

**Abstract** Extending the description of canonical rings from [Rei78] we show that every Gorenstein stable Godeaux surface with torsion of order at least 3 is smoothable.

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# Canonical rings of Gorenstein stable Godeaux surfaces

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## 1 Introduction

Surfaces of general type with the smallest possible invariants, namely  $K_X^2 = 1$  and  $p_g(X) = q(X) = 0$  are called (numerical) Godeaux surfaces, in honour of L. Godeaux who constructed the first such examples.

It is well known that their algebraic fundamental group, or equivalently, the torsion subgroup  $T(X) \subset \text{Pic}(X)$ , is cyclic of order at most 5. After his seminal work [Rei78], Reid was lead to the following.

**Conjecture (Reid)** *The fundamental group of a Godeaux surface is cyclic of order  $d \leq 5$  and for every  $d$  the moduli space of Godeaux surfaces with fundamental group  $\mathbb{Z}/d$  is irreducible and (close to) rational.*

*In particular, the Gieseker moduli space  $\mathfrak{M}_{1,1}$  has exactly five irreducible components.*

Indeed, he gave explicit descriptions of Godeaux surfaces with  $|T(X)| \geq 3$  in [Rei78]. The statement about the fundamental group is clear from the construction if  $|T(X)| \geq 4$ , the case of  $T(X) = \mathbb{Z}/3$  has been treated in [CU16].

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While quite a few examples of Godeaux surfaces with torsion  $\mathbb{Z}/2$  or trivial have been constructed since then (see e.g. [Bar84], [Bar85], [CD89], [Ino94], [Wer94], [DW99], [DW01], [RTU15] [Cou16]), the conjecture is still open. For instance, it is still possible that there is a Godeaux surface with trivial algebraic fundamental group, whose fundamental group is infinite.

The purpose of this article is to extend the results from [Rei78] to Gorenstein stable Godeaux surfaces.

**Theorem 1** *The moduli space of Gorenstein stable Godeaux surfaces with order of the torsion subgroup  $|T(X)| = 3, 4, 5$  is irreducible.*

*In particular, every Gorenstein stable Godeaux surface with  $|T(X)| = 3, 4, 5$  is smoothable.*

Stable surfaces are the analogue of stable curves in dimension two: they are the surfaces that appear in a modular compactification of the Gieseker moduli space of surfaces of general type (see [Ale06], [Kol12], [Kol16]). Indeed, this was one of the motivations for the introduction of this class of singularities by Kollár and Shepherd-Barron in [KSB88]. However, this compactification, called the moduli space of stable surfaces, can have many additional irreducible or connected components containing non-smoothable surfaces. This happens also for Gorenstein stable Godeaux surfaces [Rol16].

The method of [Rei78], studying the canonical ring of a maximal finite cover (via restriction to a canonical curve), generalises quite well to Gorenstein stable surfaces. The only obstacle is that we cannot assume the general member of a base-point free linear system to be smooth or irreducible, that is, we have to deal with slightly worse curves throughout. As an example of the flavor of arguments that need to be reconsidered carefully: a non-hyperelliptic smooth curve of genus 4 embeds canonically in  $\mathbb{P}^3$  as the complete intersection of a quadric and a cubic, while the image of the canonical map of the union of an elliptic curve and a non-hyperelliptic curve of genus 3 meeting in a node is the union of a plane quartic and a point not in that plane.

In a companion paper [FPR16] we will study more in detail examples of Gorenstein stable Godeaux surfaces and in particular prove that  $T(X)$  is a cyclic group of order at most 5 and it coincides with the fundamental group in the presence of worse than canonical singularities.

This article is part of series of papers, mostly joint with Rita Pardini, exploring Gorenstein stable surfaces with  $K_X^2 = 1$ . These are tamed by the fact that 1 is a very small number but every now and then some unexpected phenomena occur.

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## 2 Notations and conventions

We work exclusively with schemes of finite type over the complex numbers.

- A curve is a projective scheme of pure dimension one but not necessarily reduced or irreducible or connected.
- A surface is a reduced, connected projective scheme of pure dimension two but not necessarily irreducible.
- We will not distinguish between Cartier divisors and invertible sheaves.
- All schemes we consider will be Cohen-Macaulay and thus admit a dualising sheaf  $\omega_X$ . We call  $X$  Gorenstein if  $\omega_X$  is an invertible sheaf. A canonical divisor is a Weil-divisor  $K_X$  whose support does not contain any component of the non-normal locus and such that  $\mathcal{O}_X(K_X) \cong \omega_X$ .
- Given a an invertible sheaf  $L \in \text{Pic}(X)$ , one defines the ring of sections

$$R(X, L) = \bigoplus_{m \geq 0} H^0(mL);$$

for  $L = K_X$ , we have the *canonical ring*  $R(K_X) := R(X, K_X)$ .

### 2.1 stable surfaces

Since we work on stable surfaces we feel compelled to give a definition. However, the precise nature of the singularities of stable surfaces will only indirectly play a role in the sequel, namely when we use Riemann–Roch and Kodaira vanishing to compute the dimension of some spaces of sections. Our main reference is [Kol13, Sect. 5.1–5.3].

Let  $X$  be a demi-normal surface, that is,  $X$  satisfies  $S_2$  and at each point of codimension one  $X$  is either regular or has an ordinary double point. We denote by  $\pi: \bar{X} \rightarrow X$  the normalisation of  $X$ . The conductor ideal  $\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$  is an ideal sheaf in both  $\mathcal{O}_X$  and  $\mathcal{O}_{\bar{X}}$  and as such defines subschemes  $D \subset X$  and  $\bar{D} \subset \bar{X}$ , both reduced and of pure codimension 1; we often refer to  $D$  as the non-normal locus of  $X$ .

The demi-normal surface  $X$  is said to have *semi-log-canonical (slc)* singularities if it satisfies the following conditions:

1. The canonical divisor  $K_X$  is  $\mathbb{Q}$ -Cartier.
2. The pair  $(\bar{X}, \bar{D})$  has log-canonical (lc) singularities.

It is called a stable surface if in addition  $K_X$  is ample. We define the geometric genus of  $X$  to be  $p_g(X) = h^0(X, \omega_X) = h^2(X, \mathcal{O}_X)$  and the irregularity as  $q(X) = h^1(X, \omega_X) = h^1(X, \mathcal{O}_X)$ , so that we have  $\chi(X) := \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$ . A Gorenstein stable surface is a stable surface such that  $K_X$  is a Cartier divisor.

We will discuss surfaces in the following hierarchy of moduli spaces of surfaces with fixed invariants  $a = K_X^2$  and  $b = \chi(X)$ .

$$\begin{array}{rcl}
 \mathfrak{M}_{a,b} & = & \text{Gieseker moduli space of surfaces of general type} \\
 \downarrow & & \\
 \overline{\mathfrak{M}}_{a,b}^{(G)} & = & \text{moduli space of Gorenstein stable surfaces} \\
 \downarrow & & \\
 \overline{\mathfrak{M}}_{a,b} & = & \text{moduli space of stable surfaces}
 \end{array}$$

For the time being there is no self-contained reference for the existence of the moduli space of stable surfaces with fixed invariants as a projective scheme, and we will not use this explicitly. A major obstacle in the construction is that in the definition of the moduli functor one needs additional conditions beyond flatness to guarantee that invariants are constant in a family. For Gorenstein surfaces these problems do not play a role; we refer to [Kol12] and the forthcoming book [Kol16] for details.

### 3 Paracanonical curves on Gorenstein stable Godeaux surfaces

In analogy with the smooth case, we define a stable (numerical) Godeaux surface as a Gorenstein stable surface  $X$  with  $K_X^2 = 1$  and  $p_g(X) = q(X) = 0$ . This is the same as asking for  $K_X^2 = \chi(X) = 1$  by [FPR15, Prop. 4.2].

For a stable Godeaux surface the group  $T(X)$  of torsion invertible sheaves coincides with  $\text{Pic}^0(X)$  and is a finite group.

For a smooth Godeaux surface it is classically known that  $T(X)$  is cyclic of order at most 5 and in [FPR16] we show that no surprises occur for stable Godeaux surfaces.

**Proposition 2 ([FPR16, Cor. 4.2])** *Let  $X$  be a Gorenstein stable Godeaux surface. Then  $\pi_1^{\text{alg}}(X)$ , and hence  $T(X)$ , is a cyclic group of order at most 5.*

We will now analyse paracanonical curves on a Gorenstein stable Godeaux surface  $X$ , that is curves numerically equivalent to  $K_X$ .

**Lemma 3** *Let  $C_i$  be distinct Cartier divisors which are numerically equivalent to  $K_X$ . Then*

1.  $C_i$  is an irreducible Gorenstein curve of arithmetic genus 2,
2.  $C_1$  and  $C_2$  intersect with multiplicity 1 in a single point  $P(C_1, C_2)$ ,
3. if  $C_1 - C_2$  is not linearly equivalent to 0, then  $P(C_1, C_3) \neq P(C_2, C_3)$ .

*Proof* By assumption we have  $K_X C_i = K_X^2 = 1$ , thus  $C_i$  is irreducible as  $K_X$  is an ample Cartier divisor and Gorenstein. Since by adjunction  $K_{C_i} = (K_X + C_i)|_{C_i}$ , the first and the second item follow. For the last item consider

$$0 \rightarrow \mathcal{O}_X(C_1 - C_2 - C_3) \rightarrow \mathcal{O}_X(C_1 - C_2) \rightarrow \mathcal{O}_{C_3}(P(C_1, C_3) - P(C_2, C_3)) \rightarrow 0.$$

In the associated long exact sequence in cohomology  $H^0(\mathcal{O}_X(C_1 - C_2)) = 0$  by assumption and  $H^1(\mathcal{O}_X(C_1 - C_2 - C_3)) = 0$  by Kodaira vanishing (see e. g. [LR14, Cor. 19]). Thus  $H^0(\mathcal{O}_{C_3}(P(C_1, C_3) - P(C_2, C_3))) = 0$  and the last claim follows.  $\square$

The existence of a paracanonical curve is ensured as soon as  $T(X) \neq 0$ .

**Lemma 4** *Let  $L_1$  be a torsion invertible sheaf on  $X$ . Then*

1.  $h^0(m(K_X + L_1)) = 1 + \binom{m}{2}$  for  $m \geq 2$ .
2. If  $L_1$  is non-trivial then  $h^0(K_X + L_1) \geq 1$  and equality holds if there exists a different non-trivial torsion invertible sheaf  $L_2$  on  $X$ .

*Proof* The first item follows directly from the Riemann-Roch formula [LR16, Thm. 3.1] and Kodaira vanishing [LR14, Cor. 19]. In the second case  $h^2(K_X + L_i) = h^0(L_i) = 0$  since  $L_i$  is non-trivial thus  $h^0(K_X + L_1) = h^1(K_X + L_1) + 1 \geq 1$ . To show equality let  $C_2$  be a section of  $H^0(K_X + L_2)$ , which exists by Riemann-Roch and is irreducible of arithmetic genus 2 by Lemma 3. Thus the invertible sheaf of degree 1  $(K_X + L_1)|_{C_2}$  has at most 1 section and the restriction sequence

$$0 \rightarrow H^0(L_1 - L_2) \rightarrow H^0(K_X + L_1) \rightarrow H^0((K_X + L_1)|_{C'}) \rightarrow \dots$$

gives  $h^0(K_X + L_1) \leq 1$ . This concludes the proof.  $\square$

For later reference we compute the dimension of some cohomology spaces.

**Lemma 5** *Let  $X$  be a Gorenstein stable Godeaux surface with  $T(X)$  cyclic of order  $d \geq 3$ ,  $L$  be a generator for  $T(X)$ . Let  $C \in |K_X + L|$  be a paracanonical curve in  $X$  and  $M = K_X|_C$ . Then  $2M + L = K_C$  and*

$$h^0(C, M + iL) = \begin{cases} 0 & i = 0, 1 \\ 1 & i = 2, \dots, d-1 \end{cases}$$

$$h^0(C, 2M + iL) = \begin{cases} 1 & i \neq 1 \\ 2 & i = 1 \end{cases}$$

$$h^0(C, mM + iL) = m - 1 \quad m \geq 3$$

*Proof* Since  $C \in |K_X + L|$  is a Gorenstein curve of genus 2 everything, follows from the restriction sequence, the invariants of  $X$ , adjunction and Riemann-Roch on  $C$ .  $\square$

We will be studying graded section rings on curves. To control multiplication maps we need Castelnuovos base-point-free pencil trick and its variants.

**Proposition 6** *Let  $C$  be a reduced and connected Gorenstein curve and let  $F, H$  be invertible sheaves on  $C$ . Assume  $W \subseteq H^0(C, F)$  is a subspace of  $\dim = r + 1$  which defines a base point free system. Then*

1. If  $H^1(C, H \otimes F^{-1}) = 0$ , then the multiplication map

$$W \otimes H^0(C, H) \rightarrow H^0(C, F \otimes H)$$

*is surjective.*

2. If  $r = 1$ , i. e.,  $W$  is a base point free pencil, then

$$\ker\{W \otimes H^0(C, H) \rightarrow H^0(C, F \otimes H)\} \cong H^0(C, H \otimes F^{-1}).$$

*Proof* The first part follows from [Fra13, Prop. 25].

In the latter case we can repeat verbatim the proof of the base-point-free pencil trick given in [ACGH85, chap. III, §3, pag. 126].  $\square$

*Remark 7* In the following sections we study canonical rings of surfaces by restriction to canonical curves  $C$ . One step in the analysis is usually to show that the Gorenstein curve  $C$  behaves like a smooth curve of the same genus. The relevant notion in this context is numerical connectedness, which was defined in [CFHR99], and by [FT14] everything would follow if  $C$  is numerically 3-connected. Checking 3-connectedness turns out to be at least as intricate as a direct proof of the properties we need in each cases. Thus we decided to avoid this extra layer of complexity.

#### 4 Canonical ring of Gorenstein stable Godeaux surfaces with large torsion

In this section we adapt the algebraic treatment of numerical Godeaux surfaces with sufficiently large torsion by Miles Reid to the Gorenstein case. Many arguments carry over unchanged from [Rei78]. The idea is that, if  $\pi_1^{\text{alg}}$  is sufficiently big, then the universal cover is simple to describe algebraically. The main issue is that due to the presence of singularities we have to work on considerably more singular curves.

##### 4.1 The case $|T(X)| = 5$

**Theorem 8** *Consider the action of  $G = \mathbb{Z}/5$  on  $S = \mathbb{C}[x_1, \dots, x_4]$  given by  $x_i \mapsto \xi^i x_i$  where  $\xi$  is a primitive fifth root of unity.*

*Let  $X$  be a stable Gorenstein surface with  $\pi_1^{\text{alg}}(X) = \mathbb{Z}/5$  and  $Y$  the (algebraic) universal cover. Then  $Y$  is a quintic surface in  $\mathbb{P}^3$  defined by a  $G$ -invariant quintic polynomial  $q \in S$  and the canonical ring of  $X$  is*

$$R(X, K_X) = (S/(q))^G.$$

*Proof* The proof in [Rei78] works without modification.  $\square$

*Remark 9* Consider a quintic given by the orbit of a plane not meeting the fixed points of the above action, for example given by

$$q = (x_1 + x_2 + x_3 + x_4)(\xi x_1 + \xi^2 x_2 + \xi^3 x_3 + \xi^4 x_4)(\xi^2 x_1 + \xi^4 x_2 + \xi x_3 + \xi^3 x_4) \\ \cdot (\xi^3 x_1 + \xi x_2 + \xi^4 x_3 + \xi^2 x_4)(\xi^4 x_1 + \xi^3 x_2 + \xi^2 x_3 + \xi x_4).$$

The surface  $Y = \{q = 0\}$  is a normal crossing divisor in  $\mathbb{P}^3$ , in particular stable. It has exactly 10 triple points, thus the quotient by the free action is a stable surface with normalisation  $\mathbb{P}^2$  and two triple points. This is the surface  $X_{1,5}$  described in [FPR15, Sect. 4.2].

4.2 The case  $|T(X)| = 4$ 

By Proposition 2 in this case  $T(X) \cong \mathbb{Z}/4$ . This case can be described explicitly.

**Theorem 10** *Consider the polynomial ring  $S = \mathbb{C}[x_1, x_2, x_3, y_1, y_3]$  with  $\deg x_j = 1$  and  $\deg y_j = 2$ . An action of  $G = \mathbb{Z}/4$  on  $S$  is defined by  $x_j \mapsto i^j x_j$  and  $y_j \mapsto i^j y_j$ . Let  $X$  be a stable Gorenstein surface with  $T(X) = \mathbb{Z}/4$  and  $f: Y \rightarrow X$  be the associated cover. Then there are two  $G$ -invariant polynomials  $q_1, q_2 \in S$  of weighted degree 4 such that*

$$R(Y, K_Y) = S/(q_1, q_2) \text{ and } R(X, K_X) = R(Y, K_Y)^G.$$

*In other words,  $X$  is the free quotient by  $G$  of a weighted complete intersection of bidegree  $(4, 4)$  in  $\mathbb{P}(1, 1, 1, 2, 2)$ .*

*Proof* Fix a generator  $L$  of  $T(X)$  and let  $C$  be the paracanonical curve corresponding to a generator  $x_1 \in H^0(X, K_X + L)$ . We set  $M = K_X|_C$ . The preimage  $\tilde{C} = f^{-1}(C)$  is a canonical curve in  $Y$  and has arithmetic genus 5.

Let  $\tilde{M} = K_Y|_{\tilde{C}} = f^*M$ . We now compute the restriction of the canonical ring of  $Y$  to  $\tilde{C}$ , which via  $f_*$  can also be interpreted as a  $\mathbb{Z} \times \mathbb{Z}/4$  graded ring on  $C$ , that is,

$$R = R(Y, K_Y)|_{\tilde{C}} = R(\tilde{C}, \tilde{M}) = \bigoplus_m H^0(m\tilde{M}) = \bigoplus_m \bigoplus_{i=0}^3 H^0(C, mM + iL).$$

This decomposition is the weight-space decomposition with respect to the  $\mathbb{Z}/4$ -action on  $\tilde{C}$ . Note that we computed all the relevant dimensions in Lemma 5.

Let  $x_2$  be a generator of  $H^0(M + 2L)$  and  $x_3$  be a generator of  $H^0(M + 3L)$ . Then  $H^0(\tilde{M}) = \langle x_2, x_3 \rangle$  defines a base-point-free pencil on  $\tilde{C}$  by Lemma 3. We now choose  $y_1 \in H^0(2M + L)$  and  $y_3 \in H^0(2M + 3L)$  such that they span a complement of  $\text{im}\{S^2 H^0(\tilde{M}) \rightarrow H^0(2\tilde{M})\}$ .

We claim that  $x_2, x_3, y_1, y_3$  generate  $R$ . Indeed for every  $k \geq 2$  consider the multiplication map

$$\mu_{1,k}: H^0(\tilde{M}) \otimes H^0(k\tilde{M}) \rightarrow H^0((k+1)\tilde{M}).$$

It is surjective for  $k = 2$  since its Kernel is isomorphic to  $H^0(\tilde{M})$  by (ii) of Proposition 6, whilst  $h^0(2\tilde{M}) = g(\tilde{C}) = 5$  and  $h^0(3\tilde{M}) = 8$  by Lemma 5. For  $k \geq 4$  it is surjective by Proposition 6, item 2. as  $H^1((k-1)\tilde{M}) = 0$ . For  $k = 3$  the image has codimension 1, and it remains to show that there is no additional generator in degree 4. Considering a second multiplication map  $\mu_{2,2}: H^0(2\tilde{M}) \otimes H^0(2\tilde{M}) \rightarrow H^0(4\tilde{M})$  our claim follows if

$$\text{im}(\mu_{2,2}) + \text{im}(\mu_{1,3}) = H^0(4\tilde{M})$$

as vector spaces.

To prove this, we first decompose image and target of  $\mu_{1,3}$  into weight spaces to identify the culprit for non-surjectivity: it is the sequence

$$0 \rightarrow H^0(K_C) \xrightarrow{\begin{pmatrix} x_2 \\ -x_3 \end{pmatrix}} \begin{matrix} H^0(M+3L) \otimes H^0(3M+3L) \\ \oplus H^0(M+2L) \otimes H^0(3M) \end{matrix} \rightarrow H^0(2K_C) \rightarrow H^1(K_C) \rightarrow 0$$

where we used  $K_C = 2M + L$ .



Therefore it suffices to show that the image of  $\mu_{2,2}$  contains  $H^0(2K_C)$ . But since  $C$  is an integral Gorenstein curve of genus 2 the canonical linear system is also a base-point free pencil and the long exact sequence for

$$0 \rightarrow \mathcal{O}_C \rightarrow K_C \otimes H^0(K_C) \rightarrow 2K_C \rightarrow 0$$

shows that the multiplication map  $H^0(K_C) \otimes H^0(K_C) \rightarrow H^0(2K_C)$  is surjective.

Therefore  $R$  is generated in degree at most two and hence by  $x_2, x_3, y_1, y_3$ . Counting dimensions one checks that the kernel of  $\mathbb{C}[x_2, x_3, y_1, y_3] \twoheadrightarrow R$  is generated by two relations in weighted degree 4 (of weight 0 and 2).

Thus,  $R(Y, K_Y)|_{\tilde{C}} = R(\tilde{C}, \tilde{M})$  realises  $\tilde{C}$  as a complete intersection of degree  $(4, 4)$  in  $\mathbb{P}(1, 1, 2, 2)$  which implies that the canonical ring of  $Y$  realises  $Y$  as a complete intersection of degree  $(4, 4)$  in  $\mathbb{P}(1, 1, 1, 2, 2)$ .

The rest of the statements carries over verbatim from [Rei78, §2].  $\square$

#### 4.3 The case $|T(X)| = 3$

We now consider the following situation. Let  $X$  be a Gorenstein stable numerical Godeaux surface with  $T(X) = \mathbb{Z}/3$ , generated by  $L$ , and let  $C$  be the paracanonical curve corresponding to a generator  $x_1 \in H^0(X, K_X + L)$ . Let  $f: Y \rightarrow X$  be the associated étale cover of  $X$ . As in the previous case we set

$$\begin{aligned} M &= K_X|_C, \quad L = L|_C, \\ \tilde{C} &= f^{-1}(C) \in |K_Y|, \\ \tilde{M} &= f^*M = K_Y|_{\tilde{C}}. \end{aligned}$$

We will analyse the canonical ring of  $Y$  via restriction to  $\tilde{C}$ .

We now introduce some further notation. Let  $R = R(\tilde{C}, \tilde{M})$  be the graded ring of sections associated to  $\tilde{M}$ . Push-forward to  $C$  induces an additional  $\mathbb{Z}/3$  grading on this ring. Writing

$$R_i^m = H^0(C, mM + iL), \quad R^m = (R_0^m \oplus R_1^m \oplus R_2^m)$$

we have  $R = \bigoplus R^m$ , where we consider the  $R_i^m$  weight-spaces for an appropriately normalised  $\mathbb{Z}/3$ -action.

We can choose elements  $x_2, y_0, y_1, y_2 \in R$  such that

$$\begin{aligned} R_2^1 &= H^0(\mathcal{O}_C(P)) = \langle x_2 \rangle, \\ R_0^2 &= \langle y_0 \rangle, R_1^2 = \langle x_2^2, y_1 \rangle, R_2^2 = \langle y_2 \rangle, \\ (y_0 : y_1 : y_2)(\tilde{P}) &= (1 : 1 : 1). \end{aligned} \tag{1}$$

Note that we have made use of the fact that  $C$  is reduced to ensure that  $x_2^2 \neq 0$ . Since  $C$  is integral of genus 2 the sections of  $R_1^2 = H^0(K_C)$  define a base-point-free pencil on  $C$  and we get a polynomial subring  $\text{Sym}^* R_1^2 \subset R$ . Thus  $y_1$  cannot vanish at  $P$  and also  $y_0$  and  $y_2$  cannot vanish at  $P$  because  $R_1^1 = R_0^1 = 0$ ; thus we can scale the sections to satisfy the last condition.

Denoting the canonical map of  $\tilde{C}$  by

$$\varphi = \varphi_{K_{\tilde{C}}}: \tilde{C} \dashrightarrow \mathbb{P}(R^2)^\vee \cong \mathbb{P}^3$$

the action of  $\mathbb{Z}/3$  on the canonical model is induced by an action on  $\mathbb{P}^3$ .

**Proposition 11** *Let  $R(\tilde{C}, K_{\tilde{C}}) = R^{[2]} = \bigoplus_{m \geq 0} R^{2m}$  be the subring of even elements of  $R$ , which coincides with the canonical ring of  $\tilde{C}$  (with degrees multiplied by 2). Choosing appropriate elements satisfying the conditions (1) we have*

$$R^{[2]} \cong \mathbb{C}[x_2^2, y_0, y_1, y_2]/(f_2, h_0)$$

where

$$\begin{aligned} f_2 &= y_0 y_2 - y_1^2 + x_2^4, \\ h_0 &= y_0^3 - 2y_0 y_1 y_2 + y_2^3 + x_2^2 F_2, \\ &\text{with } F_2 = \alpha x_2^4 + \beta x_2^2 y_1 + \gamma y_0 y_2, \quad \alpha, \beta, \gamma \in \mathbb{C}. \end{aligned}$$

*In particular, the canonical map embeds  $\tilde{C}$  as a complete intersection of a cubic and a quadric in  $\mathbb{P}^3$ .*

*Proof* We first show that  $R^{[2]}$  is indeed generated by the elements of degree 2. First, let us consider  $R^4 = H^0(4M) \oplus H^0(4M+L) \oplus H^0(4M+2L)$ . We have

$$\begin{aligned} \langle x_2^2 y_2, y_1 y_2, y_0^2 \rangle &\subset H^0(4M) \\ \langle x_2^2 y_0, y_0 y_1, y_2^2 \rangle &\subset H^0(4M+L) \\ \langle x_2^4, y_1^2, x_2^2 y_1 \rangle &\subset H^0(4M+2L) \end{aligned}$$

and we claim that in each case there is no relation between the elements on the left hand side. This is clear for  $H^0(4M+2L) = \text{Sym}^2 H^0(2M+L)$ . Since  $H^0(2M+L)$  defines a base-point-free pencil every relation in  $\langle x_2^2 y_2, y_1 y_2, y_0^2 \rangle$  can be written as  $y_0^2 = y_1(\alpha x_2^2 + \beta y_1)$ . But  $\text{div} \frac{y_0}{y_1} \sim 2L$  is non-trivial, so the left hand side and the right hand side can never have the same zeros, a contradiction. The argument in the other case is the same. Then counting dimensions shows that we have equality in all three cases, that is,  $R^2 \otimes R^2 \rightarrow R^4$ .

Now let us consider  $R^6 = H^0(6M) \oplus H^0(6M+L) \oplus H^0(6M+2L)$ .

From the base-point-free pencil trick (Proposition 6) we get

$$\begin{aligned} H^0(2M+L) \otimes H^0(4M) &\rightarrow H^0(6M+L) \\ H^0(2M+L) \otimes H^0(4M+L) &\rightarrow H^0(6M+2L) \end{aligned}$$

since  $H^1(2M-L) = 0$  and  $H^1(2M) = 0$  because  $2M+L \cong K_C$ . To complete the remaining summand observe that

$$\ker\{H^0(2M+L) \otimes H^0(4M+2L) \rightarrow H^0(6M)\} \cong H^0(C, 2M+L) = H^0(C, K_C).$$

Hence the cokernel is one-dimensional since  $h^0(6M) = 5$ . By an argument similar to the above we see that  $(y_0)^3$  is not in the image of this multiplication map and thus

$$H^0(6M) = H^0(2M) \cdot H^0(4M) + H^0(2M+L) \cdot H^0(4M+2L)$$

is generated by products of elements in  $R^2$ .

For higher degrees Proposition 6 implies the surjectivity of

$$H^0(2M+L) \otimes H^0((2hM+iL) \rightarrow H^0((2h+2)M+(i+1)L)$$

for every  $h \geq 3$  and every  $i \in \mathbb{Z}/3$  because  $H^1((2h-2)M + (i-1)L) = 0$  for degree reasons.

Therefore  $R^2 \otimes R^{2h} \twoheadrightarrow R^{2h+2}$  for every  $h \geq 3$ , which shows that  $R^{[2]}$  is generated in degree 2. In particular, the surjection  $R^2 \cong \mathbb{C}[x_2^2, y_0, y_1, y_2] \rightarrow R^{[2]}$  defines the canonical embedding  $\varphi = \varphi_{K_{\tilde{C}}} : \tilde{C} \rightarrow \mathbb{P}^3$ ,  $\varphi(\tilde{C})$  a reduced curve of degree 6 and  $p_a(\tilde{C}) = 4$ , and  $\varphi(\tilde{P}) = (0 : 1 : 1 : 1)$  because of (1).

Counting dimensions we see that  $\varphi(\tilde{C})$  is contained in a unique quadric. Looking more closely at the basis for  $R^4$  given above we see that there is a relation  $f_2$  involving the missing element  $y_0 y_2 \in R_2^4$ . Noting that  $f_2(\varphi(\tilde{P})) = f_2(0 : 1 : 1 : 1) = 0$  we can arrange  $f_2$  to be of the form above by completing the square with  $y_1$  and rescaling  $x_2$ . This does not affect the choices made in (1).

As in the classical case,  $h^0(\mathbb{P}^3, \mathcal{I}_{\varphi(\tilde{C})}(3)) = 5$ , so we can find an irreducible cubic surface vanishing on  $\varphi(\tilde{C})$ , which together with  $f_2$  generates the ideal. More precisely, dimension counting shows that the relation  $h_0$  occurs among the monomials of degree 6 and weight 0. Since the relation cannot be contained in  $y_0 \cdot H^0(4M)$  or  $y_2 \cdot H^0(4M + L)$  modulo  $x_2$  it has to involve  $y_0^3, y_2^3$  and  $y_0 y_2 y_2$  with non-zero coefficients. Note that  $y_0 y_1 y_2 \equiv y_1^3 \pmod{x_2}$  so we can eliminate  $y_1^3$ . We rescale the equation such that  $h_0 \equiv y_0^3 - (a+1)y_0 y_1 y_2 + a y_2^3 \pmod{x_2^2}$ . On  $\tilde{C}$  the divisor of  $x_2^2$  is  $f^* 2P$ . Cutting the  $\varphi(\tilde{C})$  with the plane  $\{x_2^2 = 0\}$  thus gives three double points defined by  $(f_2, h_0) \pmod{x_2^2}$ . Restricting the cubic equation to the quadric and computing the derivative we see that this happens if and only if  $a = 1$  and thus  $h_2 \pmod{x_2^2}$  is of the claimed form.  $\square$

**Lemma 12** *Let  $S = \bigoplus_{m \geq 0} S^m$  be the subring of  $R$  generated by  $R^1$  and  $R^2$ .*

1. *With the choices made in Proposition 11 we have*

$$R^{[2]} \subset S \cong \mathbb{C}[x_0, y_0, y_1, y_2]/J \subset R,$$

where  $J = (f_2, h_0)$ .

2. *There exist elements  $z_1 \in R_1^3$  and  $z_2 \in R_2^3$  such that  $R^3 = S^3 \oplus \langle z_1, z_2 \rangle$  and the relations*

$$f_0 = x_2 z_1 + y_0^2 - y_1 y_2 = 0 \text{ and } f_1 = x_2 z_2 + y_0 y_1 - y_2^2 = 0$$

*hold in  $R^4$ . Moreover,  $z_1$  and  $z_2$  do not vanish at  $P$ .*

For convenient reference we collect generators and relations of  $R$  in small degree in Table 1.

*Proof* The first item is clear, because adding a square root of  $x_2^2$  in  $R^{[2]}$  cannot introduce new relations.

The existence of the  $z_i$  follows from comparing  $\dim R_i^3$  and  $\dim x_2 R_{i+1}^2$ , where we use again that  $x_0$  cannot be a zero-divisor. We prove the statement for  $z_1$ , the argument for  $z_2$  is the same.

By the first item we have  $x_2 z_1 \in S_0^4 = R_0^4$ , that is,  $0 \neq x_2 z_1 = a y_0^2 + b y_1 y_2 + c x_2^2 y_2$  for some coefficients  $a, b, c$ . Replacing  $z_1$  by  $z_1 - c x_2 y_2$  we may assume that  $c = 0$ . Since the relation has to vanish at  $\tilde{P}$  we also have  $a = -b$  and both have to be non-zero. Rescaling  $z_1$  we can choose  $a = -1$  as claimed.

For the final claim assume on the contrary that  $z_1$  vanishes at  $P$ . Then  $z_1/x_2 \in H^0(2M + 2L) = R_2^2$ , which is spanned by  $y_2$ . This is impossible, since  $x_2 y_2$  and  $z_1$  are linearly independent.  $\square$

**Table 1** Canonical ring of the  $\mathbb{Z}/3$ -cover of a Godeaux restricted to a canonical curve.

degree	weights	monomials in $S$	monomials not in $S$	relations in $J$	relations not in $J$
1	$R_0^1$				
	$R_1^1$				
	$R_2^1$	$x_2$			
2	$R_0^2$	$y_0$			
	$R_1^2$	$x_2^2, y_1$			
	$R_2^2$	$y_2$			
3	$R_0^3$	$x_2^3, x_2y_1$			
	$R_1^3$	$x_2y_2$	$z_1$		
	$R_2^3$	$x_2y_0$	$z_2$		
4	$R_0^4$	$x_2^2y_2, y_0^2, y_1y_2$	$x_2z_1$		$f_0 = x_2z_1 + y_0^2 - y_1y_2$
	$R_1^4$	$x_2^2y_0, y_0y_1, y_2^2$	$x_2z_2$		$f_1 = x_2z_2 + y_0y_1 - y_2^2$
	$R_2^4$	$x_2^4, x_2^2y_1, y_0y_2, y_1^2$		$f_2 = x_2^4 + y_0y_2 - y_1^2$	
5	$R_0^5$	$x_2^3y_0, x_2y_0y_1, x_2y_2^2$	$x_2^2z_2, y_1z_2, y_2z_1$		$x_2f_1, g_0 \equiv y_1z_2 - y_2z_1 \pmod{x_2}$
	$R_1^5$	$x_2^5, x_2^3y_1, x_2y_0y_2, x_2y_1^2$	$y_0z_1, y_2z_2$	$x_2f_2$	$g_1 \equiv y_0z_1 - y_2z_2 \pmod{x_2}$
	$R_2^5$	$x_2^3y_2, x_2y_0^2, x_2y_1y_2$	$x_2^2z_1, y_0z_2, y_1z_1$		$x_2f_0, g_2 \equiv y_0z_2 - y_1z_1 \pmod{x_2}$
6	$R_0^6$	$x_2^2 \cdot R_2^4,$ $y_0^3, y_0y_1y_2, y_1^3, y_2^3$	$\dots$	$x_2^2f_2, y_1f_2,$ $h_0 = y_0^3 - 2y_0y_1y_2 + y_2^3 + x_2^4F_2$	$y_0f_0, y_2f_1$ $x_2g_1, H_0$
	$R_1^6$	$x_2^2 \cdot R_0^4,$ $y_0^2y_1, y_0y_2^2, y_1^2y_2$	$\dots$	$y_2f_2$	$x_2^2f_0, y_0f_1, y_1f_0,$ $x_2g_2, H_1$
	$R_2^6$	$x_2^2 \cdot R_1^4,$ $y_0^2y_2, y_0y_1^2, y_1y_2^2$	$\dots$	$y_0f_2$	$x_2^2f_1, y_1f_1, y_2f_0,$ $x_2g_0, H_2$

We will now show, that  $R$  is generated in degree at most three and determine the relations in degree 5.

**Lemma 13** *The natural map  $\bar{R} = \mathbb{C}[x_2, y_0, y_1, y_2, z_1, z_2] \rightarrow R$  is surjective. Denoting its kernel by  $I$  the relations in degree 5 are*

$$I^5 = \langle x_2 f_1, g_0 \rangle \oplus \langle x_2 f_2, g_1 \rangle \oplus \langle x_2 f_0, g_2 \rangle$$

where the  $g_i$  can be chosen to be

$$\begin{aligned} g_0 &= y_1 z_2 - y_2 z_1 + x_2^3 y_0, \\ g_1 &= y_0 z_1 - y_2 z_2 - x_2^2 y_2, \\ g_2 &= y_0 z_1 - y_2 z_2 + y_2 x_2^3. \end{aligned}$$

*Proof* Proposition 6 implies that on  $\tilde{C}$  the multiplication map

$$H^0(m\tilde{M}) \otimes H^0(2\tilde{M}) \rightarrow H^0((m+2)\tilde{M})$$

is surjective for  $m \geq 7$ . Since we know that the subring of even elements is generated in degree 2 and have dealt with degree 3 in Lemma 12 we only need to show that there is no new generator in degree 5.

Let us discuss in detail the map

$$\bar{R}_0^5 = x_2 \langle x_2^2 y_0, y_0 y_1, y_2^2 \rangle + z_1 \langle y_2 \rangle + z_2 \langle x_2^2, y_1 \rangle \rightarrow R_0^5,$$

whose kernel contains the known relation  $x_2 f_1 = x_2^2 z_2 + x_2 (y_0 y_1 - y_2^2)$  and at least one other relation  $g_0$ , which cannot be contained in  $x_2 \langle x_2^2 y_0, y_0 y_1, y_2^2 \rangle$ . Thus  $g_0$  has to contain at least one of the monomials  $y_2 z_1$  or  $y_1 z_2$ . Since none of the  $y_i, z_i$  vanish at  $P$  while necessarily  $g_0(P) = 0$  we can normalise to get

$$g_0 \equiv y_1 z_2 - a y_2 z_1 \pmod{x_2} \quad \text{where } a = \frac{y_1 z_2}{y_2 z_1}(P) = \frac{z_2}{z_1}(P).$$

If  $r$  is any other non-zero relation in the kernel of the map, which is not a multiple of  $x_2 f_1$  then by the same argument it coincides with  $g_0$  modulo  $x_2$  (up to multiplication with scalars). Then  $g_0 - r$  gives a relation divisible by  $x_2$  which has to be a multiple of  $x_2 f_1$ . Hence there is no further relation and the map is surjective with kernel spanned by  $x_2 f_1$  and  $g_0$ .

The argument for the other weight spaces is analogous and gives  $g_0 = y_1 z_2 - a y_2 z_1 + \tilde{g}_0$ ,  $g_1 = y_0 z_1 - \frac{1}{a} y_2 z_2 + \tilde{g}_1$ , and  $g_2 = y_0 z_2 - a y_1 z_1 + \tilde{g}_2$  for some  $\tilde{g}_i$  divisible by  $x_2$ .

To determine  $a$  consider in  $\bar{R}_2^6$  the relation

$$x_2 g_0 - y_1 f_1 + a y_2 f_0 \equiv (a-1) y_1 y_2^2 + a y_0^2 y_2 - y_0 y_1^2 \pmod{x_2^2}. \quad (2)$$

which does not contain  $z_1, z_2$  modulo  $x_2^2$  and thus is in

$$J_2^6 = \langle y_0 f_2 = y_0^2 y_2 - y_0 y_1^2 + x_2^4 y_0 \rangle$$

modulo  $x_2^2$ . This is only possible if  $a = 1$ .

It remains to show that the polynomials  $\tilde{g}_i$  can be normalised as claimed. Substituting  $a = 1$  in (2) we see that the relation  $x_2 g_0 - y_0 f_2 - y_1 f_1 + y_2 f_0$  in  $\bar{R}_2^6$  is divisible by  $x_2^2$ . Since  $I_1^4 = \langle f_1 \rangle$  there exists an  $\alpha$  such that

$$x_2 g_0 - y_0 f_2 - y_1 f_1 + y_2 f_0 = \alpha x_2^2 f_1$$

Replacing  $g_0$  by  $g_0 - \alpha x_2 f_1$  gives

$$g_0 = y_1 z_2 - y_2 z_1 + x_2^3 y_0.$$

The same argument works for  $g_2$  looking at the relation

$$x_2 g_2 - y_0 f_1 + y_1 f_0 - y_2 f_2$$

and for  $g_1$  looking at the relation

$$x_2 g_1 - y_0 f_0 + y_2 f_1 + h_0.$$

This concludes the proof.  $\square$

*Remark 14* One can easily check that with these choices we have the following syzygies of degree 6:

$$\begin{aligned} x_2 g_0 - y_0 f_2 - y_1 f_1 + y_2 f_0 &= 0, \\ x_2 g_1 - y_0 f_0 + y_2 f_1 + h_0 &= 0, \\ x_2 g_2 - y_0 f_1 + y_1 f_0 - y_2 f_2 &= 0. \end{aligned}$$

**Lemma 15** *In  $R^6$  the quadratic monomials in  $z_1, z_2$  satisfy relations*

$$\begin{aligned} H_0 &= z_1 z_2 - x_2^2 y_0 y_2 + y_1 F_2 = 0, \\ H_1 &= z_2^2 - x_2^2 y_0^2 + y_2 F_2 = 0, \\ H_2 &= z_1^2 - x_2^2 y_2^2 + y_0 F_2 = 0 \end{aligned}$$

*Proof* Since  $R^6 = S^6$  we know that we can express quadratic polynomials in  $z_1$  and  $z_2$  as elements in  $S$ . To compute one such expression we use that  $x_2$  is a non-zero-divisor. Thus

$$\begin{aligned} x_2^2 z_1^2 &\equiv (-y_0^2 + y_1 y_2)^2 && \text{mod } f_0 \\ &= y_0(y_0^3 - 2y_0 y_1 y_2 + y_2^3) - y_2^2(y_0 y_2 - y_1^2) \\ &= y_0 h_0 - y_0 x_2^2 F_2 - y_2^2 f_2 + x_2^4 y_2^2 \\ &\equiv x_2^4 y_2^2 - x_2^2 y_0 F_2 && \text{mod } f_2, h \\ &\Rightarrow z_1^2 - x_2^2 y_2^2 + y_0 F_2 \in I_2^6. \end{aligned}$$

We repeat the calculation for  $z_2^2$ ,

$$\begin{aligned} x_2^2 z_2^2 &\equiv (-y_2^2 + y_0 y_1)^2 && \text{mod } f_1 \\ &= y_2(y_0^3 - 2y_0 y_1 y_2 + y_2^3) - y_0^2(y_0 y_2 - y_1^2) \\ &\equiv x_2^4 y_0^2 - x_2^2 y_2 F_2 && \text{mod } f_2, h \\ &\Rightarrow z_2^2 - x_2^2 y_0^2 + y_2 F_2 \in I_1^6, \end{aligned}$$

and for  $z_1 z_2$ ,

$$\begin{aligned} x_2^2 z_1 z_2 &\equiv (-y_0^2 + y_1 y_2)(-y_0 y_1 + y_2^2) && \text{mod } f_0, f_1 \\ &= y_0 y_2(y_1^2 - y_0 y_2) + y_1(y_0^3 - 2y_0 y_1 y_2 + y_2^3) \\ &\equiv x_2^4 y_0 y_2 - x_2^2 y_1 F_2 && \text{mod } f_2, h \\ &\Rightarrow z_1 z_2 - x_2^2 y_0 y_2 + y_1 F_2 \in I_0^6 \end{aligned}$$

$\square$

**Theorem 16** *Let  $X$  be a Gorenstein stable Godeaux surface with torsion group  $T(X) \cong \mathbb{Z}/3$  and let  $f: Y \rightarrow X$  be the corresponding triple cover. If  $\tilde{C} = f^{-1}C$  is the preimage of a paracanonical curve in  $X$  and  $\tilde{M} = K_Y|_{\tilde{C}}$  then*

$$R(\tilde{C}, \tilde{M}) = R = \mathbb{C}[x_2, y_0, y_1, y_2, z_1, z_2]/(f_0, f_1, f_2, g_0, g_1, g_2, h_0, H_0, H_1, H_2),$$

with generators and relations chosen as above.

Therefore the canonical ring of  $X$  can be described as in [Rei78] by lifting relations and syzygies to  $\mathbb{C}[x_1, x_2, y_0, y_1, y_2, z_1, z_2]$ .

A more conceptual approach to this ring is explained in [Rei15].

*Proof* By the above results there is a surjection from the ring on the right hand side onto  $R$  which is a bijection in even degrees and in degree up to 5.

Assume that some polynomial  $r(x_2, \dots, z_2)$  of odd degree at least 7 is zero in  $R$ . By Lemma 12 it cannot be contained in  $S$  and thus has to involve  $z_1$  or  $z_2$ . Using the relations in  $I$  we see that

$$r \equiv z_1 r_1(y_0, y_1, y_2) + r_2(x_2, y_0, y_1, y_2) \pmod{I}.$$

Reducing modulo  $z_1$  we see that  $r_2 \in J$ , so actually  $r \equiv r \equiv z_1 r_1(y_0, y_1, y_2) \pmod{I}$ . However,  $z_1$  is not a zero-divisor, thus  $r_1 \in I$  and consequently  $r \in I$  as claimed.  $\square$

#### 4.4 Remarks on the case $|T(X)| = 2$

The description of the universal cover of Godeaux surfaces with torsion  $\mathbb{Z}/2$  has been treated in [CD89] and from a slightly different point of view in [Cou16]. We did not attempt to extend this description to Gorenstein stable Godeaux surfaces but believe it should go through: In the smooth case, the following is the starting point of both constructions.

**Lemma 17** *Let  $f: Y \rightarrow X$  be the natural double cover of a Gorenstein stable Godeaux surface with torsion  $\mathbb{Z}/2$ . Then a canonical curve  $D$  of  $Y$  is honestly hyperelliptic of arithmetic genus 3.*

*Proof* The canonical curve is a double cover of a paracanonical curve  $C \subset X$ , which is irreducible of arithmetic genus 2 by Lemma 3. Thus the canonical pencil of  $C$  defines a polynomial subring of  $R(D, K_D)$  and since the image of the multiplication map  $H^0(K_D) \otimes H^0(K_D) \rightarrow H^0(2K_D)$  is contained in  $H^0(2K_C)$  we are done.  $\square$

### 5 Proof of Theorem 1

In this short section we quickly deduce Theorem 1 from the descriptions of the canonical rings in the previous section.

Let  $X$  be a Gorenstein stable Godeaux surface with torsion of order  $3 \leq d \leq 5$  and  $f: Y \rightarrow X$  the cover associated to  $T(X) \cong \mathbb{Z}/d$ . Then we have seen in the preceding section that the canonical ring of  $Y$ , including the action of  $\mathbb{Z}/d$ , is uniquely determined up to the choice of some parameters parametrised by an

open set in some projective space. We know from [Rei78] that the parameter space is non-empty in each case and thus the image of each of these families is a uni-rational irreducible component.

Since the general element corresponds to a smooth Godeaux surface and being smooth is an open condition every surface in the family is smoothable.

## 6 The canonical ring of a simply connected stable Godeaux surface

The canonical rings of simply-connected (stable) Godeaux surfaces remain elusive (see however [CP00]). When we consider degenerations, the constructions become much more explicit and thus actual computations are possible. In this section we give a description of the canonical ring of a simply connected Gorenstein stable Godeaux surface which however rather serves as an indication of the complexity of the problem than as a starting point for a general structure theory. The computation was carried out by the second author in discussion with Roberto Pignatelli.

Let  $a, b, c$  be homogeneous coordinates in the plane  $\bar{X} = \mathbb{P}^2$  and consider the plane quartic  $\bar{D}$  which is union of

$$\begin{aligned} C &= \{f = a^2 - 6ab + b^2 - c^2 = 0\}, \\ L &= \{a = 0\} = \text{Proj}(\mathbb{C}[b, c]), \\ L' &= \{b = 0\} = \text{Proj}(\mathbb{C}[a, c]). \end{aligned}$$

We define an involution  $\tau$  on the normalisation  $\bar{D}^\nu$  of  $\bar{D}$  that preserves the conic and interchanges the lines by its action on functions:

$$\begin{aligned} \text{on } C & & \tau^*(a, b, c) &= (-a, -b, c), \\ L \cong L' & & \tau^*(b, c) &= -\frac{1}{2}(a + c, 3a - c). \end{aligned}$$

Then as explained in [FPR16, Section 3.B, Case  $(P_2)$ ; Prop. 3.16] the triple  $(\bar{X}, \bar{D}, \tau)$  gives rise to a non-normal Gorenstein stable Godeaux surface  $X$ , by glueing  $\bar{D}$  to itself as prescribed by  $\tau$ .

Let  $\pi: \bar{X} \rightarrow X$  be the normalisation of  $X$ , and  $D \subset X$  the non-normal locus. Then  $\pi^{-1}(D) = \bar{D}$  and  $D$  consists of two irreducible components: a rational curve with one node, which is the image of the conic, and a rational curve with a triple point, which is the image of the lines. These two components meet transversally at the singular points, that is,  $D$  has a unique singular point which (analytically locally) looks like the coordinate axes in  $\mathbb{C}^5$ .

Our computation, which we will only sketch, is based on the following result of Kollár, which we state in a simplified version.

**Proposition 18** ([Kol13, Prop. 5.8]) *Let  $X$  be a Gorenstein stable surface. Define the different  $\Delta = \text{Diff}_{\bar{D}^\nu}(0)$  by the equality  $(K_{\bar{X}} + \bar{D})|_{\bar{D}} = K_{\bar{D}} + \Delta$ .*

*Then a section  $s \in H^0(\bar{X}, m(K_{\bar{X}} + \bar{D}))$  descends to a section in  $H^0(X, mK_X)$  if and only if the image of  $s$  in  $H^0(\bar{D}^\nu, m(K_{\bar{D}} + \Delta))$  under the Residue map is  $\tau$ -invariant if  $m$  is even respectively  $\tau$ -anti-invariant if  $m$  is odd.*



To do the actual computation we need to pick explicit generators of the respective bundles and compute the residue maps. We choose as generator of  $\omega_{\bar{X}}(\bar{D})$  the rational 2-form

$$\bar{\omega} = \frac{abc}{abf} \left( \frac{da \wedge db}{ab} + \frac{db \wedge dc}{bc} + \frac{dc \wedge da}{ca} \right) = \frac{c}{f} \left( \frac{da \wedge db}{ab} + \frac{db \wedge dc}{bc} + \frac{dc \wedge da}{ca} \right)$$

which gives an isomorphism  $R(\bar{X}, K_{\bar{X}} + \bar{D}) \cong \mathbb{C}[a, b, c]$ . As generators for  $\omega_L(\bar{D} - L)$ ,  $\omega_{L'}(\bar{D} - L')$ , respectively  $\omega_C(L + L')$  we choose the forms

$$\begin{aligned} \omega &= \text{Res}_L(\bar{\omega}) = \frac{c}{f} \left( \frac{db}{b} - \frac{dc}{c} \right) \\ \omega' &= \text{Res}_{L'}(\bar{\omega}) = \frac{c}{f} \left( -\frac{da}{a} + \frac{dc}{c} \right) \\ \eta &= \text{Res}_C(\bar{\omega}) \end{aligned}$$

Note that  $\tau^*\omega = \omega'$ , that is, with the choices made, the residue maps are compatible with the identifications  $R(L, \omega_L(C + L')) \cong \mathbb{C}[b, c]$  and  $R(L', \omega_{L'}(C + L)) \cong \mathbb{C}[a, c]$  and we have by Proposition 18

$$R(X, K_X) \cong \left\{ g \in \mathbb{C}[a, b, c] \mid \begin{array}{l} g \text{ is contained in } \mathbb{C}[a, b, c^2] \text{ modulo } f \\ g(a, 0, c) = g(0, 1/2(a+c), 1/2(3a+c)) \end{array} \right\},$$

where the apparent change of signs in the involutions is due to the fact that we take anti-invariant sections in odd degrees and invariant sections in even degrees. The generators of the resulting ring can be computed with a computer algebra system and we get:

**Proposition 19** *The canonical ring of  $X$  is generated as a subring of  $\mathbb{C}[a, b, c]$  by*

$$\begin{aligned} &ab, 3a^2 + 3b^2 + c^2, \\ &ab^2, a^2b, \\ &12b^3 - a^2c + 6abc - b^2c + 8ac^2 - 4bc^2 + c^3, \\ &12a^3 - a^2c + 6abc - b^2c - 4ac^2 + 8bc^2 + c^3, \\ &abc^2, ab^3, \\ &9a^3c - 45a^2bc - 45ab^2c + 9b^3c + 15a^2c^2 + 15b^2c^2 - 9ac^3 - 9bc^3 + c^4, \\ &27b^4 + 9a^2bc - 54ab^2c + 9b^3c + 30a^2c^2 + 3b^2c^2 - 9bc^3 + 2c^4, \\ &ab^2c^2, \\ &18a^2b^2c - 108ab^3c + 18b^4c + 39a^3c^2 + 33b^3c^2 - a^2c^3 \\ &\quad + 6abc^3 - 19b^2c^3 - 7ac^4 - bc^4 + c^5, \\ &3a^4c - 612ab^3c + 105b^4c + 233a^3c^2 + 199b^3c^2 - 9a^2c^3 \\ &\quad + 18abc^3 - 111b^2c^3 - 41ac^4 - 7bc^4 + 6c^5, \end{aligned}$$

and there are 54 relations in degrees  $(6^6, 7^{12}, 8^{18}, 9^{12}, 10^6)$ . In particular,  $X$  embeds canonically as a codimension 10 subvariety in  $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$ .

The numbers and degrees of generators and relations in this ring are the same as in the other example that was computed in [Rol16], which makes hope that a general structure theory is lurking in the background.

Note that by [LR14, Example 47] the unique degenerate cusp of  $X$  is a base point of the 2-canonical map, and neither the 3-canonical nor the 4-canonical map embed the non-normal locus of  $X$ .

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