Stochastic Shocks in a Two-Sector Solow Model

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Abstract

We study a stochastic, discrete-time, two-sector growth model á-la Solow (1956) characterized by perpetual growth. Assuming that exogenous i.i.d. shocks hit the physical production sector, we show that the capital dynamics can be converted, through an appropriate log-transformation, into an Iterated Function System converging to an invariant distribution supported on a Cantor set.

Keywords: Fractals, Iterated Function System, Cantor Set, Stochastic Growth
JEL classification: C61, O41

1 Introduction

Mandelbrot (1982) firstly presents the description of self-similar sets, namely sets that may be expressed as unions of rescaled copies of themselves. He called such sets as fractals, because their fractional dimensions exceeded their topological dimensions. The Cantor set is probably the most famous example of such sets. Hutchinson (1981), Barnsley and Demko (1985) and Barnsley (1989) show how systems of contractive maps with associated probabilities (iterated function systems, IFS), can be used to construct fractals, self-similar sets and measures supported on such sets. Such sets and measures are attractive fixed points of fractal transform operators. Applications of IFS theory in several fields have been widely developed, including economics.

Boldrin and Montrucchio (1986), more than twenty years ago, firstly show that complicated optimal dynamics can occur in deterministic concave intertemporal economic models. Their seminal work started a huge literature aiming at analyzing complexity and chaos in economics. Part of this literature focusses on stochastic concave intertemporal models, showing that the optimal trajectories of standard stochastic growth models are simply random processes converging to singular invariant distributions supported on some fractal sets (see Montrucchio and Privileggi, 1999; Mitra et al., 2004; Mitra and Privileggi, 2004, 2006; La Torre et al., 2011). Despite the evidence of these results, economists are still nowadays somehow reluctant to the idea that economic dynamics may generate fractals. More works are therefore needed in order to raise economists' awareness of the issue.

Iterated function systems allow to formalize the notion of self-similarity of any mathematical object. Hutchinson (1981) and Barnsley and Demko (1985) show how systems of contractive maps with associated probabilities can be used to construct self-similar sets and measures. We do not review here the theory of iterated function systems, but we simply show how this can be used in order to characterize the dynamics of a standard economic growth model; for whom can be interested in, a concise and comprehensive survey can be found in La Torre and Vrscay (2011). Next section describes the economic framework and the growth model we are interested in, and shows that the capital ratio can be converted, through an appropriate

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log-transformation, into an iterated function system converging to an invariant distribution supported on a Cantor set. Section 3 as usual concludes.

2 The Model

We study a discrete time growth model under uncertainty in which two-different sectors exist and uncertainty affects only one of them. In particular, we assume that physical capital, k, and another kind of capital, x, are combined in order to produce a unique final good (equivalent to physical capital) and another good (equivalent to the other kind of capital). Therefore, two different production processes use the same inputs in order to generate different outputs. Since we abstract for the sake of simplicity from agents' optimal behavior, no consumption and endogenous allocation of resources across sector is taken into account[1]. Given such assumptions, the accumulation of both types of capital equals investment n each of these sector: $k_{t+1} = i_k$ and $x_{t+1} = i_x$. We assume that exogenous saving rates, s_k and s_x (such that $0 < s_k, s_x, s_k + s_x < 1$), determine the investment in the physical and non-physical sector, respectively. This implies that the accumulation of physical and non-physical capital is simply equal to a share of the output of each sector: $k_{t+1} = s_k y_k$ and $x_{t+1} = s_x y_x$.

The production technology in each of the two sectors is Cobb-Douglas, as follow: $y_k = ak_t^{\alpha} x_t^{1-\alpha}$ and $y_x = bx_t^{\beta} k_t^{1-\beta}$, where a, b are scale parameters while $0 < \alpha, \beta < 1$ are efficiency parameters representing the intensity of the diminishing marginal returns in each sector. We assume that the final sector is affected by an exogenous shock, z_t, which affect multiplicatively its production function; it is assumed to be independent and identically distributed, and take on finite values: $z \in \{r, 1\}$, with $0 < r < 1$, as in Mitra et al. (2004). These two different shock configurations represent respectively the case in which physical production is negatively affected by some external factor (a financial crisis, a real business cycle or an environmentally related component can be thought to have direct effect on the output of the physical sector) and the case in which the economy evolves accordingly to its full capacity (no external factor lowers the production potential of the final sector). The economic dynamics can be therefore summarized by the following planar system of non-linear difference equations:

$$k_{t+1} = s_k a z_t k_t^{\alpha} x_t^{1-\alpha} \tag{1}$$
$$x_{t+1} = s_x b x_t^{\beta} k_t^{1-\beta}. \tag{2}$$

Equation (1) underlines the importance of physical capital in driving the evolution of modern economies in the long-run. The importance of this kind of capital (represented by machines, that is equipment and structures) is widely accepted in economics, since the seminal work of Solow (1956). Equation (2) says that also another type of capital plays a crucial role in determining economic dynamics in the long-run. This kind of capital can be thought of as any other factor identified by the economic literature as a relevant engine of growth (education, knowledge, culture), produced with a constant returns to scale (Cobb-Douglas) technology. For example, Rebelo (1991) assumes education (human capital) is produced by combining physical and human capital; La Torre and Marsiglio (2010) describe knowledge (technological capital) as obtained by processing of human capital and knowledge; Bucci and Segre (2011) claim the cultural capital is produced with human capital and culture. Our formulation of non-physical capital, x, allows us to encompass all such different views in a general framework.

It is straightforward to verify that the system (1) - (2) is not converging to any equilibrium, since both physical and non-physical capital grow without any bound. Therefore, in such a framework we define a

[1]Notice that introducing agents with optimizing behavior does not lead to much different results. In fact, if the utility function is logarithmic and the allocation of resources is determined as the share of input x to be allocated to each sector, optimal consumption will be a constant share of production and the share of input employed in each sector will be constant (see for example La Torre et a., 2011). In our framework, this would mean that s_k and s_x will be endogenously determined, and this will not affect the dynamic properties of our model economy.

steady state equilibrium as a situation in which some other variable is converging to a proper equilibrium. The natural candidate for this seems to be the ratio of the physical and non-physical capital, $v = \frac{k}{x}$. In fact, if this new variable converges to a finite value, this will mean that the original variables (k and x) are growing at the same rate; such a situation can be directly interpreted as a balanced growth equilibrium. Notice that capital ratio variable, v, evolves over time according to the following difference equation:

$$v_{t+1} \quad = \quad \mu z_t v_t^{\alpha+\beta-1} \equiv \phi(v_t), \tag{3}$$

where $\mu = \frac{s_k a}{s_x b}$. It is straightforward to see that, since $0 < \alpha, \beta < 1$, if $\alpha + \beta > 1$, the map $\phi(v_t)$ will be contractive, and therefore the random system (3) will converge to an invariant distribution. From now onward, we assume that such a condition holds. In order to show this in the simplest way, we now look for a transformation allowing us to deal with a linear system. Because of the specific form of (3), a logarithmic transformation may yield an equivalent conjugate random system which is linear in the transformed variable. The following proposition shows that a linear system conjugate to (3) exists defining a iterated function system that converges to an invariant distribution supported on a Cantor set.

Proposition 1 *Consider (3) and assume $\alpha + \beta \in (1, 2)$. Then the one-to-one logarithmic transformation $\{v_t\} \to \{\eta_t\}$ defined by:*

$$\eta_t = \rho_1 v_t + \rho_2 \tag{4}$$

with

$$\rho_1 = -\frac{2 - \alpha - \beta}{\ln r} \qquad \rho_2 = -\frac{(2 - \alpha - \beta)}{\alpha + \beta - 1}\left[1 + \frac{\ln \mu}{\ln r}\right]$$

defines a contractive linear iterated function system which is equivalent to the nonlinear dynamics in (3) and is composed of the two maps $f_0, f_1 : [0, 1] \to [0, 1]$ given by:

$$\begin{cases} f_0(\eta) = (\alpha + \beta - 1)\eta_t & \textit{with probability } p \\ f_1(\eta) = (\alpha + \beta - 1)\eta_t + 2 - \alpha - \beta & \textit{with probability } 1 - p \end{cases}$$

Such an iterated function system converges to an invariant distribution supported on a Cantor set.

Proof We look for a transformation of the form $\eta_t = \rho_1 v_t + \rho_2$ such that the dynamics of new variable will be linear as follows $\eta_{t+1} = (\alpha+\beta-1)\eta_t + \gamma_t$. This implies that $\rho_1 \ln v_{t+1} + \rho_2 = (\alpha+\beta-1)[\rho_1 \ln v_t + \rho_2] + \gamma_t$; by substituting (3) into this last equation yields: $\rho_1 \ln \mu - (\alpha + \beta - 1)\rho_2 = \gamma_t - \rho_1 \ln z_t$. Since the LHS is constant we can use the two values $\gamma_t = 0$ and $\gamma_t = 2 - \alpha - \beta$, corresponding respectively to $z_t = r$ and $z_t = 1$ to write $\rho_1 \ln \mu - (\alpha + \beta - 1)\rho_2 = 2 - \alpha - \beta = -\rho_1 \ln r$. From the second equation we determine ρ_1, while from the first one ρ_2. As $0 < \alpha + \beta - 1 < 1$, the IFSP (5) is a contraction mapping; hence, this is sufficient to show (see Hutchinson, 1981; and Barnsley et al., 2008) that the conjugate dynamics of the random system (3) describing the evolution of the capital ratio in our economy has a unique invariant distribution supported on a Cantor set to which the economy converges in the long run. ■

Rewriting the random system as

$$\eta_{t+1} = (\alpha + \beta - 1)\eta_t + \gamma_t,$$

it is straightforward to see that the two values, 0 and $2 - \alpha - \beta$, taken on by the (conjugate) random vector, γ_t, correspond respectively to the two values, r and 1, of the original random vector, z_t. Notice that the required condition, $1 < \alpha + \beta < 2$, is verified if the degree of the marginal returns in one of the two sector is not too small. Empirical estimates of α, representing the physical capital share in modern economies is around one third; this means that, in order for the condition to be met, we need to have a value of β higher than two third. Clear estimates of this parameter does not exist since they depends on the interpretation we want to use; however, the requirement $\beta \in (0.667, 1)$ does not seem to be too restrictive.

Notice that the result in Proposition 1 is very similar to that of Mitra et al. (2004). However, there is a big difference. Mitra et al. show that the capital converges to a unique invariant distribution supported on a Cantor set, and this is ensured by the fact that marginal returns in their one-sector economy are decreasing; this also implies that in the long-run (a part from the exogenous shocks) the economy is not growing at all. In our framework, what converges to an invariant distribution is the capital ratio, x, and again the presence of marginal returns both in the physical and non-physical sector ensured such a convergence; however, since this result is related to the capital ratio, in the long-run both capital stocks are growing, and this means that our result is able to represent also the case in which perpetual growth is verified. To the best of our knowledge, this is the first example of an endogenous growth model (namely a model generating sustained growth in the long-run) converging to some fractal set.

3 Conclusions

We study a simple framework of perpetual growth, namely a two-sector discrete time Solow model with physical and non-physical capital, showing that the capital ratio dynamic equation can be transformed into an iterated function system converging to an invariant distribution supported on a Cantor set. This is the first example illustrating how even a perpetual growing economy affected by random shocks can be read as an IFS, and this allows us to characterize directly its long-run properties. What can still be interesting to study is whether and under which parameter conditions the invariant distribution turns out to be singular with respect to Lebesgue measure. This exercise is left for future research.

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