# SCATTERING FOR NLS WITH A DELTA POTENTIAL 

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#### Abstract

We prove $H^{1}$ scattering for defocusing NLS with a delta potential and masssupercritical nonlinearity, hence extending in an inhomogeneous setting the classical $1-D$ scattering results first proved by Nakanishi in the translation invariant case.


## Contents

| 1. Introduction |  |
| :---: | :---: |
| 2. Profile decomposition | 4 |
| 2.1. The general case | 4 |
| 2.2. The case $A=H_{q}$. | 12 |
| 3. Preliminary results | 14 |
| 3.1. Strichartz estimates | 14 |
| 3.2. Perturbative nonlinear results | 14 |
| 3.3. The nonlinear profiles | 16 |
| 4. Proof of Theorem 1.1 | 18 |
| 4.1. Existence and compactness of a minimal element. | 19 |
| 4.2. Rigidity of compact solutions. | 22 |
| 5. Appendix | 24 |
| References | 26 |

## 1. Introduction

We consider the defocusing Schrödinger equation on the line with a delta potential of strength $q>0$ :

$$
\left\{\begin{array}{c}
i \partial_{t} u-H_{q} u-u|u|^{\alpha}=0, \quad \alpha>4,  \tag{1.1}\\
u_{\mid t=0}=\varphi
\end{array}\right.
$$

where $H_{q}=-\frac{1}{2} \partial_{x x} u+q \delta$ is a self-adjoint operator on the domain

$$
\mathcal{D}\left(H_{q}\right)=\left\{f \in \mathcal{C}(\mathbb{R}) \cap H^{2}(\mathbb{R} \backslash\{0\}), f^{\prime}\left(0^{+}\right)-f^{\prime}\left(0^{-}\right)=2 q f(0)\right\} .
$$

The quadratic form associated with $H_{q}$ is $\frac{1}{4}\left\|\partial_{x} f\right\|_{L^{2}}^{2}+\frac{q}{2}|f(0)|^{2}$, on the energy space $H^{1}(\mathbb{R})$ (see for instance Adami-Noja AN09). We underline that in the case $q=0$ the operator
$H_{0}$ is the classical Laplace operator $-\frac{1}{2} \partial_{x x}$ on the domain $H^{2}(\mathbb{R})$ and (1.1) reduces to

$$
\left\{\begin{array}{c}
i \partial_{t} u+\frac{1}{2} \partial_{x x} u-u|u|^{\alpha}=0,  \tag{1.2}\\
u_{\mid t=0}=\varphi .
\end{array}\right.
$$

The operator $H_{q}$ describes a $\delta$-interaction of strength $q$ centered at $x=0$. On the one hand, this kind of interaction, known also as Fermi pseudopotential, give rise to many currently used models in physiscs. We refer to the monograph of Albeverio-Gesztesy-Høegh-Krohn-Holden AGHKH05. We remark also that (1.1) is the simplest case of the nonlinear Schrödinger equation posed on a metric graph with delta-conditions at the vertices, namely when the graph has only one vertex and two edges. On the other hand, the qualitative properties of the solutions of the nonlinear Schrödinger equation with a potential is a subject of current interest. First a series of studies dealt with the dispersive properties of the perturbed linear operator in $1-D$ (Christ-Kiselev [CK02], D'AnconaFanelli DF06, Goldberg-Schlag [GS04], Weder Wed99], Yajima Yaj95 to quote a few of them...). Also a huge literature has been developed around the corresponding perturbed nonlinear equations in $1-D$, to quote the most recent results we mention Carles Car14, Cuccagna-Georgiev-Visciglia CGV14, Germain-Hani-Walsh [GHW15, and all the references therein.
Let us recall now the facts known about (1.1). In the repulsive case $q \geq 0$, the free solutions can be computed explicitly (see Gaveau-Schulman [GS86]), yielding the classical dispersion estimate $\left\|e^{-i t H_{q}} f\right\|_{L^{\infty}} \leq C t^{-\frac{1}{2}}\|f\|_{L^{1}}$ and therefore classical Strichartz estimates. These estimates remain valid in the attractive case $q<0$, up to projecting outside the discrete spectrum, composed by the unique eigenvalue $-\frac{q^{2}}{4}$ associated with the eigenvector $u_{q}(x)=\sqrt{\frac{|q|}{2}} e^{q|x|}$ (see Adami-Sacchetti AS05). The nonlinear problem (1.1) is therefore globally well-posed in $H^{1}$ and that the mass $\int_{\mathbb{R}}|u(t, x)|^{2} d x$ and the energy

$$
E(u(t))=\frac{1}{4} \int_{\mathbb{R}}\left|\partial_{x} u(t, x)\right|^{2} d x+\frac{q}{2}|u(t, 0)|^{2}+\frac{1}{\alpha+2} \int_{\mathbb{R}}|u(t, x)|^{\alpha+2} d x
$$

are two conserved in time quantities. Let us mention also that (1.1) in the focusing cubic case, i.e. opposite sign in front of the nonlinearity, slow and fast solitons evolutions have been studied in a series of papers Goodman-Holmes-Weinstein [GHW04, Holmer-Zworsky [HZ07, Holmer-Marzuola-Zworsky HMZ07b, HMZ07a, Datchev-Holmer DH09]. Also, stability results for bound states were obtained in Adami-Noja-Visciglia ANV13, Fukuizumi-Ohta-Ozawa [FOO08], Le Coz-Fukuizumi-Fibish-Ksherim-Sivan [LCFF ${ }^{+}$08], Holmer-Zworsky [HZ09, Deift-Park DP11]. Finally, let us note that in both cubic cases with repulsive potential small data long-range wave operators in $L^{2}$ were recently proved by Segata Seg14.

Our main contribution is the proof of the asymptotic completeness for (1.1) in $H^{1}(\mathbb{R})$ for $\alpha>4$, in the repulsive case $q>0$. We recall that in the case $q=0$ this result was first proved by Nakanishi in Nak99 by using a weighted in space and time Morawetz inequality. New proofs have been provided via interaction Morawetz estimates in the papers

Colliander-Holmer-Visan-Zhang [CHVZ08, Colliander-Grillakis-Tzirakis CGT09], PlanchonVega PV09.
Next we state our result.
Theorem 1.1. Let $\varphi \in H^{1}(\mathbb{R})$ be given and $u(t, x) \in \mathcal{C}\left(\mathbb{R} ; H^{1}(\mathbb{R})\right)$ be the unique global solution to (1.1) with $q>0$ and $\alpha>4$. Then there exist $\varphi_{ \pm} \in H^{1}(\mathbb{R})$ such that

$$
\begin{equation*}
\left\|e^{-i t H_{q}} \varphi_{ \pm}-u(t, x)\right\|_{H^{1}(\mathbb{R})} \xrightarrow{t \rightarrow \pm \infty} 0 . \tag{1.3}
\end{equation*}
$$

In the sequel we use the following compact notation for any $\varphi \in H^{1}(\mathbb{R})$ :

$$
S c(\varphi) \text { occurs } \Longleftrightarrow \text { (1.3) is true for suitable } \varphi_{ \pm} \in H^{1}(\mathbb{R})
$$

where $u(t, x) \in \mathcal{C}\left(\mathbb{R} ; H^{1}(\mathbb{R})\right)$ is the unique global solution to (1.1).
The proof is heavily based on the concentration-compactness/rigidity technique first introduced by Kenig-Merle in KM06 and borrows arguments from Duyckaerts-HolmerRoudenko DHR08 and Fang-Xie-Cazenave [FXC11]. In fact the main difficulty in our context is the lack of translation invariance of the equation, due to the delta interaction. The same difficulty appears in the paper Hon14 by Hong where he considers NLS in 3-D with a potential type perturbation. In this case the lack of homogeneity is solved thanks to the choice of a suitable Strichartz couple that allows to prove smallness of a suitable reminder. This technique seems to be non useful in the 1-D case because of a numerology problem. To give an idea of the main difference between Strichartz estimates in 1D and 3D, recall that in 1-D Strichartz estimates are far from reaching the $L^{2}$ time summability that is available in 3-D.
Moreover the delta interaction is a singular perturbation and hence the profile decomposition proof, as well as the construction of the minimal element, cannot be given in the perturbative spirit as in Hong proof, where linear scattering is at hand. We believe that the proof of the profile decomposition associated with a delta type interaction, given along this paper, has its own interest. In particular it does not rely on the corresponding profile decomposition available in the free case.

In the focusing cases $q<0$ or an opposite sign in front of the nonlinearity in (1.1) the above arguments can be used to prove scattering up to the natural threshold, given in terms of ground states, between global existence and blow-up.

Notation. We shall use the following notations without any further comments:

$$
L^{p}=L^{p}(\mathbb{R}), H^{s}=H^{s}(\mathbb{R}), L^{p} L^{q}=L^{p}\left(\mathbb{R} ; L^{q}(\mathbb{R}), \mathcal{C} H^{s}=\mathcal{C}\left(\mathbb{R} ; H^{s}(\mathbb{R})\right)\right.
$$

We also denote by (.,.) the usual $L^{2}$ scalar product and by $(., .)_{H^{1}}$ the scalar product in $H^{1}$, i.e. $(f, g)_{H^{1}}=\int_{\mathbb{R}} f(x) \bar{g}(x) d x+\int_{\mathbb{R}} f^{\prime}(x) \bar{g}^{\prime}(x) d x$. We denote by $\tau_{x}$ the translation operator, i.e. $\tau_{x} f(y)=f(y-x)$. Given a sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ we denote by $x_{n} \xrightarrow{X} x$ and $x_{n} \xrightarrow{X} x$ respectively the strong and weak convergence in the topology of $X$ as $n \rightarrow \infty$.

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## 2. Profile decomposition

2.1. The general case. The aim of this section is the proof of profile decomposition associated with a general family of propagators $e^{-i t A}$. From now on $A$ will denote a selfadjoint operator

$$
A: L^{2} \supset D(A) \ni u \mapsto A u \in L^{2}
$$

that satisfies suitable assumptions. More precisely we assume the following:

- there exist $c, C>0$ such that

$$
\begin{equation*}
c\|u\|_{H^{1}}^{2} \leq(A u, u)+\|u\|_{L^{2}}^{2} \leq C\|u\|_{H^{1}}^{2}, \quad \forall u \in D(A) ; \tag{2.1}
\end{equation*}
$$

- let $B: D(A) \times D(A) \ni(f, g) \mapsto B(f, g) \in \mathbb{C}$ be defined as follows:

$$
\begin{equation*}
(A u, v)=(u, v)_{H^{1}}+B(u, v), \quad \forall u, v \in D(A) \times D(A), \tag{2.2}
\end{equation*}
$$

then

$$
B\left(\tau_{x_{n}} \psi, \tau_{x_{n}} h_{n}\right) \xrightarrow{n \rightarrow \infty} 0, \quad \forall \psi \in H^{1}
$$

provided that:

$$
\begin{aligned}
\text { either } x_{n} & \xrightarrow{n \rightarrow \infty} \pm \infty, \quad \sup _{n}\left\|h_{n}\right\|_{H^{1}}<\infty, \\
& \text { or } x_{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}, \quad h_{n} \xrightarrow{H^{1}} 0 ;
\end{aligned}
$$

- let $\left(t^{n}\right)_{n \in \mathbb{N}},\left(x^{n}\right)_{n \in \mathbb{N}}$ be sequences of real numbers, then we have the following implications:

$$
\begin{align*}
& t^{n} \xrightarrow{n \rightarrow \infty} \pm \infty \Longrightarrow\left\|e^{i t^{n} A} \tau_{x^{n}} \psi\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0, \quad 2<p<\infty, \quad \forall \psi \in H^{1} ;  \tag{2.4}\\
& t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty \Longrightarrow  \tag{2.5}\\
& \forall \psi \in H^{1} \quad \exists \tilde{\psi} \in H^{1}, \quad\left\|\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} \psi-\tilde{\psi}\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0 ; \\
& t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R} \Longrightarrow\left\|e^{i t^{n} A} \tau_{x^{n}} \psi-e^{i \bar{t} A} \tau_{\bar{x}} \psi\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0, \quad \forall \psi \in H^{1} . \tag{2.6}
\end{align*}
$$

We can now state the main result of this section.
Theorem 2.1. Let $\left(u_{n}\right)_{n \in \mathbb{N}}$ be a sequence bounded in $H^{1}$ and let $A$ be a self-adjoint operator that satisfies (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6). Then, up to subsequence, we can write

$$
u_{n}=\sum_{j=1}^{J} e^{i t_{j}^{n} A} \tau_{x_{j}^{n}} \psi_{j}+R_{n}^{J}, \quad \forall J \in \mathbb{N}
$$

where,

$$
t_{j}^{n} \in \mathbb{R}, \quad x_{j}^{n} \in \mathbb{R}, \quad \psi_{j} \in H^{1}
$$

are such that:

- for any fixed $j$ we have:

$$
\begin{align*}
& \text { either } t_{j}^{n}=0, \quad \forall n, \quad \text { or } \quad t_{j}^{n} \xrightarrow{n \rightarrow \pm \infty} \pm \infty,  \tag{2.7}\\
& \text { either } \quad x_{j}^{n}=0, \quad \forall n, \quad \text { or } \quad x_{j}^{n} \xrightarrow{n \rightarrow \infty} \pm \infty ;
\end{align*}
$$

- orthogonality of the parameters:

$$
\begin{equation*}
\left|t_{j}^{n}-t_{k}^{n}\right|+\left|x_{j}^{n}-x_{k}^{n}\right| \xrightarrow{n \rightarrow \infty} \infty, \quad \forall j \neq k ; \tag{2.8}
\end{equation*}
$$

- smallness of the reminder:

$$
\begin{equation*}
\forall \epsilon>0 \exists J=J(\epsilon) \in \mathbb{N} \text { such that } \limsup _{n \rightarrow \infty}\left\|e^{-i t A} R_{n}^{J}\right\|_{L^{\infty} L^{\infty}} \leq \epsilon ; \tag{2.9}
\end{equation*}
$$

- orthogonality in Hilbert norms:

$$
\begin{align*}
& \left\|u_{n}\right\|_{L^{2}}^{2}=\sum_{j=1}^{J}\left\|\psi_{j}\right\|_{L^{2}}^{2}+\left\|R_{n}^{J}\right\|_{L^{2}}^{2}+o(1), \quad \forall J \in \mathbb{N}  \tag{2.10}\\
& \left\|u_{n}\right\|_{H}^{2}=\sum_{j=1}^{J}\left\|\tau_{x_{j}^{n}} \psi_{j}\right\|_{H}^{2}+\left\|R_{n}^{J}\right\|_{H}^{2}+o(1), \quad \forall J \in \mathbb{N} \tag{2.11}
\end{align*}
$$

where $\|v\|_{H}^{2}=(A v, v)$.
Moreover we have

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{p}}^{p}=\sum_{j=1}^{J}\left\|e^{i t_{j}^{n} A} \tau_{x_{j}^{n}} \psi_{j}\right\|_{L^{p}}^{p}+\left\|R_{n}^{J}\right\|_{L^{p}}^{p}+o(1), \quad p \in(2, \infty), \quad \forall J \in \mathbb{N} \tag{2.12}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
E\left(u_{n}\right)=\sum_{j=1}^{J} E\left(e^{i t_{j}^{n} A} \tau_{x_{j}^{n}} \psi_{j}\right)+E\left(R_{n}^{J}\right)+o(1), \quad \forall J \in \mathbb{N}, \tag{2.13}
\end{equation*}
$$

where $E(u)=\frac{1}{2}\|u\|_{H}^{2}+\frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}$.
In order to prove the theorem, we need first the following lemma, where we implicitly assume the same assumptions as in Theorem 2.1 .

Lemma 2.1. Let $\left(h_{n}\right)_{n \in \mathbb{N}}$ be bounded in $H^{1}$ and $\left(t^{n}, t_{1}^{n}, t_{2}^{n}, t^{n}, x_{1}^{n}, x_{2}^{n}\right)_{n \in \mathbb{N}}$ be sequences of real numbers. Then we have the following implications:

$$
\begin{align*}
h_{n} \stackrel{H^{1}}{\longrightarrow} 0, \quad \tau_{-x_{2}^{n}} e^{i\left(t_{2}^{n}-t_{1}^{n}\right) A} \tau_{x_{1}^{n}} h_{n} \stackrel{H^{1}}{\rightharpoonup} \psi \neq 0 \Longrightarrow\left|t_{1}^{n}-t_{2}^{n}\right|+\left|x_{1}^{n}-x_{2}^{n}\right| \xrightarrow{n \rightarrow \infty} \infty ;  \tag{2.14}\\
h_{n} \xrightarrow{H^{1}} 0, \quad t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty \Longrightarrow \tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} h_{n} \xrightarrow{H^{1}} 0 . \tag{2.15}
\end{align*}
$$

Proof. For $\psi \in H^{1}$ we get

$$
\left(\tau_{-x_{2}^{n}} e^{i\left(t_{2}^{n}-t_{1}^{n}\right) A} \tau_{x_{1}^{n}} h_{n}, \psi\right)=\left(h_{n}, \tau_{-x_{1}^{n}} e^{-i\left(t_{2}^{n}-t_{1}^{n}\right) A} \tau_{x_{2}^{n}} \psi\right)
$$

where (.,.) denotes the $L^{2}$-scalar product. Hence by (2.5) we have $\tau_{-x_{2}^{n}} e^{i\left(t_{2}^{n}-t_{1}^{n}\right) A} \tau_{x_{1}^{n}} h_{n} \stackrel{L^{2}}{\rightharpoonup} 0$, and up to subsequence we obtain (2.15).

Concerning (2.14) it is equivalent to prove that

$$
\begin{gathered}
h_{n} \xrightarrow{H^{1}} 0, \quad s^{n} \xrightarrow{n \rightarrow \infty} \bar{s} \in \mathbb{R}, \quad y^{n}-z^{n} \xrightarrow{n \rightarrow \infty} \bar{z} \in \mathbb{R} \\
\Longrightarrow \tau_{-z^{n}} e^{i s^{n} A} \tau_{y_{n}} h_{n} \xrightarrow{H^{1}} 0 .
\end{gathered}
$$

This fact is equivalent to

$$
\tau_{-z^{n}} e^{i s^{n} A} \tau_{z_{n}} g_{n} \stackrel{H^{1}}{\rightharpoonup} 0
$$

where $g_{n}=\tau_{\bar{z}} h_{n} \xrightarrow{H^{1}} 0$. Hence we conclude by (2.15) in the case $z^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$; in the case $z^{n} \xrightarrow{n \rightarrow \infty} \bar{z} \in \mathbb{R}$ we conclude by the strong convergence of the sequence of operators $\left(\tau_{-z^{n}} e^{i s^{n} A} \tau_{z_{n}}\right)_{n \in \mathbb{N}}$ to the operator $\tau_{-z^{*}} e^{i \bar{s} A} \tau_{\bar{z}}$.

Lemma 2.2. Let $\left(v_{n}\right)_{n \in \mathbb{N}}$ be bounded in $H^{1}$ and

$$
\Gamma\left(v_{n}\right)=\left\{w \in L^{2} \quad \mid \quad \exists\left(x_{k}\right)_{k \in \mathbb{N}} \in \mathbb{R},\left(n_{k}\right)_{k \in \mathbb{N}} \in \mathbb{N} \text { with } n_{k} \nearrow \infty \text { s.t. } \tau_{x_{k}}\left(v_{n_{k}}\right) \stackrel{L^{2}}{ } w\right\}
$$

Then there exist $M=M\left(\sup _{n}\left\|v_{n}\right\|_{H^{1}}\right)>0$, such that:

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\|v_{n}\right\|_{L^{\infty}} \leq M\left(\gamma\left(v_{n}\right)\right)^{1 / 3}, \text { where } \gamma\left(v_{n}\right)=\sup _{w \in \Gamma\left(v_{n}\right)}\|w\|_{L^{2}} \tag{2.16}
\end{equation*}
$$

Proof. We introduce the Fourier multipliers $\chi_{R}(|D|)$ and $\tilde{\chi}_{R}(|D|)$ associated with the function $\chi\left(\frac{\xi}{R}\right)$ and $1-\chi\left(\frac{\xi}{R}\right)$ where

$$
\chi(\xi) \in C^{\infty}(\mathbb{R}), \quad \chi(x)=1 \text { for } \quad|x|<1, \quad \chi(x)=0 \text { for } \quad|x|>2
$$

Recall that $H^{3 / 4} \subset L^{\infty}$ and hence

$$
\begin{equation*}
\left\|\tilde{\chi}_{R}(|D|) v_{n}\right\|_{L^{\infty}} \leq C R^{-\frac{1}{4}}\left\|v_{n}\right\|_{H^{1}} \tag{2.17}
\end{equation*}
$$

In order to estimate $\left\|\chi_{R}(|D|) v_{n}\right\|_{L^{\infty}}$ we select $\left\{y_{n}\right\} \in \mathbb{R}$ such that

$$
\begin{equation*}
\left\|\chi_{R}(|D|) v_{n}\right\|_{L^{\infty}} \leq 2\left|\chi_{R}(|D|) v_{n}\left(y_{n}\right)\right| \tag{2.18}
\end{equation*}
$$

Notice that

$$
\left|\chi_{R}(|D|) v_{n}\left(y_{n}\right)\right|=R\left|\int \eta(R x) v_{n}\left(x-y_{n}\right) d x\right|
$$

where $\hat{\chi}=\eta$. Moreover for every subsequence $\left(n_{k}\right)_{k \in \mathbb{N}}$ we can select another subsequence $\left\{n_{k_{h}}\right\}$ such that $v_{n_{k_{h}}}\left(x-y_{n_{h_{k}}}\right) \stackrel{L^{2}}{\rightharpoonup} w \in \Gamma\left(v_{n}\right)$ and hence

$$
\limsup _{h \rightarrow \infty}\left|\chi_{R}(|D|) v_{n_{k_{h}}}\left(y_{n_{k_{h}}}\right)\right|=R\left|\int \eta(R x) w d x\right| \leq C R\|\eta(R x)\|_{L^{2}}\|w\|_{L^{2}} \leq C \sqrt{R} \gamma\left(v_{n}\right)
$$

which implies

$$
\limsup _{n \rightarrow \infty}\left|\chi_{R}(|D|) v_{n}\left(y_{n}\right)\right| \leq C \sqrt{R} \gamma\left(v_{n}\right)
$$

By combining this estimate with (2.17) and (2.18) we get:

$$
\limsup _{n \rightarrow \infty}\left\|v_{n}\right\|_{L^{\infty}} \leq C R^{-\frac{1}{4}} \sup _{n}\left\|v_{n}\right\|_{H^{1}}+C \sqrt{R} \gamma\left(v_{n}\right)
$$

and we conclude by choosing $R=C\left(\sup _{n}\left\|v_{n}\right\|_{H^{1}}\right)\left(\gamma\left(v_{n}\right)\right)^{-\frac{4}{3}}$.
Lemma 2.3. Let $\left(v_{n}\right)_{n \in \mathbb{N}}$ be bounded in $H^{1}$. Then up to subsequence there exist $\psi \in H^{1}$, $\left(x^{n}\right)_{n \in \mathbb{N}},\left(t^{n}\right)_{n \in \mathbb{N}}$ sequences of real numbers and $L=L\left(\sup _{n}\left\|v_{n}\right\|_{H^{1}}\right)>0$ such that,

$$
\begin{equation*}
\tau_{-x^{n}}\left(e^{-i t^{n} A} v_{n}\right)=\psi+W_{n} \tag{2.19}
\end{equation*}
$$

where:

$$
\begin{align*}
& W_{n} \stackrel{H^{1}}{\stackrel{1}{2}} 0 ;  \tag{2.20}\\
& \limsup _{n \rightarrow \infty}\left\|e^{-i t A} v_{n}\right\|_{L^{\infty} L^{\infty}} \leq L\|\psi\|_{L^{2}}^{1 / 3} ;  \tag{2.21}\\
& \left\|v_{n}\right\|_{L^{2}}^{2}=\|\psi\|_{L^{2}}^{2}+\left\|W_{n}\right\|_{L^{2}}^{2}+o(1) ;  \tag{2.22}\\
& \left\|v_{n}\right\|_{H}^{2}=\left\|\tau_{x^{n}} \psi\right\|_{H}^{2}+\left\|\tau_{x^{n}} W_{n}\right\|_{H}^{2}+o(1), \text { where }\|v\|_{H}^{2}=(A v, v) ;  \tag{2.23}\\
& \left\|v_{n}\right\|_{L^{p}}^{p}=\left\|e^{i t^{n} A} \tau_{x^{n}} \psi\right\|_{L^{p}}^{p}+\left\|e^{i t^{n} A} \tau_{x^{n}} W_{n}\right\|_{L^{p}}^{p}+o(1), \quad \forall p \in(2, \infty) . \tag{2.24}
\end{align*}
$$

Moreover we can assume (up to subsequence):

$$
\begin{align*}
& \text { either } t^{n}=0, \quad \forall n, \quad \text { or } \quad t^{n} \xrightarrow{n \rightarrow \pm \infty} \pm \infty,  \tag{2.25}\\
& \text { either } \quad x^{n}=0, \quad \forall n, \quad \text { or } \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty .
\end{align*}
$$

Proof. First we give the definition of $\psi,\left(W_{n}\right)_{n \in \mathbb{N}},\left(t^{n}\right)_{n \in \mathbb{N}},\left(x^{n}\right)_{n \in \mathbb{N}}$ in (2.19). Let $\left(t^{n}\right)_{n \in \mathbb{N}}$ be a sequence of real numbers such that

$$
\begin{equation*}
\left\|e^{-i t^{n} A} v_{n}\right\|_{L^{\infty}}>\frac{1}{2}\left\|e^{-i t A} v_{n}\right\|_{L^{\infty} L^{\infty}} . \tag{2.26}
\end{equation*}
$$

Note that we get the boundedness of $\left(e^{-i t A} v_{n}\right)_{n \in \mathbb{N}}$ in $H^{1}$ by using (2.1) and the assumption on the boundedness of $\left(v_{n}\right)_{n \in \mathbb{N}}$ in $H^{1}$. Following the notations of Lemma 2.2 we introduce $\Gamma\left(e^{-i t^{n} A} v_{n}\right) \subset L^{2}$ and also $\gamma\left(e^{-i t^{n} A} v_{n}\right) \in[0, \infty)$. Then up to subsequence we get the existence of $\left(x^{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\psi \in H^{1}$ such that

$$
\begin{equation*}
\tau_{-x^{n}}\left(e^{-i t^{n} A} v_{n}\right) \stackrel{H^{1}}{\rightharpoonup} \psi \tag{2.27}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\psi\|_{L^{2}} \geq \frac{1}{2} \gamma\left(e^{-i t^{n} A} v_{n}\right) \tag{2.28}
\end{equation*}
$$

On the other hand by combining Lemma 2.2 with (2.28) we get

$$
\limsup _{n \rightarrow \infty}\left\|e^{-i t^{n} A} v_{n}\right\|_{L^{\infty}} \leq 2^{1 / 3} M\|\psi\|_{L^{2}}^{1 / 3}
$$

By combining this estimate with (2.26) we get (2.21).
The proof of (2.20) follows by (2.27) together with the definition of $W_{n}$ in (2.19).
To prove (2.22) we combine (2.19), (2.20) and the Hilbert structure of $L^{2}$ in order to get

$$
\left\|v_{n}\right\|_{L^{2}}^{2}=\left\|\tau_{-x^{n}}\left(e^{-i t^{n} A} v_{n}\right)\right\|_{L^{2}}^{2}=\|\psi\|_{L^{2}}^{2}+\left\|W_{n}\right\|_{L^{2}}^{2}+o(1),
$$

where we used that $\tau_{x} e^{-i t A}$ is an isometry in $L^{2}$ for every $(t, x)$.
Next we prove (2.23). By (2.19) we get

$$
v_{n}=e^{i t^{n} A} \tau_{x^{n}} \psi+e^{i t^{n} A} \tau_{x^{n}} W_{n}
$$

and hence (2.23) follows provided that

$$
\left(e^{i t^{n} A} \tau_{x^{n}} \psi, e^{i t^{n} A} \tau_{x^{n}} W_{n}\right)_{H} \xrightarrow{n \rightarrow \infty} 0 .
$$

Notice that we have

$$
\left(e^{i t^{n} A} \tau_{x^{n}} \psi, e^{i t^{n} A} \tau_{x^{n}} W_{n}\right)_{H}=\left(\tau_{x^{n}} \psi, \tau_{x^{n}} W_{n}\right)_{H}=\left(\tau_{x^{n}} \psi, \tau_{x^{n}} W_{n}\right)_{H^{1}}+B\left(\tau_{x^{n}} \psi, \tau_{x^{n}} W_{n}\right)
$$

where we used (2.2). Therefore

$$
\left(e^{i t^{n} A} \tau_{x^{n}} \psi, e^{i t^{n} A} \tau_{x^{n}} W_{n}\right)_{H}=\left(\psi, W_{n}\right)_{H^{1}}+B\left(\tau_{x^{n}} \psi, \tau_{x^{n}} W_{n}\right) .
$$

Up to subsequence we have either $x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$ or $x^{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}$, and in both cases we conclude by (2.3).

Next we prove (2.24). We can assume that, up to subsequence, we are in one of the following cases:

First case: $t^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$.
Since

$$
v_{n}=e^{i t^{n} A} \tau_{x^{n}} \psi+e^{i t^{n} A} \tau_{x^{n}} W_{n}
$$

and $W_{n}$ is uniformly bounded in $H^{1}$, we conclude by assumption (2.4).
Second case: $t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}$.
Notice that we have

$$
\begin{equation*}
v_{n}-e^{i t^{n} A} \tau_{x^{n}} \psi=e^{i t^{n} A} \tau_{x^{n}} W_{n} \xrightarrow{n \rightarrow \infty} 0 \text { a.e. } x \in \mathbb{R} . \tag{2.29}
\end{equation*}
$$

The last property follows by

$$
\left(e^{i t^{n} A} \tau_{x^{n}} W_{n}, \varphi\right)_{L^{2}}=\left(W_{n}, \tau_{-x^{n}} e^{-i t^{n} A} \varphi\right)_{L^{2}}=\left(W_{n}, \tau_{-\bar{x}} e^{-i \bar{t} A} \varphi\right)_{L^{2}}+o(1)=o(1)
$$

that implies $e^{i t^{n} A} \tau_{x^{n}} W_{n} \stackrel{L^{2}}{\sim} 0$. Hence by $H^{1}$-boundedness $e^{i t^{n} A} \tau_{x^{n}} W_{n} \stackrel{H^{1}}{\sim} 0$ (up to subsequence). We conclude by Rellich Theorem the convergence in $L_{l o c}^{2}$ which in turn implies pointwise convergence. By combining (2.6) with (2.29) we get, up to subsequence,

$$
v_{n}-e^{i \bar{t} A} \tau_{\bar{x}} \psi=e^{i t^{n} A} \tau_{x^{n}} W_{n}+h_{n}(x) \xrightarrow{n \rightarrow \infty} 0 \text { a.e. } x, \quad\left\|h_{n}\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0,
$$

and hence by the Brézis-Lieb Lemma (see [BL83]) we get

$$
\begin{gathered}
\left\|e^{i t^{n} A} \tau_{x^{n}} W_{n}\right\|_{L^{p}}^{p}=\left\|v_{n}\right\|_{L^{p}}^{p}-\left\|e^{i \bar{\epsilon} A} \tau_{\bar{x}} \psi\right\|_{L^{p}}^{p}+o(1) \\
=\left\|v_{n}\right\|_{L^{p}}^{p}-\left\|e^{i t^{n} A} \tau_{x^{n}} \psi\right\|_{L^{p}}^{p}+o(1) .
\end{gathered}
$$

Third case: $t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$.
We have by (2.19) and (2.15)

$$
\begin{equation*}
\tau_{-x^{n}} v_{n}-\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} \psi=\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} W_{n} \stackrel{H^{1}}{\nu} 0 \tag{2.30}
\end{equation*}
$$

so as before we obtain pointwise convergence towards zero. Moreover, by (2.5) we get $\tilde{\psi} \in H^{1}$ such that

$$
\begin{equation*}
\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} \psi \xrightarrow{H^{1}} \tilde{\psi} \tag{2.31}
\end{equation*}
$$

By combining the pointwise convergence and (2.31) with the Brézis-Lieb Lemma and with the translation invariance of the $L^{p}$ norm we get

$$
\begin{gathered}
\left\|e^{i t^{n} A} \tau_{x^{n}} W_{n}\right\|_{L^{p}}^{p}=\left\|v_{n}\right\|_{L^{p}}^{p}-\|\tilde{\psi}\|_{L^{p}}^{p}+o(1) \\
=\left\|v_{n}\right\|_{L^{p}}^{p}-\left\|\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} \psi\right\|_{L^{p}}+o(1)=\left\|v_{n}\right\|_{L^{p}}^{p}-\left\|e^{i t^{n} A} \tau_{x^{n}} \psi\right\|_{L^{p}}+o(1) .
\end{gathered}
$$

Finally we focus on (2.25). First we consider the case $\left(t^{n}\right)_{n \in \mathbb{N}}$ is bounded, when we get up to subsequence $t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}$. Next we consider three cases (that can occur up to subsequence).

First case: $t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$.
In this case we claim that we can write a new identity of the type (2.19) by replacing $\psi$ by another $\tilde{\psi}$, the sequence $\left(W_{n}\right)_{n \in \mathbb{N}}$ by another sequence $\left(\tilde{W}_{n}\right)_{n \in \mathbb{N}}$ and the parameters $\left(t^{n}, x^{n}\right)$ by $\left(0, x^{n}\right)$, where $\tilde{W}_{n} \stackrel{H^{1}}{\rightharpoonup} 0$.
Then the proof of (2.22), (2.23), (2.24) (where we replace $\psi$ by $\tilde{\psi}$ and $W_{n}$ by $\tilde{W}_{n}$ ) follows as above. Moreover the proof of (2.21), with $\psi$ replaced by $\tilde{\psi}$, is trivial and follows by the fact that by the construction below we get $\|\tilde{\psi}\|_{L^{2}}=\|\psi\|_{L^{2}}$.

Recall that thanks to (2.5) we get the existence of $\tilde{\psi} \in H^{1}$ such that

$$
\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} \psi \xrightarrow{H^{1}} \tilde{\psi}
$$

Hence, in view of (2.19) (with the parameters $\left(t^{n}, x^{n}\right)$ and functions $\psi,\left(W_{n}\right)_{n \in \mathbb{N}}$ constructed above)

$$
v_{n}=\tau_{x^{n}} \tilde{\psi}+e^{i t^{n} A} \tau_{x^{n}} W_{n}+r_{n}(x), \quad\left\|r_{n}\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Hence we get the decomposition

$$
\tau_{-x^{n}} v_{n}=\tilde{\psi}+\tilde{W}_{n}
$$

where

$$
\tilde{W}_{n}=\tau_{-x^{n}} e^{i t^{n} A} \tau_{x^{n}} W_{n}+\tau_{-x^{n}} r_{n} .
$$

Notice that by (2.15) and $\left\|r_{n}\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0$ we obtain $\tilde{W}_{n} \xrightarrow{H^{1}} 0$.
Second case: $t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}$.
We argue as in the previous case and we look for suitable $\tilde{\psi},\left(\tilde{W}_{n}\right)_{n \in \mathbb{N}},\left(\tilde{t}^{n}, \tilde{x}^{n}\right)=(0,0)$. We select $\tilde{\psi}$ as follows:

$$
\tilde{\psi}=e^{i \bar{t} A} \tau_{\bar{x}} \psi .
$$

We then have

$$
\tilde{W}_{n}=e^{i t^{n} A} \tau_{x^{n}} \psi-\tau_{-\bar{x}} e^{i \bar{t} A} \tau_{\bar{x}} \psi+e^{i t^{n} A} \tau_{x^{n}} W_{n} .
$$

Notice that in this case the property $\tilde{W}_{n} \stackrel{H^{1}}{\sim} 0$ follows by the fact that the sequence of operators $\left(e^{i t^{n} A} \tau_{x^{n}}\right)_{n \in \mathbb{N}}$ converge in strong topology sense to the operator $e^{i \bar{\tau} A} \tau_{\bar{x}}$. We conclude as in the previous case.

Third case: $\left(t^{n}\right)_{n \in \mathbb{N}}$ is unbounded.
Up to subsequence, we can suppose $t^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$. If $\left(x^{n}\right)_{n \in \mathbb{N}}$ is unbounded too, then up to subsequence $x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$ and we are done. In the case $\left(x^{n}\right)_{n \in \mathbb{N}}$ is bounded, then up to subsequence we can suppose $x \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}$, and we choose $\tilde{\psi}=\tau_{\bar{x}} \psi,\left(\tilde{t}^{n}, \tilde{x}^{n}\right)=\left(t^{n}, 0\right)$, $\tilde{W}_{n}=\left(\tau_{x^{n}}-\tau_{\bar{x}}\right) \psi+\tau_{x^{n}} W_{n}$. We conclude as in the previous cases.

Proof of Theorem 2.1. We iterate several times Lemma 2.3,
First step: construction of $\psi_{1}$.
By Lemma 2.3 we get

$$
\begin{equation*}
u_{n}=e^{i t_{1}^{n} A}\left(\tau_{x_{1}^{n}} \psi_{1}\right)+R_{n}^{1} \tag{2.32}
\end{equation*}
$$

where $\psi=\psi_{1},\left(t^{n}\right)_{n \in \mathbb{N}}=\left(t_{1}^{n}\right)_{n \in \mathbb{N}},\left(x_{1}^{n}\right)_{n \in \mathbb{N}}=\left(x^{n}\right)_{n \in \mathbb{N}},\left(R_{n}^{1}\right)_{n \in \mathbb{N}}=\left(e^{i t_{1}^{n} A}\left(\tau_{x_{1}^{n}} W_{n}^{1}\right)\right)_{n \in \mathbb{N}}$, $\left(W_{n}^{1}\right)_{n \in \mathbb{N}}=\left(W_{n}\right)_{n \in \mathbb{N}}$ and $\psi,\left(t^{n}\right)_{n \in \mathbb{N}},\left(x^{n}\right)_{n \in \mathbb{N}},\left(W_{n}\right)_{n \in \mathbb{N}}$ are given by Lemma 2.3 for $\left(u_{n}\right)_{n \in \mathbb{N}}$ equal to $\left(v_{n}\right)_{n \in \mathbb{N}}$. Moreover by (2.22) we get

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{2}}^{2}=\left\|\psi_{1}\right\|_{L^{2}}^{2}+\left\|W_{n}^{1}\right\|_{L^{2}}^{2}+o(1)=\left\|\psi_{1}\right\|_{L^{2}}^{2}+\left\|R_{n}^{1}\right\|_{L^{2}}^{2}+o(1), \tag{2.33}
\end{equation*}
$$

and by (2.20)

$$
\begin{equation*}
\tau_{-x_{1}^{n}}\left(e^{-i t_{1}^{n} A} R_{n}^{1}\right)=W_{n}^{1} \stackrel{H^{1}}{\rightharpoonup} 0 . \tag{2.34}
\end{equation*}
$$

The proof of (2.11) for $J=1$ follows by (2.23), and we get

$$
\begin{equation*}
\left\|u_{n}\right\|_{H}^{2}=\left\|\tau_{x_{1}^{n}} \psi_{1}\right\|_{H}^{2}+\left\|R_{n}^{1}\right\|_{H}^{2}+o(1) . \tag{2.35}
\end{equation*}
$$

Moreover (2.12) follows by (2.24).
Second step: construction of $\psi_{2}$.

We apply again Lemma 2.3 to the sequence $v_{n}=R_{n}^{1}=e^{i t_{1}^{n} A}\left(\tau_{x_{1}^{n}} W_{n}^{1}\right)$ and we get

$$
\begin{equation*}
R_{n}^{1}=e^{i t_{2}^{n} A}\left(\tau_{x_{2}^{n}} \psi_{2}\right)+R_{n}^{2} \tag{2.36}
\end{equation*}
$$

where

$$
R_{n}^{2}=e^{i t_{2}^{n} A}\left(\tau_{x_{2}^{n}} W_{n}^{2}\right)
$$

Moreover we get

$$
\begin{equation*}
\left\|R_{n}^{1}\right\|_{L^{2}}^{2}=\left\|\psi_{2}\right\|_{L^{2}}^{2}+\left\|W_{n}^{2}\right\|_{L^{2}}^{2}+o(1) \tag{2.37}
\end{equation*}
$$

and

$$
\begin{equation*}
W_{n}^{2} \stackrel{H^{1}}{\longrightarrow} 0 \tag{2.38}
\end{equation*}
$$

Summarizing by (2.32) and (2.36) we get

$$
u_{n}=e^{i t_{1}^{n} A}\left(\tau_{x_{1}^{n}} \psi_{1}\right)+e^{i t_{2}^{n} A}\left(\tau_{x_{2}^{n}} \psi_{2}\right)+R_{n}^{2} .
$$

By combining (2.33) and (2.37) we get

$$
\left\|u_{n}\right\|_{L^{2}}^{2}=\left\|\psi_{1}\right\|_{L^{2}}^{2}+\left\|\psi_{2}\right\|_{L^{2}}^{2}+\left\|W_{n}^{2}\right\|_{L^{2}}^{2}+o(1)=\left\|\psi_{1}\right\|_{L^{2}}^{2}+\left\|\psi_{2}\right\|_{L^{2}}^{2}+\left\|R_{n}^{2}\right\|_{L^{2}}^{2}+o(1)
$$

Arguing as in the first step we can also prove

$$
\left\|R_{n}^{1}\right\|_{H}^{2}=\left\|\tau_{x_{n}^{2}} \psi_{2}\right\|_{H}^{2}+\left\|R_{n}^{2}\right\|_{H}^{2}
$$

and hence by (2.35)

$$
\left\|u_{n}\right\|_{H}^{2}=\left\|\tau_{x_{1}^{n}} \psi_{1}\right\|_{H}^{2}+\left\|\tau_{x_{2}^{n}} \psi_{2}\right\|_{H}^{2}+\left\|R_{n}^{2}\right\|_{H}^{2}+o(1)
$$

Notice also that (for $J=2$ ) (2.12) follows by (2.24).
Next we prove that $\left(t_{1}^{n}\right)_{n \in \mathbb{N}},\left(t_{2}^{n}\right)_{n \in \mathbb{N}},\left(x_{1}^{n}\right)_{n \in \mathbb{N}},\left(x_{2}^{n}\right)_{n \in \mathbb{N}}$ satisfy (2.8). By combining (2.36) and (2.38) we get

$$
\tau_{-x_{2}^{n}} e^{-i t_{2}^{n} A} e^{i t_{1}^{n} A} \tau_{x_{1}^{n}} W_{n}^{1}=\tau_{-x_{2}^{n}}\left(e^{-i t_{2}^{n} A} R_{n}^{1}\right)=\psi_{2}+W_{n}^{2} \stackrel{H^{1}}{\rightharpoonup} \psi_{2}
$$

hence either $\psi_{2}=0$ and we conclude the proof, or $\psi_{2} \neq 0$ and we get (2.8) by (2.14) and (2.34).

Third step: construction of $\psi_{J}$.

By iteration of the construction above we get

$$
u_{n}=e^{i t_{1}^{n} A}\left(\tau_{x_{1}^{n}} \psi_{1}\right)+\ldots+e^{i t_{J}^{n} A}\left(\tau_{x_{J}^{n}} \psi_{J}\right)+R_{J}^{n}
$$

where

$$
R_{J}^{n}=e^{i t_{J}^{n} A}\left(\tau_{x_{J}^{n}} W_{n}^{J}\right)
$$

By repeating the computations above we obtain (2.10), (2.11) and (2.12).
Next we prove (2.9). Notice that by (2.10) and since $\sup _{n}\left\|u_{n}\right\|_{L^{2}}<\infty$ we get that $\left\|\psi_{J}\right\|_{L^{2}} \xrightarrow{J \rightarrow \infty} 0$ and hence by (2.21) we get $\lim \sup _{n \rightarrow \infty}\left\|e^{i t A} R_{n}^{J-1}\right\|_{L^{\infty} L^{\infty}} \xrightarrow{J \rightarrow \infty} 0$.
The proof of (2.8) for generic $j, k$ is similar to the proof given in the second step in the case $j=1, k=2$. We skip the details.
Finally notice that (2.7) follows by (2.25) in Lemma 2.3 ,
2.2. The case $A=H_{q}$. Along this section we verify the abstract assumptions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) required on the operator $A$ along section 2 in the specific case

$$
A=H_{q}=-\frac{1}{2} \partial_{x}^{2}+q \delta_{0} .
$$

Notice that we get in this specific context $B(f, g)=f(0) \bar{g}(0)$. The verification of (2.1), (2.2) and (2.3) are straightforward and follow by classical properties of the space $H^{1}$. Also the verification of (2.6) is trivial.

Next we shall verify (2.4) and (2.5) and we shall make extensively use of the following identity (see Lemma 2.1 in HMZ07a) available for any initial datum $f \in L^{1}$ and supported in $(-\infty, 0]$ :

$$
\begin{align*}
e^{-i t H_{q}} f(x) & =e^{-i t H_{0}} f(x)  \tag{2.39}\\
+\left(e^{-i t H_{0}}\left(f \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x) & +\left(e^{-i t H_{0}}\left(f \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x),
\end{align*}
$$

where $\rho_{q}(x)=-q e^{q x} \cdot 1_{x \leq 0}(x)$.
We check the validity of (2.5). Since $\left[e^{-i t H_{0}}, \tau_{x}\right]=0$ it is sufficient to prove that

$$
\left\|\tau_{-x^{n}} e^{-i t^{n} H_{q}} \tau_{x^{n}} \psi(x)-\tau_{-x^{n}} e^{-i t^{n} H_{0}} \tau_{x^{n}} \psi(x)\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0
$$

and since $\tau_{-x^{n}}$ are isometries on $H^{1}$ it is equivalent to

$$
\begin{equation*}
\left\|e^{-i t^{n} H_{q}} \tau_{x^{n}} \psi(x)-e^{-i t^{n} H_{0}} \tau_{x^{n}} \psi(x)\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0, \text { if } \quad t^{n} \xrightarrow{n \rightarrow \infty} \bar{t} \in \mathbb{R}, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty . \tag{2.40}
\end{equation*}
$$

In order to prove (2.40) we use formula (2.39). First notice that by a density argument we can assume $\psi \in C_{0}^{\infty}(\mathbb{R})$. In particular in the case $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ we can assume $\tau_{x^{n}} \psi \subset$ $(-\infty, 0)$ and in the case $x^{n} \xrightarrow{n \rightarrow \infty}+\infty$ we can assume $\tau_{x^{n}} \psi \subset(0, \infty)$. In the first case we can combine (2.39) and the translation invariance of the $H^{1}$ norm, and hence (2.40) becomes:

$$
\left\|\left(e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x)+\left(e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Notice that

$$
\left(e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x)=\tau_{x^{n}} \varphi(x) \cdot 1_{x \geq 0}(x)+\tau_{x^{n}} r_{n}(x) \cdot 1_{x \geq 0}(x)
$$

and

$$
\left(e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)=\left(\tau_{x^{n}} \varphi\right)(-x) \cdot 1_{x \leq 0}(x)+\tau_{x^{n}} r_{n}(x) \cdot 1_{x \leq 0}(x)
$$

where $\varphi=e^{-i \bar{t} H_{0}}\left(\psi \star \rho_{q}\right)$ and $r_{n}(x)=\left(e^{-i t^{n} H_{0}}-e^{-i \bar{t} H_{0}}\right)\left(\psi \star \rho_{q}\right)$. Notice that by continuity property of the flow $e^{-i t H_{0}}$ we get $r_{n} \xrightarrow{H^{1}} 0$ and since we are assuming $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$, we get

$$
\left\|\left(\tau_{x^{n}} \varphi\right)(x) \cdot 1_{x \geq 0}(x)\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0, \quad\left\|\left(\tau_{x^{n}} \varphi\right)(-x) \cdot 1_{x \leq 0}(x)\right\|_{H^{1}} \xrightarrow{n \rightarrow \infty} 0 .
$$

In the second case (i.e. $x^{n} \xrightarrow{n \rightarrow \infty}+\infty$ ) we get

$$
e^{-i t^{n} H_{q}} \tau_{x^{n}} \psi(x)=R e^{-i t^{n} H_{q}} R \tau_{x^{n}} \psi
$$

where $R f(x)=f(-x)$ and we used $R^{2}=I d$ and $\left[e^{-i t^{n} H_{q}}, R\right]=0$. Hence (2.40) follows since we have the following identity

$$
e^{-i t^{n} H_{q}} \tau_{x^{n}} \psi-e^{-i t^{n} H_{0}} \tau_{x^{n}} \psi=R\left(e^{-i t^{n} H_{q}} R \tau_{x^{n}} \psi-e^{-i t^{n} H_{0}} R \tau_{x^{n}} \psi\right)
$$

$R$ is an isometry in $H^{1}$ and moreover $R \tau_{x^{n}} \psi=\tau_{-x^{n}} R \psi(x)$ where $-x^{n} \xrightarrow{n \rightarrow \infty}-\infty$. Hence we are reduced to the previous case (i.e. $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ ).

Next we prove (2.4), i.e.

$$
\begin{equation*}
t^{n} \xrightarrow{n \rightarrow \infty} \pm \infty \Longrightarrow \text { (up to subsequence) }\left\|e^{i t^{n} H_{q}} \tau_{x^{n}} \psi\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0, \quad \forall \psi \in H^{1}, \tag{2.41}
\end{equation*}
$$

where $p \in(2, \infty)$. We treat only the case $t^{n} \xrightarrow{n \rightarrow \infty}+\infty$, the other case is equivalent. By combining the time decay estimate $\left\|e^{-i t H_{q}}\right\|_{\mathcal{L}\left(L^{1}, L^{\infty}\right)} \leq C t^{-1 / 2}$ with the uniform bound $\left\|e^{-i t H_{q}}\right\|_{\mathcal{L}\left(H^{1}, L^{p}\right)} \leq C$ and with a density argument, we deduce $e^{i t H_{q}} \xrightarrow{t \rightarrow \infty} 0$ in the strong topology of the operators $\mathcal{L}\left(H^{1}, L^{p}\right)$. Hence we conclude (2.41) in the case $x^{n} \xrightarrow{n \rightarrow \infty} \bar{x} \in \mathbb{R}$ by a compactness argument. Hence it is sufficient to prove

$$
\begin{equation*}
\left\|e^{-i t^{n} H_{q}} \tau_{x_{n}} \psi(x)-e^{-i t^{n} H_{0}} \tau_{x_{n}} \psi(x)\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0, \text { if } \quad t^{n} \xrightarrow{n \rightarrow \infty}-\infty, \quad x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty, \tag{2.42}
\end{equation*}
$$

and to conclude by the decay properties of the group $e^{i t H_{0}}$. The proof of (2.42) can be done via a density argument by assuming $\psi \in C_{0}^{\infty}(\mathbb{R})$ and $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ (the case $x^{n} \xrightarrow{n \rightarrow \infty}+\infty$ can be reduced to the previous one via the reflexion operator $R$, exactly as we did above along the proof of $(\overline{2.401)})$. Hence we can rely on (2.39) and we are reduced to prove

$$
\begin{aligned}
& \left.\| e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x) \|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0 \\
& \left.\| e^{-i t^{n} H_{0}}\left(\tau_{x^{n}} \varphi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x) \|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0
\end{aligned}
$$

which is equivalent to

$$
\begin{gathered}
\left\|\tau_{x^{n}}\left(e^{-i t^{n} H_{0}}\left(\psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x)\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0, \\
\left\|\tau_{x^{n}}\left(e^{-i t^{n} H_{0}}\left(\varphi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)\right\|_{L^{p}} \xrightarrow{n \rightarrow \infty} 0 .
\end{gathered}
$$

Notice that the facts above follow by the translation invariance of the $L^{p}$ norm and by the property $e^{-i t H_{0}} \xrightarrow{t \rightarrow-\infty} 0$ in the strong topology of the operators $\mathcal{L}\left(H^{1}, L^{p}\right)$.

## 3. Preliminary results

Since now on we shall use the following notations:

$$
r=\alpha+2, \quad p=\frac{2 \alpha(\alpha+2)}{\alpha+4}, \quad q=\frac{2 \alpha(\alpha+2)}{\alpha^{2}-\alpha-4},
$$

and $\alpha>4$ is the same parameter as in (1.1).
3.1. Strichartz estimates. We recall the following homogeneous and inhomogeneous Strichartz estimates:

$$
\begin{align*}
\left\|e^{-i t H_{q}} \varphi\right\|_{L^{p} L^{r}} & \leq C\|\varphi\|_{H^{1}} ;  \tag{3.1}\\
\left\|e^{-i t H_{q}} \varphi\right\|_{L^{\alpha} L^{\infty}} & \leq C\|\varphi\|_{H^{1}} ;  \tag{3.2}\\
\left\|\int_{0}^{t} e^{-i(t-s) H_{q}} F(s) d s\right\|_{L^{p} L^{r}} & \leq C\|F\|_{L^{q^{\prime}} L^{r^{\prime}}} ;  \tag{3.3}\\
\left\|\int_{0}^{t} e^{-i(t-s) H_{q}} F(s) d s\right\|_{L^{\alpha} L^{\infty}} & \leq C\|F\|_{L^{q^{\prime} L^{r^{\prime}}}} . \tag{3.4}
\end{align*}
$$

Since $p, r>6$, the first one is obtained by $\left\|e^{-i t H_{q}} \varphi\right\|_{L^{p} L^{2 p /(p-4)}} \leq C\|\varphi\|_{L^{2}}$ in conjunction with a Sobolev embedding. Since $\alpha>4$, the second one is obtained by interpolating between the 1-d admissible space $L^{4} L^{\infty}$ and $L^{\infty} L^{\infty}$. The third one enters the frame of non-admissible inhomogeneous Strichartz estimates in Lemma 2.1 in Cazenave-Weissler CW92. The last one is contained in Theorem 1.4 of Foschi [Fos05], who extends the non-admissible inhomogeneous Strichartz exponents.

### 3.2. Perturbative nonlinear results.

Proposition 3.1. Let $\varphi \in H^{1}$ be given and assume that the unique global solution to (1.1) $u(t, x) \in \mathcal{C} H^{1}$ satisfies $u(t, x) \in L^{p} L^{r}$. Then $S c(\varphi)$ occurs.

Proof. We first prove that

$$
u(t, x) \in L^{\infty} H^{1} \cap L^{p} L^{r} \Longrightarrow u(t, x) \in L^{\alpha} L^{\infty} .
$$

It follows by the following chain of inequalities

$$
\|u\|_{L^{\alpha} L^{\infty}} \leq C\left(\|\varphi\|_{H^{1}}+\left\|u|u|^{\alpha}\right\|_{L^{q^{\prime}} L^{r^{\prime}}}\right) \leq C\left(\|\varphi\|_{H^{1}}+\|u\|_{L^{r} L^{r}}^{\alpha+1}\right)<\infty
$$

where we have used the Strichartz estimates above. We shall exploit also the following trivial estimate:

$$
\left\|\int_{t_{1}}^{t_{2}} e^{i s H_{q}} F(s) d s\right\|_{H^{1}} \leq\|F(s)\|_{L^{1}\left(\left(t_{1}, t_{2}\right) ; H^{1}\right)}, \quad \forall t_{1}, t_{2} .
$$

Hence by the integral equation

$$
\begin{aligned}
& \left\|e^{i t_{1} H_{q}} u\left(t_{1}, .\right)-e^{i t_{2} H_{q}} u\left(t_{2}, .\right)\right\|_{H^{1}}=\left\|\int_{t_{1}}^{t_{2}} e^{i s H_{q}}\left(u(s)|u(s)|^{\alpha}\right) d s\right\|_{H^{1}} \\
& \leq\left\|u(s)|u(s)|^{\alpha}\right\|_{L^{1}\left(\left(t_{1}, t_{2}\right) ; H^{1}\right)} \leq C\|u\|_{L^{\infty} H^{1}}\|u\|_{L_{\left(t_{1}, t_{2}\right)}^{\alpha}}^{\alpha} L^{\infty} \xrightarrow{t_{1}, t_{2} \rightarrow \pm \infty} 0 .
\end{aligned}
$$

Hence we get scattering via a standard argument.

Proposition 3.2. There exists $\epsilon_{0}>0$ such that :

$$
\varphi \in H^{1}, \quad\|\varphi\|_{H^{1}}<\epsilon_{0} \Longrightarrow\|u\|_{L^{p} L^{r}} \leq C_{\epsilon_{0}}\|\varphi\|_{H^{1}}, \quad\|v\|_{L^{p} L^{r}} \leq C_{\epsilon_{0}}\|\varphi\|_{H^{1}}
$$

where $u, v$ are the solutions of (1.1) and (1.2) respectively.
Proof. It is sufficient to check that if $u(t, x) \in \mathcal{C} H^{1}$ is the unique global solution to (1.1), then $u(t, x) \in L^{p} L^{q}$. In fact by the Strichartz estimates we get

$$
\|u\|_{L^{p}\left((-T, T) ; L^{r}\right)} \leq C\left(\|\varphi\|_{H^{1}}+\left\|u|u|^{\alpha}\right\|_{L^{q^{\prime}}\left((-T, T) ; L^{r^{\prime}}\right)}\right) \leq C\left(\epsilon+\|u\|_{L^{p}\left((-T, T) ; L^{r}\right)}^{\alpha+1}\right) .
$$

We conclude by a continuity argument that if $\epsilon$ is small enough, then $\sup _{T}\|u\|_{L^{p}\left((-T, T) ; L^{r}\right)}<$ $\infty$ and hence $u \in L^{p} L^{r}$. The proof goes the same for $v$.

We also need the following perturbation result.
Proposition 3.3. For every $M>0$ there exists $\epsilon=\epsilon(M)>0$ and $C=C(M)>0$ such that the following occurs. Let $v \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ be a solution of the integral equation with source term $e(t, x)$ :

$$
v(t, x)=e^{-i t H_{q}} \varphi-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(v(s)|v(s)|^{\alpha}\right)(x) d s+e(t, x)
$$

with $\|v\|_{L^{p} L^{r}}<M$ and $\|e\|_{L^{p} L^{r}}<\epsilon$. Assume moreover that $\varphi_{0} \in H^{1}$ is such that $\left\|e^{-i t H_{q}} \varphi_{0}\right\|_{L^{p} L^{r}}<\epsilon$, then the solution $u(t, x)$ to (1.1) with initial condition $\varphi+\varphi_{0}$ :

$$
u(t, x)=e^{-i t H_{q}}\left(\varphi+\varphi_{0}\right)-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(u(s)|u(s)|^{\alpha}\right) d s
$$

satisfies $u \in L^{p} L^{r}$ and moreover $\|u-v\|_{L^{p} L^{r}}<C \epsilon$.
Proof. It is contained in [FXC11], setting the space dimension $N=1$.

### 3.3. The nonlinear profiles.

Proposition 3.4. Let $\left(x^{n}\right)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\left|x^{n}\right| \rightarrow+\infty, \psi \in H^{1}$ and $U(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ be the unique solution to (1.2) with initial data $\psi$. Then we have

$$
U_{n}(t, x)=e^{-i t H_{q}} \psi_{n}+i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(U_{n}(s)\left|U_{n}(s)\right|^{\alpha}\right) d s+g_{n}(t, x)
$$

where

$$
\psi_{n}=\tau_{x^{n}} \psi \text { and } U_{n}=U\left(x-x^{n}, t\right)
$$

and

$$
\left\|g_{n}(t, x)\right\|_{L^{p} L^{2}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Proof. We are reduced to show:

$$
\begin{align*}
&\left\|e^{-i t H_{q}} \psi_{n}-e^{-i t H_{0}} \psi_{n}\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 ;  \tag{3.5}\\
&\left\|\int_{0}^{t} e^{-i(t-s) H_{q}}\left(U_{n}(s)\left|U_{n}(s)\right|^{\alpha}\right) d s-\int_{0}^{t} e^{-i(t-s) H_{0}}\left(U_{n}(s)\left|U_{n}(s)\right|^{\alpha}\right) d s\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 . \tag{3.6}
\end{align*}
$$

First we prove (3.5) via the formula (2.39). Notice that by a density argument we can assume that $\psi$ is compactly supported. Moreover modulo subsequence we can assume $x^{n} \xrightarrow{n \rightarrow \infty} \pm \infty$. In the case $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ we get $\operatorname{supp}\left(\tau_{x^{n}} \varphi\right) \subset(-\infty, 0)$ and in the case $x_{n} \xrightarrow{n \rightarrow \infty}+\infty$ we get $\operatorname{supp} \tau_{x_{n}} \varphi \subset(0, \infty)$, for large $n$. In the first case we can use (2.39) and hence (3.5) becomes:

$$
\left\|\left(e^{-i t H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x)+\left(e^{-i t H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0
$$

Notice that

$$
\left(e^{-i t H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(x) \cdot 1_{x \geq 0}(x)=\tau_{x^{n}}\left(e^{-i t H_{0}}\left(\psi \star \rho_{q}\right)\right) \cdot 1_{x \geq 0}(x)
$$

and

$$
\left(e^{-i t H_{0}}\left(\tau_{x^{n}} \psi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)=\left(\tau_{x^{n}} e^{-i t H_{0}}\left(\psi \star \rho_{q}\right)\right)(-x) \cdot 1_{x \leq 0}(x)
$$

hence we conclude since by the usual Strichartz estimate we have $e^{-i t H_{0}}\left(\psi \star \rho_{q}\right) \in L^{p} L^{r}$ and moreover we are assuming $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$.
In the case $x^{n} \xrightarrow{n \rightarrow \infty}+\infty$ we can reduce to the case $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ via the reflection operator $R$ (see the proof of (2.40)).

Next we focus on the proof of (3.6). As above we shall assume $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$ (the other case $x^{n} \xrightarrow{n \rightarrow \infty}+\infty$ can be handled via the reflection operator $R$ ). We shall prove the following fact:
$\left\|\int_{0}^{t} e^{-i(t-s) H_{q}} F\left(s, x-x^{n}\right) d s-\int_{0}^{t} e^{-i(t-s) H_{0}} F\left(s, x-x^{n}\right) d s\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0, \quad \forall F(t, x) \in L^{1} H^{1}$.
Notice that this fact is sufficient to conclude since by the classical scattering theory for NLS with constant coefficients we have $U|U|^{\alpha} \in L^{1} H^{1}$.

By a density argument we can assume the existence of a compact $K \subset \mathbb{R}$ such that $\operatorname{supp} F(t, x) \subset K, \quad \forall t$. In particular for $n$ large enough we get $\operatorname{supp} F\left(t, x-x^{n}\right) \subset$ $(-\infty, 0), \quad \forall t$. Hence we can use formula (2.39) and we are reduce to prove

$$
\begin{aligned}
& \left\|\left(\int_{0}^{t} e^{-i(t-s) H_{0}}\left(\tau_{x^{n}} F(s) \star \rho_{q}\right)(x) d s\right) \cdot 1_{x \geq 0}(x)\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 ; \\
& \left\|\left(\int_{0}^{t} e^{-i(t-s) H_{0}}\left(\tau_{x^{n}} F(s) \star \rho_{q}\right)(-x) d s\right) \cdot 1_{x \leq 0}(x)\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 .
\end{aligned}
$$

Next notice that

$$
\begin{aligned}
& \left(\int_{0}^{t} e^{-i(t-s) H_{0}}\left(\tau_{x^{n}} F(s) \star \rho_{q}\right)(x) d s\right) \cdot 1_{x \geq 0}(x) \\
& \quad=\tau_{x^{n}}\left(\int_{0}^{t}\left(e^{-i(t-s) H_{0}}\left(F(s) \star \rho_{q}\right)\right)(x) d s\right) \cdot 1_{x \geq 0}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\int_{0}^{t} e^{-i(t-s) H_{0}}\right. & \left.\left.\left(\tau_{x_{n}} F(s) \star \rho_{q}\right)\right)(-x) d s\right) \cdot 1_{x \leq 0}(x) \\
& =\tau_{x^{n}}\left(\int_{0}^{t}\left(e^{-i(t-s) H_{0}}\left(F(s) \star \rho_{q}\right)\right) d s\right)(-x) \cdot 1_{x \leq 0}(x) .
\end{aligned}
$$

Since $F(t, x) \star \rho_{q} \in L^{1} H^{1}$ we get by Strichartz estimates $\left.\int_{0}^{t} e^{-i(t-s) H_{0}}\left(F(s) \star \rho_{q}\right)\right) d s \in L^{p} L^{r}$. We conclude since $x^{n} \xrightarrow{n \rightarrow \infty}-\infty$.

Proposition 3.5. Let $\varphi \in H^{1}$, then there exist $W_{ \pm} \in \mathcal{C} H^{1} \cap L_{\mathbb{R}^{ \pm}}^{p} L^{r}$ solution to (1.1) and such that

$$
\begin{equation*}
\left\|W_{ \pm}(t, .)-e^{-i t H_{q}} \varphi\right\|_{H^{1}} \xrightarrow{t \rightarrow \pm \infty} 0 . \tag{3.7}
\end{equation*}
$$

Moreover, if $\left(t^{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}$ is such that $t^{n} \xrightarrow{n \rightarrow \infty} \mp \infty$, then

$$
W_{ \pm, n}(t, x)=e^{-i t H_{q}} \varphi_{n}-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(W_{ \pm, n}(s)\left|W_{ \pm, n}(s)\right|^{\alpha}\right) d s+f_{ \pm, n}(t, x)
$$

where

$$
\varphi_{n}=e^{i t^{n} H_{q}} \varphi \text { and } W_{ \pm, n}(t, x)=W_{ \pm}\left(t-t^{n}, x\right)
$$

and

$$
\left\|f_{ \pm, n}(t, x)\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Proof. The first part of the statement concerning the existence of wave operators is classical, since $e^{-i t H_{q}}$ enjoys Strichartz estimates as $e^{-i t H_{0}}$ and since we are in the defocusing case insuring global existence. For the second part of the statement we notice that by the translation invariance with respect to time we get $f_{ \pm, n}(t, x)=e^{-i t H_{q}}\left(W_{ \pm}\left(-t^{n}\right)-\varphi_{n}\right)$. We conclude by combining Strichartz estimates with (3.7).

Proposition 3.6. Let $\left(t^{n}\right)_{n \in \mathbb{N}},\left(x^{n}\right)_{n \in \mathbb{N}}$ be sequences of numbers such that $t^{n} \xrightarrow{n \rightarrow \infty} \mp \infty$ and $\left|x^{n}\right| \xrightarrow{n \rightarrow \infty}+\infty, \varphi \in H^{1}$ and $V_{ \pm}(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ be a solution to (1.2) such that

$$
\begin{equation*}
\left\|V_{ \pm}(t, .)-e^{-i t H_{0}} \varphi\right\|_{H^{1}} \xrightarrow{t \longrightarrow \pm \infty} 0 \tag{3.8}
\end{equation*}
$$

Then we have

$$
V_{ \pm, n}(t, x)=e^{-i t H_{q}} \varphi_{n}-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(V_{ \pm, n}(s)\left|V_{ \pm, n}(s)\right|^{\alpha}\right) d s+e_{ \pm, n}(t, x)
$$

where

$$
\varphi_{n}=e^{i t^{n} H_{q}} \tau_{x_{n}} \varphi \text { and } V_{ \pm, n}(t, x)=V_{ \pm}\left(t-t^{n}, x-x^{n}\right)
$$

and

$$
\left\|e_{ \pm, n}(t, x)\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Proof. By combining (3.8) with the integral equation solved by $V_{ \pm}(t, x)$, it is sufficient to prove:

$$
\begin{gathered}
\left\|e^{-i\left(t-t^{n}\right) H_{q}} \tau_{x^{n}} \varphi-e^{-i\left(t-t^{n}\right) H_{0}} \tau_{x^{n}} \varphi\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 ; \\
\left\|\int_{0}^{t} e^{-i(t-s) H_{q}} V_{ \pm, n}(s)\left|V_{ \pm, n}(s)\right|^{\alpha} d s-\int_{0}^{t} e^{-i(t-s) H_{0}} V_{ \pm, n}(s)\left|V_{n}(s)\right|^{\alpha} d s\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 .
\end{gathered}
$$

The first one reduces to (3.5) by the change of variable $t-t^{n} \rightarrow t$. By the same change of variable the second one reduces to the estimate in (3.6) with integral between $-t^{n}$ and $t$; its proof is similar to the one of (3.6).

## 4. Proof of Theorem 1.1

In this section we prove the scattering result in Theorem 1.1. For this aim we introduce the critical energy level defined as follows:

$$
E_{c}=\sup \left\{E>0 \quad \mid \quad \forall \varphi \in H^{1}, \quad E(\varphi)<E \Longrightarrow u(t, x) \in L^{p} L^{r}\right\},
$$

where $u(t, x)$ denotes the unique solution to (1.1) with initial data $\varphi$. Our aim is to show that $E_{c}=+\infty$, then we can conclude by Proposition 3.1. Notice also that due to Proposition 3.2 we have $E_{c}>0$.
The main strategy is to prove that if by the absurd $E_{c}<\infty$, then $E_{c}$ is achieved by a suitable critical initial data $\varphi_{c} \in H^{1}$ whose corresponding solution that does not scatter and moreover enjoys suitable compactness properties. The existence of such an object will be excluded via a rigidity argument by Proposition 4.2. Therefore we shall conclude that $E_{c}=+\infty$.

### 4.1. Existence and compactness of a minimal element.

Proposition 4.1. Assume that $E_{c}<+\infty$, then there exists a non trivial initial data $\varphi_{c} \in$ $H^{1}$ such that the corresponding solution $u_{c}(t, x)$ to (1.1) has the property that $\left\{u_{c}(t, x), t \in\right.$ $\mathbb{R}\}$ is relatively compact in $H^{1}$.

Proof. Since we are assuming $E_{c}<+\infty$ then we can select a sequence $\varphi_{n} \in H^{1}$ such that $E\left(\varphi_{n}\right) \xrightarrow{n \rightarrow \infty} E_{c}$ and $u_{n}(t, x) \notin L^{p} L^{r}$ where $u_{n}(t, x)$ is the corresponding solution to (1.1). First we shall prove that under these hypotheses there exists a subsequence converging in $H^{1}$ to a function with energy $E_{c}$, whose nonlinear evolution by (1.1) does not scatter. For this purpose we use the profile decomposition for the $H^{1}$ uniformly bounded sequence $\varphi_{n}$ :

$$
\begin{equation*}
\varphi_{n}=\sum_{j=1}^{J} e^{i t_{j}^{n} H_{q}} \tau_{x_{j}^{n}} \psi_{j}+R_{n}^{J} \tag{4.1}
\end{equation*}
$$

where $\psi_{1}, \ldots, \psi_{J} \in H^{1}$. We fix $J$ large enough in a sense to be specified later. From the energy estimate (2.13) we recall that

$$
\begin{equation*}
E_{c} \geq \limsup _{n \rightarrow \infty} \sum_{j=1}^{J} E\left(e^{i t_{j}^{n} H_{q}} \tau_{x_{j}^{n}} \psi_{j}\right) \tag{4.2}
\end{equation*}
$$

Notice that in view of (2.7), modulo rearrangement we can choose $0 \leq J^{\prime} \leq J^{\prime \prime} \leq J^{\prime \prime \prime} \leq$ $J^{i v} \leq J$ such that:

$$
\begin{aligned}
\left(t_{j}^{n}, x_{j}^{n}\right)=(0,0), \quad \forall n, \quad 1 \leq j \leq J^{\prime} \\
t_{j}^{n}=0, \quad \forall n \text { and } \quad\left|x_{j}^{n}\right| \xrightarrow{n \rightarrow \infty} \infty, \quad J^{\prime}+1 \leq j \leq J^{\prime \prime} \\
x_{j}^{n}=0, \quad \forall n \text { and } \lim _{n \rightarrow \infty} t_{j}^{n}=+\infty, \quad J^{\prime \prime}+1 \leq j \leq J^{\prime \prime \prime \prime} \\
x_{j}^{n}=0, \quad \forall n \text { and } \lim _{n \rightarrow \infty} t_{j}^{n}=-\infty, \quad J^{\prime \prime \prime}+1 \leq j \leq J^{i v} \\
\lim _{n \rightarrow \infty}\left|x_{j}^{n}\right|=+\infty \text { and } \lim _{n \rightarrow \infty} t_{j}^{n}=+\infty, \quad J^{i v}+1 \leq j \leq J^{v} \\
\lim _{n \rightarrow \infty}\left|x_{j}^{n}\right|=+\infty \text { and } \lim _{n \rightarrow \infty} t_{j}^{n}=-\infty, \quad J^{v}+1 \leq j \leq J
\end{aligned}
$$

Above we are assuming that if $a>b$ then there is no $j$ such that $a \leq j \leq b$. Notice that by the condition (2.8) we have that $J^{\prime} \in\{0,1\}$.

Next we shall prove that in (4.1) we have $J=1$ and the remainder can be assumed arbitrary small in Strichartz norm. To this purpose we shall suppose by absurd that $J>1$ and we can consider two cases:

- $J^{\prime}=1$;
- $J^{\prime}=0$.

We shall treat only the first case which is the most complicated one - the other case is a simplified version of the first case. Then we have $\left(t_{1}^{n}, x_{1}^{n}\right)=(0,0)$ and we also have (recall that we are assuming by the absurd $J>1$ ) by (4.2) that $E\left(\psi_{1}\right)<E_{c}$. Hence by definition
of $E_{c}$ we get the existence of $N(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ such that

$$
N(t, x)=e^{-i t H_{q}} \psi_{1}-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(N(s)|N(s)|^{\alpha}\right) d s
$$

For every $j$ such that $J^{\prime}+1 \leq j \leq J^{\prime \prime}$ we associate with the profile $\psi_{j}$ the function $U_{j}(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ according with Proposition 3.4. In particular we introduce $U_{j, n}=U_{j}\left(t, x-x_{j}^{n}\right)$.

For every $j$ such that $J^{\prime \prime}+1 \leq j \leq J^{\prime \prime \prime}$ we associate with the profile $\psi_{j}$ the function $W_{-, j}(t, x) \in \mathcal{C} H^{1} \cap L_{\mathbb{R}^{-}}^{p} L^{r}$ according with Proposition 3.5. We claim that $W_{-, j}(t, x) \in$ $\mathcal{C} H^{1} \cap L^{p} L^{r}$. In fact by (4.2) we get $E\left(e^{t_{j}^{n} H_{q}} \psi_{j}\right)<E_{c}-\frac{\left\|\nabla \psi_{j^{\prime}}\right\|_{L^{2}}^{2}}{4}$ for some $1 \leq j^{\prime} \neq$ $j \leq J$, whose existence is insured by the hypothesis $J>1$. Hence $E\left(W_{-, j}(t, x)\right)=$ $\lim _{n \rightarrow \infty} E\left(e^{t_{j}^{n} H_{q}} \psi_{j}\right)<E_{c}$, so $W_{-, j}$ scatters both forward and backwards in time and therefore $W_{-, j}(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$. In the sequel we shall denote $W_{-, j, n}=W_{-, j}\left(t-t_{j}^{n}, x\right)$.

For every $j$ such that $J^{\prime \prime \prime}+1 \leq j \leq J^{i v}$ we introduce in a similar way following Proposition 3.5 the nonlinear solutions $W_{+, j}(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ and also $W_{+, j, n}=W_{+, j}\left(t-t_{j}^{n}, x\right)$.

For every $j$ such that $J^{i v}+1 \leq j \leq J^{v}$ we associate with $\psi_{j}$ the function $V_{-, j}(t, x) \in$ $\mathcal{C} H^{1} \cap L^{p} L^{r}$ according with Proposition 3.6 and also $V_{-, j, n}=V_{-, j}\left(t-t_{j}^{n}, x-x_{j}^{n}\right)$.

Finally, for every $j$ such that $J^{v}+1 \leq j \leq J$ we associate with $\psi_{j}$ the function $V_{+, j}(t, x) \in$ $\mathcal{C} H^{1} \cap L^{p} L^{r}$ according with Proposition 3.6 and also $V_{+, j, n}=V_{+, j}\left(t-t_{j}^{n}, x-x_{j}^{n}\right)$.

Our aim is to apply the perturbative result of Proposition 3.3 to $u_{n}$ and to $Z_{J, n}$ defined as follows:

$$
Z_{J, n}=N+\sum_{j=J^{\prime}+1}^{J^{\prime \prime}} U_{j, n}+\sum_{j=J^{\prime \prime}+1}^{J^{\prime \prime \prime}} W_{-, j, n}+\sum_{j=J^{\prime \prime \prime}+1}^{J^{i v}} W_{+, j, n}+\sum_{j=J^{i v}+1}^{J^{v}} V_{-, j, n}+\sum_{j=J^{v}+1}^{J} V_{+, j, n}
$$

Notice that by combining Propositions 3.4, 3.5 and 3.6 the function $Z_{J, n}$ satisfies:

$$
Z_{J, n}(t)=e^{-i t H_{q}}\left(\varphi_{n}-R_{n}^{J}\right)-i z_{J, n}+r_{J, n}
$$

where $\left\|r_{J, n}\right\|_{L^{p} L^{q}} \xrightarrow{n \rightarrow \infty} 0$ and

$$
\begin{aligned}
& z_{J, n}(t, x)=\int_{0}^{t} e^{-i(t-s) H_{q}}\left(N(s)|N(s)|^{\alpha}\right) d s+\sum_{j=J^{\prime}+1}^{J^{\prime \prime}} \int_{0}^{t} e^{-i(t-s) H_{q}}\left(U_{j, n}(s)\left|U_{j, n}(s)\right|^{\alpha}\right) d s \\
+ & \sum_{j=J^{\prime \prime}+1}^{J^{\prime \prime \prime}} \int_{0}^{t} e^{-i(t-s) H_{q}}\left(W_{-, j, n}(s)\left|W_{-, j, n}(s)\right|^{\alpha}\right) d s+\sum_{j=J^{\prime \prime \prime}+1}^{J^{i v}} \int_{0}^{t} e^{-i(t-s) H_{q}}\left(W_{+, j, n}(s)\left|W_{+, j, n}(s)\right|^{\alpha}\right) d s
\end{aligned}
$$

$$
+\sum_{j=J^{i v}+1}^{J^{v}} \int_{0}^{t} e^{-i(t-s) H_{q}}\left(V_{-, j, n}(s)\left|V_{-, j, n}(s)\right|^{\alpha}\right) d s+\sum_{j=J^{v}+1}^{J} \int_{0}^{t} e^{-i(t-s) H_{q}}\left(V_{+, j, n}(s)\left|V_{+, j, n}(s)\right|^{\alpha}\right) d s
$$

We note that in Lemma 6.3 of [FXC11], the estimates in $L^{p} L^{r}$ needed to apply the perturbative result are proved first in a space $L^{\gamma} L^{\gamma}$, and then concluded by an uniform bound in $L^{\infty} H^{1}$ of the approximate solutions. This last uniform bound, proved in Corollary 4.4 of [FXC11, is more delicate in our case. Therefore we prove in the appendix estimates directly for the $L^{p} L^{r}$ norm. In view of Corollary 5.1 in the appendix, based on the orthogonality condition (2.8), we have

$$
\left\|z_{J, n}(t, x)-\int_{0}^{t} e^{-i(t-s) H_{q}}\left(Z_{J, n}(s)\left|Z_{J, n}(s)\right|^{\alpha}\right) d s\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Summarizing we get:

$$
Z_{J, n}(t)=e^{-i t H_{q}}\left(\varphi_{n}-R_{n}^{J}\right)-i \int_{0}^{t} e^{-i(t-s) H_{q}}\left(Z_{J, n}(s)\left|Z_{J, n}(s)\right|^{\alpha}\right) d s+s_{J, n}
$$

with $\left\|s_{J, n}\right\|_{L^{p} L^{r}} \xrightarrow{n \rightarrow \infty} 0$. In order to apply the perturbative result of Proposition 3.3, we need also a bound on $\sup _{J}\left(\limsup _{n \rightarrow \infty}\left\|Z_{J, n}\right\|_{L^{p} L^{r}}\right)$. Corollary 5.2 ensures us that

$$
\begin{gathered}
\limsup _{n \rightarrow \infty}\left(\left\|Z_{J, n}\right\|_{L^{p} L^{r}}\right)^{1+\alpha} \leq 2\|N\|_{L^{p} L^{r}}^{1+\alpha}+2 \sum_{j=J^{\prime}+1}^{J^{\prime \prime}}\left\|U_{j}\right\|_{L^{p} L^{r}}^{1+\alpha} \\
+2 \sum_{j=J^{\prime \prime}+1}^{J^{\prime \prime \prime}}\left\|W_{-, j}\right\|_{L^{p} L^{r}}^{1+\alpha}+2 \sum_{j=J^{\prime \prime \prime}+1}^{J^{i v}}\left\|W_{+, j}\right\|_{L^{p} L^{r}}^{1+\alpha}+2 \sum_{j=J^{i v}+1}^{J^{v}}\left\|V_{-, j}\right\|_{L^{p} L^{r}}^{1+\alpha}+2 \sum_{j=J^{v}+1}^{J}\left\|V_{+, j}\right\|_{L^{p} L^{r}}^{1+\alpha} .
\end{gathered}
$$

By using the defocusing conserved energy we obtain that the initial data of the wave operators $V_{ \pm, j}, W_{ \pm, j}$ are upper-bounded in $H^{1}$ by $C\left\|\varphi_{j}\right\|_{H^{1}}$. In view of the orthogonality relation (2.11) we obtain the existence of $J_{0}$ such that for any $J \geq J_{0}$ we have

$$
\left\|\varphi_{j}\right\|_{H^{1}}<\epsilon_{0}
$$

where $\epsilon_{0}$ is the universal constant in Proposition 3.2, Then by Proposition 3.2, the fact that $N, U_{j}, V_{ \pm, j}, W_{ \pm, j}$ belong to $L^{p} L^{r}$ and by Corollary 5.2 we get

$$
\begin{equation*}
\sup _{J}\left(\limsup _{n \rightarrow \infty}\left\|Z_{J, n}\right\|_{L^{p} L^{r}}\right)=M<\infty . \tag{4.3}
\end{equation*}
$$

Due to (4.3) we are in position to apply Proposition 3.3 to $Z_{J, n}$ provided that we choose $J$ large enough in such a way that $\lim \sup _{n \rightarrow \infty}\left\|e^{-i t H_{q}} R_{n}^{J}\right\|<\epsilon$, where $\epsilon=\epsilon(M)>0$ is the one given in Proposition 3.3. As a by-product we get that $S c\left(\varphi_{n}\right)$ occurs for $n$ large, and hence we get a contradiction.

Therefore we have obtained that $J=1$ so

$$
\begin{equation*}
\varphi_{n}=e^{i t_{1}^{n} H_{q}} \tau_{x_{1}^{n}} \psi_{1}+R_{n}^{1} \tag{4.4}
\end{equation*}
$$

where $\psi_{1} \in H^{1}$ and $\lim \sup _{n \rightarrow \infty}\left\|e^{-i t H_{q}} R_{n}^{1}\right\|_{L^{p} L^{r}}=0$. Following the same argument as in Lemma 6.3 of [FXC11] one can deduce that $\left(t_{1}^{n}\right)_{n \in \mathbb{N}}$ is bounded and hence we can assume $t_{1}^{n}=0$. Moreover arguing by the absurd and by combining Propositions 3.4 and 3.3 we get $x_{1}^{n}=0$ (otherwise $S c\left(\varphi_{n}\right)$ occurs for $n$ large enough, and it is a contradiction). We obtain then that $S c\left(\psi_{1}\right)$ does not occurs, so $E\left(\psi_{1}\right) \geq E_{c}$. Equality occurs by the energy estimate (4.2), and in particular there is a subsequence of $\left(\varphi_{n}\right)_{n \in \mathbb{N}}$ converging in $H^{1}$ to $\psi_{1}$. We have as a critical element $\varphi_{c}=\psi_{1}$.

The compactness of the trajectory $u_{c}(t, x) \in H^{1}$ follows again by standard arguments. More precisely, for $\left(t^{n}\right)_{n \in \mathbb{N}}$ a sequence of times, $\left(u_{c}\left(t^{n}, x\right)\right)_{n \in \mathbb{N}}$ satisfies the same hypothesis as $\varphi_{n}$ at the beggining of the proof above so we conclude that there is a subsequence converging in $H^{1}$.
4.2. Rigidity of compact solutions. We shall get now a constraint on the solution $u_{c}(t, x)$ constructed above.

Proposition 4.2. Assume $u$ solves (1.1) with $q \geq 0$ and satisfies the property:

$$
\begin{equation*}
\{u(t, x), t \in \mathbb{R}\} \text { is compact in } H^{1} . \tag{4.5}
\end{equation*}
$$

Then $u=0$.
Proof. We start with the following virial computation.
Lemma 4.1. Let $u(t, x) \in \mathcal{C} H^{1}$ be a global solution to (1.1) and $\lambda(x)$ a weight such that $\partial_{x} \lambda(0)=0$. Then

$$
\begin{array}{r}
\frac{d^{2}}{d t^{2}} \int \lambda|u|^{2} d x=\frac{d}{d t}\left(\operatorname{Im} \int \partial_{x} \lambda \partial_{x} u \bar{u} d x\right)  \tag{4.6}\\
=\int \lambda^{\prime \prime}\left|u^{\prime}\right|^{2} d x-\frac{1}{4} \int \lambda^{i v}|u|^{2} d x+q \lambda^{\prime \prime}(0)|u(t, 0)|^{2}+\frac{\alpha}{\alpha+2} \int \lambda^{\prime \prime}|u|^{\alpha+2} d x
\end{array}
$$

Proof. We shall use the notations $u^{\prime}=\partial_{x} u, \lambda^{\prime}=\partial_{x} \lambda, u_{t}=\partial_{t} u$. We compute, for a weight in space $\lambda(x)$ :

$$
\begin{aligned}
\frac{d}{d t} \int \lambda|u|^{2} d x & =2 \operatorname{Re} \int \lambda u_{t} \bar{u} d x \\
=2 \operatorname{Re} \int \lambda\left(\frac{i}{2} u^{\prime \prime} \bar{u}-i u \bar{u}|u|^{\alpha}\right) d x+2 \operatorname{Re} \lambda(0) i|u(t, 0)|^{2} & =-\operatorname{Re} \int i \lambda^{\prime} u^{\prime} \bar{u} d x .
\end{aligned}
$$

Next we compute, due to the previous identity:

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \int \lambda|u|^{2} d x=\operatorname{Im} \int \lambda^{\prime}\left(u^{\prime} \bar{u}\right)_{t} d x \tag{4.7}
\end{equation*}
$$

We get, by using integrations by parts and $\lambda^{\prime}(0)=0$ :

$$
\begin{align*}
& \operatorname{Im} \int \lambda^{\prime}\left(u^{\prime} \bar{u}\right)_{t} d x=\operatorname{Im} \int \lambda^{\prime} u^{\prime} \bar{u}_{t} d x+\operatorname{Im} \int \lambda^{\prime} u_{t}^{\prime} \bar{u} d x  \tag{4.8}\\
&=2 \operatorname{Im} \int \lambda^{\prime} u^{\prime} \bar{u}_{t} d x-\operatorname{Im} \int \lambda^{\prime \prime} u_{t} \bar{u} d x \\
&=-\operatorname{Re} \int \lambda^{\prime} u^{\prime} \bar{u}^{\prime \prime} d x+2 \operatorname{Re} \int \lambda^{\prime} u^{\prime} \bar{u}|u|^{\alpha} d x-\frac{1}{2} \operatorname{Re} \int \lambda^{\prime \prime} u^{\prime \prime} \bar{u} d x+\operatorname{Re} \int \lambda^{\prime \prime}\left(u|u|^{\alpha}\right) \bar{u} d x+q \lambda^{\prime \prime}(0)|u(t, 0)|^{2} .
\end{align*}
$$

Next notice that

$$
\begin{gathered}
-\operatorname{Re} \int \lambda^{\prime} u^{\prime} \bar{u}^{\prime \prime} d x-\frac{1}{2} \operatorname{Re} \int \lambda^{\prime \prime} u^{\prime \prime} \bar{u} d x \\
=\frac{1}{2} \int \lambda^{\prime \prime}\left|u^{\prime}\right|^{2} d x++\frac{1}{2} \operatorname{Re} \int \lambda^{\prime \prime \prime} u^{\prime} \bar{u} d x+\frac{1}{2} \operatorname{Re} \int \lambda^{\prime \prime}\left|u^{\prime}\right|^{2} d x \\
=\int \lambda^{\prime \prime}\left|u^{\prime}\right|^{2} d x-\frac{1}{4} \int \lambda^{i v}|u|^{2} d x
\end{gathered}
$$

and

$$
\begin{gathered}
2 \operatorname{Re} \int \lambda^{\prime} u^{\prime} \bar{u}|u|^{\alpha} d x+\operatorname{Re} \int \lambda^{\prime \prime}\left(u|u|^{\alpha}\right) \bar{u} d x \\
=-\frac{2}{\alpha+2} \int \lambda^{\prime \prime}\left(|u|^{\alpha+2}\right)+\int \lambda^{\prime \prime}|u|^{\alpha+2} d x=\frac{\alpha}{\alpha+2} \int \lambda^{\prime \prime}|u|^{\alpha+2} d x .
\end{gathered}
$$

We conclude by combining the computations above with (4.7) and (4.8).

We continue the proof of Proposition 4.2 and we assume by the absurd the existence of a non-trivial solution $u(t, x)$ that satisfies (4.5). We fix a cut-off $\chi$ vanishing outside $B(0,2)$ and equal to one on $B(0,1)$. Let $R>0$ to be chosen later. By using Lemma 4.1 for $\lambda(x)=x^{2} \chi\left(\frac{|x|}{R}\right)$ then we get:

$$
\begin{array}{r}
\frac{d}{d t}\left(\operatorname{Im} \int\left(x^{2} \chi\left(\frac{|x|}{R}\right)\right)^{\prime} u^{\prime} \bar{u} d x\right) \\
\geq \int_{|x|<R}\left|u^{\prime}\right|^{2} d x+\frac{\alpha}{\alpha+2} \int_{|x|<R}|u|^{\alpha+2} d x-C \int_{|x|>R}\left(|u|^{2}+\left|u^{\prime}\right|^{2}+|u|^{\alpha+2}\right) d x \\
\geq \delta-C \int_{|x|>R}\left(|u|^{2}+\left|u^{\prime}\right|^{2}+|u|^{\alpha+2}\right) d x
\end{array}
$$

for some $\delta>0$. Notice that the existence of a positive $\delta$ comes from the fact that $u(t, x)$ is assumed to be non trivial and moreover satisfies (4.5). By integrating from 0 to $t$ and using Cauchy-Schwartz inequality, then we obtain

$$
C(R)\|u\|_{L^{\infty} H^{1}} \geq t \delta-C \int_{0}^{t} \int_{|x|>R}\left(|u|^{2}+\left|u^{\prime}\right|^{2}+|u|^{\alpha+2}\right) d x .
$$

By using again the compacteness hypothesis (4.5) then we get a contradiction as $t$ goes to infinity, provided $R>0$ is large enough.

As a conclusion, the existence of the solution $u_{c}(t, x)$ constructed in Proposition 4.1 is constrained by Proposition 4.2 to be the null function. Since $E\left(u_{c}\right)=E_{c}>0$ we get a constradiction, so the hypothesis $E_{c}<+\infty$ made in Proposition 4.2 cannot hold. Therefore we conclude that $E_{c}=+\infty$, so all solutions of (1.1) scatter.

## 5. Appendix

Proposition 5.1. Let $W_{i}(t, x) \in \mathcal{C} H^{1} \cap L^{p} L^{r}$ for $i=1,2$ be space-time functions and $\left(t_{n}, s_{n}, x_{n}, y_{n}\right)_{n \in \mathbb{N}}$ be sequences of real numbers. Assume that $\left|t_{n}-s_{n}\right|+\left|x_{n}-y_{n}\right| \xrightarrow{n \rightarrow \infty}+\infty$ then we get

$$
\left\|\left|W_{1}\left(t-t_{n}, x-x_{n}\right)\right|^{\alpha} \times\left|W_{2}\left(t-s_{n}, x-y_{n}\right)\right|\right\|_{L^{q^{\prime}} L^{r^{\prime}}} \xrightarrow{n \rightarrow \infty} 0 .
$$

Proof. First assume that $\left|t_{n}-s_{n}\right| \xrightarrow{n \rightarrow \infty} \infty$. Then in this case we get:

$$
\begin{aligned}
&\left\|\left|W_{1}\left(t-t_{n}, x-x_{n}\right)\right|^{\alpha} \times\left|W_{2}\left(t-s_{n}, x-y_{n}\right)\right|\right\|_{L^{q^{\prime} L^{r^{\prime}}}} \\
& \leq\| \| W_{1}\left(t-t_{n}, x-x_{n}\right)\left\|_{L_{x}^{r}}^{\alpha} \times\right\| W_{2}\left(t-s_{n}, x-y_{n}\right)\left\|_{L_{x}^{r}}\right\|_{L_{t}^{q^{\prime}}}
\end{aligned}
$$

The conclusion follows by the following elementary fact:

$$
\left|t_{n}-s_{n}\right| \xrightarrow{n \rightarrow \infty}+\infty \Longrightarrow| |\left|f_{1}\left(t-t_{n}\right)\right|^{\alpha} \times\left|f_{2}\left(t-s_{n}\right)\right| \|_{L_{t}^{q^{\prime}}} \xrightarrow{n \rightarrow \infty} 0,
$$

where $f_{i}(t)=\left\|W_{i}(t, x)\right\|_{L^{r}} \in L_{t}^{p}, i=1,2$.
Next we assume that $\left(\left|t_{n}-s_{n}\right|\right)_{n \in \mathbb{N}}$ is bounded and $\left|x_{n}-y_{n}\right| \xrightarrow{n \rightarrow \infty} \infty$. First notice that we have

$$
\begin{aligned}
& \left\|\left|W_{1}\left(t, x-x_{n}\right)\right|^{\alpha} \times\left|W_{2}\left(t+t_{n}-s_{n}, x-y_{n}\right)\right|\right\|_{L_{|t|>T}^{q^{\prime}} L^{r^{\prime}}} \\
\leq & \left\|W_{1}(t, x)\right\|_{L_{|t|>T}^{p}}^{\alpha} L^{r}\left\|W_{2}\left(t+t_{n}-s_{n}, x\right)\right\|_{L_{|t|>T}^{p}} L^{r} \xrightarrow{T \rightarrow \infty} 0 .
\end{aligned}
$$

Hence it is sufficient to prove

$$
\begin{equation*}
\left|\left|\left|W_{1}\left(t, x-x_{n}\right)\right|^{\alpha} \times\left|W_{2}\left(t+t_{n}-s_{n}, x-y_{n}\right)\right| \|_{L_{|t|<T}^{q^{\prime}}}{L^{r^{\prime}}}^{n \rightarrow \infty} 0\right.\right. \tag{5.1}
\end{equation*}
$$

for every fixed $T$. We notice that for every fixed $t$ we get:

$$
\begin{aligned}
&\left\|\left|W_{1}\left(t, x-x_{n}\right)\right|^{\alpha} \times\left|W_{2}\left(t+t_{n}-s_{n}, x-y_{n}\right)\right|\right\|_{L_{x}^{r^{\prime}}} \\
&=\left\|\left|W_{1}(t, x)\right|^{\alpha} \times\left|W_{2}\left(t+t_{n}-s_{n}, x+x_{n}-y_{n}\right)\right|\right\|_{L_{x}^{r^{\prime}}} \xrightarrow{n \rightarrow \infty} 0
\end{aligned}
$$

where we used at the last step the following facts (below we use the property $W_{i}(t, x) \in \mathcal{C} H^{1}$ to give a meaning to the function $W_{i}(t, x)$ for every fixed $t$ ):

$$
\begin{gathered}
\left|W_{1}(t, x)\right|^{\alpha} \in L^{\frac{r}{\alpha}}, \quad \forall t \\
\left\{W_{2}\left(t+t_{n}-s_{n}, x\right), \quad n \in \mathbb{N}\right\} \text { is compact in } L^{r}, \quad \forall t
\end{gathered}
$$

and

$$
\left|x_{n}-y_{n}\right| \xrightarrow{n \rightarrow \infty} \infty .
$$

Indeed the first property above follows by the Sobolev embedding $H^{1} \subset L^{r}$, and second one follows from the fact that $\left(\left|t_{n}-s_{n}\right|\right)_{n \in \mathbb{N}}$ is bounded and the function $\mathbb{R} \ni t \rightarrow W_{2}(t, x) \in H^{1}$ is continuous. On the other hand we have

$$
\begin{aligned}
& \sup _{t \in(-T, T)}\left\|\left|W_{1}(t, x)\right|^{\alpha} \times\left|W_{2}\left(t+t_{n}-s_{n}, x+x_{n}-y_{n}\right)\right|\right\|_{L_{x}^{r^{\prime}}} \\
& \leq \sup _{t \in(-T, T)}\left\|W_{1}(t, x)\right\|_{L_{x}^{r}}^{\alpha} \times \| W_{2}\left(\left(t+t_{n}-s_{n}, x\right) \|_{L_{x}^{r}}<\infty,\right.
\end{aligned}
$$

where we used again the Sobolev embedding $H^{1} \subset L^{r}$ and the assumption $u(t, x) \in \mathcal{C} H^{1}$. We deduce (5.1) by the Lebesgue dominated convergence theorem.

As a consequence we get the following corollary.
Corollary 5.1. Let $W_{j}(t, x) \in L^{p} L^{r} \cap \mathcal{C} H^{1}, j=1, \ldots, N$ be a family of space-time functions and let $\left(t_{j}^{n}, x_{j}^{n}\right)_{n \in \mathbb{N}}, j=1, \ldots, N$ be sequences of real numbers that satisfy the ortogonality condition:

$$
\left|t_{j}^{n}-t_{k}^{n}\right|+\left|x_{j}^{n}-x_{k}^{n}\right| \xrightarrow{n \rightarrow \infty}+\infty, \quad j \neq k .
$$

Then we have

$$
\left\|\sum_{j=1}^{N} W_{j, n}(t, x)\left|W_{j, n}(t, x)\right|^{\alpha}-\left(\sum_{j=1}^{N} W_{j, n}(t, x)\right)\left(\left|\sum_{j=1}^{N} W_{j, n}(t, x)\right|^{\alpha}\right)\right\|_{L^{q^{\prime} L^{r^{\prime}}}} \xrightarrow{n \rightarrow \infty} 0,
$$

where $W_{j, n}(t, x)=W_{j}\left(t-t_{j}^{n}, x-x_{j}^{n}\right)$.
Proof. It follows by Proposition 5.1 in conjunction with the following elementary inequality

$$
\begin{equation*}
\left.\left|\sum_{j=1}^{N} a_{j}\right| a_{j}\right|^{\alpha}-\left.\left(\sum_{j=1}^{N} a_{j}\right)\left|\sum_{j=1}^{N} a_{j}\right|^{\alpha}\left|\leq C(N, \alpha) \sum_{j \neq k}\right| a_{j}| | a_{k}\right|^{\alpha}, \quad \forall a_{1}, \ldots, a_{N} \in \mathbb{C} . \tag{5.2}
\end{equation*}
$$

Corollary 5.2. Let $W_{j}(t, x) \in L^{p} L^{r} \cap \mathcal{C} H^{1}, j=1, \ldots, N$ be a family of space-time functions and let $\left(t_{j}^{n}, x_{j}^{n}\right)_{n \in \mathbb{N}}, j=1, \ldots, N$ be sequences of real numbers that satisfy the ortogonality condition:

$$
\left|t_{j}^{n}-t_{k}^{n}\right|+\left|x_{j}^{n}-x_{k}^{n}\right| \xrightarrow{n \rightarrow \infty}+\infty, \quad j \neq k .
$$

Then we have

$$
\limsup _{n \rightarrow \infty}\left(\left\|\sum_{j=1}^{N} W_{j, n}(t, x)\right\|_{L^{p} L^{r}}\right)^{1+\alpha} \leq 2 \sum_{j=1}^{N}\left\|W_{j}\right\|_{L^{p} L^{r}}^{1+\alpha}
$$

where $W_{j, n}(t, x)=W_{j}\left(t-t_{j}^{n}, x-x_{j}^{n}\right)$.
Proof. We have

$$
\left\|\sum_{j=1}^{N} W_{j, n}(t, x)\right\|_{L^{p} L^{r}} \leq\left(\left\|\left(\sum_{j=1}^{N}\left|W_{j, n}(t, x)\right|\right)^{1+\alpha}\right\|_{L^{q^{\prime}} L^{r^{\prime}}}\right)^{\frac{1}{1+\alpha}}
$$

$$
\leq\left(\left\|\left(\sum_{j=1}^{N}\left|W_{j, n}(t, x)\right|\right)^{1+\alpha}-\sum_{j=1}^{N}\left|W_{j, n}(t, x)\right|^{1+\alpha}\right\|_{L^{q^{\prime}} L^{r^{\prime}}}+\left\|\sum_{j=1}^{N}\left|W_{j, n}(t, x)\right|^{1+\alpha}\right\|_{L^{q^{\prime}} L^{r^{\prime}}}\right)^{\frac{1}{1+\alpha}} .
$$

The conclusion follows by combining (5.2) with Proposition 5.1.

## References

[AGHKH05] Sergio Albeverio, Fritz Gesztesy, Raphael Høegh-Krohn, and Helge Holden. Solvable models in quantum mechanics. AMS Chelsea Publishing, Providence, RI, second edition, 2005. With an appendix by Pavel Exner.
[AN09] Riccardo Adami and Diego Noja. Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect. J. Phys. A, 42(49):495302, 19, 2009.
[ANV13] Riccardo Adami, Diego Noja, and Nicola Visciglia. Constrained energy minimization and ground states for NLS with point defects. Discrete Contin. Dyn. Syst. Ser. B, 18(5):11551188, 2013.
[AS05] Ricardo Adami and Andrea Sacchetti. The transition from diffusion to blow-up for a nonlinear Schrödinger equation in dimension 1. J. Phys. A, 38(39):8379-8392, 2005.
[BL83] Haïm Brézis and Elliott Lieb. A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc., 88(3):486-490, 1983.
[Car14] Rémi Carles. Sharp weights in the cauchy problem for nonlinear Schrödinger equations with potential. arXiv:1409.5759, 2014.
[CGT09] James Colliander, Manoussos Grillakis, and Nikolaos Tzirakis. Tensor products and correlation estimates with applications to nonlinear Schrödinger equations. Comm. Pure Appl. Math., 62(7):920-968, 2009.
[CGV14] Scipio Cuccagna, Vladimir Georgiev, and Nicola Visciglia. Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p>3$ and with a potential. Comm. Pure Appl. Math., 67(6):957-981, 2014.
[CHVZ08] James Colliander, Justin Holmer, Monica Visan, and Xiaoyi Zhang. Global existence and scattering for rough solutions to generalized nonlinear Schrödinger equations on $\mathbb{R}$. Commun. Pure Appl. Anal., 7(3):467-489, 2008.
[CK02] Michael Christ and Alexander Kiselev. Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials. Geom. Funct. Anal., 12(6):1174-1234, 2002.
[CW92] Thierry Cazenave and Fred B. Weissler. Rapidly decaying solutions of the nonlinear Schrödinger equation. Comm. Math. Phys., 147(1):75-100, 1992.
[DF06] Piero D'Ancona and Luca Fanelli. $L^{p}$-boundedness of the wave operator for the one dimensional Schrödinger operator. Comm. Math. Phys., 268(2):415-438, 2006.
[DH09] Kiril Datchev and Justin Holmer. Fast soliton scattering by attractive delta impurities. Comm. Partial Differential Equations, 34(7-9):1074-1113, 2009.
[DHR08] Thomas Duyckaerts, Justin Holmer, and Svetlana Roudenko. Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. Math. Res. Lett., 15(6):1233-1250, 2008.
[DP11] Percy Deift and Jungwoon Park. Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data. Int. Math. Res. Not. IMRN, (24):5505-5624, 2011.
[FOO08] Reika Fukuizumi, Masahito Ohta, and Tohru Ozawa. Nonlinear Schrödinger equation with a point defect. Ann. Inst. H. Poincaré Anal. Non Linéaire, 25(5):837-845, 2008.
[Fos05] Damiano Foschi. Inhomogeneous Strichartz estimates. J. Hyperbolic Differ. Equ., 2(1):1-24, 2005.
[FXC11] DaoYuan Fang, Jian Xie, and Thierry Cazenave. Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. Sci. China Math., 54(10):2037-2062, 2011.
[GHW04] Roy H. Goodman, Philip J. Holmes, and Michael I. Weinstein. Strong NLS soliton-defect interactions. Phys. D, 192(3-4):215-248, 2004.
[GHW15] Pierre Germain, Zaher Hani, and Samuel Walsh. Nonlinear resonances with a potential: Multilinear estimates and an application to NLS. Int. Math. Res. Not., to appear, 2015.
[GS86] Bernard Gaveau and Lawrence S. Schulman. Explicit time-dependent Schrödinger propagators. J. Phys. A, 19(10):1833-1846, 1986.
[GS04] Michael Goldberg and Wilhelm Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. Comm. Math. Phys., 251(1):157-178, 2004.
[HMZ07a] Justin Holmer, Jeremy Marzuola, and Maciej Zworski. Fast soliton scattering by delta impurities. Comm. Math. Phys., 274(1):187-216, 2007.
[HMZ07b] Justin Holmer, Jeremy Marzuola, and Maciej Zworski. Soliton splitting by external delta potentials. J. Nonlinear Sci., 17(4):349-367, 2007.
[Hon14] Younghun Hong. Scattering for a nonlinear Schrödinger equation with a potential. arXiv:1403.3944, 2014.
[HZ07] Justin Holmer and Maciej Zworski. Slow soliton interaction with delta impurities. J. Mod. Dyn., 1(4):689-718, 2007.
[HZ09] Justin Holmer and Maciej Zworski. Breathing patterns in nonlinear relaxation. Nonlinearity, 22(6):1259-1301, 2009.
[KM06] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math., 166(3):645-675, 2006.
$\left[\mathrm{LCFF}^{+} 08\right]$ Stefan Le Coz, Reika Fukuizumi, Gadi Fibich, Baruch Ksherim, and Yonatan Sivan. Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential. Phys. D, 237(8):1103-1128, 2008.
[Nak99] Kenji Nakanishi. Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2. J. Funct. Anal., 169(1):201-225, 1999.
[PV09] Fabrice Planchon and Luis Vega. Bilinear virial identities and applications. Ann. Sci. Éc. Norm. Supér. (4), 42(2):261-290, 2009.
[Seg14] Jun-Ichi Segata. Final state problem for the cubic nonlinear Schrödinger equation with repulsive delta potential. arXiv:1402.5185, 2014.
[Wed99] Ricardo Weder. The $W_{k, p}$-continuity of the Schrödinger wave operators on the line. Comm. Math. Phys., 208(2):507-520, 1999.
[Yaj95] Kenji Yajima. The $W^{k, p}$-continuity of wave operators for Schrödinger operators. J. Math. Soc. Japan, 47(3):551-581, 1995.
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