly in this case.

Define, for every $k \in \mathbf{N}$, the process $\beta_k(t) = \frac{1}{\alpha_k} M_t^{e_k}$ (and $\beta_k = 0$ if $\alpha_k = 0$). Under any weak martingale solution P, the $(\beta_k)_{k\in\mathbf{N}}$ are a sequence of independent one-dimensional standard Brownian motions.

Similarly, the process

$$W(t) = \sum \alpha_k \beta_k(t) e_k \tag{2.2}$$

is, under any weak martingale solution P, a $\mathcal{Q}$-Wiener process and the process[1]

$$Z(t) = \sum_{k\in\mathbf{N}} \alpha_k \int_0^t e^{(t-s)\lambda_k}\, d\beta_k(s) e_k \tag{2.3}$$

[1]The process Z can be equivalently defined as

$$Z(t,\omega) = W(t,\omega) + \int_0^t A e^{A(t-s)} W(s,\omega)\, ds.$$

The process Z is thus defined, in some sense, *path-wise*. Note that stochastic integrals usually only have versions (or modification) that are continuous in time. But we sometimes need an explicitly definition of Z as a map from $\Omega \to \Omega$. Thus a version of the process cannot be used.

is the associated Ornstein–Uhlenbeck process starting at 0. As the β_k are i.i.d. Brownian motions and the e_k are an orthonormal system in $\mathcal{L}^2$, the sum above is convergent in $L^2(\Omega, \mathcal{L}^2)$ (see, e.g., Da Prato and Zabczyk [12]).

Notice that, obviously, Z and W are random variables on Ω. In the following lemma we summarize all the regularity results for Z that we shall use throughout the paper.

LEMMA 2.4. *Given a weak martingale solution P, let Z be the process defined in* (2.3).

1. *For every $p \geq 1$ and $T > 0$, $Z \in L^p(\Omega \times (0,T); \mathcal{W}^{1,4})$. Moreover for some $\lambda > 0$,*
$$\sup_{T>0} \frac{1}{T}\mathbb{E}^P\left[\int_0^T \exp\{\lambda |Z(t)|^2_{\mathcal{W}^{1,4}}\}\,dt\right] < \infty.$$
2. *For every $p \geq 1$ and $\gamma \in [0, \frac{3}{2})$, $Z \in L^p(\Omega; L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{H}^\gamma))$.*
3. *Z is P-a.s. weakly continuous with values in $\mathcal{H}^\gamma$, for every $\gamma \in [0, \frac{3}{2})$.*

PROOF. Statements (1) and (2) are proved, respectively, in Lemmas 6.2 and 6.3 of Section 6. The last statement follows from (2) and the continuity in time of Z in $\mathcal{H}^{-4}$, due to the fact that Z is defined on Ω. $\square$

DEFINITION 2.5 (Energy martingale solution). Given $\mu_0 \in \mathrm{Pr}(\mathcal{L}^2)$, a probability measure P on $(\Omega, \mathcal{B})$ is an *energy martingale solution* to (1.1) starting at μ_0 if:

[E1] P is a weak martingale solution starting at μ_0,
[E2] $P[V \in L^\infty_{\mathrm{loc}}([0,\infty); \mathcal{L}^2) \cap L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^2)] = 1$,
[E3] there is a set $T_P \subset (0,\infty)$ of null Lebesgue measure such that for all $s \notin T_P$ and all $t \geq s$,

$$P[\mathcal{E}_t(V,Z) \leq \mathcal{E}_s(V,Z)] = 1,$$

where $V(t,\omega) = \xi(t,\omega) - Z(t,\omega)$, for $t \geq 0$, and the energy functional $\mathcal{E}$ is defined as

$$\begin{aligned}\mathcal{E}_t(v,z) &= \tfrac{1}{2}|v(t)|^2_{\mathcal{L}^2} \\ &\qquad + \int_0^t \big(|v_{xx}|^2_{\mathcal{L}^2} - |v_x|^2_{\mathcal{L}^2} - \langle v_x, z_x\rangle_{\mathcal{L}^2} - \langle 2v_x z_x + (z_x)^2, v_{xx}\rangle_{\mathcal{L}^2}\big)\,ds\end{aligned}$$

for $v \in L^\infty([0,t]; \mathcal{L}^2) \cap L^2([0,t]; \mathcal{H}^2)$ and $z \in L^4([0,t]; \mathcal{W}^{1,4}) \cap L^\infty([0,t]; \mathcal{H}^1)$.

REMARK 2.6 (The equation for V). Let P be an *energy martingale solution*. Then it is easy to see that, by definition, $M^\varphi_t = \langle W(t), \varphi\rangle_{\mathcal{L}^2}$ for all $\varphi \in \mathcal{D}^\infty$. Moreover,

$$\langle Z(t), \varphi\rangle_{\mathcal{L}^2} + \int_0^t \langle Z(s), \varphi_{xxxx}\rangle_{\mathcal{L}^2}\,ds = \langle W(t), \varphi\rangle_{\mathcal{L}^2}$$

and thus

$$\langle V(t) - \xi(0), \varphi\rangle_{\mathcal{L}^2} + \int_0^t \bigl(\langle V, \varphi_{xxxx} + \varphi_{xx}\rangle_{\mathcal{L}^2} + \langle Z - (V_x + Z_x)^2, \varphi_{xx}\rangle_{\mathcal{L}^2}\bigr)\, ds = 0$$

or, in other words, V is a weak solution (i.e., in the sense of distributions) to the equation,

$$\dot{V} + V_{xxxx} + V_{xx} + Z_{xx} = [(V_x + Z_x)^2]_{xx},$$

with an initial condition $V(0) = \xi(0)$.

REMARK 2.7 (Finiteness of the energy). Given an *energy martingale solution* P, we aim to show that, under P, the energy $\mathcal{E}_t$ is almost surely finite. Indeed, by [E2], it follows that $V(t)$ is P-a.s. weakly continuous in $\mathcal{L}^2$ (see, e.g., Lemma 3.1.4 of Temam [33]), and so the function $|V(t)|^2_{\mathcal{L}^2}$ is defined point-wise in the energy estimate. Similarly, the other terms are also P-a.s. finite by [E2] and the regularity properties of Z under P (see Lemma 2.4).

REMARK 2.8 (Measurability of the energy and equivalent formulations). This last remark is concerned with the measurability issues related to the energy inequality and with some equivalent formulations of property [E3] of the above definition. We first prove in the next lemma that property [E3] is quite strong and that, in a sense that will be clarified below, the energy inequality is an intrinsic property of the solution to the original problem (1.1), and does not depend on the splitting $V + Z$. A similar result was proved in Romito [29] for the Navier–Stokes equations. We then show measurability of the energy balance functional and give some equivalent formulations of the energy inequality.

Before stating the lemma, we introduce some notation. Let $z_0 \in \mathcal{H}^1$ and $\alpha \geq 0$, and let $\widetilde{Z} = \widetilde{Z}_{\alpha,z_0}$ be the solution to

$$\dot{\widetilde{Z}} = -\widetilde{Z}_{xxxx} - \alpha\widetilde{Z} + \eta, \qquad \widetilde{Z}(0) = z_0. \tag{2.4}$$

The process $\widetilde{Z}$ is given by $\widetilde{Z} = Z + w$, where w solves the (deterministic) problem

$$\dot{w} = -w_{xxxx} - \alpha w - \alpha Z, \qquad w(0) = z_0 \tag{2.5}$$

and so it is well defined P-a.s., for every martingale solution P. Define suitably $\widetilde{V} = \widetilde{V}_{\alpha,z_0}$ as $\widetilde{V} = \xi - \widetilde{Z}$. It follows that $V - \widetilde{V} = w$ and $\widetilde{V}$ solves

$$\dot{\widetilde{V}} + \widetilde{V}_{xxxx} + \widetilde{V}_{xx} = \alpha\widetilde{Z} - \widetilde{Z}_{xx} + [(\widetilde{V}_x + \widetilde{Z}_x)^2]_{xx}. \tag{2.6}$$

The corresponding energy functional is given by

$$\begin{aligned}\mathcal{E}^\alpha_t(v, z) = \tfrac{1}{2}|v(t)|^2_{\mathcal{L}^2} + \int_0^t \bigl(&|v_{xx}|^2_{\mathcal{L}^2} - |v_x|^2_{\mathcal{L}^2} - \alpha\langle v, z\rangle_{\mathcal{L}^2} \\ &- \langle v_x, z_x\rangle_{\mathcal{L}^2} - \langle 2v_x z_x + (z_x)^2, v_{xx}\rangle_{\mathcal{L}^2}\bigr)\, ds\end{aligned}$$

and in particular $\mathcal{E}^0_t = \mathcal{E}_t$.

LEMMA 2.9. *Let P be an energy martingale solution, then for every $z_0 \in \mathcal{H}^1$ and $\alpha \geq 0$,*

$$P[\mathcal{E}_t^\alpha(\widetilde{V}, \widetilde{Z}) \leq \mathcal{E}_s^\alpha(\widetilde{V}, \widetilde{Z})] = 1 \tag{2.7}$$

for almost every $s \geq 0$ (including $s = 0$) and every $t \geq s$, where $\widetilde{V}$, $\widetilde{Z}$ have been defined above.

PROOF. The proof works as in [29], Theorem 2.8, and we give just a sketch. Since $\widetilde{V} = V - w$, it follows that

$$|\widetilde{V}(t)|_{\mathcal{L}^2}^2 = |V(t)|_{\mathcal{L}^2}^2 + |w(t)|_{\mathcal{L}^2}^2 - 2\langle V(t), w(t)\rangle_{\mathcal{L}^2}$$

and, since by assumptions the energy inequality holds for V, it is sufficient to prove a balance equality for w and $\langle V(t), w(t)\rangle_{\mathcal{L}^2}$. Indeed, it is easy to show by regularization that

$$\tfrac{1}{2}|w(t)|_{\mathcal{L}^2}^2 + \int_s^t (|w_{xx}|_{\mathcal{L}^2}^2 + \alpha\langle \widetilde{Z}, w\rangle_{\mathcal{L}^2})\,dr = \tfrac{1}{2}|w(s)|_{\mathcal{L}^2}^2, \tag{2.8}$$

P-a.s. for all $s \geq 0$ and $t \geq s$. We only need to show that for almost all $s \geq 0$ and $t \geq s$,

$$\begin{aligned} &\langle V(t), w(t)\rangle_{\mathcal{L}^2} - \langle V(s), w(s)\rangle_{\mathcal{L}^2} \\ &\quad = -2\int_s^t \langle V_{xx}, w_{xx}\rangle_{\mathcal{L}^2}\,dr + \int_s^t \langle w_x, V_x + \widetilde{Z}_x\rangle_{\mathcal{L}^2}\,dr \\ &\quad = -\alpha\int_s^t \langle V, \widetilde{Z}\rangle_{\mathcal{L}^2}\,dr + \int_s^t \langle w_{xx}, (V_x + \widetilde{Z}_x)^2\rangle_{\mathcal{L}^2}\,dr. \end{aligned} \tag{2.9}$$

We sketch the proof of the above formula. Since we know that almost surely $V \in L^\infty_{\mathrm{loc}}([0,\infty); \mathcal{L}^2) \cap L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^2)$, it follows by Lemma 6.5 of Section 6 that $\dot{V} \in L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^{-3})$. Moreover, we know that $z_0 \in \mathcal{H}^1$ and $Z_x \in L^4_{\mathrm{loc}}([0,\infty); \mathcal{L}^4)$ and so it is easy to see (by writing the energy balance for $|w_x|_{\mathcal{L}^2}^2$) that $w \in L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^3)$, hence $\dot{w} \in L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^{-2})$. By slightly adapting Lemma 1.2 of Temam [33], Section 3, this implies that $\langle V, w\rangle_{\mathcal{L}^2}$ is differentiable in time with derivative $\langle \dot{V}, w\rangle_{\mathcal{H}^{-3},\mathcal{H}^3} + \langle \dot{w}, V\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}$. Integration by parts then gives (2.9).

Finally, [E3], (2.8) and (2.9) together provide (2.7). □

PROPOSITION 2.10. *Given $z_0 \in \mathcal{H}^1$ and $\alpha \geq 0$, denote by $\widetilde{V}$ and $\widetilde{Z}$ the processes defined above corresponding to z_0 and α. Then the map $(t, \omega) \in [0,\infty) \times \Omega \mapsto \mathcal{E}_t^\alpha(\widetilde{V}(\omega), \widetilde{Z}(\omega))$ is progressively measurable and*:

(i) *For all $0 \leq s \leq t$, the sets $E_{s,t}(z_0, \alpha) = \{\mathcal{E}_t^\alpha(\widetilde{V}, \widetilde{Z}) \leq \mathcal{E}_s^\alpha(\widetilde{V}, \widetilde{Z})\}$ are $\mathcal{B}_t$-measurable.*

(ii) *For all* $t>0$, *the sets*

$$E_t(z_0,\alpha)=\{\mathcal{E}_t^\alpha(\widetilde{V},\widetilde{Z})\le\mathcal{E}_s^\alpha(\widetilde{V},\widetilde{Z})\ \textit{for a.e. } s\le t\ (\textit{including } 0)\}$$

are $\mathcal{B}_t$-*measurable*.

(iii) *The set*

$$E(z_0,\alpha)=R\cap\{\mathcal{E}_t^\alpha(\widetilde{V},\widetilde{Z})\le\mathcal{E}_s^\alpha(\widetilde{V},\widetilde{Z})\ \textit{for a.e. } s\ge 0\ (\textit{including } 0),\ \textit{all } t\ge s\}$$

is $\mathcal{B}$-*measurable*, *where*

$$R=\{Z\in L^4_{\mathrm{loc}}([0,\infty);\mathcal{W}^{1,4}),\qquad V\in L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{L}^2)\cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}^2)\}.$$

Moreover, given an energy martingale solution P, *property* [E3] *is equivalent to each of the following*:

[E3a] *There are* $z_0\in\mathcal{H}^1$ *and* $\alpha\ge 0$ *such that for each* $t>0$ *there is a set* $T\subset(0,t]$ *of null Lebesgue measure and* $P[E_{s,t}(z_0,\alpha)]=1$ *for all* $s\notin T$.

[E3b] *There are* $z_0\in\mathcal{H}^1$ *and* $\alpha\ge 0$ *such that for each* $t>0$, $P[E_t(z_0,\alpha)]=1$.

[E3c] *There are* $z_0\in\mathcal{H}^1$ *and* $\alpha\ge 0$ *such that* $P[E(z_0,\alpha)]=1$.

PROOF. Measurability of the map $\mathcal{E}^\alpha$ follows from the semi-continuity properties of the various terms of $\mathcal{E}^\alpha$ with respect to the topology of Ω (see also Lemma 2.1 of Flandoli and Romito [19]).

The measurability of each $E_{s,t}(z_0,\alpha)$ now follows easily from measurability of the map $\mathcal{E}^\alpha$. As it regards (ii), fix $t>0$ and notice that the Borel σ-algebra of the interval $(0,t)$ is countably generated, so that if $\mathcal{T}_t$ is a countable basis,

$$E_t(z_0,\alpha)=E_{0,t}(z_0,\alpha)\cap\bigcap_{T\in\mathcal{T}_t}\left\{\int_0^t\mathbf{1}_T(s)\big(\mathcal{E}_t^\alpha(\widetilde{V},\widetilde{Z})-\mathcal{E}_s^\alpha(\widetilde{V},\widetilde{Z})\big)\,ds\le 0\right\}$$

and all sets $\{\int_0^t\mathbf{1}_T(s)(\mathcal{E}_t^\alpha(\widetilde{V},\widetilde{Z})-\mathcal{E}_s^\alpha(\widetilde{V},\widetilde{Z}))\,ds\le 0\}$ are $\mathcal{B}_t$-measurable by the measurability of $\mathcal{E}^\alpha$.

We next show (iii). Let $J\subset[0,\infty)$ be a countable dense subset and define

$$R_t=\{Z\in L^4_{\mathrm{loc}}([0,t);\mathcal{W}^{1,4}),\ V\in L^\infty(0,t;\mathcal{L}^2)\cap L^2(0,t;\mathcal{H}^2)\}$$

(notice that the regularity of Z and V implies that of $\widetilde{V}$ and $\widetilde{Z}$), then $R_t\in\mathcal{B}_t$ and, by the lower semi-continuity of the various terms of $\mathcal{E}_t^\alpha(\widetilde{V},\widetilde{Z})-\mathcal{E}_s^\alpha(\widetilde{V},\widetilde{Z})$ with respect to t, it follows that

$$E(z_0,\alpha)=\bigcap_{t\in J}\big(R_t\cap E_t(z_0,\alpha)\big)$$

is $\mathcal{B}$-measurable. The last statement of the lemma is now obvious from the above equalities, property [E2] and Lemma 2.4. $\square$

3. Existence of Markov solutions. This section is devoted to the existence of Markov solutions for (1.1). With such an aim, define for each $x \in \mathcal{L}^2$,

$$\mathcal{C}(x) = \{P : P \text{ is an energy martingale solution starting at } \delta_x\}.$$

We state the main theorem of this part.

THEOREM 3.1. *There exists a family $(P_x)_{x\in\mathcal{L}^2}$ of probability measures on $(\Omega, \mathcal{B})$ such that for each $x \in \mathcal{L}^2$, P_x is an energy martingale solution with initial distribution δ_x, and the a.s. Markov property holds: There is a set $T_P \subset (0,\infty)$ with null Lebesgue measure such that for all $s \notin T_P$, all $t \geq s$ and all bounded measurable $\phi : \mathcal{L}^2 \to \mathbf{R}$,*

$$\mathbb{E}^P[\phi(\xi_t)|\mathcal{B}_t] = \mathbb{E}^{P_{\xi_s}}[\phi(\xi_{t-s})].$$

PROOF. We use the method developed in Flandoli and Romito [19] (cf. Theorem 2.8). It is sufficient to show that the family $(\mathcal{C}(x))_{x\in\mathcal{L}^2}$ defined above is an *a.s. pre-Markov family*. We recall now the various properties of an a.s. pre-Markov family, which we need to verify in order to prove the theorem (see also Definition 2.5 of Flandoli and Romito [19]).

1. Each $\mathcal{C}(x)$ is nonempty, compact and convex, and the map $x \to \mathcal{C}(x)$ is measurable with respect to the Borel σ-fields of the space of compact subsets of $\Pr(\Omega)$ (endowed with the Hausdorff measure).
2. For each $x \in \mathcal{L}^2$ and all $P \in \mathcal{C}(x)$, $P[C([0,\infty); \mathcal{L}^2_{\text{weak}})] = 1$, where $\mathcal{L}^2_{\text{weak}}$ is the space $\mathcal{L}^2$ with the weak topology.
3. For each $x \in \mathcal{L}^2$ and $P \in \mathcal{C}(x)$ there is a set $T \subset (0,\infty)$ with null Lebesgue measure, such that for all $t \notin T$ the following properties hold:
   (a) (*Disintegration*). There exists $N \in \mathcal{B}_t$ with $P(N) = 0$ such that for all $\omega \notin N$

   $$\omega(t) \in \mathcal{L}^2 \quad \text{and} \quad P|^{\omega}_{\mathcal{B}_t} \in \Phi_t \mathcal{C}(\omega(t)).$$

   (b) (*Reconstruction*). For each $\mathcal{B}_t$-measurable map $\omega \mapsto Q_\omega : \Omega \to \Pr(\Omega^t)$ such that there is $N \in \mathcal{B}_t$ with $P(N) = 0$ and for all $\omega \notin N$

   $$\omega(t) \in \mathcal{L}^2 \quad \text{and} \quad Q_\omega \in \Phi_t \mathcal{C}(\omega(t)),$$

   we have that $P \otimes_t Q \in \mathcal{C}(x)$.

The validity of these properties is verified in the following lemmas. Properties (1) and (2) are proved in Lemmas 3.3 and 3.4, while Lemmas 3.5 and 3.6 show disintegration and reconstruction, respectively. $\square$

3.1. *The core lemmas for the proof of Theorem* 3.1. This section contains the key results used in the proof of Theorem 3.1. To this aim, we first state a tightness result for sequences of energy martingale solutions that we shall use in the proof of the core lemmas. The proof of this theorem (restated as Theorem 6.7) is given in Section 6.

THEOREM 3.2. *Let $(P_n)_{n\in\mathbf{N}}$ be a family of energy martingale solutions with each P_n starting in μ_n and*

$$\int_{\mathcal{L}^2} [\log(|x|_{\mathcal{L}^2}+1)]^\kappa \mu_n(dx) \le K \qquad \text{for all } n \in \mathbf{N}$$

for some $\kappa > 0$ and $K > 0$. Then $(P_n)_{n\in\mathbf{N}}$ is tight on $\Omega \cap L^2([0,\infty), \mathcal{H}^1)$.

Furthermore, there is a constant depending only on $T > 0$, $z_0 \in \mathcal{H}^1$, $K > 0$, and $\kappa > 0$, such that

$$\mathbb{E}^{P_n}\biggl[\log\biggl(1+\int_0^T |\xi_x(s)|^2_{\mathcal{L}^2}\,ds\biggr)\biggr]^\kappa \le C, \tag{3.1}$$

$$\begin{aligned}&\mathbb{E}^{P_n}\biggl[\log\biggl(1+\int_0^T |V_{xx}(s)|^2_{\mathcal{L}^2}\,ds\biggr)\biggr]^\kappa \\ &\quad + \mathbb{E}^{P_n}\Bigl[\sup_{t\in[0,T]} \log\bigl(1+|V(t)|^2_{\mathcal{L}^2}\bigr)\Bigr]^\kappa \le C.\end{aligned} \tag{3.2}$$

LEMMA 3.3 (Continuity lemma). *For each $x \in \mathcal{L}^2$, the set $\mathcal{C}(x)$ is nonempty, convex and for all $P \in \mathcal{C}(x)$,*

$$P[C([0,\infty); \mathcal{L}^2_{\mathrm{weak}}] = 1.$$

PROOF. Existence of weak martingale mild solutions is proved in Blömker and Gugg [3], using standard spectral Galerkin methods. This is similar to Lemma 3.4.

By Remark 2.3, this implies existence of weak martingale solutions according to Definition 2.2. In order to prove the energy inequality of Definition 2.5, one can proceed as in the next lemma (where it is proved in a slightly more general situation).

Next, it is easy to show that $\mathcal{C}(x)$ is convex, since all requirements of both Definitions 2.2 and 2.5 are linear with respect to measures $P \in \mathcal{C}(x)$. Finally, if $P \in \mathcal{C}(x)$, we know by statement (3) of Lemma 2.4 that, under P, the process Z is weakly continuous. Moreover, by property [E2] of Definition 2.5, V is also weakly continuous and, in conclusion, $C([0,\infty); \mathcal{L}^2_{\mathrm{weak}})$ is a full set. $\square$

LEMMA 3.4 (Compactness lemma). *For each $x \in \mathcal{L}^2$, the set $\mathcal{C}(x)$ is compact and the map $x \mapsto \mathcal{C}(x)$ is Borel measurable.*

PROOF. Following Lemma 12.1.8 of Stroock and Varadhan [32], it is sufficient to prove that for each sequence $(x_n)_{n\in\mathbf{N}}$ converging to x in $\mathcal{L}^2$ and for each $P_n \in \mathcal{C}(x_n)$, the sequence $(P_n)_{n\in\mathbf{N}}$ has a limit point P, with respect to weak convergence of measures, in $\mathcal{C}(x)$.

Let $x_n \to x$ in $\mathcal{L}^2$ and let $P_n \in \mathcal{C}(x_n)$. By Theorem 3.2, $(P_n)_{n\in\mathbf{N}}$ is tight on $\Omega \cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}^1)$. Hence, up to a sub-sequence that we keep denoting by $(P_n)_{n\in\mathbf{N}}$, it follows that $P_n \rightharpoonup P$, for some P. To complete the proof it remains to show that $P \in \mathcal{C}(x)$. Therefore, we need to verify that the limit point P satisfies properties [W1], [W2] and [W3] of Definition 2.2 and properties [E2] and [E3] of Definition 2.5.

We start by proving [W2] for P. Given $\varphi \in \mathcal{D}^\infty$, we know that for each $n \in \mathbf{N}$ the process $(|\mathcal{Q}^{1/2}\varphi|^{-1}_{\mathcal{L}^2} M^\varphi_t, \mathcal{B}_t, P_n)_{t\geq 0}$ is a one-dimensional standard Brownian motion. Now, since $P_n \rightharpoonup P$ and M^φ, as a function from Ω with values in $C([0,\infty);\mathbf{R})$, is continuous, it follows that M^φ has the law of a standard Brownian motion under P. Indeed, continuity of M^φ allows it to pass to the limit in the characteristic functions. Since moreover M^φ is adapted and has continuous paths, we can conclude that $(|\mathcal{Q}^{1/2}\varphi|^{-1}_{\mathcal{L}^2} M^\varphi_t, \mathcal{B}_t, P)_{t\geq 0}$ is a standard Brownian motion.

Property [W3] is obvious, since the marginals of P_n at time 0 converge, by assumption, to both δ_x and the marginal of P at time 0, hence they coincide and P is started at δ_x.

Before proving the other properties (namely, [W1], [E2] and [E3]) for P, we need to make the following statement, which will be crucial for the conclusion of the proof. By using the tightness from Theorem 3.2 with $K = \log(1+|x|^2_{\mathcal{L}^2})^\kappa$, the boundedness of the stochastic convolution in $L^{16/3}_{\mathrm{loc}}([0,\infty);\mathcal{W}^{1,4})$ from statement (1) of Lemma 2.4, and the classical Skorokhod theorem (see, e.g., Ikeda and Watanabe [23]), we know that there exists a probability space $(\Sigma, \mathcal{F}, \mathbb{P})$ and random variables $(h^{(n)}, z^{(n)})_{n\in\mathbf{N}}$ and $(h^{(\infty)}, z^{(\infty)})_{n\in\mathbf{N}}$ such that:

1. Each $(h^{(n)}, z^{(n)})$ has the same law of (ξ, Z) under P_n,
2. $(h^{(\infty)}, z^{(\infty)})$ has the same law of (ξ, Z) under P,
3. $h^{(n)} \to h^{(\infty)}$ in $\Omega \cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}^1)$, $\mathbb{P}$-a.s.,
4. $z^{(n)} \to z^{(\infty)}$ in $L^{16/3}(0,T;\mathcal{W}^{1,4})$, $\mathbb{P}$-a.s.

In particular, $v^{(n)} = h^{(n)} - z^{(n)}$ has the same law of V under P_n (and so is for $v^{(\infty)} = h^{(\infty)} - z^{(\infty)}$ and V under P).

In the rest of the proof we shall verify properties [W1], [E2] and [E3] on the processes $(h^{(\infty)}, z^{(\infty)})$. Since they have the same law as (ξ, Z) under P (as stated above), we shall conclude that these properties hold for P.

In order to prove [W1], it is sufficient to show that

$$\mathbb{P}[\|h^{(n)}\|_{L^2(0,T;\mathcal{H}^1)} > K] \to 0 \qquad \text{as } K \uparrow \infty \text{ for all } T > 0.$$

By (3.1), we know that $\mathbb{E}^{\mathbb{P}}[\log(1 + \int_0^T |h^{(n)}|^2_{\mathcal{H}^1}\,ds)] \leq C_T$, so that Fatou's lemma implies a similar estimate for $h^{(\infty)}$ and Chebyshev inequality gives the result.

One can proceed similarly to prove [E2], using (3.2) and the fact that norms in $L^\infty(0,T;\mathcal{L}^2)$ and in $L^2(0,T;\mathcal{H}^2)$ are lower semi-continuous with respect to the topology where $v^{(n)} \to v^{(\infty)}$.

In order to prove [E3], we show that property [E3a] (with $z_0 = 0$ and $\alpha = 0$) of Proposition 2.10 holds true. Fix $t > 0$. Before proving [E3a], we state two useful remarks.

The first useful fact is that $v^{(n)}$ converges weakly in $L^2(0,t;\mathcal{H}^2)$ to $v^{(\infty)}$. Indeed, we can use [E3], applied to each $v^{(n)}$, and the bounds of $z^{(n)}$ in the space $L^{16/3}_{\mathrm{loc}}([0,\infty);\mathcal{W}^{1,4})$, ensured by statement (1) of Lemma 2.4, and in $L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{H}^1)$, ensured by statement (2) of Lemma 2.4, to show that $(v^{(n)})_{n\in\mathbf{N}}$ is bounded in $L^2(0,t;\mathcal{H}^2)$, $\mathbb{P}$-a.s. The bound follows from an inequality for each $v^{(n)}$ which can be obtained from the energy inequality in the same way as (6.1) in Lemma 6.6. It follows then that $v^{(n)} \rightharpoonup v^{(\infty)}$, in $L^2(0,t;\mathcal{H}^2)$, since we already know that $v^{(n)}$ converges to $v^{(\infty)}$ in $L^2(0,t;\mathcal{H}^1)$.

A second useful fact is that there is a null Lebesgue set $S \subset (0,t]$ such that for all $s \notin S$,

$$\mathbb{P}\big[|v^{(n')}(s)|_{\mathcal{L}^2} \to |v^{(\infty)}(s)|_{\mathcal{L}^2} \text{ for a subsequence } v^{(n')}\big] = 1. \tag{3.3}$$

Note that this does not imply a.s. convergence for a subsequence, as the subsequence may depend on $\sigma \in \Sigma$.

To prove (3.3) note that $v^{(n)} \to v^{(\infty)}$, $\mathbb{P}$-a.s. in $L^2(0,t;\mathcal{L}^2)$, and so

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(1 + \frac{1}{t}\int_0^t |v^{(n)} - v^{(\infty)}|^2_{\mathcal{L}^2}\,ds\right)\right] \to 0.$$

This follows from uniform bounds on higher moments from (3.2) with $\kappa > 1$. By the Jensen inequality,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{t}\int_0^t \log(1 + |v^{(n)} - v^{(\infty)}|^2_{\mathcal{L}^2})\,ds\right] \le \mathbb{E}^{\mathbb{P}}\left[\log\left(1 + \frac{1}{t}\int_0^t |v^{(n)} - v^{(\infty)}|^2_{\mathcal{L}^2}\,ds\right)\right]$$

and so there are a set $S \subset (0,t]$ [notice that $0 \notin S$ since we already know that $v^{(n)}(0) \to v^{(\infty)}(0)$] and a subsequence $v^{(n')}$ such that

$$\mathbb{E}^{\mathbb{P}}[\log(1 + |v^{(n')}(s) - v^{(\infty)}(s)|^2_{\mathcal{L}^2})] \to 0 \qquad \text{for all } s \notin S.$$

From this claim (3.3) now easily follows, possibly by taking a further sub-sequence depending on $\sigma \in \Sigma$.

We are now able to prove [E3a] for P (with $z_0 = 0$ and $\alpha = 0$). We know that for each $n \in \mathbf{N}$ there is a null Lebesgue set $T_n \subset (0,t]$ such that $\mathbb{P}[\mathcal{E}_t(v^{(n)}, z^{(n)}) \le \mathcal{E}_s(v^{(n)}, z^{(n)})] = 1$, for all $s \notin T_n$. Let $T = S \cup \bigcup T_n$ and consider $s \notin T$, so that $\mathcal{E}_t(v^{(n)}, z^{(n)}) \le \mathcal{E}_s(v^{(n)}, z^{(n)})$ holds $\mathbb{P}$-a.s. for all $n \in \mathbf{N}$.

Now we can use all the convergence information we have collected. Recall that for $v^{(n)}$ we have (3.3), strong convergence in $L^2_{\mathrm{loc}}(0,T;\mathcal{H}^1)$ and weak convergence in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}^2)$. Furthermore, for $z^{(n)}$ we can rely on strong convergence in $L^{16/3}_{\mathrm{loc}}([0,\infty);\mathcal{W}^{1,4}$ and boundedness in $L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{H}^1)$. Actually, strong convergence of $z^{(n)}$ in any $L^p_{\mathrm{loc}}([0,\infty);\mathcal{W}^{1,4}$ is true, but in the analysis $p = 16/3$ is sufficient to control cubic terms like $\int_0^t \langle v_{xx}^{(n)}, (z_x^{(n)})^2\rangle_{\mathcal{L}^2}$. See also the proof of Lemma 6.6, where these terms are bounded. For simplicity of presentation, we omit the technical detail of this analysis, which is similar to [5], where convergence of statistical quantities similar to the energy for the spectral Galerkin approximation were studied, and we are able to pass to the limit $n \to \infty$ in the energy.

In conclusion $\mathbb{P}[\mathcal{E}_t(v^{(\infty)}, z^{(\infty)}) \le \mathcal{E}_s(v^{(\infty)}, z^{(\infty)})] = 1$. □

Before stating the next two lemmas (which contain the multi-valued form of the Markov property), we need to analyze what happens to processes W, Z and V under the action of the forward shift Φ_{t_0} for a given $t_0 \ge 0$. First, given $s \ge 0$ and $z_0 \in \mathcal{H}^1$, denote by $Z(t,\cdot|s,z_0)$ the Ornstein–Uhlenbeck process starting in z_0 at time s, namely

$$Z(t,\cdot|s,z_0) = e^{A(t-s)}z_0 + \sum \alpha_k \int_s^t e^{(t-r)\lambda_k}\,d\beta_k(r)e_k.$$

In particular, we have that $Z(t,\cdot|0,0) = Z(t,\cdot)$. Set moreover $V(t,\cdot|s,z_0) = \xi - Z(t,\cdot|s,z_0)$. Now, from [W2] and (2.2) it is easy to verify that, for all $\omega \in \Omega^{t_0}$,

$$W(t, \Phi_{t_0}^{-1}(\omega)) = W(t+t_0,\omega) - W(t_0,\omega)$$

and it depends only on the values of ω in $[t_0, t_0+t]$. Similarly,

$$\begin{aligned} Z(\Phi_{t_0}^{-1}(\omega), t|s,z_0) &= Z(\omega, t+t_0|s+t_0, z_0),\\ V(\Phi_{t_0}^{-1}(\omega), t|s,z_0) &= V(\omega, t+t_0|s+t_0, z_0). \end{aligned} \tag{3.4}$$

LEMMA 3.5 (Disintegration lemma). *For every $x \in \mathcal{L}^2$ and $P \in \mathcal{C}(x)$, there is a set $T \subset (0,\infty)$, with null Lebesgue measure, such that for all $t \notin T$ there is $N \in \mathcal{B}_t$, with $P[N] = 0$, such that for all $\omega \notin N$,*

$$\omega(t) \in \mathcal{L}^2 \quad \textit{and} \quad P|^{\omega}_{\mathcal{B}_t} \in \Phi_t\mathcal{C}(\omega(t)).$$

PROOF. Fix $x \in \mathcal{L}^2$ and $P \in \mathcal{C}(x)$, let T_P be the set of exceptional times of P, as given by [E3] of Definition 2.5, and fix $t_0 \notin T_P$. Let $(P|^{\omega}_{\mathcal{B}_{t_0}})_{\omega\in\Omega}$ be a regular conditional probability distribution of P given $\mathcal{B}_{t_0}$. We aim to show that there is a P-null set $N \in \mathcal{B}_{t_0}$ such that $\omega(t_0) \in \mathcal{L}^2$ and $P|^{\omega}_{\mathcal{B}_{t_0}} \in \Phi_{t_0}\mathcal{C}(\omega(t_0))$ for all $\omega \notin N$. In order to prove that $P|^{\omega}_{\mathcal{B}_{t_0}} \in \Phi_{t_0}\mathcal{C}(\omega(t_0))$, we need to find the P-null set N and verify that $P|^{\omega}_{\mathcal{B}_{t_0}}$ satisfies properties [E1] (hence [W1], [W2] and [W3] of Definition 2.2), [E2] and [E3] of Definition 2.5. We shall find the null set N as $N = N_{[E1]} \cup N_{[E2]} \cup N_{[E3]}$, where $N_{[E1]}$ is the P-null set such that [E1] holds for $P|^{\omega}_{\mathcal{B}_{t_0}}$ for all $\omega \notin N_{[E1]}$, and similarly for $N_{[E2]}$, $N_{[E3]}$.

The proof of property [E1] for the conditional distributions $P|^{\omega}_{\mathcal{B}_{t_0}}$ is entirely similar to Lemma 4.4 of Flandoli and Romito [19], so we shall focus on the proof of properties [E2] and [E3].

We start by [E2]. We need to show that $P|^{\omega}_{\mathcal{B}_{t_0}}[V(\cdot, \Phi_{t_0}^{-1}(\cdot)) \in S_{[0,\infty)}] = 1$ or, equivalently, by (3.4), that $P|^{\omega}_{\mathcal{B}_{t_0}}[V(\cdot, \cdot|t_0, 0) \in S_{[t_0,\infty)}] = 1$, where we have set, for brevity, $S_J = L^{\infty}_{\mathrm{loc}}(J; \mathcal{L}^2) \cap L^2_{\mathrm{loc}}(J; \mathcal{H}^2)$, for any interval $J \subset [0, \infty)$. Set

$$\begin{aligned} \mathsf{Reg}_{t_0} &= \{V \in S_{[0,t_0]} \text{ and } e^{At} Z(t_0, \cdot) \in S_{[0,\infty)}\}, \\ \mathsf{Reg}^{t_0} &= \{V(\cdot, \cdot|t_0, 0) \in S_{[t_0,\infty)}\}, \end{aligned} \tag{3.5}$$

then $\mathsf{Reg}_{t_0} \in \mathcal{B}_{t_0}$ and $\mathsf{Reg}^{t_0} \in \mathcal{B}^{t_0}$, since by definition V and Z are adapted. Moreover, since $V(t + t_0, \omega) = V(t + t_0, \omega|t_0, 0) - e^{At} Z(t_0, \omega)$, it follows from [E2] for P, statement (2) of Lemma 2.4 and the regularity properties of the semigroup e^{At}, that $\mathsf{Reg}_{t_0} \cap \mathsf{Reg}^{t_0}$ is a P-full set and so, by disintegration,

$$1 = P[\mathsf{Reg}_{t_0} \cap \mathsf{Reg}^{t_0}] = \int_{\mathsf{Reg}_{t_0}} P|^{\omega}_{\mathcal{B}_{t_0}}[\mathsf{Reg}^{t_0}] P(d\omega).$$

Thus, there is a P-null set $N_{[\mathrm{E2}]} \in \mathcal{B}_{t_0}$ such that $P|^{\omega}_{\mathcal{B}_{t_0}}[\mathsf{Reg}^{t_0}] = 1$ for all $\omega \notin N_{[\mathrm{E2}]}$ and [E2] for the conditional probabilities is true.

Next, we prove [E3] for the conditional probabilities. Indeed, it is sufficient to verify condition [E3c] of Proposition 2.10. Set

$$\begin{aligned} A &= \{\mathcal{E}_t(V, Z) \le \mathcal{E}_s(V, Z) \text{ for a.e. } s \ge 0 \text{ (including } 0, t_0), \text{ all } t \ge s\}, \\ A_{t_0} &= \{\mathcal{E}_t(V, Z) \le \mathcal{E}_s(V, Z) \text{ for a.e. } s \in [0, t_0] \text{ (including } 0, t_0), \text{ all } t \in [s, t_0]\}, \end{aligned}$$

where, for the sake of simplicity, in the definitions of the above sets we have omitted the information on regularity for V and Z, which are essential to ensure measurability (cf. with Proposition 2.10). They can be treated as in the proof of [E2] above. We have $A_{t_0} \in \mathcal{B}_{t_0}$ and $P[A] = P[A_{t_0}] = 1$, since $t_0 \notin T_P$. Now, if $\overline{\omega} \in A_{t_0} \cap \{Z \in \mathcal{H}^1\}$ [which is again a P-full set by statement (2) of Lemma 2.4], set

$$B(\overline{\omega}) = A \cap \{\omega : \omega = \overline{\omega} \text{ on } [0, t_0]\}$$

and notice that, for such $\overline{\omega}$, $B(\overline{\omega})$ is equal to

$$\{\mathcal{E}_t(V_{\overline{\omega}}, Z_{\overline{\omega}}) \le \mathcal{E}_s(V_{\overline{\omega}}, Z_{\overline{\omega}}) \text{ for a.e. } s \ge t_0 \text{ (including } t_0), \text{ all } t \ge s\},$$

since $V(t + t_0, \omega) = V(t + t_0, \omega|t_0, Z(t_0, \omega))$ (a similar relation holds for Z as well), and we have set $V_{\overline{\omega}}(\cdot) = V(\cdot|t_0, Z(t_0, \overline{\omega}))$ and $Z_{\overline{\omega}}(\cdot) = Z(\cdot|t_0, Z(t_0, \overline{\omega}))$. Moreover, the map

$$\omega \to \mathbf{1}_{A_{t_0} \cap \{Z \in \mathcal{H}^1\}}(\omega) P|^{\omega}_{\mathcal{B}_{t_0}}[B(\omega)]$$

is $\mathscr{B}_{t_0}$-measurable, since $P|^{\omega}_{\mathscr{B}_{t_0}}[B(\omega)] = P|^{\omega}_{\mathscr{B}_{t_0}}[A]$ for all $\omega \in A_{t_0} \cap \{Z \in \mathscr{H}^1\}$. Now, by [E3c] for P (with $z_0 = 0$ and $\alpha = 0$) and disintegration,

$$1 = P[A] = \mathbb{E}^P\big[\mathbf{1}_{A_{t_0} \cap \{Z \in \mathscr{H}^1\}}(\cdot) P|^{\cdot}_{\mathscr{B}_{t_0}}[B(\cdot)]\big]$$

and so there is $N_{[\mathrm{E3}]} \in \mathscr{B}_{t_0}$ such that $P|^{\omega}_{\mathscr{B}_{t_0}}[B(\omega)] = 1$ for all $\omega \notin N_{[\mathrm{E3}]}$ or, in different words, such that [E3c] holds [with $z_0 = Z(t_0, \omega)$ and $\alpha = 0$] for $P|^{\omega}_{\mathscr{B}_{t_0}}$ for all $\omega \notin N_{[\mathrm{E3}]}$. □

LEMMA 3.6 (Reconstruction lemma). *For every $x \in \mathcal{L}^2$ and $P \in \mathcal{C}(x)$, there is a set $T \subset (0, \infty)$, with null Lebesgue measure, such that for each $t \notin T$, for each $\mathscr{B}_t$-measurable map $\omega \mapsto Q_\omega : \Omega \to \Pr(\Omega^t)$ such that there is $N \in \mathscr{B}_t$ with $P[N] = 0$, and for all $\omega \notin N$ with*

$$\omega(t) \in \mathcal{L}^2 \quad \textit{and} \quad Q_\omega \in \Phi_t \mathcal{C}(\omega(t)),$$

we have $P \otimes_t Q \in \mathcal{C}(x)$.

PROOF. Let $x \in \mathcal{L}^2$, $P \in \mathcal{C}(x)$, T_P be the set of exceptional times of P and fix $t_0 \notin T_P$. Let $(Q_\omega)_{\omega \in \Omega}$ be a $\mathscr{B}_{t_0}$-measurable map and N_Q a P-null set such that $\omega(t_0) \in \mathcal{L}^2$ and $Q_\omega \in \Phi_{t_0} \mathcal{C}(\omega(t_0))$ for all $\omega \notin N_Q$. In order to verify that $P \otimes_{t_0} Q \in \mathcal{C}(x)$, we only check properties [E2] and [E3], since the proof of [E1] can be carried on as in Flandoli and Romito [19], Lemma 4.5.

We start by [E2]. Consider again sets $\mathsf{Reg}_{t_0} \in \mathscr{B}_{t_0}$ and $\mathsf{Reg}^{t_0} \in \mathscr{B}^{t_0}$ defined in (3.5) and notice that, by [E2] for Q_ω, for each $\omega \notin N_Q$ we have that $Q_\omega[\mathsf{Reg}^{t_0}] = 1$. Moreover, by [E2] for P, statement (2) of Lemma 2.4 and the regularity properties of the semigroup e^{At}, it follows that $P[\mathsf{Reg}_{t_0}] = 1$. Finally, since we know that $V(t + t_0, \omega) = V(t + t_0, \omega | t_0, 0) - e^{At} Z(t_0, \omega)$, it follows easily that $\mathsf{Reg}_{t_0} \cap \mathsf{Reg}^{t_0} = \{V \in S_{[0,\infty)}\}$ and so

$$\begin{aligned}(P \otimes_{t_0} Q)\big[V \in S_{[0,\infty)}\big] &= (P \otimes_{t_0} Q)[\mathsf{Reg}_{t_0} \cap \mathsf{Reg}^{t_0}] \\ &= \int_{\mathsf{Reg}_{t_0}} Q_\omega[\mathsf{Reg}^{t_0}] P(d\omega) = 1.\end{aligned}$$

We next prove [E3]. Again, we prove it by means of [E3c], thanks to Proposition 2.10. Define A and A_{t_0} as in the proof of the previous lemma (the regularity conditions on Z and V are again omitted). Since $t_0 \notin T_P$ and $A_{t_0} \in \mathscr{B}_{t_0}$, we know that $(P \otimes_{t_0} Q)[A_{t_0}] = P[A_{t_0}] = 1$. Moreover, by statement (2) of Lemma 2.4, there is a P-null set $N \in \mathscr{B}_{t_0}$ such that $Z(t_0, \omega) \in \mathscr{H}^1$ for all $\omega \notin N$. For each $\overline{\omega} \notin N$, define $B(\overline{\omega}) = A \cap \{\omega : \omega = \overline{\omega} \text{ on } [0, t_0]\}$ and notice that, if $\omega \in A_{t_0} \cap (N \cap N_Q)^c$ [which is again a $\mathscr{B}_{t_0}$-measurable $(P \otimes_{t_0} Q)$-full set], then by [E3c] [with $z_0 = Z(t_0, \omega)$ and $\alpha = 0$] for Q_ω it follows that $Q_\omega[B(\omega)] = 1$. The map $\omega \mapsto \mathbf{1}_{A_{t_0} \cap (N \cap N_Q)^c}(\omega) Q_\omega[B(\omega)]$ is then trivially $\mathscr{B}_{t_0}$-measurable and equal to 1, P-a.s. Moreover, we have that $Q_\omega[A] = Q_\omega[B(\omega)] = 1$ for all $\omega \in A_{t_0} \cap (N \cap N_Q)^c$ and so

$$(P \otimes_{t_0} Q)[A] = \mathbb{E}^P[\mathbf{1}_{A_{t_0} \cap (N \cap N_Q)^c} Q.[B(\cdot)]] = P[A_{t_0} \cap (N \cap N_Q)^c] = 1.$$

In conclusion, [E3c] (with $z_0 = 0$ and $\alpha = 0$) holds true for $P \otimes_{t_0} Q$. $\square$

4. The strong Feller property. Throughout this section we shall assume that the noise is nondegenerate. This is summarized by the following assumption:

ASSUMPTION 4.1. The operator $\mathcal{Q}^{-1/2}$ is bounded, where $\mathcal{Q}$ is the covariance of the noise. In other words,

$$\alpha_k \geq \delta > 0$$

for some constant δ, where α_k^2 are the eigenvalues of $\mathcal{Q}$.

THEOREM 4.2. *Under the above assumption, any a.s. Markov family $(P_x)_{x \in \mathcal{L}^2}$ of energy martingale solutions defines a Markov semigroup that has the $\mathcal{H}^1$-strong Feller property.*

PROOF. We mainly rely on [19, 20]. Let $(P_x)_{x \in \mathcal{L}^2}$ be an a.s. Markov family of energy martingale solution and denote by $(\mathcal{P}_t)_{t \geq 0}$ the corresponding (a.s.) semigroup generated by P_x. Then the claim follows from the following lemma:

LEMMA 4.3. *There is an $\epsilon = \epsilon(|g|_{\mathcal{H}^1}, R) \to 0$ for $g \to 0$ such that*

$$|\mathcal{P}_\epsilon \varphi(x+g) - \mathcal{P}_\epsilon \varphi(x)| \leq C|g|_{\mathcal{H}^1} \log(1/|g|_{\mathcal{H}^1}) \tag{4.1}$$

for all $|g|_{\mathcal{H}^1} \leq 1$, all $\varphi \in L^\infty(\mathcal{H}^1)$ with $|\varphi|_{\mathcal{L}^\infty} \leq 1$, and all $|x|_{\mathcal{H}^1} \leq R/4$ for some sufficiently large R.

With this lemma at hand, we define for $\varphi \in L^\infty(\mathcal{H}^1)$ with $|\varphi|_{\mathcal{L}^\infty} = 1$ and g (i.e., ϵ) sufficiently small $\varphi_* = \mathcal{P}_{t-\epsilon}\varphi \in L^\infty(\mathcal{H}^1)$ with $|\varphi_*|_{\mathcal{L}^\infty} \leq 1$. Thus

$$\begin{aligned} |\mathcal{P}_t\varphi(x+h) - \mathcal{P}_t\varphi(x)| &\leq |\mathcal{P}_\epsilon\varphi_*(x+g) - \mathcal{P}_\epsilon\varphi_*(x)| \\ &\leq C|g|_{\mathcal{H}^1} \log(1/|g|_{\mathcal{H}^1}). \end{aligned} \tag{4.2}$$

This implies strong Feller for $\mathcal{P}_t$. $\square$

Following the arguments of [19, 20] it is enough to prove strong Feller for the following *regularized* problem:

$$\partial_t \widetilde{h} = -\widetilde{h}_{xxxx} + \big(-\widetilde{h} + (\widetilde{h}_x)^2\big)_{xx} \chi_\rho(|\widetilde{h}|^2_{\mathcal{H}^1}) + \partial_t W, \tag{4.3}$$

where $\chi_\rho \in C^\infty$ is a cut-off function such that $\chi_\rho \equiv 1$ on $[0, \rho^2]$ and $\chi_\rho \equiv 0$ on $[2\rho^2, \infty)$. For all $\zeta \geq 0$ we have

$$|\chi_\rho(\zeta)| \leq 1, \qquad |\chi'_\rho(\zeta)| \leq C\rho^{-2},$$
$$|\chi_\rho(\zeta^2)\zeta^p| \leq C\rho^p, \qquad |\chi'_\rho(\zeta^2)\zeta^p| \leq C\rho^{p-2}.$$

Let $P_x^{(\rho)}$ be the (unique) Markov energy martingale solution of the regularized problem (4.3). This is well defined, as we can solve (4.3) path-wise. The mild solution of (4.3) is given by

$$\widetilde{h}(t) = e^{tA}\widetilde{h}(0) - \int_0^t \partial_x^2 e^{(t-s)A} F(\widetilde{h}(s))\, ds + Z(t), \tag{4.4}$$

where Z has been defined in (2.3) and

$$F(\widetilde{h}) = \bigl(-\widetilde{h} + (\widetilde{h}_x)^2\bigr)\chi_\rho(|\widetilde{h}|^2_{\mathcal{H}^1}).$$

Using the embedding of L^1 into $H^{-1+4\gamma}$ for $\gamma \in (0, \frac{1}{8})$, we can easily check that

$$\begin{gathered} |F(h_1) - F(h_2)|_{\mathcal{H}^{-1+4\gamma}} \leq C_\rho |h_1 - h_2|_{\mathcal{H}^1}, \\ |F(h)|_{\mathcal{H}^{-1+4\gamma}} \leq C(\rho + \rho^2). \end{gathered} \tag{4.5}$$

Now uniqueness for (4.3) in $C^0([0,\infty), \mathcal{H}^1)$ follows from standard path-wise fixed point arguments. The proof is straightforward as we can rely on one hand on F being Lipschitz and bounded. On the other hand e^{tA} generates an analytic semigroup such that

$$|e^{tA}w|_{\mathcal{H}^1} \leq |w|_{\mathcal{H}^1} \quad \text{and} \quad |\partial_x^2 e^{tA} w|_{\mathcal{H}^1} \leq M(1 + t^{\gamma-1})|w|_{\mathcal{H}^{-1+4\gamma}}$$

(see, e.g., Henry [22], Pazy [26] or Lunardi [25]).

Next, define

$$\tau_\rho = \inf\{t > 0\colon \text{ solution of (4.3) is bounded in } \mathcal{H}^1 \text{ on } [0,t] \text{ by } \rho\}. \tag{4.6}$$

Thus the solution of the regularized problem coincides with the energy solution up to τ_ρ and in view of (4.1) we have

$$\begin{aligned} |\mathcal{P}_\epsilon\varphi(x+g) - \mathcal{P}_\epsilon\varphi(x)| \leq{}& 2(P_x[\tau_\rho < \epsilon] + P_{x+g}[\tau_\rho < \epsilon]) \\ &+ \bigl|\mathcal{P}_\epsilon^{(\rho)}\varphi(x+g) - \mathcal{P}_\epsilon^{(\rho)}\varphi(x)\bigr|, \end{aligned} \tag{4.7}$$

where $\mathcal{P}^{(\rho)}$ is the semigroup generated by (4.3) or (4.4), respectively.

In order to prove Lemma 4.3 we need the following two lemmas:

LEMMA 4.4. *There is a $p > 1$ sufficiently large, such that for $\rho \geq 1$ and $t \leq 1$*

$$\bigl|\mathcal{P}_t^{(\rho)}\varphi(x+g) - \mathcal{P}_t^{(\rho)}\varphi(x)\bigr| \leq \frac{C}{t}|h|_{\mathcal{H}^{-1}} e^{ct\rho^p}$$

for all $x, g \in \mathcal{H}^1$.

LEMMA 4.5. *There is a small constant c_τ depending on γ, and M such that for all $\rho \geq 1$, $\epsilon \in (0,1]$, $\widetilde{h}_0$ such that $|\widetilde{h}_0|_{\mathcal{H}^1} \leq \rho/4 + 1$, we have*

$$P_{\widetilde{h}_0}[\tau_\rho \geq \epsilon] \geq P_{\widetilde{h}_0}\Big[\sup_{t\in[0,\epsilon]} |Z(t)|_{\mathcal{H}^1} \leq \rho/4\Big]$$

for all $\epsilon \leq C_\tau \rho^{-2/\gamma}$.

Using arguments analogous to [20], Proposition 15, we immediately obtain:

COROLLARY 4.6. *There are two constants c, $C > 0$ depending on γ and M such that for all $\rho \geq 1$, $\epsilon \in (0,1]$, $\widetilde{h}_0$ such that $|\widetilde{h}_0|_{\mathcal{H}^1} \leq \rho/4 + 1$, we have*

$$P_{\widetilde{h}_0}[\tau_\rho \geq \epsilon] \leq Ce^{-c\rho^2/\epsilon}$$

for all $\epsilon \leq c_\tau \rho^{-2/\gamma}$.

PROOF OF LEMMA 4.3. For $g, x \in \mathcal{H}^1$ such that $|x|_{\mathcal{H}^1} \leq \rho/4$ and $|g|_{\mathcal{H}^1} \leq 1$, we can apply Corollary 4.6 for $\widetilde{h}_0 = x$ and $\widetilde{h}_0 = x + g$. From (4.7) together with Lemma 4.4 and the embedding of $\mathcal{H}^1$ into $\mathcal{H}^{-1}$ for $\epsilon \leq \min\{1, c_\tau \rho^{-2/\gamma}\}$, $\rho \geq \max\{4|x|_{\mathcal{H}^1}, 1\}$, $t \leq 1$,

$$|\mathcal{P}_\epsilon \varphi(x+g) - \mathcal{P}_\epsilon \varphi(x)| \leq Ce^{-c\rho^2/\epsilon} + C|g|_{\mathcal{H}^1} \frac{1}{t} e^{ct\rho^p}. \tag{4.8}$$

Thus, if we fix for a suitable constant $C > 0$

$$\epsilon = \min\left\{1; \frac{C}{\rho^q \ln(1/|g|_{\mathcal{H}^1})}\right\} \qquad \text{for some } q > \max\{p, 2/\gamma\},$$

then we obtain

$$|\mathcal{P}_\epsilon \varphi(x+g) - \mathcal{P}_\epsilon \varphi(x)| \leq C|g|_{\mathcal{H}^1} \ln(1/|g|_{\mathcal{H}^1}). \qquad \square$$

The remainder of the section is devoted to the proof of the two remaining lemmas.

PROOF OF LEMMA 4.5. First from (4.4) for $t \leq 1$

$$|\widetilde{h}(t)|_{\mathcal{H}^1} \leq |\widetilde{h}(0)|_{\mathcal{H}^1} + C\int_0^t (t-s)^{\gamma-1} |F(\widetilde{h})|_{\mathcal{H}^{-1+4\gamma}}\, ds + |Z(t)|_{\mathcal{H}^1}.$$

Thus from (4.5) for $t \leq \min\{1, \tau_\rho\}$ and $\rho \geq 1$

$$|\widetilde{h}(t)|_{\mathcal{H}^1} \leq \rho/4 + C\tau_\rho^\gamma \rho^2 + |Z(t)|_{\mathcal{H}^1},$$

which easily implies the claim. $\square$

PROOF OF LEMMA 4.4. We proceed in a manner analogous to the proof of [19], Proposition 5.13. For every $x \in \mathcal{H}^1$, let $\widetilde{h}(t,x)$ be the solution to (4.3) with $\widetilde{h}(0,x) = x$. By the Bismut, Elworthy and Li formula,

$$D_y\big(\mathcal{P}_t^{(\rho)}\varphi\big)(x) = \frac{1}{t}\mathbb{E}\Big[\varphi(\widetilde{h}(t,x))\int_0^t \langle \mathcal{Q}^{-1} D_y\widetilde{h}(s,x), dW(s)\rangle_{\mathcal{L}^2}\Big].$$

Now the Burkholder, Davis and Gundy inequality states

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \langle f(s), dW(s)\rangle_{\mathcal{L}^2}\right|^p \le C\mathbb{E}\Big(\int_0^T |\mathcal{Q}^{1/2} f(t)|_{\mathcal{L}^2}^2\,dt\Big)^{p/2}$$

and thus, for $|\varphi|_\infty \le 1$,

$$\begin{aligned} &\big|\big(\mathcal{P}_t^{(\rho)}\varphi\big)(x+g) - \big(\mathcal{P}_t^{(\rho)}\varphi\big)(x)\big| \\ &\quad\le \frac{C}{t}\sup_{\eta\in[0,1]}\mathbb{E}\Big[\Big(\int_0^t |\mathcal{Q}^{-1/2} D_g\widetilde{h}(s,x+\eta g)|_{\mathcal{L}^2}^2\,ds\Big)^{1/2}\Big]. \end{aligned} \tag{4.9}$$

Now $\psi(t) = D_g\widetilde{h}(t,x+\eta g)$ with $\psi(0) = \eta g$ solves

$$\partial_t\psi = -\psi_{xxxx} + \partial_x^2 DF(\widetilde{h})[\psi] \tag{4.10}$$

with

$$DF(\widetilde{h})[\psi] = -(\psi + 2\widetilde{h}_x\psi_x)\chi_\rho(|\widetilde{h}|_{\mathcal{H}^1}^2) - 2\big(\widetilde{h} + (\widetilde{h}_x)^2\big)\chi_\rho'(|\widetilde{h}|_{\mathcal{H}^1}^2)\langle\widetilde{h},\psi\rangle_{\mathcal{H}^1}.$$

The following arguments are only formal, but as we are working with unique solutions they can all be made rigorous by Galerkin approximations. Multiplying (4.10) with $\langle\cdot,\psi\rangle_{\mathcal{H}^{-1}}$ yields for $\rho \ge 1$

$$\begin{aligned} \tfrac{1}{2}\partial_t|\psi|_{\mathcal{H}^{-1}}^2 + |\psi|_{\mathcal{H}^1}^2 &\le |DF(\widetilde{h})[\psi]|_{\mathcal{L}^1}|\psi|_{\mathcal{L}^\infty} \\ &\le C|\psi|_{\mathcal{L}^\infty}(|\psi|_{\mathcal{L}^1} + |\widetilde{h}|_{\mathcal{H}^1}|\psi|_{\mathcal{H}^1})\chi_\rho(|\widetilde{h}|_{\mathcal{H}^1}^2) \\ &\quad + C|\psi|_{\mathcal{L}^\infty}(|\widetilde{h}|_{\mathcal{L}^1} + |\widetilde{h}|_{\mathcal{H}^1}^2)\chi_\rho'(|\widetilde{h}|_{\mathcal{H}^1}^2)|\widetilde{h}|_{\mathcal{H}^1}|\psi|_{\mathcal{H}^1} \\ &\le C\rho|\psi|_{\mathcal{H}^1}|\psi|_{\mathcal{L}^\infty}, \end{aligned}$$

where we used Hölder, Sobolev embedding and the definition of the cut-off χ.

Using Sobolev embedding of $\mathcal{L}^\infty$ into $\mathcal{H}^\delta$ for some $\delta > \frac{1}{2}$ together with interpolation and the Young inequality yields for some sufficiently large $p > 1$ and some constant $c > 0$

$$\partial_t|\psi|_{\mathcal{H}^{-1}}^2 + |\psi|_{\mathcal{H}^1}^2 \le \tfrac{1}{2}|\psi|_{\mathcal{H}^1}^2 + c\rho^p|\psi|_{\mathcal{H}^{-1}}^2.$$

First, by the Gronwall lemma

$$|\psi(t)|_{\mathcal{H}^{-1}}^2 \le |\psi(0)|_{\mathcal{H}^{-1}}^2 e^{ct\rho^p}$$

and then

$$\int_0^t |\psi|_{\mathcal{H}^1}^2\,dt \le |\psi(0)|_{\mathcal{H}^{-1}}^2 + c\rho^p\int_0^t |\psi(s)|_{\mathcal{H}^{-1}}^2\,ds \le |\psi(0)|_{\mathcal{H}^{-1}}^2 e^{ct\rho^p}.$$

This together with (4.9) and the assumption on $\mathcal{Q}$ completes the proof. □

4.1. *Some consequences.* It is well known that the strong Feller property implies that the laws $P(t, x, \cdot)$ are mutually equivalent for all x and t. A less obvious fact, which follows from Theorem 13 of Flandoli and Romito [20], is that the same property holds between different selections. In detail, if $P^{(1)}(t, x, \cdot)$ and $P^{(2)}(t, x, \cdot)$ are the Markov kernels associated with two different selections, then $P^{(1)}(t, x, \cdot)$ and $P^{(2)}(t, x, \cdot)$ are mutually equivalent for all x and t.

Before enumerating all other properties following from strong Feller, we need to show a technical result on the support of the measures $P(t, x, \cdot)$. Following Flandoli and Romito [19], we say that a Borel probability measure μ is *fully supported* on $\mathcal{H}^1$ if $\mu[A] > 0$ for every open set A in $\mathcal{H}^1$.

PROPOSITION 4.7 (Support theorem). *Under the Assumption 4.1, let $(P_x)_{x\in\mathcal{L}^2}$ be an a.s. Markov family. For every $x \in \mathcal{H}^1$ and $T > 0$ the image measure of P_x at time T is fully supported on $\mathcal{H}^1$.*

PROOF. The proof is rather technical but straightforward; we only give a sketch of it. To this purpose, we follow the same steps of Flandoli [15] (see also Proposition 6.1 of [19]). By Assumption 4.1 the Wiener measure driving the equation is fully supported on Ω, or any smaller space, where W is still defined. Thus it turns out that we only have to analyze the following control problem

$$\dot{h} + h_{xxxx} = [-h_{xx} + (h_x{}^2)_{xx}]\chi_\rho + \dot{w}, \qquad h(0) = x, \tag{4.11}$$

where w is the control. More precisely, we need to prove the following two statements:

1. Given $T > 0$, there is $\lambda \in (0, 1)$ such that for $\rho > 0$, $x \in \mathcal{H}^1$, $y \in \mathcal{H}^4$ with $|x|_{\mathcal{H}^1} \le \lambda\rho$ and $|y|_{\mathcal{H}^1} \le \lambda\rho$, there are $w \in \mathrm{Lip}([0, T]; \mathcal{H}^1)$ and $h \in C([0, T]; \mathcal{H}^1)$ that solve (4.11) with $h(T) = y$ and $\tau_\rho(w) > T$, where τ_ρ is defined as in (4.6).
2. Let $w_n \to w$ in $W^{\gamma,p}([0, T]; D(A^\beta))$, with $\gamma \in (\frac{3}{8}, \frac{1}{2})$, $p > 1$ such that $\gamma p > 1$ and $\beta \in (\frac{1}{4} - \gamma, -\frac{1}{8})$. Let h_n, h be the solutions to (4.11) corresponding to w_n, w and let $\tau_n = \tau_\rho(w_n)$ and $\tau = \tau_\rho(w)$. If $\tau > T$, then $\tau_n > T$ for sufficiently large n and $h_n \to h$ in $C([0, T]; \mathcal{H}^1)$.

For the first claim, one uses (4.4) with $w = 0$ to get a time $T_* < T$ such that $h(T_*) \in \mathcal{H}^4$ and $|h(T_*)|_{\mathcal{H}^1} \le \rho$ (here we choose λ, using the estimates on the semigroup e^{tA}). Then h is given in $[T_*, T]$ by linear interpolation from $h(T_*)$ to y and w in such a way that (4.11) is satisfied.

For the second claim, γ, p and β are chosen so that the Wiener measure corresponding to the random perturbation gives probability 1 to the space $W^{\gamma,p}([0, T]; D(A^\beta))$ and the convergence of w_n implies that $z_n \to z$ in $C([0, T]; \mathcal{H}^1)$, where z_n, z are the solutions to $\dot{z} = -z_{xxxx} + \dot{w}$ corresponding to w_n and w (this also gives a common bound to τ_n and τ, as in Lemma 4.5). From this, it is easy to see, by the mild formulation (4.4), that $h_n \to h$. $\square$

PROPOSITION 4.8 (Local regularity). *Let $(P_x)_{x\in\mathcal{L}^2}$ be an a.s. Markov family and assume Assumption 4.1. Then for each $x \in \mathcal{H}^1$ and all times $t > 0$,*

$$P_x\big[\textit{there is } \varepsilon > 0 \textit{ such that } \xi \in C\big((t-\varepsilon, t+\varepsilon); \mathcal{H}^1\big)\big] = 1.$$

Moreover, for each $x \in \mathcal{H}^1$, the set T_{P_x} of property [E3] *is empty, that is, the energy inequality holds for all times.*

PROOF. Let $(\mathcal{P}_t)_{t\geq 0}$ be the transition semigroup defined by the given Markov family and set $\widetilde{\nu} = \int_0^1 (\mathcal{P}_s^* \delta_0)\,ds$. Moreover, define the two sets $\widetilde{\Omega}_{a,b} = \{\xi \in C((a,b); \mathcal{H}^1)\}$ and $\widetilde{\Omega}_t = \bigcup \widetilde{\Omega}_{t-\varepsilon,t+\varepsilon}$. We first observe that by (3.1),

$$\widetilde{P}[|\xi_t|^2_{\mathcal{H}^1} \geq \rho] = \int_t^{t+1} P_0[|\xi_s|^2_{\mathcal{H}^1} \geq \rho]\,ds \leq \frac{C}{\log(1+\rho)},$$

where in particular the constant C depends on t (but it is increasing in t). Now, by the Markov property, for all $\rho > 0$,

$$\begin{aligned}\widetilde{P}[\widetilde{\Omega}_{t-\varepsilon,t+\varepsilon}] &= \int P_y[\widetilde{\Omega}_{\varepsilon,3\varepsilon}]\pi_{t-2\varepsilon}\widetilde{P}(dy)\\ &\geq \Big(\inf_{|y|_{\mathcal{H}^1}\leq\rho} P_y[\widetilde{\Omega}_{\varepsilon,3\varepsilon}]\Big)\Big(1 - \frac{C}{\log(1+\rho)}\Big),\end{aligned}$$

where $\pi_s \widetilde{P}$ is the marginal of $\widetilde{P}$ at time s. By Lemma 4.5 we know that $\inf_{|y|_{\mathcal{H}^1}\leq\rho} P_y[\widetilde{\Omega}_{\varepsilon,3\varepsilon}] \uparrow 1$ as $\varepsilon \to 0$ and, in conclusion, $\widetilde{P}[\widetilde{\Omega}_t] = 1$.

By disintegration, $P_x[\widetilde{\Omega}_t] = 1$ for $\widetilde{\nu}$-a.e. x, hence for a dense set of $\mathcal{H}^1$ by Proposition 4.7 and, in conclusion, for all $x \in \mathcal{H}^1$ by the strong Feller property. □

The previous proposition and Theorem 6.7 of [19] (suitably adapted to this framework) improve our knowledge on the Markov property as follows:

COROLLARY 4.9. *Under Assumption 4.1, if $(P_x)_{x\in\mathcal{L}^2}$ is an a.s. Markov family of solutions to (1.1), then $(P_x)_{x\in\mathcal{H}^1}$ is a Markov process. Namely*

$$\mathbb{E}^{P_x}[\varphi(\xi_t)|\mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\varphi(\xi_{t-s})], \qquad P_x\textit{-a.s.}$$

for all $x \in \mathcal{H}^1$, $\varphi \in C_b(\mathcal{L}^2)$ and $0 \leq s \leq t$.

5. Existence and uniqueness of invariant measures. Existence of an invariant measure for (1.1) is straightforward for trace-class noise, as one can rely on Itô formula applied to the energy balance given by $|h(t)|^2_{\mathcal{L}^2}$. The standard approximation is then tight, since we can control $\mathbb{E}[\int_0^T |h_{xx}|^2_{\mathcal{L}^2}\,dt]$.

In this section we prove the existence of an invariant measure for more general noise (such as space–time white noise) under the assumption (which will be valid for the whole section) that the equation has no linear instability, namely

$$\dot{h} = -h_{xxxx} + (h_x)^2_{xx} + \eta. \tag{5.1}$$

In order to take the linear instability into account, gauge functions have to be used, as in Blömker and Hairer [6], Collet et al. [8] or Temam [34], but up to now this is quite technical and only applicable to Dirichlet or Neumann boundary conditions. For periodic boundary conditions this question is still open.

THEOREM 5.1. *Let $(P_x)_{x\in\mathcal{L}^2}$ be any a.s. Markov family of energy martingale solutions to* (5.1). *Then there exists an invariant measure for the transition semigroup associated to $(P_x)_{x\in\mathcal{L}^2}$ with support contained in $\mathcal{H}^\gamma$, for some $\gamma \in (\frac{5}{4}, \frac{3}{2})$.*

REMARK 5.2. Note that the upper bound $\gamma < \frac{3}{2}$ is stated only for convenience. The crucial restriction is $\gamma > \frac{5}{4}$, as in the proof of this theorem we shall need that $\widetilde{Z}_{\alpha,\cdot} \in \mathcal{W}^{1,4}$, which is implied by $\widetilde{Z}_{\alpha,\cdot} \in \mathcal{H}^\gamma$, where $\widetilde{Z}_{\alpha,\cdot}$ is the process defined in (2.4).

By the results of the previous section we can immediately conclude that the invariant measure is unique (via the strong Feller property and Doob's theorem) and that it is fully supported on $\mathcal{H}^1$ (by means of Proposition 4.7).

COROLLARY 5.3. *Under Assumption* 4.1, *the invariant measure provided by Theorem* 5.1 *above is unique and fully supported on $\mathcal{H}^1$.*

So far we know that each Markov solution has its own unique invariant measure. In principle, these invariant measures come from different transition semigroups and do not need to be equal, even though they have something in common. For example, we know from [20], Theorem 13, that they are mutually equivalent. At this stage, the problem of uniqueness of the invariant measure over all selection is open, as well as the well-posedness of the martingale problem.

5.1. *The proof of Theorem* 5.1. Existence of an invariant measure will be proved by means of the Krylov–Bogoliubov method. Let $(P_x)_{x\in\mathcal{L}^2}$ be a Markov solution and consider the following family of measures on $\mathcal{L}^2$:

$$\mu_T = \frac{1}{T}\int_0^T P_0[\xi(s) \in \cdot\,]\,ds, \qquad T \geq 1,$$

where P_0 is the energy martingale solution starting at $x = 0$.

It is sufficient to show that the family $(\mu_T)_{T\geq 1}$ is compact in $\mathcal{H}^\gamma$, for some γ (see, e.g., Da Prato and Zabczyk [13], Theorem 3.1.1). Thus we need to show that for all $\varepsilon > 0$ there is $R > 0$ such that

$$\mu_T[x : |x|_{\mathcal{H}^\gamma} > 2R] < \varepsilon \qquad \text{for all } T \geq 1. \tag{5.2}$$

The proof of this statement is divided into several steps. In the first step we divide the estimate of (5.2) into two pieces corresponding to terms V and Z. In the second step we estimate the term in V by means of a disintegration in smaller intervals. In the third step we show that the quantities obtained in the previous step can be bounded by solutions to some simpler ODEs. The fourth step contains the analysis of these one-dimensional ODEs, while the fifth step gives a bound on stochastic convolutions for large α. Finally, in the sixth step we pack all the estimates together and we average with respect to the initial condition of the linear process Z, thus obtaining a uniform estimate that finally proves (5.2).

STEP 1 (Splitting in V and Z). Fix a value $\alpha > 0$ chosen later in the proof in Step 5, and consider for every $z_0 \in \mathcal{H}^\gamma$ processes $\widetilde{Z} = \widetilde{Z}_{\alpha,z_0}$ defined in (2.4) and $\widetilde{V} = \xi - \widetilde{Z}_{\alpha,z_0}$. As in Remark 2.6, $\widetilde{V}$ satisfies

$$\dot{\widetilde{V}} + \widetilde{V}_{xxxx} = [(\widetilde{V}_x + \widetilde{Z}_x)^2]_{xx} + \alpha \widetilde{Z}, \qquad \widetilde{V}(0) = -z_0.$$

Now we can bound

$$\begin{aligned}
&\mu_T[x : |x|_{\mathcal{H}^\gamma} > 2R] \\
&\quad = \frac{1}{T}\int_0^T P_0[|\xi(s)|_{\mathcal{H}^\gamma} > 2R]\,ds \\
&\quad \leq \frac{1}{T}\int_0^T P_0[|\widetilde{V}(s)|_{\mathcal{H}^\gamma} + |\widetilde{Z}(s)|_{\mathcal{H}^\gamma} > 2R]\,ds \\
&\quad \leq \frac{1}{T}\int_0^T P_0[|\widetilde{V}(s)|_{\mathcal{H}^\gamma} > R]\,ds + \frac{1}{T}\int_0^T P_0[|\widetilde{Z}(s)|_{\mathcal{H}^\gamma} > R]\,ds \\
&\quad \leq \frac{1}{T}\int_0^T P_0[|\widetilde{V}_{xx}(s)| > R]\,ds + \frac{1}{T}\int_0^T P_0[|\widetilde{Z}(s)|_{\mathcal{H}^\gamma} > R]\,ds
\end{aligned}$$

and we can estimate the two terms separately. The term in $\widetilde{Z}$ converges to 0 as $R \uparrow \infty$ from Chebyshev's inequality and statement (2) of Lemma 2.4, so everything boils down to an estimate of the term $\frac{1}{T}\int_0^T P_0[|\widetilde{V}_{xx}(s)| > R]\,ds$.

STEP 2 (The φ-moment). A standard way to estimate terms like $\frac{1}{T} \times \int_0^T P_0[|\widetilde{V}_{xx}(s)| > R]\,ds$ is to use the Chebyshev inequality and the information that some moment of $|\widetilde{V}_{xx}(s)|$ is finite. We are not able to bound any moments uniformly in time. We are also unable to bound any log-moments. Blömker and Hairer [6] give a different proof of the existence of an invariant measure, which relies on Galerkin approximations and the estimate of a log-moment of $|\widetilde{V}_{xx}(s)|^2$. Here we consider any arbitrary solution to the equation, which in principle is not a limit of Galerkin approximations, since it is not known if the solution is unique. All we can show is that there is a suitable function φ such that the φ-moment of $|\widetilde{V}_{xx}(s)|$ is bounded uniformly in time.

Let $\varphi:[0,\infty)\to\mathbf{R}$ be a function, which we will determine at the end of the proof (see Step 5), such that φ is increasing, concave, with $\varphi(r)\uparrow\infty$ as $r\uparrow\infty$, and for every $x, y\geq 0$,

$$\varphi(x+y)\leq C+\varphi(x)+\log(y+1)\leq C+\varphi(x)+y. \tag{5.3}$$

By using Chebyshev's inequality and concavity of φ we get

$$\begin{aligned}\frac{1}{T}\int_0^T P_0[|\widetilde{V}_{xx}(s)|_{\mathcal{L}^2}>R]\,ds &= \frac{1}{T}\int_0^T P_0[\varphi(|\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2})>\varphi(R^2)]\,ds\\ &\leq \frac{1}{\varphi(R^2)}\mathbb{E}^{P_0}\biggl[\frac{1}{T}\int_0^T \varphi(|\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2})\,ds\biggr]\\ &\leq \frac{1}{\varphi(R^2)T}\sum_{k=0}^{[T]}\mathbb{E}^{P_0}\biggl[\int_k^{k+1}\varphi(|\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2})\,ds\biggr]\\ &\leq \frac{1}{\varphi(R^2)T}\sum_{k=0}^{[T]}\mathbb{E}^{P_0}\biggl[\varphi\biggl(\int_k^{k+1}|\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2}\,ds\biggr)\biggr]\end{aligned}$$

and so it is sufficient to bound

$$\mathbb{E}^{P_0}\biggl[\varphi\biggl(\int_k^{k+1}|\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2}\,ds\biggr)\biggr], \tag{5.4}$$

independently of k.

STEP 3 (Gronwall's estimates). In this step we use the energy inequality for $\widetilde{V}$ [cf. (2.7)], the Gronwall's estimate given by Proposition 7.3 and comparison theorems for ODE to simplify the estimate of (5.4). From the energy inequality we know that for all t and almost every $s\in[0,t]$,

$$\begin{aligned}|\widetilde{V}(t)|^2_{\mathcal{L}^2}+\int_s^t|\widetilde{V}_{xx}(r)|^2_{\mathcal{L}^2}\,dr &\leq |\widetilde{V}(s)|^2_{\mathcal{L}^2}+C\int_s^t|\widetilde{Z}_x(r)|^{16/3}_{\mathcal{L}^4}|\widetilde{V}(r)|^2_{\mathcal{L}^2}\,dr\\ &\qquad +C\int_s^t\bigl(|\widetilde{Z}_x(r)|^4_{\mathcal{L}^4}+\alpha^2|\widetilde{Z}(r)|^2_{\mathcal{L}^2}\bigr)\,dr\\ &\leq |\widetilde{V}(s)|^2_{\mathcal{L}^2}+\int_s^t\bigl(a_1(r)|\widetilde{V}(r)|^2_{\mathcal{L}^2}+b_1(r)\bigr)\,dr,\end{aligned} \tag{5.5}$$

where we have set

$$a_1(t)=C|\widetilde{Z}_x(t)|^{16/3}_{\mathcal{L}^4}\quad\text{and}\quad b_1(t)=C(1+\alpha^2)\bigl(|\widetilde{Z}_x(t)|^4_{\mathcal{L}^4}+|\widetilde{Z}(t)|^2_{\mathcal{L}^2}\bigr). \tag{5.6}$$

By using the Poincaré inequality $\lambda\|\cdot\|_{\mathcal{L}^2} \le \|\cdot\|_{\mathcal{H}^2}$, it follows that

$$|\widetilde{V}(t)|^2_{\mathcal{L}^2} + \int_s^t \big(\lambda - a_1(r)\big)|\widetilde{V}(r)|^2_{\mathcal{L}^2}\,dr \le |\widetilde{V}(s)|^2_{\mathcal{L}^2} + \int_s^t b_1(r)\,dr.$$

By the modified Gronwall's lemma (see Proposition 7.3), we know that $|\widetilde{V}(t)|^2_{\mathcal{L}^2} \le u_1(t)$, where u_1 solves the following one-dimensional problem,

$$\dot{u}_1 + (\lambda - a_1)u_1 = b_1, \qquad u_1(0) = |z_0|^2_{\mathcal{L}^2}.$$

Moreover, by a standard comparison principle for ODE, $u_1^2(t) \le u_2(t)$, where u_2 solves

$$\dot{u}_2 + (\lambda - a_2)u_2 = b_2, \qquad u_2(0) = u_1(0)^2 = |z_0|^4_{\mathcal{L}^2} \tag{5.7}$$

with $a_2(t) = 2a_1(t)$ and $b_2(t) = \lambda^{-1}b_1^2(t)$, since

$$\frac{d}{dt}(u_1^2) + 2(\lambda - a_1)u_1^2 = 2u_1b_1 \le \lambda u_1^2 + \lambda^{-1}b_1^2.$$

In conclusion, $|\widetilde{V}(t)|^4_{\mathcal{L}^2} \le u_2(t)$ and so, using (5.5), we get

$$\begin{aligned}
\int_k^{k+1} |\widetilde{V}(s)|^2_{\mathcal{L}^2}\,ds &\le |\widetilde{V}(k)|^2_{\mathcal{L}^2} + \int_k^{k+1}\big(a_1(r)|\widetilde{V}(r)|^2_{\mathcal{L}^2} + b_1(r)\big)\,dr \\
&\le 1 + \sup_{r\in[k,k+1]} |\widetilde{V}(r)|^4_{\mathcal{L}^2} + \int_k^{k+1}\big(a_1(r)^2 + b_1(r)\big)\,dr \\
&\le 1 + \sup_{r\in[k,k+1]} u_2(r) + \int_k^{k+1}\big(a_1(r)^2 + b_1(r)\big)\,dr.
\end{aligned}$$

By applying φ to the previous inequality and using (5.3), we finally get

$$\begin{aligned}
&\mathbb{E}^{P_0}\Big[\varphi\Big(\int_k^{k+1} |\widetilde{V}_{xx}(s)|^2_{\mathcal{L}^2}\,ds\Big)\Big] \\
&\quad\le C + \mathbb{E}^{P_0}\Big[\int_k^{k+1}\big(a_1(r)^2 + b_1(r)\big)\,dr\Big] \\
&\qquad + \mathbb{E}^{P_0}\Big[\varphi\Big(\sup_{r\in[k,k+1]} u_2(r)\Big)\Big] \\
&\quad\le C(1 + |z_0|^{32/3}_{\mathcal{W}^{1,4}}) + \mathbb{E}^{P_0}\Big[\varphi\Big(\sup_{r\in[k,k+1]} u_2(r)\Big)\Big],
\end{aligned} \tag{5.8}$$

since we know by Lemma 2.4 that all moments of a_1 and b_1 are bounded by some constant and the initial condition z_0.

STEP 4 (Estimate of the auxiliary function u_2). In this step we analyze the term $\mathbb{E}^{P_0}[\varphi(\sup_{r\in[k,k+1]} u_2(r))]$, in order to get an estimate independent of k. By its definition (5.7),

$$
\begin{aligned}
u_2(t) &= |z_0|^4_{\mathcal{L}^2} e^{\int_0^t(-\lambda+a_2(r))\,dr} + \int_0^t e^{\int_s^t(-\lambda+a_2(r))\,dr} b_2(s)\,ds \\
&\le |z_0|^4_{\mathcal{L}^2} e^{\int_0^t(-\lambda+\theta(r))\,dr} + (1+\alpha^2)^2 \int_0^t \frac{d}{ds}\big[-e^{\int_s^t(-\lambda+\theta(r))\,dr}\big]\,ds \\
&\qquad + \lambda(1+\alpha^2)^2 \int_0^t e^{\int_s^t(-\lambda+\theta(r))\,dr}\,ds \\
&\le \big((1+\alpha^2)^2 + |z_0|^4_{\mathcal{L}^2}\big) e^{\int_0^t(-\lambda+\theta(r))\,dr} + \lambda(1+\alpha^2)^2 u_3(t),
\end{aligned}
\tag{5.9}
$$

where $\theta(t) = a_2(t) + \frac{b_2(t)}{(1+\alpha^2)^2}$ [so that θ depends on α only through $\widetilde{Z}$, see (5.6)] and

$$
u_3(t) = \int_0^t e^{\int_s^t(-\lambda+\theta(r))\,dr}\,ds,
$$

which is a solution of $\dot{u}_3 + (\lambda - \theta)u_3 = 1$, with an initial condition $u_3(0) = 0$. Since

$$
\begin{aligned}
u_3(t) &\le \sup_{s\in[0,t]} \exp\Big(\int_s^t \Big(-\frac{\lambda}{2} + \theta(r)\Big)\,dr\Big) \int_0^t \exp\Big(-\frac{\lambda}{2}(t-s)\Big)\,ds \\
&\le \frac{2}{\lambda} \sup_{s\in[0,t]} \exp\Big(\int_s^t \Big(-\frac{\lambda}{2} + \theta(r)\Big)\,dr\Big),
\end{aligned}
\tag{5.10}
$$

by using inequalities (5.9) and (5.10) and property (5.3) of φ, we finally get

$$
\begin{aligned}
&\varphi\Big(\sup_{r\in[k,k+1]} u_2(r)\Big) \\
&\quad\le C + \sup_{[k,k+1]} \int_0^t \big(-\lambda + \theta(r)\big)\,dr + \log\big(1 + (1+\alpha^2)^2 + |z_0|^4_{\mathcal{L}^2}\big) \\
&\qquad + \varphi\Big((1+\alpha^2)^2 \exp\Big[\sup_{t\in[k,k+1]} \sup_{s\in[0,t]} \int_s^t \Big(-\frac{\lambda}{2} + \theta(r)\Big)\,dr\Big]\Big) \\
&\quad\le C + \lambda + \int_0^{k+1} \big(-\lambda + \theta(r)\big)\,dr + (1+\alpha^2)^2 + |z_0|^4_{\mathcal{L}^2} \\
&\qquad + \varphi\Big((1+\alpha^2)^2 \exp\Big[\frac{\lambda}{2} + \sup_{s\in[0,k+1]} \int_s^{k+1} \Big(-\frac{\lambda}{2} + \theta(r)\Big)\,dr\Big]\Big).
\end{aligned}
\tag{5.11}
$$

STEP 5 (A bound for θ for large α). Now the choice of α becomes crucial. We will first bound the terms in (5.11) that are outside of φ uniformly in time. We first use the Young and Hölder inequality to derive

$$\theta(t) \le C_\lambda \|\widetilde{Z}(t)\|_{\mathcal{W}^{1,4}}^{16/3} + \frac{\lambda}{8},$$

where the constant $C_\lambda > 0$ depends on λ, but not on t or α. Recall that $\widetilde{Z}(t) = e^{t(A-\alpha)} z_0 + W_{A-\alpha}(t)$, with $W_{A-\alpha}(t) = \int_0^t e^{(t-s)(A-\alpha)}\, dW(s)$. Thus using that e^{tA} is a bounded semigroup on $\mathcal{W}^{1,4}$, we obtain

$$\mathbb{E}^{P_0}[\|\widetilde{Z}(t)\|_{\mathcal{W}^{1,4}}^{16/3}] \le C e^{-16t\alpha/3} \|z_0(t)\|_{\mathcal{W}^{1,4}}^{16/3} + C\mathbb{E}^{P_0}[\|W_{A-\alpha}(t)\|_{\mathcal{W}^{1,4}}^{16/3}].$$

As $A - \alpha$ is a strictly negative operator, we can always bound moments of the stochastic convolution uniformly in time. Using Sobolev embedding for some sufficiently large $\gamma < 3/2$ yields for all $t > 0$

$$\begin{aligned} \mathbb{E}^{P_0}[\|W_{A-\alpha}(t)\|_{\mathcal{W}^{1,4}}^{16/3}] &\le \big(\mathbb{E}^{P_0}[\|(1-\Delta)^{\gamma/2} W_{A-\alpha}(t)\|_{\mathcal{L}^2}^2]\big)^{8/3} \\ &\le \big(\operatorname{trace}\{(1-\Delta)^{\gamma} Q(\alpha - A)^{-1}\}\big)^{8/3} \\ &\to 0 \qquad \text{for } \alpha \to \infty. \end{aligned}$$

In the following, we choose α sufficiently large such that

$$\mathbb{E}^{P_0}[\theta(t)] \le \frac{\lambda}{4} + C e^{-16t\alpha/3} \|z_0(t)\|_{\mathcal{W}^{1,4}}^{16/3} \qquad \text{for all } t > 0.$$

STEP 6 (Average over all z_0 and conclusion of the proof). In Step 2 we have shown that Theorem 5.1 is proved if we can bound (5.4) independently of k. Putting the conclusion of Step 3 (5.8) and the conclusion of Step 4 (5.11) together with the bound from the previous step, we get

$$\begin{aligned} &\mu_T[x : |x|_{\mathcal{H}^\gamma} > 2R] \\ &\quad \le C(1 + |z_0|_{\mathcal{L}^2}^4 + |z_0|_{\mathcal{W}^{1,4}}^{32/3} + |z_0|_{H^\gamma}^2) \\ &\qquad + \frac{1}{\varphi(R^2)T} \sum_{k=0}^{[T]} \mathbb{E}^{P_0} \varphi\Big(C_{\alpha,\lambda} \exp\Big[\sup_{s\in[0,k+1]} \int_s^{k+1} \Big(-\frac{\lambda}{2} + \theta(r)\Big)\, dr\Big]\Big). \end{aligned}$$

Now we integrate the inequality above over z_0, with respect to the invariant measure of $\widetilde{Z}$ and, by virtue of Lemma 5.5 below, we obtain

$$\begin{aligned} &\mu_T[x : |x|_{\mathcal{H}^\gamma} > 2R] \\ &\quad \le C_{\mathrm{IM}} + \frac{1}{\varphi(R^2)T} \sum_{k=0}^{[T]} \mathbb{E}^{P_0} \varphi\Big(C_{\alpha,\lambda} \exp\Big[\sup_{s\in[0,k+1]} \int_s^{k+1} \Big(-\frac{\lambda}{2} + \Theta(r)\Big)\, dr\Big]\Big), \end{aligned}$$

where $(\Theta(t))_{t\in\mathbf{R}}$ is the process defined as in Step 4 with $\widetilde{Z}$ replaced by the stationary solution $\widetilde{Z}_{\mathrm{st}}$ of problem (2.4). Due to stationarity we have

$$\begin{aligned}\sup_{s\in[0,k+1]}\int_s^{k+1}\Big(-\frac{\lambda}{2}+\Theta(r)\Big)\,dr &\overset{(L)}{=} \sup_{s\in[-(k+1),0]}\int_s^{0}\Big(-\frac{\lambda}{2}+\Theta(r)\Big)\,dr\\ &\le \sup_{s\in(-\infty,0]}\int_s^{0}\Big(-\frac{\lambda}{2}+\Theta(r)\Big)\,dr\\ &\overset{(L)}{=} \sup_{t\in[0,\infty)}\int_0^{t}\Big(-\frac{\lambda}{2}+\Theta(r)\Big)\,dr.\end{aligned}$$

Therefore, if we define the random variable

$$\widetilde{X}=\sup_{t\in[0,\infty)}\int_0^{t}\Big(-\frac{\lambda}{2}+\Theta(r)\Big)\,dr,$$

we only have to prove that there exists a function φ as above such that

$$\mathbb{E}^{P_0}[\varphi(C_{\alpha,\lambda}e^{\widetilde{X}})]<\infty.$$

Since $\widetilde{X}$ is finite with probability 1 by the ergodic theorem, such a φ exists by Lemma 7.1. The proof of Theorem 5.1 is complete.

REMARK 5.4. In the previous proof, we were only able to bound some moment of V_{xx}, but using the trick of Debussche and Da Prato [11], where α is allowed to be random, it is possible to bound arbitrary polynomial moments on bounded time intervals.

LEMMA 5.5. *Let $\delta>0$ and let ϕ be a positive map defined on the probability space Ω. If for all z_0*

$$\delta\le\mathbb{E}^{P_x}[\phi(\widetilde{Z}_{\alpha,z_0})]\qquad \textit{for } P_x\textit{-almost every } \xi\in\Omega,$$

where $\widetilde{Z}_{\alpha,z_0}$ is the Ornstein–Uhlenbeck process starting in z_0, as defined in (2.4), *then*

$$\delta\le\int_{\mathcal{H}^1}\phi(z)\mu^*_{OU}(dz),$$

where μ^*_{OU} is the law of the stationary Ornstein–Uhlenbeck process.

The lemma is easily proved by averaging both sides with respect to z_0 with the stationary Ornstein–Uhlenbeck process and using Tonelli theorem.

6. A priori estimates. In this section we state all regularity results on processes Z and V. The first part contains the results on Z under an arbitrary weak martingale solution (from Definition 2.2). Similarly, the second part contains the results on V under an arbitrary energy martingale solution (from Definition 2.5).

6.1. *Weak martingale solution.* Here we will present some lemmas on the regularity of Z without using equivalent versions, since our approach forces us to keep the canonical process.

LEMMA 6.1. *Given a weak martingale solution P, then for every $T > 0$,*

$$\mathbb{E}^P \int_0^T |Z(t)|_{\mathcal{W}^{1,4}}^4 \, dt < \infty.$$

PROOF. It is enough to verify that $(Z_x)^2 \in L^2(\Omega \times (0,T), \mathcal{L}^2)$. From the definition, we can write $Z(t)$ as a complex Fourier series, such that

$$Z_x = \sum_{k \neq 0} I_k e^{ikx},$$

where I_k is a time dependent Gaussian real valued random variable with $\mathbb{E}^P I_k^2 \leq C|k|^{-2}$. Thus, $\mathbb{E}^P I_k^4 \leq C|k|^{-4}$, too. Now,

$$(Z_x)^2 = \sum_{n \in \mathbf{Z}} \sum_{k \neq 0, n} I_k I_{n-k} e^{inx}.$$

We derive

$$\begin{aligned} \mathbb{E}^P |(Z_x)^2|_{\mathcal{L}^2}^2 &= \sum_{n \in \mathbf{Z}} \mathbb{E}^P \left(\sum_{k \neq 0, n} I_k I_{n-k} \right)^2 \\ &\leq \sum_{n \in \mathbf{Z}} \sum_{k \neq 0, n} \sum_{l \neq 0, n} \mathbb{E}^P [|I_k||I_{n-k}||I_l||I_{n-l}|] \\ &\leq \sum_{n \in \mathbf{Z}} \left(\sum_{k \neq 0, n} \frac{1}{|k||n-k|} \right)^2, \end{aligned}$$

where we used Hölder's inequality in the last step. It is an elementary exercise to check that the series in the last equation converges. Thus integration in time yields the result. $\square$

LEMMA 6.2. *Let P be a weak martingale solution. Then for some $\lambda > 0$ there is a constant C such that*

$$\int_0^T \mathbb{E}^P \exp\{\lambda \|Z_x(t)\|_{\mathcal{L}^4}^2\} \, dt \leq CT \qquad \textit{for all } T > 0.$$

Thus, for some constant C depending only on q, p and T,

$$\sup_{T \geq 0} \frac{1}{T} \mathbb{E}^P \|Z\|_{L^p([0,T], \mathcal{W}^{1,4})}^p < \infty \quad \textit{and} \quad \sup_{T \geq 0} \frac{1}{T} \mathbb{E}^P \|Z\|_{L^p([0,T], \mathcal{W}^{1,4})}^q \leq C.$$

PROOF. Using Lemma 6.1 we know that $\mathbb{E}^P \|Z_x(t)\|_{\mathcal{L}^4}^4 \le C$ for all $t \ge 0$. As $Z_x(t)$ is a Gaussian random variable in $\mathcal{L}^4$, Fernique's theorem (see Da Prato and Zabczyk [12]) implies that

$$\sup_{t\ge 0} \mathbb{E}^P \exp\{\lambda \|Z_x(t)\|_{\mathcal{L}^4}^2\} < \infty$$

for some $\lambda > 0$. Thus

$$\mathbb{E}^P \|Z\|^p_{L^p([0,T],\mathcal{W}^{1,4})} \le C \int_0^T \mathbb{E}^P \exp\{\lambda \|Z_x(t)\|^2_{\mathcal{L}^4}\}\,dt \le CT,$$

where the constant does not depend on T. The last claim follows from Hölder inequality. □

The following lemma on the $L^\infty([0,\infty),\mathcal{L}^2)$-regularity is necessary to transfer weak continuity in $\mathcal{L}^2$ from V to Z. Note again that we cannot prove continuity of Z, as we are not using continuous versions of the canonical process Z.

LEMMA 6.3. *Let P be a weak martingale solution. Then for $0 \le \gamma < \frac{3}{2}$, $p > 1$ and $T > 0$*

$$Z \in L^p(\Omega, L^\infty([0,T],\mathcal{H}^\gamma))$$

and thus

$$P\big[Z \in L^\infty([0,\infty),\mathcal{H}^\gamma)\big] = 1.$$

Due to $Z \in \Omega$, we have Z is P-a.s. weakly continuous with values in $\mathcal{H}^\gamma$.

PROOF. Using the factorization method (see Da Prato and Zabczyk [12], Chapter 5),

$$Z(t) = C_\alpha \int_0^t e^{(t-\tau)A}(t-\tau)^{\alpha-1} Y(\tau)\,d\tau,$$

where

$$Y(\tau) = \int_0^\tau e^{(\tau-s)A}(\tau-s)^{-\alpha}\,dW(s).$$

We fix $T > 0$, $\alpha \in (0, \frac{3-2\gamma}{8})$ and $m > \frac{1}{\alpha} > \frac{8}{3}$, and let the constants depend on them. Now, using Hölder's inequality,

$$\sup_{t\in[0,T]} |Z(t)|_{\mathcal{H}^\gamma} \le C \sup_{t\in[0,T]} \int_0^t (t-\tau)^{\alpha-1} |Y(\tau)|_{\mathcal{H}^\gamma}\,d\tau \le C\left(\int_0^T |Y(\tau)|^m_{\mathcal{H}^\gamma}\,d\tau\right)^{1/m}.$$

Thus using that Y is Gaussian,

$$\mathbb{E}^P \sup_{t\in[0,T]} |Z(t)|^m_{\mathcal{H}^\gamma} \le C \int_0^T (\mathbb{E}^P |Y(\tau)|^2_{\mathcal{H}^\gamma})^{m/2}\,d\tau \le C\left(\sum_{k=1}^\infty k^{2\gamma}\alpha_k^2 |\lambda_k|^{2\alpha-1}\right)^{m/2}.$$

The last series converges, as $\alpha_k^2 \le C$ and $\lambda_k \sim -k^4$. Taking $T \in \mathbf{N}$ concludes the proof. □

6.2. *Energy martingale solution.* This part is devoted to the proof of the *tightness* property for sequences of energy martingale solutions, essentially by means of bounds on the process V.

LEMMA 6.4. *Let $(P_n)_{n\in\mathbf{N}}$ be a family of energy Markov solutions. Then the sequence of laws of V under P_n is tight in $L^2(0,T,\mathcal{H}^1)$, if and only if $(P_n)_{n\in\mathbf{N}}$ is tight in $L^2(0,T,\mathcal{H}^1)$.*

The same result is true for any space in which Z is defined, for example, $C(0,T,\mathcal{H}^{-4})$.

PROOF. We prove only one direction, the other one is the same. As $\mathrm{Law}_{Z,n} = P_n[Z\in\cdot]$ is by Definition 2.2 and Lemma 6.3 the law of the stochastic convolution in $L^2([0,T],\mathcal{H}^1)$ and thus independent of n. Hence, the family of measures $(\mathrm{Law}_{Z,n})_{n\in\mathbf{N}}$ is tight in $L^2([0,T],\mathcal{H}^1)$. Thus there is a compact subset $K_{\varepsilon,1}\subset L^2([0,T],\mathcal{H}^1)$ with $P_n[Z\in K_{\varepsilon,1}]>1-\varepsilon$. Furthermore, by the tightness of $P_n[V\in\cdot]$, there is a compact set $K_{\varepsilon,2}\subset L^2([0,T],\mathcal{H}^1)$ such that $P_n[V\in K_{\varepsilon,2}]>1-\varepsilon$.

Define now the compact subset

$$K_{\varepsilon,3}=K_{\varepsilon,1}+K_{\varepsilon,2}=\{u=u_1+u_2|u_i\in K_{\varepsilon,i}\},$$

then by $\xi=V+Z$ we have

$$P_n[K_{\varepsilon,3}]\geq P_n[Z\in K_{\varepsilon,1},V\in K_{\varepsilon,2}]\geq 1-2\varepsilon,$$

which concludes the proof. □

LEMMA 6.5. *Let P be an energy martingale solution. Then for all $T>0$*

$$\|\partial_t V\|_{L^2([0,T],\mathcal{H}^{-3})}\leq C\|V\|_{L^2([0,T],\mathcal{H}^2)}\bigl(1+\|V\|_{L^\infty([0,T],\mathcal{L}^2)}\bigr)+C\|Z\|_{L^4([0,T],\mathcal{H}^1)},$$

P-almost surely, with constants independent of P.

PROOF. From Remark 2.6, we know that for $\varphi\in\mathcal{H}^3$ with $|\varphi|_{\mathcal{H}^3}=1$ we have

$$\partial_t\langle V,\varphi\rangle_{\mathcal{L}^2}=-\langle V_{xx}+V,\varphi_{xx}\rangle_{\mathcal{L}^2}-\langle (V_x+Z_x)^2,\varphi_{xx}\rangle_{\mathcal{L}^2}.$$

Thus, using the embedding of L^1 into H^{-1} and an interpolation inequality,

$$\begin{aligned}|\partial_t V|_{\mathcal{H}^{-3}}&\leq |V_{xx}|_{\mathcal{L}^2}+|V|_{\mathcal{L}^2}+|(V_x+Z_x)^2|_{L^1}\\&\leq |V|_{\mathcal{H}^2}+C|V|_{L^2}|V|_{\mathcal{H}^2}+2|Z|^2_{\mathcal{H}^1}.\end{aligned}$$

Integrating the square in time yields the result. □

LEMMA 6.6. *Let $(P_n)_{n\in\mathbf{N}}$ be a family of energy martingale solutions. Define*

$$G(R) = \{u : \|u\|_{L^\infty([0,T],\mathcal{L}^2)} < R \text{ and } \|u\|_{L^2([0,T],\mathcal{H}^2)} < R\}.$$

Suppose that P_n is started at a probability measure μ_n such that

$$\int_{\mathcal{L}^2} \big(\log(|x|_{\mathcal{L}^2} + 1)\big)^\kappa \mu_n(dx) \le K$$

for all $n \in \mathbf{N}$ and for some $\kappa > 0$, then

$$\sup_{n\in\mathbf{N}} P_n[V \in G(R)] \ge 1 - \frac{C}{\log(1+R)^\kappa}.$$

PROOF. By property [E3], we have that, P_n-almost surely,

$$\begin{aligned}
&|V(t)|^2_{\mathcal{L}^2} + \int_0^t |V_{xx}|^2_{\mathcal{L}^2}\,ds \\
&\quad \le |V(0)|^2_{\mathcal{L}^2} + \int_0^t (|V_x|^2_{\mathcal{L}^2} + 2|V_x|_{\mathcal{L}^4}|Z_x|_{\mathcal{L}^4}|V_{xx}|_{\mathcal{L}^2} + |Z_x|^2_{\mathcal{L}^4}|V_{xx}|_{\mathcal{L}^2})\,ds \\
&\quad \le |V(0)|^2_{\mathcal{L}^2} + \int_0^t \tfrac{1}{2}|V_x|^2_{\mathcal{L}^2} + C(1 + |Z_x|^{16/3}_{\mathcal{L}^4})|V|^2_{\mathcal{L}^2} + C|Z_x|^4_{\mathcal{L}^4}\,ds,
\end{aligned}$$

where we have used the Sobolev embedding of $\mathcal{H}^1$ into $\mathcal{L}^4$, interpolation, Young and Poincaré inequalities. Now from Gronwall's inequality it follows that, for all $t \in [0,T]$,

$$\begin{aligned}
&|V(t)|^2_{\mathcal{L}^2} + \int_0^t |V_{xx}|^2_{\mathcal{L}^2}\,ds \\
&\quad \le C\big(|V(0)|^2_{\mathcal{L}^2} + \|Z\|^4_{L^4([0,T],\mathcal{W}^{1,4})}\big)\exp\big(C\|Z\|^{16/3}_{L^{16/3}([0,T],\mathcal{W}^{1,4})}\big),
\end{aligned} \tag{6.1}$$

where the constants might depend on T. Applying $(\log(x+1))^\kappa$ and using the inequality

$$\log(x+y+1)^\kappa \le C\big(\log(x+1)^\kappa + \log(y+1)^\kappa\big) \qquad \text{for } x, y \ge 0,$$

leads to

$$\mathbb{E}^{P_n}\Big[\sup_{t\in[0,T]} \log\big(1 + |V(t)|^2_{\mathcal{L}^2}\big)\Big]^\kappa \le C$$

and

$$\mathbb{E}^{P_n}\Big[\log\Big(1 + \int_0^t |V_{xx}(s)|^2_{\mathcal{L}^2}\,ds\Big)\Big]^\kappa \le C,$$

where the constant is independent of n. Now the Chebyshev inequality yields the result. □

We are now able to prove Theorem 3.2. We rewrite the statement for the convenience of readers.

THEOREM 6.7 (Restatement of Theorem 3.2). *Let $(P_n)_{n\in\mathbf{N}}$ be a family of energy martingale solutions with each P_n starting in μ_n and*

$$\int_{\mathcal{L}^2}[\log(|x|_{\mathcal{L}^2}+1)]^\kappa \mu_n(dx) \le K \qquad \text{for all } n \in \mathbf{N}$$

for some $\kappa > 0$ and $K > 0$. Then $(P_n)_{n\in\mathbf{N}}$ is tight on $\Omega \cap L^2([0,\infty), \mathcal{H}^1)$.

Furthermore, there is a constant depending only on $T > 0$, $z_0 \in \mathcal{H}^1$, $K > 0$ and $\kappa > 0$, such that

$$\mathbb{E}^{P_n}\left[\log\left(1 + \int_0^T |\xi_x(s)|^2_{\mathcal{L}^2}\,ds\right)\right]^\kappa \le C,$$

$$\mathbb{E}^{P_n}\left[\log\left(1 + \int_0^T |V_{xx}(s)|^2_{\mathcal{L}^2}\,ds\right)\right]^\kappa + \mathbb{E}^{P_n}\left[\sup_{t\in[0,T]} \log\bigl(1 + |V(t)|^2_{\mathcal{L}^2}\bigr)\right]^\kappa \le C.$$

PROOF. For the bounds on logarithmic moments of V we use the bounds obtained at the end of the proof of the previous Lemma 6.6. Using the bounds on Z from Lemma 6.3 yields the bound on logarithmic moments of ξ.

For the tightness of the law of V under P_n we use Lemmas 6.6 and 6.5 for the bound for $\partial_t V$, together with the compact embeddings of $H^1([0,T], \mathcal{H}^{-3})$ into $C([0,T], \mathcal{H}^{-4})$ and of $L^2([0,T], \mathcal{H}^2) \cap H^1([0,T], \mathcal{H}^{-3})$ into $L^2([0,T], \mathcal{H}^1)$ (see, e.g., Temam [33]).

For the tightness of P_n we use Lemma 6.4 on the transfer of tightness in the spaces $L^2([0,\infty), \mathcal{H}^1)$ and $C([0,T], \mathcal{H}^{-4})$. $\square$

7. Some useful technical tools.

7.1. *A suitable concave moment.* We aim to prove the following proposition:

PROPOSITION 7.1. *Let X be a random variable with values in $[0,\infty)$. Then there is a concave and nondecreasing map $\phi:[0,\infty) \to [0,\infty)$ such that $\phi(x) \uparrow \infty$ and*

$$\mathbb{E}[\phi(X)] < \infty.$$

Moreover, ϕ can be chosen in such a way that for some constant C,

$$\phi(x+y) \le \phi(x) + Cy \qquad \text{for all } x, y \in [0,\infty).$$

REMARK 7.2. Notice that the last condition on ϕ given in the proposition above can be replaced by

$$\phi(x+y) \le \phi(x) + C\log(1+y)$$

for some constant $C > 0$ and for all $x, y \in [0,\infty)$. Indeed, let φ be the map given by the proposition, then $\phi(x) = \varphi(\log(1+x))$ has exactly the same properties of φ and $\phi(x+y) = \varphi(\log(1+x+y)) \le \phi(x) + C\log(1+y)$, since $\log(1+x+y) \le \log(1+x) + \log(1+y)$.

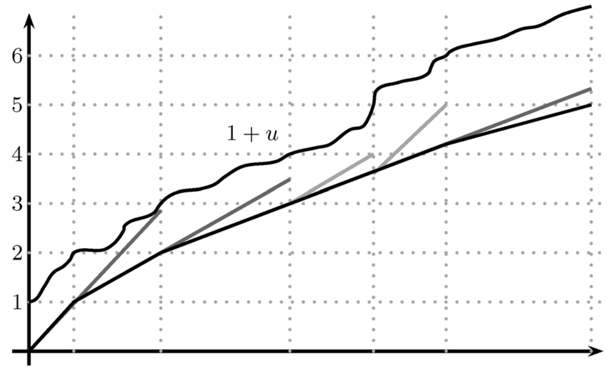

FIG. 1. *An example of the construction.*

PROOF OF PROPOSITION 7.1. We first show that there is a nondecreasing continuous map $u:[0,\infty)\to[0,\infty)$ such that $u(0)=0$, $u(x)\uparrow\infty$ as $x\to\infty$ and $\mathbb{E}[u(X)]<\infty$. Choose a sequence $(x_n)_{n\in\mathbf{N}}$ such that $x_0=0$, $x_n\uparrow\infty$ and $4^n\mathbb{P}[x_n\le X<x_{n+1}]\longrightarrow 1$. This can always been done, since X is a.s. finite. Now, let $\widetilde{u}$ be the piecewise constant function that on each interval $[x_n,x_{n+1})$ takes the value 2^n. We finally set $u(t)=\frac{1}{t}\int_0^t[\widetilde{u}(t)-\inf_{s\ge0}\widetilde{u}(s)]\,ds$.

Next, we show how to construct a map ϕ as in the statement of the proposition such that $\phi\le 1+u$. Define the sequence $(y_n)_{n\in\mathbf{N}}$ as $y_0=0$ and $y_n=\max\{x\in[0,\infty):u(x)=n\}$, for $n\ge1$. The sequence $(y_n)_{n\in\mathbf{N}}$ is increasing and $y_n\uparrow\infty$. Define ϕ as $\phi(y_0)=0$, $\phi(y_1)=1$,

$$\phi(y_n)=\min\left\{n,\phi(y_{n-2})+\frac{\phi(y_{n-1})-\phi(y_{n-2})}{y_{n-1}-y_{n-2}}(y_n-y_{n-2})\right\}$$

and by linear interpolation for all other values of $x\in[0,\infty)$. In other words, at each point y_n the map is defined either as the continuation of the line $y_{n-2}\longrightarrow y_{n-1}$ or as $u(y_n)$, depending on which is the smallest value. The construction is shown in Figure 1. All properties of ϕ are apparent from the picture; we only show that $\phi(y_n)\uparrow\infty$. Let $A=\{n:\phi(y_n)=n\}$. If A is infinite, we are done; otherwise, let N be the largest value in A. Then for $x\ge y_N$,

$$\phi(x)=\phi(y_{N-1})+\frac{N-\phi(y_{N-1})}{y_N-y_{N-1}}(x-y_{N-1})$$

and $\phi(x)\uparrow\infty$, since $\phi(x_{N-1})\le N-1<N$. $\square$

7.2. *A slight variation of Gronwall's lemma.* Here we give a detailed proof of the variation of Gronwall's lemma used in Section 5.1. The result is elementary and probably well known; it is given here only for the sake of completeness. The main differences are the following: We do not assume that the term $a(\cdot)$ is positive and the inequality holds only for a.e. time, but then it holds starting from arbitrary initial times.

PROPOSITION 7.3. *Let a, $b \in L^1(0,T)$, with $b \geq 0$ and let $u:[0,T] \to \mathbf{R}$ be a lower semi-continuous and positive function. Assume that there exists a set $S \subset (0,T]$ (thus, not containing 0) with null Lebesgue measure, such that for all $s \notin S$ and all $t \in [s,T]$,*

$$u(t) \leq u(s) + \int_s^t a(r)u(r)\,dr + \int_s^t b(r)\,dr.$$

Then

$$u(T) \leq u(0)e^{\int_0^T a(s)\,ds} + \int_0^T b(s)e^{\int_s^T a(r)\,dr}\,ds.$$

PROOF. We only need to prove the proposition if $a(\cdot)$ is piecewise constant. Indeed, if this claim is true and $a \in L^1(0,T)$, there are piecewise constant functions a_n such that $a_n \longrightarrow a$ and, without loss of generality, we can assume that each a_n is constant on a finite number of intervals whose extreme points do not belong to S (but possibly for the last one). By the usual Gronwall's lemma we can deduce that u is bounded by some constant M. We then set $b_n(s) = b(s) + M|a(s) - a_n(s)|$, and we apply the claim with a_n and b_n. As $n \to \infty$, we recover the original statement.

Assume then that $a = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{J_k}$, where the intervals $J_k = [t_k, t_{k+1})$, $0 = t_0 < t_1 < \cdots < t_n = T$ and $t_0, t_1, \ldots, t_{n-1} \notin S$. If $\alpha_k \geq 0$, since $t_k \in S$, we know by the usual Gronwall's lemma and semi-continuity of u that

$$u(t_{k+1}) \leq u(t_k)e^{\alpha_k(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}} b(s)e^{\alpha_k(t_{k+1}-s)}\,ds.$$

If $\alpha_k < 0$, we reverse time as it is done in the proof of Theorem 5 of Flandoli and Romito [17] and we apply again Gronwall's lemma to get

$$u(t_{k+1}) \leq u(t_k)e^{\alpha_k(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}} b(s)e^{\alpha_k(t_{k+1}-s)}\,ds.$$

It is then easy to prove by induction on $k \leq n$ that

$$u(t_k) \leq u(0)e^{\int_0^{t_k} a(s)\,ds} + \int_0^{t_k} b(s)e^{\int_s^{t_k} a(r)\,dr}\,ds$$

and in particular $k = n$ is exactly what we aimed to prove. $\square$

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