The final publication is available at Springer via
https://link.springer.com/article/10.1007/s10649-013-9492-4

Discernment in dynamic geometry environments

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Abstract
In this paper, we discuss discernment of invariants in dynamic geometry environments (DGE) based on a combined perspective that puts together the lens of variation and the maintaining dragging strategy developed previously by the authors. We interpret and describe a model of discerning invariants in DGE through types of variation awareness and simultaneity, and sensorimotor perception leading to awareness of dragging control. In this model, level-1 invariants and level-2 invariants are distinguished. We discuss the connection between these two levels of invariants through the concept of path that can play an important role during explorations in DGE, leading from discernment of level-1 invariants to discernment of level-2 invariants. The emergence of a path and the usefulness of the model will be illustrated by analysing two students' DGE exploration episodes. We end the paper by discussing a possible pathway between the phenomenal world of DGE and the axiomatic world of Euclidean geometry by introducing a dragging exploration principle.

Keywords
Dynamic geometry, Discernment, Variation, Perception, Dragging control

Introduction

Dynamic geometry environments (DGE) provide an epistemic domain where movement and variation together with visual and sensorimotor feedback can guide the identification of geometrical properties of figures. Even though DGE are modelled after a theoretical system like Euclidean geometry, the dynamism that characterizes DGE phenomena opens new perspectives not only for geometry as a mathematical discipline but also for geometry education (Laborde, 2000; Strässer, 2001; Lopez-Real & Leung, 2006). In particular, dragging in DGE has been studied in pedagogical settings and has been gradually understood as a pedagogical tool that is conducive to mathematical reasoning, especially in the process of generating conjectures in geometry. The epistemic potential of the drag-mode in DGE lies in its relationship with the discernment of invariants.

Identifying invariants is a major activity in mathematical thinking. Invariants concern what remains the same when different aspects of a phenomenon vary, and a sensorial aspect of discerning invariants is to perceive them visually and to separate them out during variation. For those who have a keen mathematical sense, especially in the domain of geometry, this may take place through a mental simulation. For example, it is possible to discern the symmetry of a given geometrical object by mentally rotating it or reflecting it. DGE are hinged on visual variation, and through dragging, they introduce the user to a pseudo-reality that helps to facilitate visualization of such mental simulation.

visually *separating them out* while associated phenomena vary. For those who have a keen mathematical sense, especially in the domain of geometry, this may take place through a mental simulation. For example, it is possible to discern the symmetry of a given geometrical object by mentally rotating it or reflecting it. DGE are rooted in visual variation, especially through dragging. In such environments when a figure is constructed it will vary while keeping all the constructed properties unchanged together with all the properties that are their consequences according to Euclidean Geometry. This opens up an environment where geometrical properties (as invariant patterns) can be visually perceived and hence linked to the theoretical control that is behind them. However, discussion is still wide open on how to link the phenomenological world of DGE, experimental in nature, and the Euclidean world that is instead axiomatic and deductive. In other words, how to make meaningful sense of DGE "dragging phenomena"? Specifically, how can dragging strategies generate discernment of invariants that potentially correspond to geometrical properties? What kind of perception is involved when certain dragging modalities are used during explorations in DGE, and what are the basic means of discernment involved?

These are questions that contribute to the ongoing discussion on the epistemology and the related pedagogical potential of DGE. In this paper, we attempt to build a framework to address these questions, focussing on an idea of *invariant* as the fundamental object of discernment. The use of variation to interpret dragging in DGE, discussed in Leung (2008), and the role of sensorimotor perception to develop awareness of direct an indirect control, hinted at by Baccaglini-Frank and Mariotti (2010; Baccaglini-Frank, 2010, in press), will be used as basic means of discernment through which others can be constructed (this will be described and exemplified in the upcoming sequel of the present paper). These means of discernment allow the distinction of two levels of invariants. The relationship between these two levels of invariants is clarified by the concept of path, which we will illustrate by analyzing actual students' exploration episodes in DGE. We end the paper discussing an idea of time in DGE and we open a possible pathway between the phenomenology of DGE and the axiomatic world of Euclidean Geometry by introducing a Dragging Exploration Principle.

Perception and Discernment

In the introduction we have referred to *discerning* and *perceiving* invariants in DGE. Before introducing the elements of our framework we briefly clarify our choice of terminology. According to the Stanford Encyclopedia of philosophy, perception is: "the awareness or apprehension of things by sight, hearing, touch, smell and taste" (http://plato.stanford.edu/entries/perception-problem/). We focus on what seem to be three key ingredients of the notion of perception: "sensory information" and "awareness or apprehension". The relationship between perception and thinking is complex and addressed differently in the literature. For example, according to Alva Noë (2004)

> "In perception you "entertain" a judgeable content in the sense that the experience puts the question of whether the content holds into play. To have an experience is to be confronted with the possible way the world is. For this reason, the experiences themselves, although not judgment, are thoroughly *thoughtful*. Perception is a *way of thinking* about the world." (2004, p.189)

So "thinking" may also be considered an essential component of perceiving, and one could describe perception as

> *a thoughtful process in which sensory data is brought to awareness and organized, ready to be interpreted*.

In particular, Noë describes that it is through perception that it is possible for "the animal to explore the structure of the flow of sensory changes and to discern in this structure *invariant* properties of the environment." (ibid.p.21) This brings *discernment* into the picture.

According to the Oxford Dictionaries online discernment is "the ability to judge well". Usually discernment is used to describe a process that goes beyond perception and that can lead to more in-depth understanding of a phenomenon developed through detailed judgements. This makes discernment a *finalized perception*; that is, perception accompanied by intentionality towards perceiving *something in particular*, for example a structure underlying the experienced phenomenon. As stated by Bowden and Marton (1998), discernment goes hand in hand with *simultaneity* and *variation*.

> "To discern an aspect is to differentiate among the various aspects and focus on the one most relevant to the situation. Without variation there is no discernment. […] Learning in terms of changes in or widening in our ways of seeing the world can be understood in terms of discernment, simultaneity and variation." (p.7)

According to the Theory of Variation proposed by Marton (Marton et al, 2004) the notions of variation and simultaneity are central in describing when critical features of a phenomenon are observed under four patterns of variation: contrast, separation, generalization and fusion. Critical features are aspects of a phenomenon that, when put together, have the potential to describe, represent or even define the phenomenon. The four patterns of variation can be briefly described as follows:

> *Contrast*: "... in order to experience something, a person must experience something else to compare it with."
> *Generalisation*: "... in order to fully understand what ''three'' is, we must also experience varying appearances of "three",…"
> *Separation*: "In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant."

> *Fusion*: "If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously."
>
> (Marton et al., 2004, p.6)

The two forms of simultaneity considered in the theory are:

> *Synchronic Simultaneity* (spatial type): "[...] a way of seeing something as the discernment of various critical features of an instance simultaneously [...] [This] is the experience of different co-existing aspects of the same thing at the same time." (ibid. pp. 17–18).
>
> *Diachronic Simultaneity* (temporal type): "In order to experience variation in certain respect, we have to experience the different instances that vary in that respect simultaneously, i.e., we have to experience instances that we have encountered at different points in time, *at the same time*." (ibid. p. 17, italics in original).

Synchronic simultaneity happens in a real point of time while diachronic simultaneity occurs in a time sequence. Leung has used these patterns and simultaneity in what he has defined a "lens of variation" to discuss discernment in dragging explorations in DGE (Leung, 2003; 2008).

In the present paper we take inspiration from previous work, and elaborate a new perspective which is essentially cognitive, and thus distinct from phenomenography, the field where Marton's theory is situated. The building blocks of our framework combine elements both from Marton's Theory of Variation adapted to explorations in DGE as presented by Leung (2008), and from findings on sensorimotor perception from a recent study by Baccaglini-Frank and Mariotti (Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010; 2011). Our framework introduces *basic means* through which perception in DGE seems to be guided, paving the way for discernment. Since we aim to describe and analyze finalized perception (in our case the objective is the generation of conjectures) accompanied by intentionality, that is discernment as introduced above, we will refer to the means we introduce as *means of discernment*. The basic means we introduce are building blocks for more complex means of discernment which will be the focus of an upcoming paper.

Discernment during Dragging in DGE

Invariants and invariant relationships between invariants

Dragging in DGE consists in selecting an element of a dynamic figure (a figure constructed according to a set of properties within a DGE) with a pointing device (mouse, finger) and moving the pointing device with the result of "moving" the selected object (and possibly others) on the screen. Actually, such transformation of images on the screen is obtained by a production of a sequence of new images. Each image is reconstructed after the user's choice of a new position for the object selected. The high number of images in this sequence and the speed at which they are produced on the screen give a visual effect of continuity, analogous to what is seen in a movie.

The changes in the image on the screen are perceived in contrast to what simultaneously remains invariant, and this constitutes the base of the perception of “movement of the image” (Mariotti, 2010). The movement and the identification of invariants are what lies at the heart of activities that aim at exploiting the potential of DGE (for example, Laborde and Laborde 1995; Hölzl 1996; Healy and Hoyles 2001; Healy 2000; Arzarello et al. 2002; Olivero, 2002; Leung, 2008; Baccaglini-Frank and Mariotti, 2010). A dynamic figure may be considered, in Hölzl’s words, “the materialised representation of an ideal geometric figure only characterised by its internal relationships” (2001), and dragging can be considered as

> “a tool to find different representations of one and the same figure in continuous transition. Because dragging acts on a drawing with the effect being determined by the figure, a mediating function emerges.” (Hölzl, 2001)

Such mediating function of dragging has been suggested elsewhere in the literature (for example, Lopez-Real & Leung, 2006; Baccaglini-Frank & Mariotti, 2010). Interaction with dynamic figures through dragging in DGE

> “could induce a special type of reasoning (or explaining) in DGE in which a signified object in DGE could have a diachronic nature. That is, one has to conceptualize a draggable object in DGE as it varies (over time) under dragging. Hence, a whole object in DGE should be understood as a (continuous) sequence of the “same” object under variation.” (Leung & Or, 2007).

“Sameness” of a sequence of figures recognized as one is given by the perception of *invariants* that characterize each figure of the sequence. In general, and this is the case in DGE, like Cabri, the invariants are determined both, as mentioned above, by the geometrical relations defined by the commands used to construct the dynamic-figure, and by the relationship of dependence between the original relations of the construction and those that are derived as a consequence within the theory of Euclidean Geometry (Laborde & Strässer, 1990). All these invariants appear simultaneously as the dynamic-figure is acted upon through dragging. For instance, let us consider the following construction (see Mariotti & Baccaglini-Frank, 2011):

ABCD is a quadrilateral in which D is chosen on the parallel line to AB through C, and the perpendicular bisectors of AB and CD, r and s respectively, are constructed.

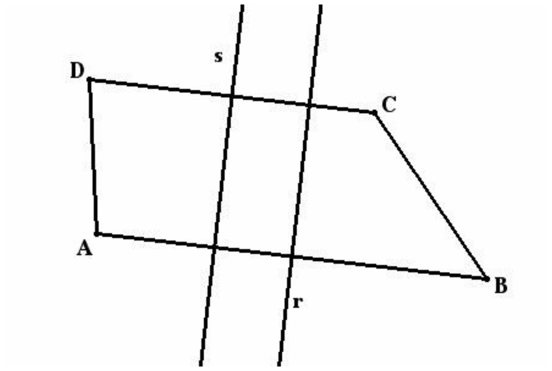


Figure 1: ABCD as a result of the construction described in the example above.

Figure 1 shows the dynamic figure derived from such construction. It has a set of *constructed* invariants (the parallelism between AB and DC, the perpendicularity of *s* to DC and of *r* to AB, and the passing of *s* and *r* through the midpoints of DC and AB respectively), and consequently all the invariants *derived* from these (for example the parallelism between *r* and *s*) which are all conserved simultaneously during dragging. The above example illustrates how, as they appear simultaneously during dragging, the different invariants have different status according to the type of control (direct or indirect) the dragger[1] will exercise on each of them. Our framework will show how different means of discernment contribute to reaching awareness of such distinction. There can be a relationship between invariants with different status. For instance, in a constructed figure there is a relationship of dependency between the constructed invariants and any that are derived from these according to Euclidean Geometry. It may happen that a *relationship between invariants* is an *invariant* itself, and such a relationship can be discerned through dragging. In our example there is an invariant relationship between invariants, that can be expressed by the following statement.

If two lines are respectively perpendicular to two parallel lines, then the first two lines are parallel.

Discerning invariants and discerning invariant properties between invariants are cognitively quite different tasks. We intend to speak extensively of both, and for this reason we introduce the following terminology.

[1] Since dragging is the fundamental activity we will be referring to within DGE, we will refer to the student, dragger, or in general person dragging, as *dragger*.

- *Level-1 invariants*: aspects of a dynamic figure, potentially corresponding to geometrical properties, that are perceived as constant during variation of the figure through dragging. For example, "AB parallel to CD" and "*s* parallel to *r*" are level-1 invariants of the dynamic figure constructed in Fig. 1.
- *Level-2 invariants*: invariant relationships between level-1 invariants. For example, "AB parallel to CD causes (or implies) *s* parallel to *r*".

Dragging Modalities

When dragging is used to explore a dynamic figure, it is possible to distinguish two broad categories based on (possibly implicit) goals of the dragger. These are in line with Hölzl's description of two principle drag modes (2001).

- *Dragging for testing*: it consists in dragging to *check the presence of desired (known) properties* in a dynamic figure. If, for example, a dragger has completed a construction and is unsure whether it has specific invariant properties, s/he can try to drag elements of the figure to check whether such properties are invariants. Dragging for testing "presupposes an expectation as to the reaction of the construction when it is being dragged" (ibid, p. 83). The movements on the screen can meet that expectation or not. The dragger will use the movement s/he induces on the figure to find the feedback (in terms of invariants) that s/he is looking for.

- *Dragging for searching/discovering*: it consists in dragging to look for *new* properties of the figure. In other words, "the changing appearance of the drawing must be evaluated under *aspects which are still unknown"* (ibid, p. 83). These may be possible configurations it might assume, invariants, and/or relationships between them. If, for example, the dragger's task is to formulate conjectures on the figure, s/he will be using this type of dragging to discover new properties through the perception of invariants and relationships between them.

Without lessening in any way the richness of dragging modalities proposed in the literature, the essential classification introduced above, based on the identification of the goal of the dragger, allows us to order and compare all the dragging modalities described in the literature, as long as they contain an identifiable goal of the dragger. For example, in the "dragging for testing" category, making the dragger's goals explicit, we would include:

- the *dragging test* as described by Arzarello, Olivero, Paola, and Robutti: moving dragable or semi-dragable points in order to see whether the drawing keeps the initial properties. If so, then the figure passes the test; if not, the drawing was not constructed according to the geometric properties you wanted it to have (Arzarello et al., 2002, p.67);
- *déplacer un point pour tester une construction* (Restrepo, 2008): drag a single point to see whether the drawing keeps the initial properties;

- *déplacer pour valider une construction* (Restrepo, 2008): drag all base-points[2] to see whether the drawing keeps the initial properties;
- the *soft dragging (conjecture) test* as described by Baccaglini-Frank & Mariotti (2010) that consists in "the dragging mode in which a base point is dragged with the intention of observing two (soft or robust) invariant properties occur simultaneously" (p. 231). This can be useful in the context of conjecture-generation tasks, because this dragging test is a way for a dragger to test a conjecture. It could be maintaining dragging when the object along which to drag the selected base point has become explicit.
- the *robust dragging test* (or an adaptation of *linked dragging* or *bound dragging* (Arzarello et al., 2002; Olivero, 2002) if this is done with the intention of performing a dragging conjecture test): redefining a point on an object and then dragging it along this object to check the simultaneity of two robust invariants.

While in the "dragging for searching/discovering" category we would consider modalities such as:

- *wandering dragging* (Arzarello et al., 2002; Olivero, 2002): dragging points randomly to look for invariants;
- *guided dragging* (Arzarello et al., 2002; Olivero, 2002) and *dragging to fit* (Lopez-Real & Leung, 2006): dragging base-points of a figure in order to obtain a particular configuration;
- *lieu muet dragging* (Arzarello et al., 2002; Olivero, 2002) or *maintaining dragging* (Baccaglini-Frank & Mariotti, 2010; Baccaglini-Frank, 2010) also similar to *déplacer un point pour identifier l'objet-trajectoire* (dragging a base-point to find a trajectory along which a certain property is maintained): dragging a base-point with the intention of maintaining some regularity of the figure (that may not directly involve the selected base-point);
- *line dragging* (Arzarello et al., 2002; Olivero, 2002): drawing new points along a line when a regularity of the figure is maintained, in order to determine geometrical properties of the line;
- *dragging with trace activated* (Baccaglini-Frank & Mariotti, 2010; Baccaglini-Frank, 2010): dragging points of a figure having activate the trace on one or more points in order to discover new properties;
- *bound dragging* (Arzarello et al., 2002; Olivero, 2002): dragging a point defined on an object if the dragger is looking for other regularities of the figure;
- *dragging to generate conjectures* (Mariotti & Baccaglini-Frank, 2011), in particular
    - *dragging to search for a premise of a (potential) conditional statement*,

[2] "Base-point" refers to a point upon which other elements of the dynamic figure depend.

- *dragging to search for a conclusion of a (potential) conditional statement.*

We do not intend the above to be a comprehensive list of all dragging modalities described in the literature, nor do we claim that all dragging modalities do fit such general classification. In fact they do not, because various dragging modalities have been described without an explicit goal. For example this is the case in Arzarello et al. (2002) since the classification was a posteriori and based on observations in which the goal was not always made explicit. Modalities like *photo-dragging*, *cinema-dragging* in Olivero, 2002, p.141, and *bound dragging* without explicit goal were observed.

When elaborating ideas and possibly conceiving new ones during a process of discernment within DGE, the feedback experienced plays a fundamental role. In addition to the feedback provided when a construction is accomplished, when an exploration of the dynamic figure starts, the modality of dragging the dragger chooses to use, whether for testing or for searching, influences the feedback s/he can perceive and discern. The main forms of perception that come into play are visual perception and sensorimotor perception. In the following sections we will introduce the basic means of discernment of our framework, applying them first to visual perception and showing how they come into play during discernment of level-1 invariants. We will then apply them then to sensorimotor perception, showing how they can come into play in discernment of level-2 invariants.

Basic Means of Discernment Applied to Visual Perception

In this section we introduce the basic means of discernment of our framework, applied to visual perception, for the discernment, in particular, of level-1 invariants. We are purposefully simplifying processes of discernment for clarity of the presentation.

To stress the intricate relationship between simultaneity and observable invariants, we further develop the patterns of variation along with types of simultaneous focus to form basic means of discernment of mathematical concepts in the context of DGE. We first associate our means of discernment to visual perception and describe how they can come into play during the discernment of level-1 invariants. We use the following exploration to introduce these basic means of discernment.

(***E***): *Explore how many circles can be constructed through any two given points and explain the construction.*

Discerning by contrast means to establish whether something satisfies a certain condition or not, that is, whether something "is" or "isn't", simultaneously focussing on distinguishing different and alike things. To fully comprehend a mathematical concept, one often resorts to finding non-examples in order to discern the critical features of the concept. Contrast is at the core of classifying to arrive at concepts. In Fig 3, a circle is constructed that passes through a free point (hence arbitrary) A with a draggable base-point C as its centre, B is another draggable base-point. C is dragged (while not moving A and B) to make the circle visually pass or not pass through B (guided dragging/drag-to-fit with direct control over C). In this case, contrast based on the diachronic simultaneous focus on the varying position of C, the circle, and B allows the dragger to discern that many circles can pass through two given points, hence *contrast via diachronic simultaneous focus* can be considered as a first means of discernment.

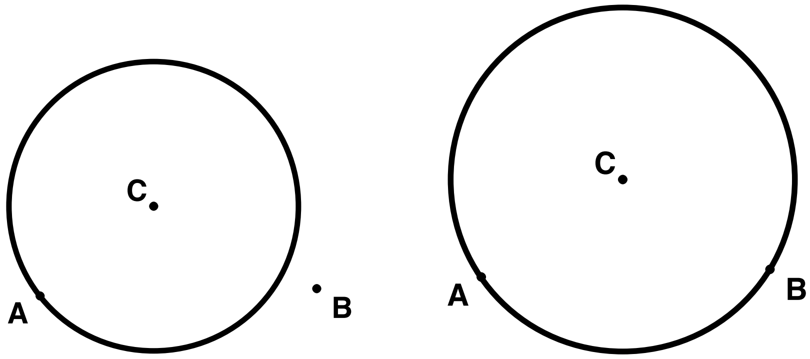


Fig. 3 Contrast via diachronic simultaneous focus: circle centred at C through B or not through B

Discerning by means of separation means *to bring to awareness* certain features as separate from others, thus making them become *critical* (or *critical features*) and potentially *features to be varied* (or *dimensions of variation*). A dimension of variation of a phenomenon is a feature of the phenomenon perceived as varying while some other aspects are being kept constant. Separation through simultaneous focus on the different appearances of certain features allows discernment of a dimension of variation and may lead to considering this dimension of variation as a potential invariant. Discernment through separation thus brings about exploration of the part-whole relationship within a phenomenon, pinpointing how to differentiate an invariant part from the whole. In Fig 4, the trace is activated for C and C is dragged while keeping point B visually on the circle (A, B are not dragged). This is a maintaining dragging modality with trace activated. It results in a visual *path* appearing on the screen. This path may provide a visualization of diachronic simultaneous focus because it is a unique simultaneous representation of a time sequence (different appearances in time) of positions that visually satisfied the desired condition. In this sense, the process of discernment can become cyclic and the path can bring to awareness a new critical feature, hence a new dimension of variation, and eventually a level-1 invariant for the exploration (***E***).

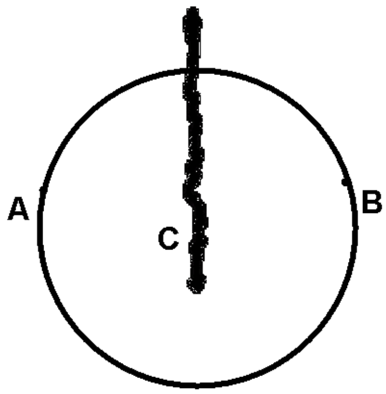


Fig. 4 Separation via awareness of critical features: a *path* is traced while maintaining a circle that passes through A and B.

Discerning by generalization means seeking to verify the invariance of a critical feature. It can be considered a means of checking the general validity of a separated out invariant. In exploration (***E***), the path separated out during maintaining dragging (Fig. 4) can be "re-generated" for different positions of A and B (Figure 5). Focussing on the re-appearance of different paths for different positions of A and B through contrast and diachronic simultaneous focus allows the discernment of an invariant property of all these paths leading to the generality of such invariant, which may also be described geometrically. In exploration (***E***) the invariant property is "C belongs to a line (the path)" and this "line" could be better described geometrically as "the perpendicular bisector of AB".

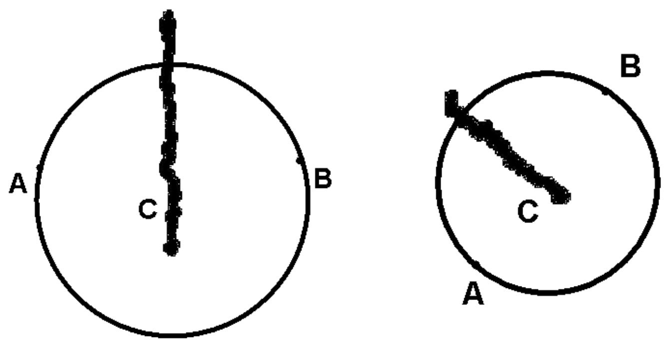


Fig. 5 Generalization: the repeated appearance of an invariant (path) for different positions of A and B

These means of discernment through variation and simultaneity enable the dragger to visually perceive level-1 invariants and the dragging experience thus produced prepares the dragger for the stage of discernment of level-2 invariants through sensorimotor perception.

Basic Means of Discernment
Applied to Sensorimotor Perception

A dynamic figure depends from its base points, and the figure's possible movements depend on the steps of the construction that induce corresponding invariant properties of the figure. This constitutes an essential aspect of the "being dynamic" of a dynamic figure. In the example depicted in Figure 1 (and as explained in Mariotti & Baccaglini-Frank, 2011), A, B, and C are base-points of the dynamic figure with two degrees of freedom. Therefore they can be dragged to any place on the screen while D can only be dragged along the parallel line to AB through C. Dependent elements of a construction, like the perpendicular bisectors in Figure 1, cannot be directly acted upon. The basic and constructed elements of a figure are determined by the steps of the construction, and their different status determines how the dynamic figure will behave during dragging (of any type). When trying to drag a dependent element, for example the perpendicular bisector *s* of CD, the dragger will perceive a mismatch between her clicking on the line and moving her hand (in the attempt to drag the line) and the system's "response": *s* will not move on the screen. It actually is possible to move *s* but *not* by acting on it *directly*. For example, dragging point C will make *s* move as a *consequence*.

Making sense of this kind of feedback, which is what we view as *sensorimotor perception*, is entirely up to the dragger who will need to interpret the "construction steps" as invariant geometrical properties, relate them to other invariants, discover new ones, and logically link the perceived geometrical properties and relationships to one another. Through sensorimotor perception the dragger can *distinguish between direct and indirect* movement of elements of a dynamic figure, and s/he may relate (or learn to relate) such feedback to the "construction steps" and hence elaborate a hierarchy among the elements of the dynamic figure. The analysis of the status of the different elements within such hierarchy can support the dragger in determining and checking properties of figures and relationships between them. However the dragger still is completely responsible for the non-trivial task of making sense of what s/he experiences and s/he may encounter various difficulties along the way. In fact this task of sense-making is neither simple nor spontaneous, and it may take a considerable amount of time and specific training (Restrepo, 2008) for the dragger to be able to conquer it (Baccaglini-Frank, 2010).

A great leap in complexity is constituted by becoming aware of the *hierarchy induced* not only on the elements but *on their properties* (that is relationships between elements) by the steps of the construction, and of the fact that such a hierarchy corresponds to logical relationships between the properties of the geometric figure defined by the construction. Thus not only can the dragger experience different types of control over elements, *direct and indirect control may also be exercised over invariants*. A fundamental means of discernment, in particular of level-2 invariants, that can be fostered by sensorimotor perception in DGE is awareness of direct and indirect control over invariants. We discuss such means of discernment in the context of inducing new invariants on a dynamic figure. In the example in Figure 1 that we have been analyzing it is possible to try to induce a new invariant like "coinciding perpendicular bisectors". In Healy's terms this is a *soft* property (2000). A dragger can also try to maintain such interesting property by dragging a base point. We refer to this type of dragging modality as *maintaining dragging* (MD), which we already mentioned in the previous section when we were discussing the variation means of separation. For the sake of this example, let's choose point C. Figure 6 shows how such property can become a *soft invariant* as a base point is dragged.

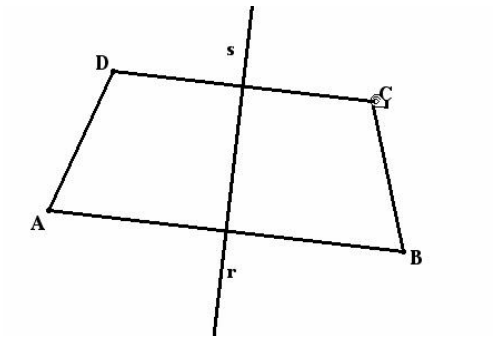


Figure 6: The figure shows a dragger trying to maintain the coincidence between the perpendicular bisectors, by dragging point C.

Of course the movement of C cannot be random as in the case of robust invariants (those in the steps of the construction and their consequences), but controlled. The *control* exercised over the movement of C is *direct* while that over the invariance of our desired property (*r* and *s* coincident) is *indirect*. Such awareness that can be originally developed through the experience of sensorimotor perception becomes fundamental when maintaining dragging is used as a means of exploration (Baccaglini-Frank, 2010; Mariotti & Baccaglini-Frank, 2011, Baccaglini-Frank & Mariotti, in press).

Going back to variation and simultaneity, sensorimotor perception plays a key role in discerning by fusion. Discerning by fusion means integrating critical features chosen as dimensions of variation into a whole during simultaneous variation. By contrasting different critical features fusion brings to awareness of a differentiated whole when its parts vary in interconnected ways. In exploration (***E***), sensorimotor perception via direct and indirect dragging control brings about fusion (a simultaneous awareness) that allows the integration and interrelation of two critical features: (1) the circle passing through B, and (2) C belonging to a path. The invariant property of all the paths discerned through generalization can be interpreted geometrically as "the perpendicular bisector of AB", while the integration and interrelation discerned through fusion can be expressed through the following conjecture:

Given two free points A and B, if a point C moves along the perpendicular bisector of AB, then the circle with C as a centre passes through A and B (Figure 7).

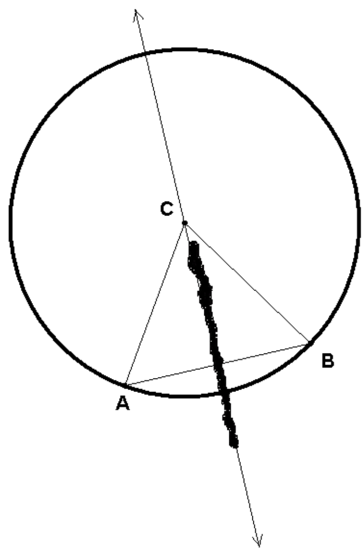


Fig. 7 Fusion: the perpendicular bisector of AB as an invariant relationship between two critical features.

Thus in this case fusion brings about the awareness of a relationship between two level-1 invariants (the two critical features), that is, a level-2 invariant. In exploration (***E***), the transition from a traced path to a geometrical path (a perpendicular bisector) holds the key to the formation of a conjecture. In the next section, we will develop a notion that we have only hinted at until now: the notion of *path*. In explorations in DGE we will show the path can play an important role, leading from discernment of level-1 to discernment of level-2 invariants, and we will use the framework we have introduced to analyze details of this process.

Discerning Paths

We have described sensorimotor perception, types of awareness of variation, and different types of simultaneous focus as means of discernment of invariants. We also described a more complex means of discernment in DGE that is awareness of direct and indirect control, which can lead to discernment of level-2 invariants through awareness of variation patterns. In the discernment of level-2 invariants a critical role can be played by the conception of a path. A path is a trajectory such that when a base point of a configuration is being dragged along it, the configuration will satisfy a certain prescribed condition. The notion was introduced by Baccaglini-Frank and Mariotti (2009b) and is consistent with Leung and Lopez-Real's (2002) notion of *locus of validity*. A path can be conceived as a visual record of a controlled variation, but at the same time it may be a record of simultaneous invariance: of the maintained invariant and of a new invariant causing it[3]. As the exploration proceeds the representation of the path (both mental and within DGE) passes through a sequence of steps that capture an evolution process that is representative of how the roles of various elements of our frame of discernment can play out. Firstly the path is envisaged (envisaged path), then a path is roughly traced (traced path), then a path is constructed and along such path a base point can be dragged (drag-along path), finally a generalized robust path is constructed (generalized robust path).

In this section we discuss the discernment of a *path* illustrated by students' productions collected during Baccaglini-Frank and Mariotti's study on MD (Baccaglini-Frank & Mariotti, 2009a, 2009b). For that study the draggers (who belonged to a 10th grade in northern Italy) had been preliminarily introduced to MD as a possible dragging modality. The exploration problem is:

> Construct three points A, B, and C on the screen, the line through A and B, and the line through A and C. Then construct the parallel line l to AC through B, and the perpendicular line to l through C. Call the point of intersection of these last two lines D. Consider the quadrilateral ABCD. Write conjectures on the kinds of quadrilaterals can it become, trying to describe all the ways it can become a particular kind of quadrilateral.

The problem asks to "write conjectures" and to do so "describing all the ways" in which certain configurations can occur. In a pilot study the authors had found that this kind of question seemed to foster students' choice of using MD to search for a condition under which a certain invariant is maintained. This kind of context has shown to be particularly favourable for emergence of a path. We will organize the analysis in terms of the developmental sequence of the four types of path mentioned above.

Envisaged Path: conception of a possible path

The students decided to use MD. In order to do this an invariant to induce intentionally (referred to as "Intentionally Induced Invariant" or III in Baccaglini-Frank & Mariotti, 2010) has to be selected, but identifying a possible soft invariant is not immediate. The bold letters indicate the student who was dragging.

1	Gio:	and when... do like maintaining dragging when it is a rectangle [Figure 8].
2	**Fab**:	Never... I mean one point and that's it.

[3] This is why a path can become the key element in a process of conjecture-generation. So analyzing how students discern paths and elaborate them can provide insights on their processes of conjecture-generation.

3 Gio: really? If you move... moving A... let's write moving ... [G starts to write the conjecture]

4 **Fab**: Moving A ...

5 Gio: Moving A... there is only one point... but are you sure, even going over there? Can't you go over there?

6 Gio: There... Already two...

7 **Fab**: two... [as F seems to find a new good position for A]

8 Gio: eh, no.

9 **Fab**: No, here...no it does funny things.

10 Gio: wait,... no that is the one from before.

11 **Fab**: Exactly. This is the one from before...

12 Gio: two...

13 **Fab**: two... I mean, one...[as he moves A back on these]

14 Gio: one...two...three, four...twenty thousand!

15 **Fab**: yes, there are really many of them [laughing]... let's do trace... we made a mistake. There are really too many.

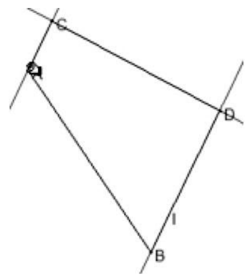
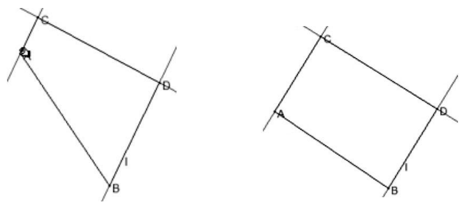


Figure 8

Randomly dragging point A let the students see instances of the property "ABCD rectangle". They began to locate possible positions where a rectangle would appear visually. However, a slight variation of a "good" position of A (contrast between "good" and "bad" positions) would instantly result in a non-rectangular configuration. This drag-sensitive visual perturbation initially prompted the students either to drag with a more controlled focus of attention (line 1) or to stay "safe" at a "good" position/zone (line 2). The students continued trying to explore the possibility of more "good" positions (a generalization) by dragging with an intention to make ABCD "look like" a rectangle. Specifically they observed two symmetric positions (lines 5 and 6) (separation) and repeated the pattern (lines 10 and 11). However, at the same time they observed "bad" positions where the quadrilateral became a triangle or a crossed figure (line 9). As students' "drag A to get a rectangle" skill gradually improved (actually, it was Fab's skill), they began to realize progressively from discrete visual images that "Yes, there are really many of them" (lines 14 and 15). We can see the emergence of a first level-1-invariant: the recognition of a geometrical property "rectangle" (line 1) as an induced soft invariant. The proposal to maintain the property "ABCD rectangle" indicates that such soft property has been potentially conceived as an III, but it is not until lines 14 and 15 that it actually achieves the status of an III, as the students succeed to generalize the "good points". The word "when" indicates use of contrast (rectangle as opposed to other things ABCD can become). Also "never" links the visual experience to a temporal experience. Seeing a second good point and recognizing it as such can be seen as use of contrast and diachronic simultaneity (they compare this position to the previous good one and seem to recognize it as if it were marked on the screen). Different types of simultaneous focuses were mobilized as the students consciously used MD. In particular the students seem to separate out good positions, which allowed them to experience diachronic simultaneity (the present position and passed positions). Hence here a path is in a stage of discernment of “good positions” being separated out. The following is a discernment table that presents a simultaneous overview of different aspects of an envisaged path.

			**DGE Phenomena** An envisaged path consists of possible “good” point positions for an observed soft invariant ABDC rectangle	
Dragging Modalities Used	**Lens of Variation**	**Type of Invariants**	**Type of Path**	**Conditional Link**
Wandering dragging with intention to search for instances of an interesting property	Contrast and generalization to separate out good positions for a potential soft invariant	Intentionally Induced level-1 soft invariant (III) Perceived Invariants: ABDC rectangle	Possible good positions for a perceived soft invariant	Certain positions seem to “cause” the appearance of a level-1 soft invariant

Traced Path

The students dragged A until they obtained a “nice rectangle” (Figure 9) and then activated the trace on A using MD with “ABCD rectangle” as their III. The focus of attention was on the shapes of the rectangle and of a traced path. Quite soon they guessed the traced path was a circle, thus perceiving the emergence of another level-1 invariant associated to the rectangle.

16 Gio: circle with...
17 **Fab**: no
18 Gio: eh, no.
19 **Fab**: look at C. C doesn't move.
20 Gio: I see a kind of circle with...
21 **Fab**: ... with radius CB, and center...
22 Gio: No, with diameter AD, I see.
23 **Fab**: Ah, wait I am...[as he goes “off track”]
24 Gio: I see it with diameter AD. like with diameter AD.
25 **Fab**: wait, no, let's...uhm...
26 Gio: with diameter CB instead, that... as a consequence...

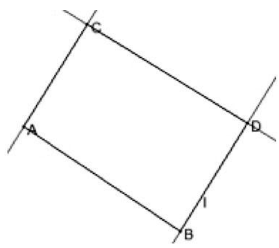


Figure 9

27 **Fab**: I would say that I made it very ugly, but... no, I would say... I would trace CB and its… [Figure 10]

28 **F&G**: midpoint

29 Gio: for the radius

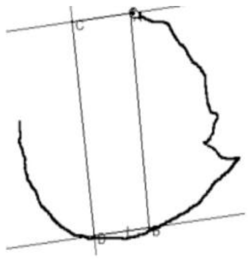
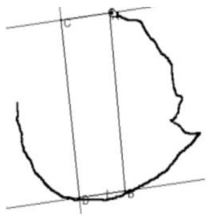


Figure 10

A was varying while B and C were fixed. D is dependent on A, B and C, and in particular, as A varies, D varies accordingly. C and B are independent points and were fixed in the students' exploration (i.e. the students didn't drag B and C so far). As A varied, D seemed to vary diametrically with respect to the traced "circle". In particular, A and D seemed to co-vary in a visually invariant pattern, i.e., the distance between A and D seemed to be approximately constant as ABDC was being kept "a rectangle". Gio "saw" AD as a diameter (lines 22 and 24) of the traced circle, in contrast with Fab's focus on the constancy of C. It seems that this circle was in the students' mind rather than on the screen. This was an interesting separation experience associated with MD, that is, Gio saw "a varying object as an invariant object under MD". This seemingly cognitive conflict reveals the "approximating" or "converging" nature of MD. However, as Fab traced out the guessed circle (even though it was a "very ugly" shape), the constancy of BC as a diameter became more and more convincing to him, possibly due to the following separation experience. When A's trace was turned on C's constancy was recognized (line 19) and it became a critical feature of this experience. The position of C was perceived as a strong invariant property. More specifically, as A was being dragged, with maintaining dragging and the trace activated on it, Fab and Gio saw the varying quadrilateral ABDC as a dynamic whole (synchronic simultaneity). Maintaining dragging to keep ABDC visually a rectangle resulted in a rotational motion about some fixed references (lines 20 and 21). "C does not move" (hence B also) was an obvious invariant property associated to the variation of A. C and B were more than just constructed free points; they were experienced as parts of a dynamic whole when the trace was activated. The students describe the path first as a circle with diameter AD, but then decide that such property would be a consequence if instead they defined the circle through its fixed diameter CB (line 26).

The students initially did not seem sure about how to handle the "unstable invariant - the varying AD as a diameter" and decided to consider BC as a diameter instead. Indeed, AD was also a diameter of the sought after circle. Students could have focused on AD to carry on the exploration, however; this would not have allowed them to construct an object that stayed fixed during the dragging of A. The students' words "that…as a consequence" referred to AD being the diameter defining the traced circle (line 26) indicate that in their minds the two properties ("BC diameter of the circle" and "AD diameter of the circle") do not have the same status. In particular, they seem to perceive a conditional link between the two properties leading to an intermediate level-2 invariant (circle with diameter BC implies circle with diameter AD). However the major level-2 invariant that starts to emerge during this episode is the relationship between dragging along the recognized circle and the invariance of ABDC rectangle.

In summary, the students concentrated on maintaining the III (ABDC rectangle) while focusing on the movement of the dragged base point A. This simultaneous focus allowed them to perceive a regularity, which they eventually interpreted as an Invariant Observed under Dragging (IOD) that "causes" the III to be visually verified. In particular, G started to separate out a traced path, that is, a mark on the screen traced out by the dragged-base-point when it is being dragged maintaining the III. The visual experience leading to a traced path prompted the students to think theoretically and propose a geometric description of the movement or of the path observed as a circle.

			**DGE Phenomena** The Trace is activated to produce a traced path under a maintaining dragging strategy, two level-1 invariants appear on screen	
Dragging Modalities Used	**Lens of Variation**	**Type of Invariants**	**Type of Path**	**Conditional Link**
Maintaining dragging and maintaining dragging with the Trace activated to produce a trace mark and propose a geometric description of such mark	Contrast and generalization driven by separation to visualize a traced path and to perceive a diameter Fusion: synchronic simultaneity of two level-1 invariants; a traced path as a diachronic simultaneity of rectangles	The emergence of a level-1-invariant (IOD): a traced path Transition from level-1 invariants to a level-2 invariant (dragging A along the path causes ACDB to be a rectangle)	Figure-specific traced path that represents a time sequence record of positions for A in order to obtain rectangles	Direct control of a level-1 invariant indirectly causing the appearance of another level-1 invariant and the the recognition of a potential level-2 invariant (A on the path causes ACDB to be a rectangle)

Drag-Along Path

Once they recognized that BC was a diameter for the circle that they sought after, the students constructed a *drag-along path* (a circle with BC as a diameter) in order to use a drag-to-fit strategy and perform a soft dragging test. Conceiving such a path corresponds to conceiving a definite trajectory along which a base point must be dragged. The fact that such a trajectory has been identified allows it to be geometrically constructed so that when a base point is being dragged along it (by eye), an III is visually verified.

1	Gio:	Circle, do circle.
2	Gio:	eh, let's choose...
3	**Fab**:	well, I would say B and C because they are the two points that don't move...here...yes, because actually now we take A.
4	Gio:	eh, we did it...cute! [Figure 11]
5	**Fab**:	yes, definitely.

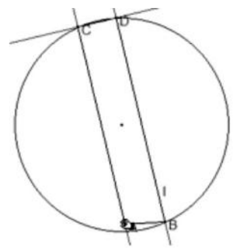
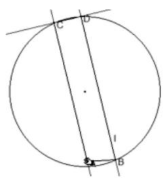


Figure 11

Here two level-1 invariants are experienced simultaneously, with a difference of control: direct over "maintaining the point on the circle" (IOD) and indirect over "obtaining that the figure remains a rectangle" (III). Such experience of simultaneity and control, elaborated through the function of *fusion*, becomes an experience of *synchronic simultaneity*, because the III and the IOD become coexisting features of a level-2-invariant which is a conditional link between the two level-1 invariants confirmed via the soft dragging test. That is, the IOD may be interpreted as a "condition under which" the III may be verified. Here is the discernment table for the soft path.

			**DGE Phenomena** The drag-along path is constructed geometrically and the appearance of the III is verified when the selected base point is dragged along it	
Dragging Modalities Used	**Lens of Variation**	**Type of Invariants**	**Type of Path**	**Conditional Link**
Verification of the conditional link through a drag-to-fit strategy, a soft dragging test.	Fusion: synchronic simultaneity during dragging along a constructed element	Soft level-2 invariant: invariant relation between level-1 invariants. The level-2 invariant is soft because the base point is not yet linked to the constructed element	Figure-specific drag-along path	Confirm a conditional link: direct induction of level-1 invariant (here A on the circle) causes the appearance of another level-1 invariant (ACDB rectangle)

Generalized Robust Path

The students linked the base point A to the circle they constructed (using the tool "redefinition of object"), performed a robust dragging test, and wrote the following conjecture: "If A belongs to the circle with diameter BC, ABCD is a rectangle." Once A was linked to the circle (the drag-along path), the circle became a *generalized robust path* (Figure 12). A generalized robust path is a geometrically constructed element to which the dragged base point is linked and on which the III and IOD are simultaneously and robustly synchronized, thus it can be seen as a *locus of invariance* and interpreted as a level-1 invariant. The newly defined level-1 invariant (in this case A on the circle) induces the III to also become a robust level-1 invariant. The process of emergence of the path, leading to a new construction, therefore constitutes in this last phase a level-2 invariant. Now students carried out a robust dragging test, through which they could experience fusion through "perfect" diachronic simultaneity. The two level-1 invariants are experienced as the fusing together of two critical features into a level-2 invariant. This experience finally became synchronic simultaneity in the final conjecture, expressed in "static" logical terms. Here is the discernment table for generalized robust path.

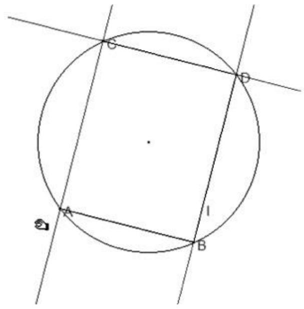
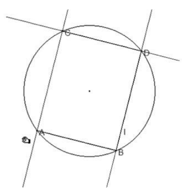


Figure 12

			**DGE Phenomena** A merging/redefining function is used to construct a robust III and IOD, resulting in a generalized robust configuration	
Dragging Modalities Used	**Lens of Variation**	**Type of Invariants**	**Type of Path**	**Conditional Link**
Robust dragging test to verify the conditional link	Generalization Fusion: perfect diachronic simultaneity which becomes synchronic simultaneity	Robust level-2-invariant: invariant relation between level-1 invariants	Figure-specific generalized robust path	Construct a robust level-1 invariant that causes the appearance of another robust level-1 invariant

In the analyses above we showed how the means of discernment we presented (types of variation awareness and awareness of direct/indirect control through sensorimotor perception) contributed to discerning something conceptually new, a *path*, through a process that can lead to discernment of level-2 invariants. In particular we described how such process evolves through four steps

Envisaged Path → Traced Path → Drag-Along Path → Generalized Robust Path

that capture an evolution process, representative of how the elements in the frame of discernment play their roles. Leung remarked that

> "the dragging tool in DGE makes possible the construction of transient signs that are not intrinsically stable but can point to critical features of the phenomenon at hand." (Leung, 2008, p.154)

The first three types of path are progressively revealing with respect to the pining down of a conditional link, and even a conjecture. The transient nature of these paths is a reflection of the transition of discernment from level-1 invariants to a level-2 invariant under discernment of sensorimotor perception. Furthermore, the transformation from a level-2 invariant to a level-1 invariant in the Generalized Robust Path step may serve as a pathway from the phenomenological world of DGE to the world of Euclidean Geometry.

Discussion

DGE are software designed to embody Euclidean Geometry in a dynamic and interactive way. Basically they are computer programs that can induce all the properties that are Euclidean consequences of the properties of construction. As Lopez-Real and Leung (2006) suggested, if dragging in DGE is accepted as a tool that can bring about structures and patterns, then "…we have new 'rules of the game', or even a new game, for experiencing geometry." (op cit, p.676)

In spite of the relationship between the realm of the software and the World of Euclidean Geometry – relationship that is intrinsic of the design of DGE (Laborde & Laborde, 1995) – a fruitful geometric exploration must take into account experiential aspects that do not have immediate conceptual counterparts in the World of Euclidean Geometry. Previous discussion highlighted the complexity of geometrical exploration as the complexity of discerning in the phenomenological realm of DGE and making sense of such discernment within the Theory of Euclidean Geometry. Through the specific framework that we developed we described explorations within DGE as discernment that can lead a solver to transform acting and perceiving to conceptual counterparts and to the fundamental theoretical aspects of the exploration.

Therefore, a basic assumption emerges from such analysis that expresses what is needed for changing the rules of the game: *exploring by dragging is a powerful tool supporting geometrical reasoning*. Such assumption usually remains implicit and almost unconscious for experts; however, it may become explicit, expressing in a clear way the relationship between the realm of DGE and the world of Euclidean Geometry. We will call it the *Dragging Exploration Principle* and we formulate it as follows:

> During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction entails in Euclidean Geometry.

This principle implicitly embraces variation, invariants (soft and robust) and sensorimotor perception and explicitly relates Geometrical figures and images in DGE. In other words it takes into account the issue of time that is not explicit in the logical world of Euclidean Geometry, but that is present in explorations in DGE and that plays an implicit role in discerning conditionality.

In this respect we conclude with some considerations that arise from the framework presented and that concern conditionality. In particular we introduce an idea of "time of a dynamic figure".

Time of a Dynamic Figure

Through our frame we analyzed the transition from a temporal sequence of phenomena in DGE to discerning conditionality in Euclidean Geometry, explaining how this can occur through the perception of causality given by awareness of direct/indirect control plus simultaneity. Such perception can lead to overcoming the temptation of stating the invariants discovered (for example the III and the IOD) in the order given by the temporal sequence (III *and then* IOD) and stating them instead as fused into a level-2 invariant (IOD *implies* III).

The analysis of the episode presented in the previous section shows how the establishment of conditional links seems to guide the transition from a dynamic to static view of the exploration. In producing a conditional statement with a premise and a conclusion (a conjecture) previous research refers to a transition from a dynamic exploration to its "crystallization" in a statement (Boero et al., 1999). The focus is on a "temporal section" of an exploration (Boero et al., 1999; Boero et al., 2007):

> "a time section in a dynamic exploration of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the condition A, hence "if A, then B".
>
> (Boero et al., 1999)

As far as dragging explorations are concerned, identifying invariants may determine a temporal section in the exploration process, characterized by sensorimotor perception that can induce discernment of conditionality. In particular, when discussing the conditional link that is developed between the IOD and the III, we introduced the hypothesis that the sensorimotor nature of the exploration may foster a feeling of direct control over one of the two invariants that occur simultaneously on the screen.

We suggested that the combination of simultaneity and a form of control (direct or indirect) over a certain invariant can help the dragger attribute to an invariant the status of "premise" or "conclusion" of her conjecture. This means that specific experiences in DGE may guide the solver's transition from dynamic to static interpretation of the experienced phenomena, moving from the phenomenological world of experience ordered through time to the crystallized world of formal axiomatic Euclidean geometry ordered by logics.

During an exploration, the properties of the figure that the dragger perceives, like any phenomena in the real world, are observed in the order imposed by time, we will denote this time as t_p. If the properties observed are, for example, an III and an IOD during maintaining dragging, according to t_p the III comes at a previous moment in t_p with respect to the IOD.

However, the dragger has control over the figure through the dragging of its base points, and she can add and take away properties, for example, by inducing or relaxing soft invariants, ad libitum. In this way she seems to travel back and forth again and again, imposing a *controllable time frame* on the dynamic figure, that we will denote *time of a dynamic figure* (t_f). We would like to think of t_f as the reversible and stoppable (with respect to t_p) time travelling back and forth with respect to the III and the IOD in which it is possible to modify the figure and associate to a certain "instant" of it a particular significant (for the dragger) configuration. Furthermore, if a dragger is aware of some property she is interested in, associated to a certain t_f, she may return to it with a new intention, for example of noticing other properties and relationships between these and the initial interesting property. This way she can transition to new t_fs (she can become aware of new level-1 and level-2 invariants) in a very brief interval of t_p. For example think of when Fab and Gio drag A with the trace activated and alternate their attention rapidly between the rectangle and what they discern to be a circle. As t_p advances, each student transitions back and forth between t_fs and all of a sudden overcomes the previous t_fs to conceive the figure as having a new potential level-2 invariant. Hence the time of a dynamic figure, t_f, is intrinsically linked to the dragger's intentionality, and therefore it interweaves such intentionality with t_p, providing a potential way of bridging the phenomenal world of DGE and the world of Euclidean Geometry. What before the advent of DGE could only be done as a mental experiment can now be *embodied* in dynamic figures.

According to these considerations, a dynamic figure possesses a controllable time frame t_f with respect to a real time sequence t_p of a dragging experience, and the discernment means that we have described guide the dragger's action in both time frames. These two time-related aspects can be seen as consequences of the dragging exploration principle, since the principle makes possible the appearance of a t_p phenomenon and serves as a rule to allow further t_f exploration to look for a cause for (or explanation of) the t_p phenomenon. In particular, the principle is behind the maintaining dragging modality that drives the path sequence that we discussed which converges to a conditional link (a conjecture) between an III and an IOD.

The Dragging Exploration Principle imbues DGE with an epistemic quality that is process-oriented and user-centred. In particular, the idea of time of a figure opens up a type of geometrical reasoning that could be distinct from deduction and induction, and possibly suggests a different type of pedagogical process. Such process should bestow on the teaching and learning of geometry an explorative and experimental nature that is complementary to deductive and inductive approaches. Equipped with this principle, learners can search, via dragging, for reasonable explanations that are consistent with the Euclidean axioms. In a sequel paper, we will discuss this type of reasoning using the means of discernment developed here to analyze more complex explorations in DGE.

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