ZERO-GENERIC INITIAL IDEALS

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Abstract. Given a homogeneous ideal I of a polynomial ring $A = \mathbb{K}[X_1, \dots, X_n]$ and a monomial order τ, we construct a new monomial ideal of A associated with I. We call it the zero-generic initial ideal of I with respect to τ and denote it with $\mathrm{gin}_0(I)$. When $\operatorname{char} \mathbb{K} = 0$, a zero-generic initial ideal is the usual generic initial ideal. We show that $\mathrm{gin}_0(I)$ is endowed with many interesting properties and, quite surprisingly, it also satisfies Green's Crystallization Principle, which is known to fail in positive characteristic. Thus, zero-generic initial ideals can be used as formal analogues of generic initial ideals computed in characteristic 0.

Introduction

After the founding paper of Galligo [Ga] and the results of [BaSt], generic initial ideals have become a central topic in Commutative Algebra. They are the subject of dedicated chapters in books and monographs, cf. [Ei], [HeHi2], [Gr], and of many research papers, cf. for instance [ArHeHi], [ChChPa], [Co], [CoSi], [Mu], [Mu2], [MuHi] and [MuPu], with topics ranging from Algebraic Geometry to Combinatorial and Computational Commutative Algebra. One of the main reasons why generic initial ideals have been studied so extensively in the literature after the work of Bayer and Stillman is that, when computed with respect to the reverse lexicographic order, they preserve many important invariants including the Castelnuovo-Mumford regularity. Furthermore several geometrical properties of projective varieties are encoded by generic initial ideals, especially when computed with respect to the lexicographic order, as shown in [Gr], [CoSi], [AhKwSo], [FlGr] and [FlSt].

Let $\mathbb{K}$ be any field, I a homogeneous ideal of the standard graded polynomial ring $A = \mathbb{K}[X_1, \dots, X_n]$, and τ a monomial order. The generic initial ideal of I with respect to τ is denoted by $\mathrm{gin}_\tau(I)$. When $\mathbb{K}$ is infinite, there exists a non-empty Zariski open set $U \subseteq \mathrm{GL}_n(\mathbb{K}) \subseteq \mathbb{K}^{n^2}$ of coordinates changes such that $\mathrm{gin}_\tau(I) = \mathrm{in}_\tau(gI)$ for all $g \in U$, [BaSt]. In particular, I and the monomial ideal $\mathrm{gin}_\tau(I)$ share the same Hilbert function. It is a consequence of a well-known upper semi-continuity argument that all graded Betti numbers and Hilbert functions of local cohomology modules do not decrease when passing from a homogeneous ideal to its initial ideal, cf. [Pa1], [Sb]. Therefore, also the Castelnuovo-Mumford regularity does not decrease when taking (generic) initial ideals.

The characteristic of the base field comes into play because generic initial ideals are Borel-fixed, and these have different combinatorial properties in characteristic zero and in positive

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characteristic; in the first case Borel-fixed ideals are strongly stable, when $\operatorname{char}\mathbb{K} = p$ they are p-Borel instead [Pa].

When $\operatorname{char}\mathbb{K} = p > 0$, some of the properties of $\operatorname{gin}_\tau(I)$ do not hold true, the combinatorics underlying its structure of p-Borel ideal becomes more intricate, see for instance [ArHe], [EnPfPo], [HePo], and Green's Crystallization Principle fails, see Example 2.6. Therefore, the common strategy of passing to the generic initial ideal of I does not work in positive characteristic that well.

Motivated by all of the above, we want to provide a tool endowed with the same properties as a generic initial ideal computed over a field of characteristic 0, which helps to overcome some of the extra difficulties one can encounter in positive characteristic.

Let $\mathbb{K}$ be any field; when we want to stress the dependence on the coefficients field $\mathbb{K}$, we shall write $A_\mathbb{K}$ instead of A. Whenever I is a monomial ideal of $A_\mathbb{K}$, we can assume that I is generated by monic monomials and let $I_{\mathbb{K}'}$ be the ideal generated by the image of these monomials in the ring $A_{\mathbb{K}'}$, where $\mathbb{K}'$ is any other field.

Our construction of zero-generic initial ideals is elementary: Let I be a homogeneous ideal of $A_\mathbb{K}$, and let τ be a monomial order. We define the *zero-generic initial ideal* of I with respect to τ to be the ideal $\operatorname{gin}_0(I) := (\operatorname{gin}_\tau((\operatorname{gin}_\tau(I))_\mathbb{Q}))_\mathbb{K}$ of $A_\mathbb{K}$. We denote by $\operatorname{Gin}_0(I)$ the zero-generic initial ideal of I with respect to the reverse lexicographic order. The reader accustomed to working with generic initial ideal immediately sees that $\operatorname{gin}_0(I)$ is invariant with respect to coordinates changes applied to I, it is Borel-fixed, it is strongly stable independently of the characteristic, it preserves the Hilbert function and if the characteristic is 0 it coincides with $\operatorname{gin}(I)$. In Proposition 2.2 we establish the main properties of $\operatorname{gin}_0(I)$. We prove that $\operatorname{gin}_0(I)$ satisfies the Crystallization Principle in Theorem 2.7. We also prove a lower bound for the Castelnuovo-Mumford regularity of general hyperplane sections in terms of restrictions of $\operatorname{Gin}_0(-)$ in Theorem 2.19. In the last section we show some applications of the results we obtained: a characteristic-free definition of symmetric algebraic shifting, a generalization of the main results of [CiLeMaRo] and an alternative proof of a well-known doubly exponential upper bound for the Castelnuovo-Mumford regularity of a homogeneous ideal in terms of its generating degree established in [CaSb].

1. Local cohomology and weakly stable ideals

In this section we develop some technical results on Hilbert functions of local cohomology modules with focus on a special class of monomial ideals called *weakly stable* and explain how the Hilbert functions of local cohomology modules of quotient rings defined by weakly stable monomial ideals are not affected by a change of the base field, see Theorem 1.3.

In the following $\mathfrak{m}_A$ will denote the graded maximal ideal of A.

Definition 1.1. A monomial ideal I of A is called *weakly stable* if $X_n, \dots, X_1$ form a filter-regular sequence for A/I.

Equivalently, a monomial ideal I of A is weakly stable if, for all monomials $u \in I$ and for all $j < m(u)$, where $m(u) = \max\{i \colon X_i | u\}$, there exists a positive integer k such that $X_j^k u / X_{m(u)}^l \in I$, where l is the largest integer such that $X_{m(u)}^l$ divides u. Furthermore, it is then easily seen that strongly stable, stable and p-Borel ideals are weakly stable ideals; in particular generic initial ideals are always weakly stable.

Remark 1.2. The definition of weakly stable ideals can be found for instance in [CaSb]. This class has also been introduced by means of equivalent definitions by other authors, see for instance that of *ideals of nested type* in [BeGi]. The name we use comes from the above exchange condition, which is weaker than those which define stable and strongly stable ideals. Another useful characterization of weakly stable ideals is that all of their associated primes are segments, i.e. of the form $(X_1, X_2, \ldots, X_i)$ for some i. We notice that $\mathfrak{m}_A$-primary ideals are weakly stable. The saturation $I : X_n^\infty$ of a weakly stable ideal I with respect to the last variable equals the saturation $I : \mathfrak{m}_A^\infty$ of I with respect to $\mathfrak{m}_A$ and the resulting ideal is again weakly stable. Finally, we observe that, if we let $A_{[j]} := \mathbb{K}[X_1, \ldots, X_j]$ and $I_{[j]}$ denote the ideal $I \cap A_{[j]}$ (so that $A_{[n]} = A$ and $I_{[n]} = I$), it descends immediately from the definition that, when I is weakly stable $I_{[j]}$ is weakly stable for all $j = 1, \ldots, n$.

Let M be a finitely generated graded A-module, let $\beta_{ij}^A(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i(M, \mathbb{K})_j$ be the graded Betti numbers of M and $H^i_{\mathfrak{m}_A}(M)$ the i^{th} (graded) local cohomology module of M with support in the graded maximal ideal $\mathfrak{m}_A$ of A.

In his Ph.D. Thesis [Pa], Pardue conjectured that the graded Betti numbers of p-Borel ideals would be independent of the characteristic of the ground field $\mathbb{K}$ or, in other words, that, for every p-stable ideal I of $A_{\mathbb{K}}$ one would have $\beta_{ij}^{A_{\mathbb{K}}}(I) = \beta_{ij}^{A_{\mathbb{Q}}}(I_{\mathbb{Q}})$ for all i, j, which is easily seen if $i = 0, 1$. Furthermore, Pardue was able to show that the Castelnuovo-Mumford regularity and projective dimension of p-Borel ideals are characteristic independent. Recently, this conjecture has been disproved in [CaKu], cf. also Remark 2.3. We prove below that the analogous statement for local cohomology holds indeed, even under the milder assumption that I is weakly stable.

Theorem 1.3. *Let $I \subseteq A$ be a weakly stable ideal. Then, for all i,* $\operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/I)\right) = \operatorname{Hilb}\left(H^i_{\mathfrak{m}_{A_{\mathbb{Q}}}}(A_{\mathbb{Q}}/I_{\mathbb{Q}})\right)$,

Before proving Theorem 1.3 we need first some technical results on weakly stable ideals.

Lemma 1.4. *Let I be a weakly stable ideal of $A_{[n]} = A$, with $n > 1$. Then,*

$$I_{[n-1]} : X_{n-1}^\infty = (I : X_n^\infty)_{[n-1]} : X_{n-1}^\infty.$$

Proof. First of all we notice that $I_{[n-1]}$ and $(I : X_n^\infty)_{[n-1]}$ are both weakly stable and that to prove the desired equality is equivalent to show that these two ideals agree in every sufficiently large degree d. This is easily seen, since I and $I : X_n^\infty = I : \mathfrak{m}_A{}^\infty$ agree in degree $d \gg 0$ and, therefore, their restrictions to $A_{[n-1]}$ agree as well for d sufficiently large. □

Lemma 1.5. *Let $I \subseteq A$ be a given weakly stable ideal. For $0 < i \leq n$, let $J := (I_{[n-i+1]} : X_{n-i+1}^\infty)_{[n-i]}$. Then, the following formula for the Hilbert function of the i-th local cohomology module of A/I holds:*

$$\operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/I)\right) = \operatorname{Hilb}\left(H^0_{\mathfrak{m}_{A_{[n-i]}}}(A_{[n-i]}/J)\right) \cdot \left(\sum_{j<0} t^j\right)^i. \tag{1.6}$$

Moreover, for every $0 < h \leq i \leq n$ *one has*

$$\mathrm{Hilb}\left(H^i_{\mathfrak{m}_A}(A/I)\right) = \mathrm{Hilb}\left(H^h_{\mathfrak{m}_{A_{[n-i+h]}}}(A_{[n-i+h]}/I_{[n-i+h]})\right) \cdot (\sum_{j<0} t^j)^{i-h}. \tag{1.7}$$

Proof. Since $i > 0$, one has that $H^i_{\mathfrak{m}_A}(A/I) \simeq H^i_{\mathfrak{m}_A}(A/(I : \mathfrak{m}_A^\infty))$. Also, $I : \mathfrak{m}_A^\infty = I : X_n^\infty = (I : X_n^\infty)_{[n-1]}A$ and, thus, by [Sb2] or [CaSb1] (3.8), (3.9), we have

$$\mathrm{Hilb}(H^i_{\mathfrak{m}_A}(A/I)) = \mathrm{Hilb}(H^{i-1}_{\mathfrak{m}_{A_{[n-1]}}}(A_{[n-1]}/(I : X_n^\infty)_{[n-1]})) \cdot (\sum_{j<0} t^j), \tag{1.8}$$

which is formula (1.6) when $i = 1$. The other cases of (1.6) follow by inducting on the cohomological index, considering the ideal $(I : X_n^\infty)_{[n-1]}$ and using Lemma 1.4.

When $i = h$, (1.7) is trivial. By using (1.8) and the same inductive argument as before, we see that, for $0 < h < i$

$$\mathrm{Hilb}(H^i_{\mathfrak{m}_A}(A/I)) = \mathrm{Hilb}(H^h_{\mathfrak{m}_{A_{[n-i+h]}}}(A_{[n-i+h]}/(I_{[n-i+h+1]} : X^\infty_{n-i+h+1})_{[n-i+h]})) \cdot (\sum_{j<0} t^j)^{i-h}.$$

The conclusion follows by a repeated use of Lemma (1.4), which implies that $I_{[n-i+h]}$ and $(I_{[n-i+h+1]} : X^\infty_{n-i+h+1})_{[n-i+h]}$ have the same saturation. □

Proof of Theorem 1.3. If $i = 0$ the conclusion is clear, since $H^0_{\mathfrak{m}_A}(A/I)) = (I : \mathfrak{m}_A^\infty)/I$ which, by hypothesis, is just $(I : X_n^\infty)/I$. When $i > 0$, use Lemma 1.5 (1.6) and the conclusion follows from the previous case. □

By [HeSb], we know that strongly stable and p-stable ideals are sequentially Cohen-Macaulay. It is easy to generalize the statement to weakly stable ideals.

Proposition 1.9. *Let* $I \subset A$ *be a weakly stable ideal. Then* A/I *is sequentially CM.*

We recall now the definition of extremal Betti numbers and corners of the Betti diagram. Following [BaChPo], we call a non-zero Betti number $\beta^A_{ij}(M)$ such that $\beta^A_{rs}(M) = 0$ whenever $r \geq i$, $s \geq j+1$ and $s - r \geq j - i$ an *extremal Betti number of* M; moreover, we call a pair of indexes $(i, j-i)$ such that $\beta^A_{ij}(M)$ is extremal a *corner of* M. One can see that the extremal Betti numbers of A/I can be computed directly from the local cohomology modules of A/I, since, by [Tr] or again by [BaChPo], for any finitely generated graded A-module M

$$\beta^A_{ij}(M) = \mathrm{Hilb}\left(H^{n-i}_{\mathfrak{m}_A}(M)\right)_{j-n}, \tag{1.10}$$

when $(i, j-i)$ is a corner of M. The following results are yielded by Theorem 1.3.

Corollary 1.11. *The extremal Betti numbers of a weakly stable ideal and, thus, of p-Borel ideals do not depend on the base field.*

Corollary 1.12. *The Castelnuovo-Mumford regularity and the projective dimension of a weakly stable ideal do not depend on the base field.*

2. Properties of $\operatorname{gin}_0(-)$ and $\operatorname{Gin}_0(-)$.

This section is entirely dedicated to define and prove a list of properties of the zero-generic initial ideal.

Definition 2.1. Let I be a homogeneous ideal of $A_{\mathbb{K}}$, and let τ be a monomial order. We define the *zero-generic initial ideal* of I with respect to τ to be the ideal of $A_{\mathbb{K}}$

$$\operatorname{gin}_0(I) := \operatorname{gin}_\tau(\operatorname{gin}_\tau(I)_{\mathbb{Q}})_{\mathbb{K}}.$$

We use the notation $\operatorname{Gin}(I)$ and $\operatorname{Gin}_0(I)$ when τ is the reverse lexicographic order.

Remark. Let τ be a monomial order on the set of (monic) mononials of $A_{\mathbb{K}}$; we recall the classical definition of generic initial ideal with respect to τ and some related basic facts. First, consider a matrix of indeterminates $\mathbf{y} = (y_{ij})_{1\le i,j\le n}$ and the extension field $\mathbb{K}(\mathbf{y})$ of $\mathbb{K}$. Let γ be the $\mathbb{K}$-algebra homomorphism $\gamma : A_{\mathbb{K}} \longrightarrow A_{\mathbb{K}(\mathbf{y})}$ defined by the assignment $X_i \mapsto \sum_{j=1}^n y_{ij}X_j$ for all $i = 1,\dots,n$ and extended by linearity. Given a homogeneous ideal I of $A_{\mathbb{K}}$, we can compute the ideal $\gamma I \subseteq A_{\mathbb{K}(\mathbf{y})}$ and its initial ideal with respect to τ, obtaining a monomial ideal J of $A_{\mathbb{K}(\mathbf{y})}$. In this way, one defines the usual generic initial $\operatorname{gin}_\tau(I)$ of I to be $J_{\mathbb{K}}$. Observe that $\mathbb{K}$ is not required to be infinite.

Recall that by [Co], one has $\operatorname{gin}_\tau(I) = \operatorname{gin}_\tau(\operatorname{gin}_\tau(I))$, and thus $\operatorname{gin}_0(I) = \operatorname{gin}_0(\operatorname{gin}_\tau(I))$ for any monomial order τ.

Proposition 2.2. *Let I be a homogeneous ideal of $A = \mathbb{K}[X_1,\dots,X_n]$.*

(i) The ideal $\operatorname{gin}_0(I)$ is a strongly stable ideal of A; I and $\operatorname{gin}_0(I)$ have the same Hilbert function; I and $\operatorname{gin}_\tau(I)$ have the same gin_0. When the characteristic of $\mathbb{K}$ is 0, $\operatorname{gin}_0(I) = \operatorname{gin}(I)$ and $\operatorname{Gin}_0(I) = \operatorname{Gin}(I)$.

(ii) For all i and j, the following inequality between Hilbert functions of local cohomology modules holds

$$\operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/I)\right)_j \le \operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/\operatorname{gin}_0(I))\right)_j.$$

In particular, when $(i, j-i)$ is a corner of $A/\operatorname{gin}_0(I)$, then $\beta_{ij}(A/I) \le \beta_{ij}(A/\operatorname{gin}_0(I))$; the projective dimension and Castelnuovo-Mumford regularity of I are bounded above by those of $\operatorname{gin}_0(I)$.

(iii) For all i, $\operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/\operatorname{Gin}(I))\right) = \operatorname{Hilb}\left(H^i_{\mathfrak{m}_A}(A/\operatorname{Gin}_0(I))\right)$; I and $\operatorname{Gin}_0(I)$ have therefore the same extremal Betti numbers, projective dimension and Castelnuovo-Mumford regularity.

Proof. Part (i) is easy. (ii): By [Sb], the Hilbert functions of local cohomology modules increase when taking initial ideals, thus the conclusion is yielded by Theorem 1.3. (iii): Note that $\operatorname{Gin}(I)$ is weakly stable, and by Proposition 1.9 it is sequentially CM. The desired equality follows from Theorem 1.3 together with (2.4). Finally, extremal Betti numbers are left unchanged after taking a generic initial ideal when the chosen monomial order is the revlex order. Thus, the equality of the extremal Betti numbers is yielded by (1.10) and the previous fact. □

Remark 2.3. For $i = 0, 1$ and all j, we have $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{gin}_0(\mathrm{I}))$, since β_{0j}, β_{1j} of a monomial ideal do not depend on the characteristic of the base field $\mathbb{K}$ and $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{gin}(I))$.
It is reasonable to ask whether such inequality is true for any homological index i. This is clear only in a few special cases, for instance: when $\mathrm{char}(\mathbb{K}) = 0$, since $\mathrm{gin}_0(I)$ is $\mathrm{gin}(I)$; when I is a stable monomial ideal, since a minimal free resolution of I is given by the Eliahou-Kervaire resolution and therefore $\beta_{ij}^{A_{\mathbb{K}}}(I) = \beta_{ij}^{A_{\mathbb{Q}}}(I_{\mathbb{Q}})$; when $\mathrm{gin}(I)$ is stable, by a similar reason; finally, when $(i, j-i)$ is a corner for $\mathrm{gin}_0(I)$, the inequality is just a special case of (ii). In general, if we assume that there exist a homogeneous ideal I in $A_{\mathbb{K}}$ and indexes i, j such that $\beta_{ij}(I) > \beta_{ij}(\mathrm{gin}_0(I))$, the characteristic of $\mathbb{K}$ is necessarily $p > 0$; moreover, if we let $J = \mathrm{gin}(I)$, then $\beta_{ij}(J) > \beta_{ij}(\mathrm{gin}_0(J))$, otherwise $\beta_{ij}(J) \leq \beta_{ij}(\mathrm{gin}_0(J)) = \beta_{ij}(\mathrm{gin}_0(I)) < \beta_{ij}(I) \leq \beta_{ij}(J)$. Thus, if there is a counterexample, this can be chosen to be a p-Borel ideal which is also a counterexample to the conjecture of Pardue discussed earlier. We believe that ideals with these properties, which can be constructed with the method found in [CaKu], could be suitable candidates.

In [HeSb], a criterion for a quotient algebra of A to be sequentially Cohen-Macaulay is given: this is the case exactly when the Hilbert functions of local cohomology modules do not change when taking the generic initial ideal with respect to the revlex order, i.e. A/I is sequentially Cohen-Macaulay if and only if

$$\mathrm{Hilb}\left(H^i_{\mathfrak{m}}(A/I)\right)_j = \mathrm{Hilb}\left(H^i_{\mathfrak{m}}(A/\,\mathrm{Gin}(I))\right)_j \quad \text{for all} \quad i, j. \tag{2.4}$$

The following result is a straightforward consequence of (2.4) and Proposition 2.2 (iii) and it provides the analogue for Gin_0 of the above. Recall that, for a monomial ideal, being sequentially CM may depend on the characteristic of the given base field, e.g. the Stanley-Reisner ideal of the minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$ is sequentially Cohen-Macaulay if and only if $\mathrm{char}\,\mathbb{K} \neq 2$.

Proposition 2.5 (Criterion for sequentially Cohen-Macaulayness)**.** *Let I be a homogeneous ideal of A. Then, A/I is sequentially Cohen-Macaulay if and only if the local cohomology modules of A/I and $A/\,\mathrm{Gin}_0(I)$ have same Hilbert functions.*

One of the most useful properties of generic initial ideals, which holds true when $\mathrm{char}(\mathbb{K}) = 0$ and is false in general, is what Green called the *Crystallization Principle* in [Gr], cf. also [Pe, 29.3].

Example 2.6. Let $\mathbb{K}$ be a field of characteristic 3, and let J be the ideal of $\mathbb{K}[X_1, X_2]$ generated by (X_1^6, X_2^6). Then, $\mathrm{gin}(J) = (X_1^6, X_1^3X_2^3, X_2^9)$ and there is a gap in degree 7 and 8, where there is no minimal generator; on the other hand, since $\mathrm{gin}_0(J)$ is strongly stable, it is equal to the lex-segment ideal with the same Hilbert function as J, that is $(X_1^6, X_1^5X_2, X_1^4X_2^3, X_1^3X_2^5, X_1^2X_2^7, X_1X_2^9, X_2^{11})$. By using the Frobenius map it is easy to find many examples of ideals for which $\mathrm{gin}(I)$ does not satisfy the Crystallization Principle.

The reason why the theorem below is unexpected is due to the fact that, by definition, computing $\mathrm{gin}_0(I)$ when the characteristic of the field is positive requires as an intermediate step the calculation of $\mathrm{gin}(I)$, which does not satisfy Crystallization.

Theorem 2.7 (Crystallization Principle for zero-generic initial ideals)**.** *Let I be a homogeneous ideal, τ be a monomial order, and d be an integer such that I has no minimal generator of degree d or larger. If $\mathrm{gin}_0(I)$ has no minimal generator in degree d, then it also has no minimal generator in degree larger than d.*

Proof. Without loss of generality, we may assume the base field $\mathbb{K}$ to be infinite, and that I is generated in degree $d-1$.

Let $J = \mathrm{in}(gI) = \mathrm{gin}(I)$ for a general linear change of coordinates g, and assume that $\mathrm{gin}_0(I) = \mathrm{gin}(J_{\mathbb{Q}})_{\mathbb{K}}$ has no minimal generator in degree d. Notice that $\mathrm{gin}(J_{\mathbb{Q}})$ and, thus, $J_{\mathbb{Q}}$ and J have no minimal generator of degree d as well.

We now denote by P the generic initial ideal of $(J_{d-1})_{\mathbb{Q}}$, and we observe that, by assumption, P agrees with $\mathrm{gin}(J_{\mathbb{Q}})$ in degree d; thus, P has no minimal generator in degree d and P is generated in degree $d-1$. Furthermore, P is strongly stable and, thus, its Castelnuovo-Mumford regularity is precisely $d-1$. Moreover, $\beta_{1,j}(P) \geq \beta_{1,j}((J_{d-1})_{\mathbb{Q}}) = \beta_{1,j}(J_{d-1})$. Hence, the first syzygies of (J_{d-1}) are linear. It follows that J is generated in degree $d-1$. Thus $\mathrm{gin}(J_{\mathbb{Q}}) = P$ and this yields that $\mathrm{gin}_0(I) = (\mathrm{gin}(J_{\mathbb{Q}}))_{\mathbb{K}} = P_{\mathbb{K}}$ is generated in degree $d-1$, as desired. □

Example 2.8. Let I be an ideal of $\mathbb{K}[X_1, \ldots, X_n]$, $n \geq 3$, generated by a codimension 2 vector space of cubics. Theorem 2.7 implies that there are only 4 possible $\mathrm{Gin}_0(I)$, namely the ideal J generated by all monomials of degree 3 except $X_{n-1}X_n^2$ and X_n^3, $J + (X_{n-1}X_n^3)$, $J + (X_{n-1}X_n^3, X_n^4)$ and $J + (X_{n-1}X_n^3, X_n^5)$.

A consequence of Theorem 2.7, and of the method used in its proof, is a characteristic-free adaptation of a result which is well-known in characteristic zero. We recall first that a homogeneous ideal I of A is *component-wise linear* if, for every degree d, the ideal (I_d) generated by I_d, has a linear graded free resolution. This is equivalent to saying that for every d, (I_d) is either 0 or has Castelnuovo-Mumford regularity equal to d.
Component-wise linear ideals have many remarkable properties. In particular, [HeReWe], [HeHi] proved that a squarefree monomial ideal is component-wise linear if and only if its Alexander dual is sequentially CM. Also, it was proved in [ArHeHi] that, when $\mathrm{char}(\mathbb{K}) = 0$, an ideal I is component-wise linear if and only if I and $\mathrm{Gin}(I)$ have the same number of minimal generators. Next, we state the analogue, in any characteristic, of this result.

Theorem 2.9 (Criterion for component-wise linearity)**.** *A homogeneous ideal I is component-wise linear if and only if I and $\mathrm{Gin}_0(I)$ have the same number of minimal generators. Moreover, when I and $\mathrm{gin}_0(I)$ have the same number of minimal generators, I is component-wise linear.*

Proof. For all d, $\mathrm{Gin}_0((I_d))$ has no minimal generator in degree $d+1$ by Proposition 2.2 and, therefore, $((I_d))_{d+1}$ and $(\mathrm{Gin}_0((\mathrm{I_d}))_{\mathrm{d+1}}$ have the same dimension; thus, the Hilbert functions of $I/\mathfrak{m}_A I$ and $\mathrm{Gin}_0(I)/\mathfrak{m}_A\mathrm{Gin}_0(\mathrm{I})$ are the same and, consequently, I and $\mathrm{Gin}_0(I)$ have same number of minimal generators.

Now, let τ be a monomial order and assume that I and $\mathrm{gin}_0(I)$ have same number of minimal generators. By Remark 2.3, $\beta_{0j}(I) = \beta_{0j}(\mathrm{gin}_0(I))$, for all j and this, together with Proposition 2.2, implies that the Hilbert functions of $\mathfrak{m}_A I$ and $\mathfrak{m}_A\,\mathrm{gin}_0(I)$ are the same. Equivalently, for every d, $\mathrm{gin}_0((I_d))$ has no minimal generator in degree $d+1$. Theorem 2.7

implies that when it is not zero, $\text{gin}_0((I_d))$ is generated in degree d; hence, $\text{gin}_0((I_d))$ has regularity d since it is strongly stable. By Proposition 2.2, the ideal (I_d) is either zero or it has regularity d, as desired. □

2.1. **Castelnuovo-Mumford regularity of general restrictions.** One important property of the reverse lexicographic order is that taking initial ideals commutes with respect to going modulo the last variables. As a consequence, generic initial ideals with respect to such an order give some information also on restrictions to general linear spaces. Throughout this section, thus, we let τ be the reverse lexicographic order. It is not restrictive to assume, and we do, that $|\mathbb{K}| = \infty$.

Let $l_n, \dots, l_{i+1}$ be linear forms of A such that $X_1, \dots, X_i, l_{i+1}, \dots, l_n$ form an ordered basis of A_1. By defining a change of coordinates g which maps this basis to $X_1, \dots, X_n$, given any homogeneous ideal I of A, we let the *restriction of I to $A_{[i]}$ with respect to $l_n, \dots, l_{i+1}$* to be the image of gI in $A_{[i]}$ via the isomorphism $A/(X_n, \dots, X_{i+1}) \simeq \mathbb{K}[X_1, \dots, X_i]$.

Definition 2.10. We say that *a general restriction of I to $A_{[i]}$ satisfies a property $\mathcal{P}$* if there exists a non-empty Zariski open set of $(\mathbb{P}^{n-1})^{n-i}$ whose points $([l_n], ..., [l_{i+1}])$ are such that the restriction of I to $A_{[i]}$ with respect to $l_n, \dots, l_{i+1}$ satisfies $\mathcal{P}$.

Remark 2.11. It is relevant for the following to notice that, for a homogeneous ideal I of A and a general restriction J of I to $A_{[i]}$, the ideal $\text{Gin}(J)$ is well-defined. In fact, the ideal $\text{Gin}(J)$ is equal to $\text{Gin}(I)_{[i]}$, which is the general restriction of $\text{Gin}(I)$ to $A_{[i]}$, see for instance [Gr] Theorem 2.30 (4). Hence,

$$\operatorname{reg} A/(I + (l_n, \dots, l_{i+1})) = \operatorname{reg} A/(\text{Gin}(I) + (X_n, \dots, X_{i+1})) = \operatorname{reg} A_{[i]}/\text{Gin}(I)_{[i]}, \tag{2.12}$$

for general linear forms $l_n, \dots, l_{i+1}$; moreover, $\operatorname{reg} J = \operatorname{reg} \text{Gin}(I)_{[i]}$.

From the above observations, we can conclude that, for a homogeneous ideal I of A and a general restriction J of I to $A_{[i]}$, also $\text{Gin}_0(J)$ is well-defined; unfortunately, though, $\text{Gin}_0(J)$ is not the general restriction to $A_{[i]}$ of $\text{Gin}_0(I)$. We observe that the latter is the ideal $\text{Gin}_0(I)_{[i]}$, since $\text{Gin}_0(I)$ is strongly stable. Therefore the second equality in (2.12) is still valid for $\text{Gin}_0(\cdot)$, whereas the first one is false in general. The following example illustrates such a situation.

Example 2.13. Let $I = (X_1^2, X_2^2, X_3^2) \subset A = \mathbb{K}[X_1, X_2, X_3]$ and $\operatorname{char} \mathbb{K} = 2$. Since the ideal I is 2-Borel, $\text{Gin}(I) = I$ and also the general restriction J of I to $A_{[2]}$ is (X_1^2, X_2^2). Moreover, $\text{Gin}_0(J) = (X_1^2, X_1X_2, X_2^3)$ whereas $\text{Gin}_0(I)_{[2]} = (X_1^2, X_1X_2, X_2^2)$. Furthermore, $2 = \operatorname{reg} A/(I + (l_3)) > \operatorname{reg} A/(\text{Gin}_0(I) + (X_3)) = 1$.

We are going to show in Theorem 2.19 that one inequality is still valid and it provides a lower bound for the regularity of general restrictions in terms of zero-generic initial ideals. To this purpose, we prove first a technical fact which will be crucial in our proof.

Proposition 2.14. *Let I be a weakly stable ideal of A. Then, for all $j = 1, \dots, n$,*

$$\text{Hilb}(H^0_{\mathfrak{m}_{A_{[j]}}}(A_{[j]}/I_{[j]})) \geq \text{Hilb}(H^0_{\mathfrak{m}_{A_{[j]}}}(A_{[j]}/\text{Gin}(I)_{[j]})), \text{ and} \tag{2.15}$$

$$\text{Hilb}(H^i_{\mathfrak{m}_{A_{[j]}}}(A_{[j]}/I_{[j]})) = \text{Hilb}(H^i_{\mathfrak{m}_{A_{[j]}}}(A_{[j]}/\text{Gin}(I)_{[j]})) \text{ for all } i > 0. \tag{2.16}$$

Proof. We first prove (2.16). If $i > j$ there is nothing to prove. Assume $0 < i \leq j$ and observe that, by Lemma 1.5 (1.7), it is enough to show that $\operatorname{Hilb}\left(H^{n+i-j}_{\mathfrak{m}_A}(A/I)\right) = \operatorname{Hilb}\left(H^{n+i-j}_{\mathfrak{m}_A}(A/(\operatorname{Gin}(I)))\right)$; by Proposition 1.9, I is sequentially Cohen-Macaulay and, thus, this is achieved by applying Herzog-Sbarra's Criterion (2.4).

Next, we show (2.15) and to do so we first recall the following formula due to Serre (see for instance [BrHe] Theorem 4.4.3). Let U be a homogeneous ideal of a standard graded $\mathbb{K}$-algebra R. Then for every degree d,

$$\dim_K(R/U)_d - \operatorname{HilbPol}_{R/U}(d) = \sum_{i\geq 0}(-1)^i \dim_K H^i_{\mathfrak{m}_R}(R/U)_d, \tag{2.17}$$

where $\operatorname{HilbPol}_{R/U}$ denotes the Hilbert polynomial of R/U. Now, notice that $\operatorname{Hilb}(A_{[j]}/I_{[j]}) = \operatorname{Hilb}(A/(I + (X_n, \dots, X_{j+1}))$ and $\operatorname{Hilb}(A_{[j]}/\operatorname{Gin}(I)_{[j]}) = \operatorname{Hilb}(A/(\operatorname{Gin}(I) + (X_n, \dots, X_{j+1}))$, since I and $\operatorname{Gin}(I)$ are monomial ideals. Furthermore, $\operatorname{Hilb}(A/(\operatorname{Gin}(I) + (X_n, \dots, X_{j+1})) = \operatorname{Hilb}(A/(\operatorname{in}(gI) + (X_n, \dots, X_{j+1}))$ where g is a general linear change of coordinates. By a well-known property of the reverse lexicographic order the latter is equal to $\operatorname{Hilb}(A/(\operatorname{in}(gI + (X_n, \dots, X_{j+1})))$. Thus, $\operatorname{Hilb}(A_{[j]}/\operatorname{Gin}(I)_{[j]}) = \operatorname{Hilb}(A/(gI + (X_n, \dots, X_{j+1}))) = \operatorname{Hilb}(A/(I + (l_n, \dots, l_{j+1}))$ for $l_n, \dots, l_{j+1}$ general linear forms. In particular, since the $l_n, \dots, l_{j+1}$ are general, we have $\operatorname{Hilb}(A/(I + (X_n, \dots, X_{j+1})) \geq \operatorname{Hilb}(A/(I + (l_n, \dots, l_{j+1}))$, which now yields

$$\operatorname{Hilb}(A_{[j]}/I_{[j]}) \geq \operatorname{Hilb}(A_{[j]}/\operatorname{Gin}(I)_{[j]}).$$

Hence, by (2.16) and (2.17), we are left to prove that $A_{[j]}/I_{[j]}$ and $A_{[j]}/\operatorname{Gin}(I)_{[j]}$ have the same Hilbert polynomial or, equivalently, it is enough to verify that for all d sufficiently large, $(A_{[j]}/I_{[j]})_d$ and $(A_{[j]}/\operatorname{Gin}(I)_{[j]})_d$ have the same dimension. To this purpose, we just need to observe that A/I and $A/\operatorname{Gin}(I)$ have the same Hilbert function and that $X_n, \dots, X_{j+1}$ is a filter-regular sequence on both rings, since I and $\operatorname{Gin}(I)$ are weakly stable. □

From the definition of regularity via local cohomology modules, we derive immediately the following corollary.

Corollary 2.18. *Let I be a weakly stable ideal of A. Then, for all $j = 1, \dots, n$,*

$$\operatorname{reg} A_{[j]}/I_{[j]} \geq \operatorname{reg} A_{[j]}/\operatorname{Gin}(I)_{[j]}.$$

We are now ready to prove the following theorem.

Theorem 2.19 (Castelnuovo-Mumford regularity and general restrictions)**.** *Let I be a homogeneous ideal of $A = \mathbb{K}[X_1, \dots, X_n]$ and let $l_n, \dots, l_{i+1}$, $0 \leq i < n$, general linear forms. Then,*

$$\operatorname{reg} A/(I + (l_n, \dots, l_{i+1})) \geq \operatorname{reg} A/(\operatorname{Gin}_0(I) + (X_n, \dots, X_{i+1})) = \operatorname{reg} A_{[i]}/\operatorname{Gin}_0(I)_{[i]}.$$

Proof. We already motivated the validity of the second equality and are left to prove the above inequality. First, by (2.12), $\operatorname{reg} A/(I + (l_n, \dots, l_{i+1})) = \operatorname{reg} A/(\operatorname{Gin}(I) + (X_n, \dots, X_{i+1}))$ and, since $\operatorname{Gin}(I)$ is weakly stable, Corollary 1.12 implies that the left-hand side of the inequality is equal to $\operatorname{reg} A_{\mathbb{Q}}/(\operatorname{Gin}(I)_{\mathbb{Q}} + (X_n, \dots, X_{i+1}))$. On the other hand, $\operatorname{reg} A/(\operatorname{Gin}_0(I) + (X_n, \dots, X_{i+1})) = \operatorname{reg} A/(\operatorname{Gin}(\operatorname{Gin}(I)_{\mathbb{Q}})_{\mathbb{K}} + (X_n, \dots, X_{i+1}))$ and, again by Corollary 1.12, equal to $\operatorname{reg} A_{\mathbb{Q}}/(\operatorname{Gin}(\operatorname{Gin}(I)_{\mathbb{Q}}) + (X_n, \dots, X_{i+1}))$. The conclusion is now a straightforward consequence of Corollary 2.18, applied to the ideal $\operatorname{Gin}(I)_{\mathbb{Q}}$ in the ring $A_{\mathbb{Q}}$. □

3. Applications

In this final section, we provide some applications of the results we proved so far.

3.1. **Algebraic shifting.** Algebraic shifting is a powerful tool introduced by Kalai, cf. [Ka], for studying combinatorics of simplicial complexes, see also [HeHi] Chapter 11. One of the most relevant shifting operation is the so-called symmetric algebraic shifting for Stanley-Reisner ideals, which is used as an analogue of taking generic initial ideal in the squarefree-case. The definition of the classic symmetric algebraic shifting relies on Kalai's shifting operator σ and the properties that $\mathrm{Gin}(I)$ has in characteristic 0: it is strongly stable and has the same extremal Betti numbers as I. We can therefore give a characteristic-free definition of symmetric shifting in the following manner.

Definition 3.1 (Characteristic-free definition of symmetric algebraic shifting)**.** Let Δ be a simplicial complex on $[n]$ and $I_\Delta \subset A_{\mathbb{K}}$ its Stanley-Reisner ideal, where $\mathbb{K}$ is a field of any characteristic. We let the symmetric algebraic shifted complex Δ^s of Δ be the simplicial complex on $[n]$ defined by the Stanley-Reisner ideal $I_{\Delta^s} = (\mathrm{Gin}_0(I_\Delta))^\sigma$.

3.2. **Bounds for the Castelnuovo-Mumford regularity.** Recently, in [CiLeMaRo], lower bounds for the Castelnuovo-Mumford regularity of saturated ideals with fixed Hilbert polynomial have been proven in characteristic zero, but the assumption on the characteristic can now be dropped: Proposition 2.2, and Corollary 1.11 yield the following remark, which, in turn, implies the next theorem.

Remark 3.2. Let $\mathbb{K}$ and $\mathbb{F}$ be any two fields. Given a homogeneous ideal I of $A_{\mathbb{K}} = \mathbb{K}[X_1, \ldots, X_n]$ there exists a strongly stable ideal J of $A_{\mathbb{F}} = \mathbb{F}[X_1, \ldots, X_n]$ such that I and J have same Hilbert function, extremal Betti numbers and, therefore, same projective dimension and Castelnuovo-Mumford regularity. Since I and J have the same projective dimension, I is saturated if and only if J is saturated.

Theorem 3.3. [CiLeMaRo, Theorem A] *holds in any characteristic.*

As a last application, we provide, as an application of $\mathrm{Gin}_0(-)$, a new characteristic-free proof of a well-known doubly exponential bounds for the Castelnuovo-Mumford regularity of an ideal in terms of its generating degree. By Theorem 2.19, our line of reasoning follows now closely that of Galligo's original proof for the characteristic zero case [Ga].

Let I be a non-zero homogeneous ideal of A. We denote by $D(I)$ the generating degree of I, i.e. the maximum degree of a minimal generator of I; we also let $\mu(I)$ be the number of minimal generators of I. In particular,

$$\mu(I) = \sum_j \beta_{0j}(I) \quad \text{and} \quad D(I) := \max\{j \colon \beta_{0j}(I) \neq 0\} \leq \operatorname{reg} I.$$

The following lemma is a straightforward consequence of the combinatorial properties of strongly stable ideals, see again [CaSb], Proposition 1.6, for a generalization to weakly stable ideals.

Lemma 3.4. *Let I be a strongly stable ideal of A. Then*

$$\mu(I) \leq \prod_{i=1}^{n-1}(D(I_{[i]}) + 1).$$

In the proof of the following theorem, given a homogeneous ideal U of $A_{[i]}$, we shall denote by $U_{\langle j\rangle}$, for $1 \leq j \leq i$ a general restriction to $A_{[j]}$ of U, so that $U_{\langle i\rangle} = U$. We notice that, by Remark 2.11, $\mathrm{Gin}_0(U_{\langle j\rangle})$ is well-defined.

Theorem 3.5. *Let I be a homogeneous ideal of A with $n \geq 2$. Then*

$$\operatorname{reg} I \leq (2D(I))^{2^{n-2}}.$$

Proof. The statement is trivial if $D(I) \leq 1$ and, thus, we may assume $D(I) \geq 2$. Let J be $\mathrm{Gin}_0(I)$, and recall that by Proposition 2.2, $\operatorname{reg} I = \operatorname{reg} J$. Since J is strongly stable, its regularity is equal to $D(J)$ and, for the same reason, $\operatorname{reg} J_{[i]} = D(J_{[i]})$ for all $1 \leq i \leq n$.

Let now $J_{(j)}$ denote the ideal $\mathrm{Gin}_0(I_{\langle j\rangle})$, for all $1 \leq j < n$. Theorem 2.19, applied to the ideal $I_{\langle j\rangle}$ for $i \leq j$, yields $\operatorname{reg}(J_{(j)})_{[i]} \leq \operatorname{reg}(I_{\langle j\rangle})_{\langle i\rangle}$. Since both $(I_{\langle j\rangle})_{\langle i\rangle}$ and $I_{\langle i\rangle}$ are general restrictions of I to $A_{[i]}$, we may rewrite the last inequality as $D((J_{(j)})_{[i]}) \leq \operatorname{reg} I_{\langle i\rangle}$.

Together with Lemma 3.4, this implies

$$\mu(J_{(j)}) \leq \prod_{i=1}^{j-1}(D((J_{(j)})_{[i]}) + 1) \leq \prod_{i=1}^{j-1}(\operatorname{reg} I_{\langle i\rangle} + 1).$$

By Proposition 2.2,

$$\operatorname{reg} I_{\langle j\rangle} = \operatorname{reg} \mathrm{Gin}_0(I_{\langle j\rangle}) = D(J_{(j)}),$$

and, furthermore,

$$D(J_{(j)}) \leq D(I_{\langle j\rangle}) + \mu(J_{(j)}) - 1 \leq D(I) + \mu(J_{(j)}) - 1 \leq D(I) - 1 + \prod_{i=1}^{j-1}(\operatorname{reg} I_{\langle i\rangle} + 1), \tag{3.6}$$

where the first inequality is a straightforward application of the Crystallization Principle, see Theorem 2.7.

As in the proof of [CaSb], Corollary 1.8, we set $B_1 = D(I)$, and recursively $B_j = D(I)-1+\prod_{i=1}^{j-1}(B_i+1)$, for all $1 < j \leq n$. It is easy to see that $B_j \leq B_{j-1}^2$ and, thus, $B_j \leq (B_2)^{2^{j-2}}$, for all $j \geq 2$. An easy induction together with (3.6) implies that $\operatorname{reg} I_{\langle j\rangle} \leq B_j$, for all $1 \leq j \leq n$. Hence, $\operatorname{reg} I = \operatorname{reg} I_{\langle n\rangle}$ is bounded above by $(B_2)^{2^{n-2}} = (2D(I))^{2^{n-2}}$, as desired. □

References

[AhKwSo] J. Ahn, S. Kwak, Y. Song. "Generic initial ideals of singular curves in graded lexicographic order". J.Algebra 372 (2012) 584-594.

[ArHe] A. Aramova, J. Herzog. "p-Borel principal ideals". Illinois J. Math. 41 (1997), no. 1, 103-121.

[ArHeHi] A. Aramova, J. Herzog, T. Hibi. "Ideals with stable Betti numbers". Adv. Math. 152 (2000), no. 1, 72-77.

[BaChPo] D. Bayer, H. Charalambous, S. Popescu, "Extremal Betti numbers and applications to monomial ideals". J. Algebra 221 (1999), no. 2, 497-512.

[BaSt] D. Bayer, M. Stillman. "A criterion for detecting m-regularity". Invent. Math. 87 (1987), no. 1, 1-11.

[BeGi] I. Bermejo, P. Gimenez. "Saturation and Castelnuovo-Mumford regularity". J. Algebra 303 (2006), 592-617.

[BrHe] W. Bruns, J. Herzog. *Cohen-Macaulay rings. Revised edition* Cambridge University Press, Cambridge, 1998.

[CaKu] G. Caviglia, M. Kummini. “Betti tables of p-Borel-fixed ideals”. J. Algebraic Combin. 39 (2014), no. 3, 711-718.

[CaSb] G. Caviglia, E. Sbarra. “Characteristic-free bounds for Castelnuovo-Mumford regularity”. Comp. Math. 141 (2005), no. 6, 1365-1373.

[CaSb1] G. Caviglia, E. Sbarra. “A Lex-Plus-Power inequality for local cohomology modules”. To appear in Mathematische Annalen, DOI:10.1007/s00208-015-1180-5.

[CaMu] G. Caviglia, S. Murai. “Sharp upper bounds for the Betti numbers of a given Hilbert polynomial”. Algebra Number Theorey 7 (2013), no. 5, 1019-1064.

[ChChPa] H. M. Cho, Y. H. Cho, J. P. Park. “Generic initial ideals of arithmetically Cohen-Macaulay projective subschemes”. Comm. Algebra 35 (2007), no. 7, 2281-2297.

[CiLeMaRo] F. Cioffi, P. Lella, M.G. Marinari, M. Roggero. “Minimal Castelnuovo-Mumford regularity fixing the Hilbert polynomial”. arXiv:1307.2707 [math.AG].

[Co] A. Conca. “Koszul homology and extremal properties of Gin and Lex”. Trans. Amer. Math. Soc. 356 (2004), no. 7, 2945-2961.

[CoSi] A. Conca, J. Sidman. “Generic initial ideals of points and curves”. J. Symbolic Comput. 40 (2005), no. 3, 1023-1038.

[Ei] D. Eisenbud. *Commutative algebra with a view towards algebraic geometry.* Springer Verlag, New York, 1995.

[EnPfPo] V. Ene, G. Pfister, D. Popescu, “Betti numbers for p-stable ideals”. Comm. Algebra 28 (2000), no. 3, 1515-1531

[FlGr] G. Fløystad, M. Green. “The information encoded in initial ideals”. Trans. Amer. Math. Soc. 353, 2001, 1427-1453.

[FlSt] G. Fløystad, M. Stillman “Geometric properties derived from generic initial spaces”. Proc. Amer. Math. Soc. 137 (2009), no. 11, 3619-3625.

[Ga] A. Galligo. *A propos du théorèm de préparation de Weierstrass*, *in* Fonctions des Plusieurs Variables Complexes, Lect. Notes in Math. 409 (1974), 543-579.

[Gr] M. Green. “Generic initial ideals”, in Six lectures on Commutative Algebra (J. Elias et. al. eds), 119-186 (2010), Birkhäuser Verlag AG, Basel, 2010.

[HeHi] J. Herzog, T. Hibi. “Componentwise linear ideals”. Nagoya Math. J. 153 (1999), 141-153.

[HeHi2] J. Herzog, T. Hibi. *Monomial ideals.* GTM 260, Springer Verlag, 2010.

[HePo] J. Herzog, D. Popescu. “On the regularity of p-Borel ideals”. Proc. Amer. Math. Soc. 129 (2001), no. 9, 2563-2570.

[HeReWe] J. Herzog, V. Reiner, V. Welker. “Componentwise linear ideals and Golod rings”. Michigan Math. 46 (1999), no. 2, 211-223.

[HeSb] J. Herzog, E. Sbarra. “Sequentially Cohen-Macaulay modules and local cohomology”. Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000) Tata Inst. Fund. Res., Bombay (2002), 327-340.

[Ka] G. Kalai. “Algebraic shifting”. *Computational commutative algebra and combinatorics (Osaka, 1999)*, 121âĂŞ163, Adv. Stud. Pure Math., 33, Math. Soc. Japan, Tokyo, 2002.

[Mu] S. Murai. “Generic initial ideals and exterior algebraic shifting of the join of simplicial complexes”. Ark. Mat. 45 (2007), no. 2, 327-336.

[Mu2] S. Murai. “Generic initial ideals and squeezed spheres. Adv”. Math. 214 (2007), no. 2, 701-729.

[MuHi] S. Murai, T. Hibi. “Gin and lex of certain monomial ideals”. Math. Scand. 99 (2006), no. 1, 76-86.

[MuPu] S. Murai, P. Singla. “Rigidity of linear strands and generic initial ideals”. Nagoya Math. J. 190 (2008), 35-61.

[Pa] K Pardue. “Nonstandard Borel-Fixed Ideals”. Ph.D. Brandeis University 1994.

[Pa1] K. Pardue. “Deformation classes of graded modules and maximal Betti numbers”. Illinois. J. Math. 40 (1996), 564-585.

[Pe] I. Peeva. *Graded Syzygies.* Algebra and Applications 14. Springer, 2011.

[Sb] E. Sbarra. "Upper bounds for local cohomology for rings with given Hilbert function". Comm. Algebra, 29 (2001), no. 12, 5383-5409.

[Sb2] E. Sbarra. "Ideals with maximal local cohomology". Rend. Sem. Mat. Univ. Padova, 111 (2004), 265-275.

[Tr] N. V. Trung. "Gröbner bases, local cohomology and reduction number". Proc. Amer. Math. Soc., 129 (2001), no. 1, 9-18.

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