Image Convexity of Generalized Systems with Infinite Dimensional Image and Applications

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Abstract Convexity properties of a generalized system with infinite dimensional image are investigated by means of the notions of image and its extensions associated with the system. Complete characterizations of (proper) linear separation in the image space are given by using the quasi relative interior, which allow one to obtain necessary and/or sufficient conditions for the impossibility of an image convex generalized system with infinite dimensional image. These new results are applied to investigate vector quasi optimization problems and vector dynamic variational inequalities.

Keywords Image space · Convexity · Generalized system · Quasi optimization problem · Dynamic variational inequality

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1 Introduction

Generalized systems provide a general framework to analyse vector equilibrium problems, vector optimization problems, complementarity systems, variational and quasi variational inequalities (see, e.g., [1–3]). Such problems can be reduced to the impossibility of a generalized system (GS) which can be studied by means of separation techniques in the image space associated with the (GS) [4–6]. In this paper, motivated by the pioneering works [2, 3] we provide a further contribution in the analysis of the property of image convexity of a (GS), giving particular emphasis to the applications to vector quasi optimization problems and vector quasi variational inequalities. Image convexity can be investigated by means of the image space analysis [1], since several of the recently introduced generalized convexity notions have shown to be equivalent to the convexity of suitable extensions of the image of the mapping involved in the (GS) [2, 7, 8]. Image convexity provides equivalent formulations of the impossibility of a (GS) in terms of the disjunction of two convex sets in the image space associated with the (GS). By using (proper) linear separation arguments related to the quasi relative interior in the image space, we obtain necessary and/or sufficient conditions for the impossibility of a (GS) and we apply these results to investigate vector quasi optimization problems and vector dynamic variational inequalities. So far, there are few results concerning optimization problems and variational inequalities with infinite dimensional image from the view of the image space.

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To the best of our knowledge, our results, especially complete characterizations of (proper) linear separation in the image space and related arguments, are new, compared with those existing in the literature, where, in general, classical separation techniques related to the interior but not to the quasi relative interior were employed.

The paper is organized as follows. Section 2 is devoted to preliminary definitions and results which will be used throughout the paper. In Section 3, the image of a (GS) is defined and the problem of stating the impossibility of the (GS) is equivalently expressed by means of the disjunction of two suitable sets in the image space associated with the (GS). Next we recall some of the main convexity properties of a mapping and we relate such properties to suitable conical extensions of the image mapping. In Section 4 we analyse the notion of image convexity of a (GS) and we show that the generalized convexity assumptions considered in Section 3 are sufficient conditions in order to guarantee the image convexity of the (GS). Applications to vector quasi optimization problems and vector dynamic variational inequalities are considered in Section 5, devoting particular attention to linear separation and saddle point optimality conditions.

2 Preliminaries

A generalized system [1, 2] is defined by

$$F(x;y) \in \mathcal{H}_y \subseteq V, \quad x \in K, \tag{1}$$

where $y \in Y$ a parameter set, V is a real topological linear space, $\mathcal{H}_y \subseteq V$, for every $y \in Y$, K is a subset of the topological linear space X, and $F : X \times Y \to V$ is a mapping.

In this paper, we consider two generalizations of the previous problems. The first one is the vector quasi optimization problem (P) defined by

$$\min_{C(y)} f(x), \quad \text{s.t.} \quad x \in R(y) := \{x \in K : g(x,y) \in D(y)\},$$

where $\mathcal{Y}$ and $\mathcal{Z}$ are topological linear spaces, $f : X \to \mathcal{Y}$, $g : X \times X \to \mathcal{Z}$, $K \subseteq X$, $C(y)$ is a closed pointed convex cone in $\mathcal{Y}$ and $D(y)$ a closed convex cone in $\mathcal{Z}$, for every $y \in Y := \{y \in X : y \in R(y)\}$.

We say that $y \in Y$ is a vector quasi minimum point (for short, v.q.m.p.) for (P) if $f(y) - f(x) \notin C(y) \setminus \{0\}$, $\forall x \in R(y)$. Put

$$F(x;y) := (f(y) - f(x), g(x,y)), \quad \mathcal{H}_y := (C(y) \setminus \{0\}) \times D(y), \tag{2}$$

where $x \in X$ and $y \in Y$. Then $y \in Y$ is a v.q.m.p. for (P) iff system (1) is impossible. Assume, additionally, that $C(y), y \in Y$, is with nonempty interior. Setting $F(x;y) := (f(y) - f(x), g(x,y))$, $\mathcal{H}_y := (\text{int}C(y)) \times D(y)$. Then $y \in Y$ is a weak v.q.m.p. for (P) iff system (1) is impossible.

Let $L(X, \mathcal{Y})$ be the set of continuous linear operators from X to $\mathcal{Y}$ and for $l \in L(X, \mathcal{Y})$, denote by $\langle l, x \rangle$ the value of l at x. The second problem that will be considered is the vector quasi variational inequality (VQVI) that consists in finding $y \in Y := \{y \in X : y \in R(y)\}$, where $R(y)$ is defined as in the previous problem, such that

$$\langle T(y), x - y \rangle \notin C(y) \setminus \{0\}, \quad \forall x \in R(y),$$

where $T : X \to L(X, \mathcal{Y})$. Put $F(x; y) := (\langle T(y), x - y \rangle, g(x, y))$, $\mathcal{H}_y := (C(y) \setminus \{0\}) \times D(y)$, where $x \in X$ and $y \in Y$. Then $y \in Y$ is a solution of (VQVI) iff (1) is impossible.

The following notations and definitions will be considered throughout the paper.

Denote by $\mathbb{N}$ the set of positive integers. Let $\mathbb{R}^n$ be the n dimensional Euclidean space, where $n \in \mathbb{N}$. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x > 0\}$, where $n \in \mathbb{N}$. Let $\mathbb{R}_+ := \mathbb{R}^1_+$, and so forth.

Let M, Q be subsets of a Hausdorff locally convex topological linear space V. The closure, the interior, the boundary and the convex hull of M are denoted by cl$M$, int$M$, bd$M$ and conv$M$, respectively. It is well known that if V is finite dimensional, then the relative interior of M, say ri$M$, is the interior of M relative to its affine hull. Let $M + Q := \{m + q \in V : \ m \in M, \ q \in Q\}$, $\mathbb{R}_+ M := \{tm \in V : \ m \in M, t \in \mathbb{R}_+\}$ and $\mathbb{R}_{++} M := \{tm \in V : \ m \in M, t \in \mathbb{R}_{++}\}$.

The set $M \subseteq V$ is said to be a cone if $\lambda M \subseteq M$, with $\lambda \geq 0$, and a convex cone if, in addition, $M + M \subseteq M$. Denote by cone$M := \mathbb{R}_+ M$ the cone generated by M and let cone$_+M := \mathbb{R}_{++} M$. $M^* := \{y \in V^* : \langle x, y \rangle \geq 0, \ \forall x \in M\}$ is the positive polar (or the dual cone) of M, where V^* is the topological dual of V. A convex cone $M \subseteq V$ is called pointed if $M \cap (-M) = \{0\}$.

Let $M \subseteq V$ be a convex set and $x \in M$. $N_M(x)$ is the normal cone to M at x and is defined by $N_M(x) := \{z \in V^* : \langle z, y - x \rangle \leq 0, \ \ \forall \ y \in M\}$. It is clear that $0 \in N_M(x)$, and if $x = 0$, then $N_M(x) = -M^*$. If $x \in$ int$M$, then $N_M(x) = \{0\}$. That is to say, if $N_M(x) \neq \{0\}$, then $x \in$ bd$M$.

Separation arguments related to quasi relative interior, introduced by Borwein and Lewis [9], are useful in investigating the generalized system (1).

Definition 2.1 (see [9]) Let M be a subset of a Hausdorff locally convex topological linear space V.

(i) We say that $x \in M$ is a quasi interior point of M, denoted by $x \in$ qi$M$, if cl cone$(M - x) = V$, or equivalently, $N_M(x) = \{0\}$;

(ii) We say that $x \in M$ is a quasi relative interior point of M, denoted by $x \in$ qri$M$, if cl cone$(M - x)$ is a linear subspace of V, or equivalently, $N_M(x)$ is a linear subspace of V^*.

For any convex set M, we have that qi$M \subseteq$ qri$M$ and, int$M \neq \emptyset$ implies int$M =$ qri$M$ [9] and int$M =$ qi$M$ [10]. Moreover, qri$\{x\} = \{x\}$, $\forall x \in V$. Similarly, if qi$M \neq \emptyset$, then qi$M =$ qri$M$ [10, 11], see also [12, 13]. Moreover, if V is a finite dimensional space, then qi$M =$ int$M$ and qri$M =$ ri$M$ [9].

We need the following lemmas.

Lemma 2.1 *Let us consider M and N two nonempty convex subsets of the Hausdorff locally convex topological linear space Y, $x \in Y$ and $t \in \mathbb{R}$. Then the following statements are true:*

(i) $\mathrm{qri}M + \mathrm{qri}N \subseteq \mathrm{qri}(M+N)$;

(ii) $\mathrm{qri}(M \times N) = \mathrm{qri}M \times \mathrm{qri}N$;

(iii) $\mathrm{qri}(M - x) = \mathrm{qri}M - x$;

(iv) *If* $\mathrm{qri}M \neq \emptyset$, *then* $\mathrm{qri}(tM) = t\,\mathrm{qri}M$;

(v) $t\,\mathrm{qri}M + (1-t)M \subseteq \mathrm{qri}M,\ \forall t \in (0,1]$; *hence,* $\mathrm{qri}M$ *is a convex set;*

(vi) *If* $\mathrm{qri}M \neq \emptyset$, *then* $\mathrm{cl}\ \mathrm{qri}M = \mathrm{cl}M$;

(vii) *If M is a convex cone, then* $\mathrm{qri}M + M = \mathrm{qri}M$;

(viii) *If* $\mathrm{qri}M \neq \emptyset$, *then* $(\mathrm{qri}M)^* = M^*$;

(ix) *If* $\mathrm{qi}M \neq \emptyset$, *then* $\mathrm{qi}M - N \subseteq \mathrm{qi}(M-N)$; *furthermore, if* $N := \{x\}$, *then* $\mathrm{qi}(M-x) = \mathrm{qi}M - x$.

Proof Statements (i)-(vi) can be found in [9, 10, 14, 15]. Statements (vii)-(ix) can be found in [4, 16]. □

Lemma 2.2 [17, Lemma 2.6] *Let $M \subseteq V$ be a nonempty set and $N \subseteq V$ a convex cone with* $\mathrm{qri}N \neq \emptyset$. *Then* $\mathrm{cl}[\mathrm{cone}(M+N)] = \mathrm{cl}[\mathrm{cone}(M+\mathrm{qri}N)] = \mathrm{cl}[\mathrm{cone}M + \mathrm{qri}N]$.

Lemma 2.3 [9, Proposition 2.21] *Let X and Y be Hausdorff locally convex spaces with M a convex subset of X and $A : X \to Y$ a linear continuous operator. Then* $A(\mathrm{qri}M) \subseteq \mathrm{qri}A(M)$.

Lemma 2.4 [18, Lemma 2.5] *Let $M \subseteq V$ be a nonempty set and $P \subseteq V$ a convex cone with* $\mathrm{int}P \neq \emptyset$. *Then* $M + \mathrm{int}P = \mathrm{int}(M+P)$.

3 Image of a Generalized System and Convexity Properties of Image Mappings

In the first part of this section, we introduce the notions of image associated with system (1) and of its extensions. In the second part, we recall some of the main convexity concepts introduced in the literature and point out that they are closely related to the convexity properties of the extended images of the functions involved.

Definition 3.1 $\mathcal{K}_y{:=}F(K;y)$, $y \in Y$, is called the *image* associated with the generalized system (1).

Let $y \in Y$; we recall that the generalized system (1) is impossible iff

$$\mathcal{K}_y \cap \mathcal{H}_y = \emptyset. \tag{3}$$

In the image space approach, (3) is proved by means of separation techniques [1]. In particular, when $\mathcal{K}_y$ and $\mathcal{H}_y$ are "linearly separable" (i.e. they admit a separating hyperplane), and, moreover, are supposed to be disjoint, we will say that the system (1) is *image convex* (see Definition 4.1).

It is evident that convexity properties of the image mapping $F(\cdot; y)$ play a crucial role in order to state the image convexity of (1). Most of the generalized convexity properties of the mapping $F(\cdot; y)$ can be related to the convexity of a suitable approximation of the image $\mathcal{K}_y$, namely its extension with respect the set $\mathcal{A}_y \subseteq V$, defined by:

$$\mathcal{E}(\mathcal{A}_y) := \mathcal{K}_y + \mathcal{A}_y, \qquad y \in Y.$$

The introduction of the extension of the image set allows us to obtain an equivalent formulation of condition (3) [2]:

Proposition 3.1 [2, Theorem 2.1] *Let $y \in Y$ and assume that*

$$\mathcal{H}_y - \mathcal{A}_y = \mathcal{H}_y. \tag{4}$$

Then the generalized system (1) *is impossible iff*

$$\mathcal{E}(\mathcal{A}_y) \cap \mathcal{H}_y = \emptyset. \tag{5}$$

As mentioned by one referee, the condition (4) is related to the so-called free disposal from the Economics. The following statements provide conditions such that (4) holds.

Proposition 3.2 *Let V be a Hausdorff locally convex topological linear space and $y \in V$. Condition* (4) *is fulfilled in the following particular cases:*

(i) $\mathcal{A}_y := -\mathcal{H}_y$ *and $\mathcal{H}_y$ is a convex cone;*

(ii) $\mathcal{A}_y := -\mathrm{cl}\mathcal{H}_y$ *and $\mathcal{H}_y$ is defined as in* (2)*;*

(iii) $\mathcal{A}_y := -\mathrm{int}\mathcal{H}_y \cup \{0\}$*, provided that $\mathcal{H}_y$ is a convex cone with* $\mathrm{int}\mathcal{H}_y \neq \emptyset$*;*

(iv) $\mathcal{A}_y$ *is a convex cone and* $\mathcal{H}_y := -\mathrm{qri}\mathcal{A}_y$*, provided that* $\mathrm{qri}\mathcal{A}_y \neq \emptyset$*.*

Proof For (i)-(iii) see [19].
(iv) The conclusion follows directly from (vii) in Lemma 2.1. □

By (iv) of the previous proposition it follows that (4) holds if $\mathcal{A}_y := -Q$, $\mathcal{H}_y := \mathrm{ri}Q$, where $Q \subseteq \mathbb{R}^n$ is a convex cone.

Hence, proving the impossibility of (1) is equivalent to show that (5) holds, which, in certain cases, may be easier to prove, because the set $\mathcal{E}(\mathcal{A}_y)$ may have some advantageous properties that $\mathcal{K}_y$ has not. For example, if (P) is a finite dimensional convex problem, i.e., f is $C(y)$-convex and $-g(\cdot, y)$ is $D(y)$-convex, for every $y \in Y$, with $\mathcal{Y}$ and $\mathcal{Z}$ finite dimensional, then $\mathcal{E}(\mathcal{A}_y)$ is convex and system (1) is image convex.

We recall that $f : X \to \mathcal{Y}$ is C-convex on K, if, for any $x_1, x_2 \in K, \alpha \in]0,1[$, $\alpha f(x_1) + (1-\alpha)f(x_2) - f(\alpha x_1 + (1-\alpha)x_2) \in C$, where K is a convex subset of X and C be a convex cone in $\mathcal{Y}$. Note that, when $\mathcal{Y} := \mathbb{R}$ and $C := \mathbb{R}_+$, the previous definition collapses to that of the classic convexity.

In the second part of this section, we analyse the convexity properties of extended images $\mathcal{E}(\mathcal{A}_y)$ in order to weaken the assumptions on $F(\cdot, y)$ which guarantee the image convexity of (1).

The concept of convexity related to a function has been generalized in several ways. Some of the most important ones are summarized in the next definition.

Definition 3.2 Let X and $\mathcal{Y}$ be real topological linear spaces, $K \subseteq X$, $\mathcal{A} \subseteq \mathcal{Y}$ be a convex cone, and $f : X \to \mathcal{Y}$ be a mapping.

(i) f is said to be $\mathcal{A}$-convexlike on K, if, for any x_1, $x_2 \in K$ and any $\alpha \in]0,1[$, there exists $x_3 \in K$, such that $\alpha f(x_1) + (1-\alpha)f(x_2) - f(x_3) \in \mathcal{A}$;

(ii) Suppose that $\text{qri}\mathcal{A} \neq \emptyset$. f is said to be $\mathcal{A}$-subconvexlike on K, if $f(K) + \text{qri}\mathcal{A}$ is convex;

(iii) f is said to be closely $\mathcal{A}$-convexlike on K, if the set $\text{cl}(f(K)+\mathcal{A})$ is convex;

(iv) f is said to be $\mathcal{A}$-preconvexlike on K, if, for any x_1, $x_2 \in K$ and $\alpha \in]0,1[$, there exists $x_3 \in K$ and $\rho > 0$, such that $\alpha f(x_1)+(1-\alpha)f(x_2)-\rho f(x_3) \in \mathcal{A}$;

(v) Suppose that $\text{qri}\mathcal{A} \neq \emptyset$. f is said to be generalized $\mathcal{A}$-subconvexlike on K, if $\text{cone}f(K) + \text{qri}\mathcal{A}$ is convex.

Remark 3.1 In (ii)-(v), we do not suppose that $\text{int}\mathcal{A} \neq \emptyset$. Under the assumption $\text{int}\mathcal{A} \neq \emptyset$, some equivalent characterizations of (ii)-(v) have been given by many authors. These results are useful in obtaining the image convexity of system (1). Convexlike mappings were considered by Fan [20]. $\mathcal{A}$-subconvexlike mappings were introduced by Jeyakumar [21] as follows: f is said to be $\mathcal{A}$-subconvexlike on K, if there exists an $a_0 \in \mathcal{A}$, such that for any x_1, $x_2 \in K$, for any $\alpha \in]0,1[$ and $\varepsilon > 0$, there exists $x_3 \in K$ such that $\varepsilon a_0 + \alpha f(x_1) + (1-\alpha)f(x_2) - f(x_3) \in \mathcal{A}$, which is equivalent to assume that the set $f(K) + \text{int}\mathcal{A}$ is convex under the assumption that $\text{int}\mathcal{A} \neq \emptyset$ (see, for example, [7, 22, 23]).

The class of closely $\mathcal{A}$-convexlike mappings was considered in [18]. If $\text{int}\mathcal{A} \neq \emptyset$, then f is $\mathcal{A}$-subconvexlike on K iff it is closely $\mathcal{A}$-convexlike on K [24].

Definitions 3.2 (iv) and (v) were introduced in [8] and [17], respectively. If $\text{int}\mathcal{A} \neq \emptyset$, then Definition 3.2 (v) collapses to the following: f is generalized $\mathcal{A}$-subconvexlike on K, if $\text{cone}f(K)+\text{int}\mathcal{A}$ is convex. This definition is equivalent to that given by Yang, Yang and Chen [25] as follows (see, for example, [7, 22, 23]): there exists an $a_0 \in \text{int}\mathcal{A}$, such that for any x_1, $x_2 \in K$, $\alpha \in]0,1[$ and $\varepsilon > 0$, there exists $x_3 \in K$ and $\rho > 0$, such that $\varepsilon a_0 + \alpha f(x_1) + (1-\alpha)f(x_2) - \rho f(x_3) \in \mathcal{A}$.

Next results ([26, Lemma 3.1], [27, Proposition 5]) state characterizations of $\mathcal{A}$-convexlike and $\mathcal{A}$-preconvexlike functions, in terms of the properties of suitable conical extensions of their images.

Proposition 3.3 *Let $\mathcal{A} \subseteq \mathcal{Y}$ be a convex cone. Then*

(i) *f is $\mathcal{A}$-convexlike on K, iff the set $f(K) + \mathcal{A}$ is ,convex.*

(ii) *$f : X \to \mathcal{Y}$ is $\mathcal{A}$-preconvexlike on K iff the set $\mathrm{cone}_+ f(K) + \mathcal{A}$ is convex.*

It is not difficult to show that the following relations hold:

$$f \ \mathcal{A}\text{-convex} \ \Rightarrow \ f \ \mathcal{A}\text{-convexlike} \ \Rightarrow \ f \ \mathcal{A}\text{-preconvexlike}.$$

where, in the first implication, K is assumed to be convex, and

$$f \ \mathcal{A}\text{-preconvexlike} \quad \Rightarrow \quad f \text{ generalized } \mathcal{A}\text{-subconvexlike}.$$

For the last implication, it is enough to observe that, since $\mathcal{A}$ is a convex cone, from Lemma 2.1 (vii) one has $\mathrm{cone}_+ f(K) + \mathrm{qri}\mathcal{A} = \mathrm{cone}_+ f(K) + \mathcal{A} + \mathrm{qri}\mathcal{A}$ and the last set is convex in view of Proposition 3.3 (ii). The convexity of $\mathrm{cone}_+ f(K) + \mathrm{qri}\mathcal{A}$ implies the convexity of $\mathrm{cone} f(K) + \mathrm{qri}\mathcal{A}$, recalling that $\mathrm{qri}\mathcal{A}$ is a convex set.

Similarly, it can be shown that

$$f \ \mathcal{A}\text{-subconvexlike} \quad \Rightarrow \quad f \text{ generalized } \mathcal{A}\text{-subconvexlike}.$$

The reverse implication does not hold as shown by the following example.

Example 3.1 Let $K := \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 > 1\}$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) := (x_1, x_2)$ and $\mathcal{A} := \mathbb{R}_+ \times \{0\}$. It is simple to see that $\mathrm{qri}\mathcal{A} = \mathbb{R}_{++} \times \{0\}$ and that:

$$\begin{aligned} f(K) + \mathrm{qri}\mathcal{A} &= \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 > 1, x_1 > 0\}, \\ \mathrm{cone} f(K) + \mathrm{qri}\mathcal{A} &= \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > 0\}. \end{aligned}$$

Therefore, f is not $\mathcal{A}$-subconvexlike on K, but it is generalized $\mathcal{A}$-subconvexlike on K.

Next theorem states a characterization of the generalized convexity concepts, introduced in Definition 3.2, in terms of the properties of the extended image associated with the generalized system (1).

Theorem 3.1 *Let $\mathcal{A}_y \subseteq V$ be a convex cone, $F : X \times Y \to V$ a mapping and $K \subseteq X$. Then*

(a) *$F(\cdot; y)$ is $\mathcal{A}_y$-convexlike on K iff the set $\mathcal{E}(\mathcal{A}_y)$ is convex;*

(b) *$F(\cdot; y)$ is $\mathcal{A}_y$-preconvexlike on K iff the set $\mathrm{cone}_+ \mathcal{E}(\mathcal{A}_y)$ is convex;*

(c) *$F(\cdot;y)$ is closely $\mathcal{A}_y$-convexlike on K iff the set* $\mathrm{cl}\mathcal{E}(\mathcal{A}_y)$ *is convex.*

(d) *$F(\cdot;y)$ is $\mathcal{A}_y$-subconvexlike on K iff the set* $\mathcal{E}(\mathrm{qri}\mathcal{A}_y)$ *is convex.*

Assume that that $\mathrm{int}\mathcal{A}_y \neq \emptyset$.

(e) *$F(\cdot;y)$ is $\mathcal{A}_y$-subconvexlike on K iff the set* $\mathrm{int}\mathcal{E}(\mathcal{A}_y)$ *is convex;*

(f) *$F(\cdot;y)$ is generalized $\mathcal{A}(y)$-subconvexlike on K iff the set* $\mathrm{int}[\mathrm{cone}\mathcal{E}(\mathcal{A}_y)]$ *is convex under the assumption that* $0 \in \mathcal{K}_y = F(K;y)$.

Proof (a) It follows from Proposition 3.3 (i).
(b) From Proposition 3.3 (ii) we have that $F(\cdot;y)$ is $\mathcal{A}_y$-preconvexlike on K iff the set $\mathrm{cone}_+ \ F(K;y) + \mathcal{A}_y$ is convex. We observe that

$$\begin{aligned} \mathrm{cone}_+\mathcal{E}(\mathcal{A}_y) &:= \{u \in V : u = \lambda(F(x;y) + a),\ x \in K, a \in \mathcal{A}_y,\ \lambda > 0\} \\ &= \{u \in V : u = \lambda F(x;y) + a,\ x \in K, a \in \mathcal{A}_y,\ \lambda > 0\} \\ &= \mathrm{cone}_+F(K;y) + \mathcal{A}_y, \end{aligned}$$

where the first equality is due to the hypothesis that $\mathcal{A}_y$ is a cone. This completes the proof of part (b).
(c) and (d) follow directly by Definition 3.2 (iii) and (ii) respectively.
(e) It follows from Definition 3.2 (ii) and Lemma 2.4.
(f) Let $M \subseteq V$ be a nonempty set with $0 \in M$ and $P \subseteq V$ a cone. We first note that $\mathrm{cone}(M + P) = \mathrm{cone}M + P$. In fact, since P is a cone, $\mathrm{cone}(M + P) \subseteq \mathrm{cone}(\mathrm{cone}M + P) = \mathrm{cone}M + P$. Let $ta + b \in \mathrm{cone}M + P$, where $a \in M, b \in P$ and $t \geq 0$. If $t = 0$, then $ta + b = b \in \mathrm{cone}(M + P)$ since $0 \in M$. If $t > 0$, then $ta + b = t(a + \frac{1}{t}b) \in \mathrm{cone}(M + P)$.

From Definition 3.2 (v) we have that $F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K iff the set $\mathrm{cone}F(K;y) + \mathrm{int}\mathcal{A}_y$ is convex. Since $0 \in F(K;y)$, by Lemma 2.4 it follows that

$$\begin{aligned} \mathrm{cone}F(K;y) + \mathrm{int}\mathcal{A}_y &= \mathrm{int}[\mathrm{cone}F(K;y) + \ \mathcal{A}_y] \\ &= \mathrm{int}[\mathrm{cone}(F(K;y) + \ \mathcal{A}_y)] = \mathrm{int}[\mathrm{cone}\mathcal{E}(\mathcal{A}_y)]. \end{aligned}$$

This completes the proof. □

4 Image Convexity of Generalized Systems

In this section, we formally recall and analyse the concept of image convexity of system (1) and we establish sufficient conditions for its fulfillment.

Definition 4.1 The generalized system (1) is said to be image convex, if for all $y \in Y$ such that (1) is impossible, the sets $\mathcal{K}_y$ and $\mathcal{H}_y$ are linearly separable.

Observe that, if $\mathcal{H}_y$ is a cone, then the image convexity is equivalent to the fact that, whenever (1) is impossible, there exists $\lambda^* \in \mathcal{H}_y^* \setminus \{0\}$ such that

$$\langle \lambda^*, F(x;y) \rangle \leq 0, \quad \forall x \in K.$$

Although, in most of the applications $\mathrm{cl}\mathcal{H}_y$ is a convex cone, for further developments, we first state some results that do not rely on such an assumption.

Proposition 4.1 *Assume that* (4) *is fulfilled. Then,* $\mathcal{K}_y$ *and* $\mathcal{H}_y$ *are (properly) linearly separable iff* $\mathcal{E}(\mathcal{A}_y)$ *and* $\mathcal{H}_y$ *are (properly) linearly separable.*

Proof We preliminary note that for given two nonempty subsets B and C in V, B and C are (properly) linearly separable iff $\{0\}$ and $B-C$ are (properly) linearly separable. Indeed, let $\lambda^* \in V^* \setminus \{0\}$. Then,

$$\begin{aligned} \sup_{b\in B}\langle \lambda^*, b\rangle \quad &\leq \quad \inf_{c\in C}\langle \lambda^*, c\rangle \;\Leftrightarrow\; \langle \lambda^*, b\rangle \leq \langle \lambda^*, c\rangle,\ \forall b\in B,\ \forall c\in C \\ &\Leftrightarrow \quad \langle \lambda^*, b-c\rangle \leq 0,\ \forall b\in B,\ \forall c\in C, \end{aligned}$$

which proves that the linear separability between B and C is equivalent to the one between $\{0\}$ and $B-C$. For the proper separability, we have further to ensure that

$$\inf_{b\in B}\langle \lambda^*, b\rangle < \sup_{c\in C}\langle \lambda^*, c\rangle \quad \Leftrightarrow \quad \inf_{\substack{b\in B\\ c\in C}}\langle \lambda^*, b-c\rangle < 0.$$

Now let $B := \mathcal{K}_y$ and $C := \mathcal{H}_y$. Then $\mathcal{K}_y$ and $\mathcal{H}_y$ are (properly) linearly separable iff $\{0\}$ and $\mathcal{K}_y - \mathcal{H}_y$ are (properly) linearly separable. By (4) it follows that

$$\mathcal{K}_y - \mathcal{H}_y = \mathcal{K}_y - \mathcal{H}_y + \mathcal{A}_y = \mathcal{E}(\mathcal{A}_y) - \mathcal{H}_y,$$

which implies that the (proper) separability between $\{0\}$ and $\mathcal{E}(\mathcal{A}_y) - \mathcal{H}_y$ is equivalent to the (proper) separability between $\{0\}$ and $\mathcal{K}_y - \mathcal{H}_y$. This completes the proof. □

Proposition 4.2 *Let* V *be a Hausdorff locally convex topological linear space and suppose that* $\mathrm{cl}\mathcal{A}_y = -\mathrm{cl}\mathcal{H}_y$*. Then* $\mathcal{K}_y$ *and* $\mathcal{H}_y$ *are linearly separable iff* $N_{\mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y))}(0) \neq \{0\}$*, that is,* $0 \notin \mathrm{qi}[\mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y)]$.

Proof Note that $\mathcal{K}_y$ and $\mathcal{H}_y$ are linearly separable iff there exists $\lambda^* \in V^* \setminus \{0\}$ such that

$$\langle \lambda^*, u-w\rangle \leq 0, \qquad \forall u\in \mathcal{K}_y, \forall w\in \mathcal{H}_y.$$

Since $\mathrm{cl}\mathcal{A}_y = -\mathrm{cl}\mathcal{H}_y$, this is equivalent to

$$\langle \lambda^*, u+w\rangle \leq 0, \qquad \forall u\in \mathcal{K}_y, \forall w\in \mathcal{A}_y.$$

The previous inequality holds iff $\langle \lambda^*, e-0\rangle \leq 0,\ \forall e\in \mathrm{conv}(\mathcal{E}(\mathcal{A}_y))$, and, in turn, iff $\langle \lambda^*, e-0\rangle \leq 0,\ \forall e \in \mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y))$, i.e., $0 \neq \lambda^* \in N_{\mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y))}(0)$ or equivalently, by Definition 2.1, $0 \notin \mathrm{qi}[\mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y))]$. □

Remark 4.1 Observe that the condition $N_{\mathrm{cone\ conv}(\mathcal{E}(\mathcal{A}_y))}(0) \neq \{0\}$ is equivalent to $[\mathrm{cone\ conv}\mathcal{E}(\mathcal{A}_y)]^* \neq \{0\}$.

Next result given in [28] states that a point and a set can be separated properly in Hausdorff locally convex topological linear spaces under suitable assumptions in terms of the quasi relative interior.

Theorem 4.1 *Let M be a nonempty subset of the Hausdorff locally convex topological linear space V. Then $0 \notin \text{qri}[\text{cone conv}M]$ iff 0 and M can be separated properly, i.e., there exists $l^* \in V^*\backslash\{0\}$ such that $\langle l^*, x\rangle \leq 0, \forall x \in M$ with strict inequality for some $\bar{x} \in M$.*

Similarly to Proposition 4.2 , we have the following:

Proposition 4.3 *Let V be a Hausdorff locally convex topological linear space and suppose that $\text{cl}\mathcal{A}_y = -\text{cl}\mathcal{H}_y$. Then the sets $\mathcal{K}_y$ and $\mathcal{H}_y$ are properly linearly separable iff $N_{\text{cone conv}(\mathcal{E}(\mathcal{A}_y))}(0)$ is not a linear subspace of V^*, that is, $0 \notin \text{qri}[\text{cone conv}(\mathcal{E}(\mathcal{A}_y))]$.*

Proof Observe that $\mathcal{K}_y$ and $\mathcal{H}_y$ are properly linearly separable iff there exists $\lambda^* \in V^* \setminus \{0\}$ such that

$$\langle \lambda^*, u - w\rangle \leq 0, \qquad \forall u \in \mathcal{K}_y, \forall w \in \mathcal{H}_y,$$

with strict inequality for some $\bar{u} \in \mathcal{K}_y$ and $\bar{w} \in \mathcal{H}_y$. Since $\text{cl}\mathcal{A}_y = -\text{cl}\mathcal{H}_y$, this is equivalent to

$$\langle \lambda^*, u + w\rangle \leq 0, \qquad \forall u \in \mathcal{K}_y, \forall w \in \mathcal{A}_y, \tag{6}$$

with strict inequality for some $\bar{u} \in \mathcal{K}_y$ and $\bar{w} \in \mathcal{A}_y$. Note that if strict inequality holds in (6) for some $\bar{u} \in \mathcal{K}_y$ and $\bar{w} \in \text{cl}\mathcal{A}_y$, then with no loss of generality we can suppose that $\bar{w} \in \mathcal{A}_y$. Finally, (6) is equivalent to $0 \notin \text{qri}[\text{cone conv}(\mathcal{E}(\mathcal{A}_y))]$ by setting $M := \mathcal{E}(\mathcal{A}_y) = \mathcal{K}_y + \mathcal{A}_y$ in Theorem 4.1. □

We aim now to state sufficient conditions that ensure the image convexity of the system (1).

Theorem 4.2 *Let $\mathcal{A}_y$ be a convex cone such that (4) holds, let $\mathcal{H}_y$ be a convex set such that $\mathcal{H}_y = \text{cone}_+\mathcal{H}_y$, and assume that $F(\cdot; y)$ is generalized $\mathcal{A}_y$-subconvexlike on K. Then, if V is finite dimensional with $0 \notin \text{ri}\mathcal{H}_y$, or if $\text{int}\mathcal{A}_y \neq \emptyset$, then system (1) is image convex.*

Proof Assume that $\mathcal{K}_y \cap \mathcal{H}_y = \emptyset$; we must show that $\mathcal{K}_y$ and $\mathcal{H}_y$ are linearly separable. By Proposition 4.1 it is enough to prove that $\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ are linearly separable. By assumptions on the set $\mathcal{H}_y$, it follows that

$$\text{cone}_+\mathcal{K}_y \cap \mathcal{H}_y = \emptyset. \tag{7}$$

Let V be finite dimensional and $0 \notin \text{ri}\mathcal{H}_y$. Then (7) yields $\text{cone}\mathcal{K}_y \cap \text{ri}\mathcal{H}_y = \emptyset$ and so

$$0 \notin \text{cone}\mathcal{K}_y - \text{ri}\mathcal{H}_y = \text{cone}\mathcal{K}_y + \text{ri}\mathcal{A}_y - \text{ri}\mathcal{H}_y,$$

since V is finite dimensional and $\text{ri}\mathcal{H}_y = \text{ri}\mathcal{H}_y - \text{ri}\mathcal{A}_y$ in view of (4). Therefore

$$[\text{cone}\mathcal{K}_y + \text{ri}\mathcal{A}_y] \cap \text{ri}\mathcal{H}_y = \emptyset$$

and being $F(\cdot;y)$ generalized $\mathcal{A}_y$-subconvexlike on K, it follows that $\text{cone}\mathcal{K}_y + \text{ri}\mathcal{A}_y$ is a convex set which is linearly separable from $\mathcal{H}_y$, by the separation theorem for finite dimensional convex sets [29]. By Lemma 2.2 we have that

$$\text{cl}[\text{cone}\mathcal{K}_y + \text{ri}\mathcal{A}_y] = \text{cl}[\text{cone}(\mathcal{K}_y + \text{ri}\mathcal{A}_y)] = \text{cl}[\text{cone}(\mathcal{K}_y + \mathcal{A}_y)] = \text{clcone}(\mathcal{E}(\mathcal{A}_y)),$$

which implies that $\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ are linearly separable, i.e., (1) is image convex.

Assume now that $\text{int}\mathcal{A}_y \neq \emptyset$. Since (4) holds and $\text{int}\mathcal{A}_y \neq \emptyset$, it follows from Lemma 2.4 that $\text{int}\mathcal{H}_y = \mathcal{H}_y - \text{int}\mathcal{A}_y \neq \emptyset$. Similar to the proof of the previous part, noticing that because of (7) $0 \notin \text{int}\mathcal{H}_y$, we have

$$0 \notin \text{cone}\mathcal{K}_y - \text{int}\mathcal{H}_y = \text{cone}\mathcal{K}_y + \text{int}\mathcal{A}_y - \mathcal{H}_y$$

and, in turn,

$$[\text{cone}\mathcal{K}_y + \text{int}\mathcal{A}_y] \cap \text{int}\mathcal{H}_y = \emptyset.$$

Since $F(\cdot;y)$ generalized $\mathcal{A}_y$-subconvexlike on K, it follows that $\text{cone}\mathcal{K}_y + \text{int}\mathcal{A}_y$ is a convex set which is linearly separable from $\mathcal{H}_y$, by the separation theorem for convex sets [12]. By Lemma 2.2 $\text{cl}[\text{cone}\mathcal{K}_y + \text{int}\mathcal{A}_y] = \text{clcone}(\mathcal{E}(\mathcal{A}_y))$, which implies that $\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ are linearly separable, i.e., (1) is image convex. □

Remark 4.2 The assumption that $F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K can be replaced by $F(\cdot;y)$ $\mathcal{A}_y$-convexlike, $\mathcal{A}_y$-subconvexlike or $\mathcal{A}_y$-preconvexlike on K. In the first two cases $\mathcal{H}_y$ can be assumed to be any nonempty convex set, while in the third case the assumption $0 \notin \text{ri}\mathcal{H}_y$ is not needed.

Next example shows that the assumption $0 \notin \text{ri}\mathcal{H}_y$ is necessary for the validity of Theorem 4.2.

Example 4.1 Let $K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\}$, $F(\cdot;y) : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x_1, x_2, x_3; y) := (x_1, x_2, x_3)$ and $\mathcal{H}_y = \mathcal{A}_y := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. It is easy to see that

$$\mathcal{K}_y := F(K;y) = K, \quad \mathcal{A}_y = \text{ri}\mathcal{A}_y, \quad \text{cone}\mathcal{K}_y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}.$$

Then, $\text{cone}\mathcal{K}_y + \text{ri}\mathcal{A}_y = \mathbb{R}^3$ and F is generalized $\mathcal{A}_y$-subconvexlike on K, but $\mathcal{K}_y$ is not linearly separable from $\mathcal{H}_y$. Note that $0 \in \text{ri}\mathcal{H}_y$ and $\text{cone}_+\mathcal{K}_y + \mathcal{A}_y = \mathbb{R}^3 \setminus \mathcal{A}_y$ so that $F(\cdot;y)$ is not $\mathcal{A}_y$-preconvexlike on K.

Next result extends the analysis to the case where V is an infinite dimensional space and the interior of the set $\mathcal{A}_y$, and therefore of $\mathcal{H}_y$, is not necessarily nonempty.

Theorem 4.3 *Let V be a Hausdorff locally convex topological linear space. Let $\mathcal{A}_y$ be a convex cone such that* $\text{qri}\mathcal{A}_y \neq \emptyset$ *and* $\text{cl}\mathcal{A}_y = -\text{cl}\mathcal{H}_y$. *Then, if additionally $F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K, and* $0 \notin \text{qi}[\text{cl}(\text{cone } \mathcal{E}(\mathcal{A}_y))]$, *system* (1) *is image convex.*

Proof By Lemma 2.2 we have that

$$\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)] = \mathrm{cl}[\mathrm{cone}F(K;y) + \mathrm{qri}A_y].$$

Since $F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K, it follows from the previous equality that $\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)]$ is convex and so

$$\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)] = \mathrm{cl}[\mathrm{cone}\,\mathrm{conv}\mathcal{E}(A_y)]. \tag{8}$$

Indeed, the inclusion $\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)] \subseteq \mathrm{cl}[\mathrm{cone}\,\mathrm{conv}\mathcal{E}(A_y)]$ is obvious. For the reverse inclusion, we first note that if $E \subseteq V$ is a cone, so is $\mathrm{conv}E$. Let $t \geq 0$ and $a \in \mathrm{conv}E$. Then there are $a_i \in E$ and $t_i \in [0,1] (i = 1, \cdots, l)$ with $\sum_{i=1}^{l} t_i = 1$ such that $a = \sum_{i=1}^{k} t_i a_i$. Therefore, $ta = \sum_{i=1}^{l} t_i(ta_i) \in \mathrm{conv}E$ since E is a cone, which implies that $\mathrm{conv}E$ is a cone. Since $\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)]$ is convex, it follows that

$$\mathrm{cone}\,\mathrm{conv}\mathcal{E}(A_y) \subseteq \mathrm{conv}\,\mathrm{cone}\mathcal{E}(A_y) \subseteq \mathrm{conv}\,\mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)] = \mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)],$$

and taking the closure in the left-hand side, we obtain

$$\mathrm{cl}[\mathrm{cone}\ \mathrm{conv}\mathcal{E}(A_y)] \subseteq \mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)].$$

Therefore (8) is fulfilled and from the assumptions we have

$$\begin{aligned} 0 \notin \mathrm{qi}[\mathrm{cl}(\mathrm{cone}\,\mathcal{E}(\mathcal{A}_y))] \quad &\Leftrightarrow \quad \mathrm{cl}[\mathrm{cone}\mathcal{E}(A_y)] \neq V \\ &\Leftrightarrow \quad \mathrm{cl}[\mathrm{cone}\,\mathrm{conv}\mathcal{E}(A_y)] \neq V \\ &\Leftrightarrow \quad 0 \notin \mathrm{qi}[\mathrm{cone}\ \mathrm{conv}\mathcal{E}(\mathcal{A}_y)]. \end{aligned}$$

As a consequence, Proposition 4.2 yields that $\mathcal{K}_y$ and $\mathcal{H}_y$ are linearly separable, which completes the proof. □

Remark 4.3 The assumption that $F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K can be replaced by any of the following: $F(\cdot;y)$ is $\mathcal{A}_y$-convexlike on K, $F(\cdot;y)$ is $\mathcal{A}_y$-preconvexlike on K or $F(\cdot;y)$ is $\mathcal{A}_y$-subconvexlike on K. In the first two cases, the set $\mathrm{cone}\mathcal{E}(\mathcal{A}_y)$ is convex and the assumption $0 \notin \mathrm{qi}[\mathrm{cl}(\mathrm{cone}\ \mathcal{E}(\mathcal{A}_y))]$ is equivalent to $0 \notin \mathrm{qi}[\mathrm{cone}\mathcal{E}(\mathcal{A}_y)]$, which is equivalent to $0 \notin \mathrm{qi}\mathcal{E}(\mathcal{A}_y)$ if $0 \in \mathcal{E}(\mathcal{A}_y)$.

5 Applications

In this section, we shall apply the results obtained in Sections 3 and 4 to investigate vector quasi optimization problems and vector dynamic variational inequalities.

5.1 Applications to Vector Quasi Optimization Problems

One of the most important consequences of our analysis is that, for an image convex vector quasi optimization problem, the optimality of a weak vector quasi minimum point can be expressed by means of a saddle point condition under a Slater type constraint qualification.

Let $C(y), y \in X$, be a family of closed, pointed, convex cones in $\mathcal{Y}$ with nonempty quasi relative interiors, that define a variable ordering relation on the space $\mathcal{Y}$; let $K \subseteq X$, let $D(y), y \in X$ be a family of closed convex cones in $\mathcal{Z}$, which define the feasible values of the image $g(K, y)$ of the constraints, for a given y, and consider the vector quasi optimization problem (P):

$$\min_{C(y)} f(x) \quad \text{s.t.} \quad x \in R(y) := \{x \in K : g(x, y) \in D(y)\}.$$

Set $F(x; y) := (f(y) - f(x), g(x, y))$ and $\mathcal{H}_y := (\text{qri}C(y)) \times D(y)$. We say that $y \in Y := \{y \in X : y \in R(y)\}$ is a weak v.q.m.p. for (P) iff system (1) is impossible or equivalently, iff (3) holds. This definition of weak v.q.m.p. collapses to the classic one, given in Section 2, provided that $\text{int}C(y) \neq \emptyset$, $y \in Y$.

We remark that several of the results presented in this section do not require any additional assumption on the set K so that they can be applied to a wide class of problems including nonlinear integer vector optimization problems.

Vector optimization problems with a variable domination structure have extensively been considered in the literature (see e.g. [7, 30] and references therein) but in general the feasible set is fixed or it is not defined by means of explicit constraints as it happens for problem (P) . In particular, this allows us to obtain saddle point optimality conditions for the Lagrangian function $L_{y,\theta} : \mathcal{Z}^* \times K \to \mathbb{R}$ associated with (P) and defined by

$$L_{y,\theta}(\lambda, x) := \langle \theta, f(x) \rangle - \langle \lambda, g(x, y) \rangle, \quad \forall (\lambda, x) \in \mathcal{Z}^* \times K,$$

where $y \in K$ and $\theta \in \mathcal{Y}^*$.

Proposition 5.1 *Suppose that $y \in R(y)$ is a weak v.q.m.p. for* (P) *and that* (1) *is image convex. Then there exist $\theta^* \in C(y)^*$ and $\lambda^* \in D(y)^*$, $(\theta^*, \lambda^*) \neq 0$, such that (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$, i.e.,*

$$L_{y,\theta^*}(\lambda, y) \leq L_{y,\theta^*}(\lambda^*, y) \leq L_{y,\theta^*}(\lambda^*, x), \quad \forall (\lambda, x) \in D(y)^* \times K. \tag{9}$$

Proof From (viii) in Lemma 2.1, one has $(\text{qri}C(y))^* = C(y)^*$. The image convexity of (1) implies that there exist $\theta^* \in (\text{qri}C(y))^* = C(y)^*$ and $\lambda^* \in D(y)^*$, $(\theta^*, \lambda^*) \neq 0$, such that

$$\langle \theta^*, f(y) - f(x) \rangle + \langle \lambda^*, g(x, y) \rangle \leq 0, \quad \forall x \in K. \tag{10}$$

Since $y \in R(y)$, one has $\langle \lambda^*, g(y, y) \rangle \geq 0$, so that (10) implies $\langle \lambda^*, g(y, y) \rangle = 0$. This leads to the inequality $L_{y,\theta^*}(\lambda^*, y) \leq L_{y,\theta^*}(\lambda^*, x)$, $\forall x \in K$. The inequality $L_{y,\theta^*}(\lambda, y) \leq L_{y,\theta^*}(\lambda^*, y)$, $\forall \lambda \in D(y)^*$, is equivalent to $\langle \lambda, g(y, y) \rangle \geq 0$, $\forall \lambda \in D(y)^*$, which is fulfilled since $g(y, y) \in D(y)$. □

The following proposition provides a useful characterization of the saddle point condition.

Proposition 5.2 *Let $\mathcal{Z}$ be a locally convex topological linear space, let $\theta^* \in C(y)^*$, $\lambda^* \in D(y)^*$ and $y \in K$. Then (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$, iff,*

(i) $L_{y,\theta^*}(\lambda^*, y) \le L_{y,\theta^*}(\lambda^*, x), \quad \forall x \in K$;

(ii) $g(y, y) \in D(y)$;

(iii) $\langle \lambda^*, g(y, y)\rangle = 0$.

Proof It is immediate to prove that (i)-(iii) imply that (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$. Actually, (ii) and (iii) lead to

$$\begin{aligned} L_{y,\theta^*}(\lambda, y) &= \langle \theta^*, f(y)\rangle - \langle \lambda, g(y, y)\rangle \le \langle \theta^*, f(y)\rangle \\ &= \langle \theta^*, f(y)\rangle - \langle \lambda^*, g(y, y)\rangle = L_{y,\theta^*}(\lambda^*, y), \quad \forall \lambda \in D(y)^*. \end{aligned}$$

For the reverse implication, we first show that if (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$ then (ii) is fulfilled.

By the first inequality in the saddle point condition we have:

$$\langle \theta^*, f(y)\rangle - \langle \lambda, g(y, y)\rangle \le \langle \theta^*, f(y)\rangle - \langle \lambda^*, g(y, y)\rangle, \quad \forall \lambda \in D(y)^*. \tag{11}$$

Since $D(y)$ is a closed and convex cone in the locally convex topological linear space $\mathcal{Z}$, we have $D(y) = (D(y)^*)^*$ (see e.g., [12]). We declare that $g(y, y) \in D(y)$. In fact, if $g(y, y) \notin D(y)$, then there is $\bar{\lambda} \in D(y)^*$ such that $\langle \bar{\lambda}, g(y, y)\rangle < 0$. Since $D(y)^*$ is a cone, then $t\bar{\lambda} \in D(y)^*$ for all $t \ge 0$, and thus

$$-\langle t\bar{\lambda}, g(y, y)\rangle = -t\langle \bar{\lambda}, g(y, y)\rangle \to +\infty, \quad \text{as} \quad t \to +\infty,$$

which contradicts (11).

Let us prove (iii). Since $\lambda^* \in D(y)^*$ and $g(y, y) \in D(y)$, we have $\langle \lambda^*, g(y, y)\rangle \ge 0$. Setting $\lambda := 0$ in (11) leads to $\langle \lambda^*, g(y, y)\rangle \le 0$ and consequently $\langle \lambda^*, g(y, y)\rangle = 0$. □

From Proposition 5.1 it follows that the existence of a saddle point of the Lagrangian function associated with (P) is ensured by the image convexity of system (1).

The analysis developed in the previous sections leads us to state Lagrangian-type necessary optimality conditions. We consider, at first, the case where the image associated with (P) is finite dimensional or with nonempty interiors.

Theorem 5.1 *Let $F(x; y) := (f(y) - f(x), g(x, y))$ and $\mathcal{A}_y := -\mathrm{cl}\mathcal{H}_y$ and let $y \in R(y)$ be a weak v.q.m.p. for* (P)*. Suppose that $V := \mathcal{Y} \times \mathcal{Z}$ is finite dimensional or $\mathrm{int}C(y) \ne \emptyset$ and $\mathrm{int}D(y) \ne \emptyset$, and, moreover, that any of the following conditions holds:*

(a) *$F(\cdot; y)$ is $\mathcal{A}_y$-convexlike on K;*

(b) *$F(\cdot; y)$ is $\mathcal{A}_y$-subconvexlike on K;*

(c) *$F(\cdot; y)$ is $\mathcal{A}_y$-preconvexlike on K;*

(d) *$F(\cdot; y)$ is generalized $\mathcal{A}_y$-subconvexlike on K.*

Then there exist $\theta^* \in C(y)^*$ and $\lambda^* \in D(y)^*$, $(\theta^*, \lambda^*) \ne 0$, such that (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$.

Proof We preliminarily observe that any of the assumptions (a)-(c) guarantees that (d) is fulfilled. From (vi) in Lemma 2.1 it follows that $\mathcal{A}_y = -\mathrm{cl}\mathcal{H}_y = (-C(y)) \times (-D(y))$ and from (vii) in Lemma 2.1 one has

$$\mathcal{H}_y - \mathcal{A}_y = (\mathrm{qri}C(y) \times D(y)) + (C(y) \times D(y)) = \mathcal{H}_y. \tag{12}$$

Note that, since $C(y)$ is pointed, we have $0 \notin \mathrm{ri}\mathcal{H}_y = (\mathrm{ri}C(y)) \times (\mathrm{ri}D(y))$. Therefore by Theorem 4.2, we have that (1) is image convex and the result follows from Proposition 5.1. □

Remark 5.1 Let $F(x; y) := (f(y) - f(x), g(x, y))$, $\mathcal{A}_y := -\mathrm{cl}\mathcal{H}_y, \mathrm{int}C(y) \neq \emptyset$ and $\mathrm{int}D(y) \neq \emptyset$. If f and $g(\cdot, y)$ are $C(y)$-convex and $-D(y)$-convex on K, respectively, then any of the conditions (a)-(d) in Theorem 5.1 holds.

From assumptions, we have $0 \in \mathcal{A}_y$ and int $\mathcal{A}_y \neq \emptyset$. We only need to show if f and $g(\cdot, y)$ are $C(y)$-convex and $-D(y)$-convex on K, respectively, then $F(\cdot; y)$ is $\mathcal{A}_y$-convexlike and $\mathcal{A}_y$-preconvexlike on K.

We first prove the set $F(K; y) + \mathcal{A}_y = F(K; y) - [C(y) \times D(y)]$ is convex. Let $t \in]0, 1[, x_i \in K, c_i \in C(y)$ and $d_i \in D(y)(i = 1, 2)$ be given. Set

$$x := (1 - t)x_1 + tx_2, \ \ c := (1 - t)c_1 + tc_2 \text{ and } d := (1 - t)d_1 + td_2.$$

Since $K, C(y)$ and $D(y)$ are convex, $x \in K, c \in C(y)$ and $d \in D(y)$. Since f and $g(\cdot, y)$ are $C(y)$-convex and $-D(y)$-convex on K, respectively, one has $(1-t)f(x_1)+tf(x_2) \in f(x)+C(y)$ and $(1-t)g(x_1, y)+tg(x_2, y) \in g(x, y)-D(y)$. Consequently,

$$\begin{aligned}
& (1 - t)[F(x_1; y) - (c_1, d_1)] + t[F(x_2; y) - (c_2, d_2)] \\
= \ & (1 - t)[(f(y) - f(x_1), g(x_1, y)) - (c_1, d_1)] \\
& +t[(f(y) - f(x_2), g(x_2, y)) - (c_2, d_2)] \\
= \ & (f(y) - ((1 - t)f(x_1) + tf(x_2)) - ((1 - t)c_1 + tc_2), \\
& (1 - t)g(x_1, y) + tg(x_2, y) - ((1 - t)d_1 + td_2)) \\
= \ & (f(y) - ((1 - t)f(x_1) + tf(x_2)) - c, (1 - t)g(x_1, y) + tg(x_2, y) - d) \\
= \ & (f(y) - ((1 - t)f(x_1) + tf(x_2)), (1 - t)g(x_1, y) + tg(x_2, y)) - (c, d) \\
\in \ & (f(y) - f(x), g(x, y)) - [C(y) \times D(y)](\text{ since } C(y) \text{ and } D(y) \text{ are convex cones}) \\
= \ & F(x; y) - [C(y) \times D(y)] \\
\subseteq \ & F(K; y) - [C(y) \times D(y)],
\end{aligned}$$

and we obtain the convexity of $F(K; y) - [C(y) \times D(y)]$. It follows from Proposition 3.3 (i) that $F(\cdot; y)$ is $\mathcal{A}_y$-convexlike on K.

Since $F(K; y) + \mathcal{A}_y$ is convex, so is $\mathrm{cone}_+(F(K; y) + \mathcal{A}_y)$ and it follows from part (b) of Theorem 3.1 that $F(\cdot; y)$ is $\mathcal{A}_y$-preconvexlike on K, and hence, generalized $\mathcal{A}_y$-subconvexlike on K.

Next result is concerned with the case where the image associated with (P) is infinite dimensional with possibly empty interiors.

Theorem 5.2 *Let $F(x;y) := (f(y) - f(x), g(x,y))$ and $\mathcal{A}_y := -\text{cl}\mathcal{H}_y$. Suppose that $y \in R(y)$ is a weak v.q.m.p. for* (P), $\text{qri}D(y) \neq \emptyset$ *and that any of the following conditions holds:*

(a) *$F(\cdot;y)$ is $\mathcal{A}_y$-convexlike on K and $0 \notin \text{qi cone}\mathcal{E}(\mathcal{A}_y)$;*

(b) *$F(\cdot;y)$ is $\mathcal{A}_y$-preconvexlike on K and $0 \notin \text{qi cone}\mathcal{E}(A_y)$;*

(c) *$F(\cdot;y)$ is $\mathcal{A}_y$-subconvexlike on K, and $0 \notin \text{qi}[\text{cl}(\text{cone } \mathcal{E}(\mathcal{A}_y))]$;*

(d) *$F(\cdot;y)$ is generalized $\mathcal{A}_y$-subconvexlike on K, and $0 \notin \text{qi}[\text{cl}(\text{cone } \mathcal{E}(\mathcal{A}_y))]$.*

Then there exist $\theta^* \in C(y)^*$ and $\lambda^* \in D(y)^*$, $(\theta^*, \lambda^*) \neq 0$, such that (λ^*, y) is a saddle point for $L_{y,\theta^*}(\lambda, x)$ on $D(y)^* \times K$.

Proof We preliminarily observe that any of the assumptions (a)-(c) guarantees that (d) is fulfilled (see Remark 4.3). Similar to the proof of Theorem 5.1, we have that (12) is fulfilled and Theorem 4.3 yields that (1) is image convex. Thus the conclusion follows from Proposition 5.1. □

The saddle point condition does not guarantee that y is a weak v.q.m.p. of (P): to this aim, we have to ensure that the multiplier $\theta^* \in C(y)^*$ is non zero.

Next results state some important features of the quasi relative interior, which will be used in what follows. For a critical review of the properties of the quasi-relative interior of a set, see also [13].

Theorem 5.3 [9, Theorem 3.10] *Let V be a locally convex topological linear space partially ordered by a closed convex cone $P \subseteq V$ with $\text{cl}(P - P) = V$. Then*

$$y \in \text{qri}P \quad \Leftrightarrow \quad \langle y^*, y \rangle > 0, \ \forall y^* \in P^* \setminus \{0\}.$$

The following example shows that the closedness of the convex cone P in Theorem 5.3 cannot be removed.

Example 5.1 Let V be an infinite dimensional normed space, let $f : V \to \mathbb{R}$ be a non continuous linear functional and P be the null space of f, i.e., $P = \ker f$. Then P is a convex cone with $\text{cl}P = V$, which implies that $P^* = \{0\}$ and $P = \text{qi}P = \text{qri}P$. Note that, if we take $y \notin P$, then the right-hand side of the equivalence in Theorem 5.3 is fulfilled since $P^* = \{0\}$, but $y \notin \text{qri}P$.

Proposition 5.3 [28, Proposition 2.6] *Let V be a locally convex topological linear space and $P \subseteq V$ be a convex cone with $\text{cl}(P - P) = V$. Then $\text{qri}P = \text{qi}P$.*

Let $y \in Y$. We consider the following generalized Slater condition:

$$0 \in \text{qri}\,[\text{conv}(g(K,y) - D(y))] = \text{qri}\,[\text{conv}g(K,y) - D(y)]. \tag{13}$$

Note that the equality in (13) follows from the facts that $D(y)$ is convex and $\text{conv}(\mathcal{A} + \mathcal{B}) = \text{conv}\mathcal{A} + \text{conv}\mathcal{B}$ (see, e.g., [12]), where $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of $\mathcal{Z}$.

Remark 5.2 Note that (see, e.g., [11, 28]) the condition $0 \in \text{qi}\,[\text{conv}g(K,y) - D(y)]$ is equivalent to $0 \in \text{qri}\,[\text{conv}g(K,y) - D(y)]$ and $0 \in \text{qi}\,[(\text{conv}g(K,y) - D(y)) - (\text{conv}g(K,y) - D(y))]$. If $\text{qi}D(y) \neq \emptyset$, then from Lemma 2.1 (iv) and (ix) we have $\text{conv}g(K,y) - \text{qi}D(y) \subseteq \text{qi}\,[\text{conv}g(K,y) - D(y)] \neq \emptyset$ and so the generalized Slater condition (13) reduces to $0 \in \text{qi}\,[\text{conv}g(K,y) - D(y)]$, which implies that (13) is equivalent to the first two relations in Theorem 4.2 in [11]. If $\text{int}D(y) \neq \emptyset$, then by Lemma 2.4, the generalized Slater condition (13) collapses to $0 \in \text{conv}g(K,y) - \text{int}D(y)$, which is milder than the classical Slater condition $0 \in g(K,y) - \text{int}D(y)$.

Proposition 5.4 *Assume that $\mathcal{Y}$ and $\mathcal{Z}$ are locally convex topological linear spaces, $\mathcal{A}_y := -\text{cl}\mathcal{H}_y$. Suppose that $\text{qri}D(y) \neq \emptyset$, $\text{cl}(C(y)-C(y)) = \mathcal{Y}, \text{cl}(D(y)-D(y)) = \mathcal{Z}$. Consider the following statements:*

(i) *$\mathcal{K}_y$ and $\mathcal{H}_y$ are linearly separable;*

(ii) *$\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ are linearly separable;*

(iii) $0 \notin \text{qi}[\text{cone conv}(\mathcal{E}(\mathcal{A}_y))]$;

(iv) *$\mathcal{K}_y$ and $\mathcal{H}_y$ are properly linearly separable;*

(v) *$\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ are properly linearly separable;*

(vi) $0 \notin \text{qri}[\text{cone conv}(\mathcal{E}(\mathcal{A}_y))]$;

(vii) *$\mathcal{K}_y$ and $\mathcal{H}_y$ admit a regular linear separation, i.e., there exists $(\theta^*, \lambda^*) \in C(y)^* \times D(y)^*$ with $\theta^* \neq 0$ such that (10) holds, or equivalently,*

$$\langle \theta^*, u\rangle + \langle \lambda^*, v\rangle \leq 0, \quad \forall (u,v) \in \mathcal{K}_y;$$

(viii) *$\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ admit a regular linear separation, i.e., there exists $(\theta^*, \lambda^*) \in C(y)^* \times D(y)^*$ with $\theta^* \neq 0$ such that*

$$\langle \theta^*, u\rangle + \langle \lambda^*, v\rangle \leq 0, \quad \forall (u,v) \in \mathcal{E}(\mathcal{A}_y).$$

Then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi)$\Leftarrow$(vii)$\Leftrightarrow$(viii). *Furthermore, if the generalized Slater condition* (13) *holds, then* (i)-(viii) *are equivalent.*

Proof Since $\mathcal{A}_y := -\text{cl}\mathcal{H}_y$ and $\mathcal{H}_y = (\text{qri}C(y)) \times D(y)$, as in the proof of Theorem 5.1 it can be shown that (4) is fulfilled. Now Propositions 4.1-4.3 yield that (i)-(iii) are equivalent and (iv)-(vi) are equivalent. Moreover, by the proof of Proposition 4.1, it follows that a linear functional separates $\mathcal{E}(\mathcal{A}_y)$ and $\mathcal{H}_y$ iff it separates $\mathcal{K}_y$ and $\mathcal{H}_y$, so that (vii) and (viii) are equivalent. It is obvious that (vii)$\Rightarrow$(i). We show (i) and (iv) are equivalent. Clearly, (iv)$\Rightarrow$(i). Assume that (i) is true. Then there exists $(\theta^*, \lambda^*) \in \mathcal{H}_y^* \setminus \{0\} = C(y)^* \times D(y)^* \setminus \{0\}$ (by (viii) in Lemma 2.1) such that

$$\langle \theta^*, u\rangle + \langle \lambda^*, v\rangle \leq 0, \quad \forall (u,v) \in \mathcal{K}_y.$$

Since $\theta^* \neq 0$ or $\lambda^* \neq 0$ and, by assumptions, $\text{qi}\mathcal{H}_y = \text{qri}\mathcal{H}_y \neq \emptyset$, it follows from Theorem 5.3 that $\sup_{(u,v)\in\mathcal{H}_y}(\langle\theta^*, u\rangle + \langle\lambda^*, v\rangle) > 0$, which implies that (iv) holds.

Suppose that the generalized Slater condition (13) holds. We prove that (i)⇒(vii). We first note from Proposition 5.3 that $\text{qri}D(y) = \text{qi}D(y) \neq \emptyset$. Assume that (i) is true, i.e., there exists $(\theta^*, \lambda^*) \in C(y)^* \times D(y)^* \setminus \{0\}$ such that (10) holds. Ab absurdo, assume that $\theta^* = 0$ so that $\lambda^* \neq 0$. Then from (10) we have

$$\langle\lambda^*, v\rangle \leq 0, \quad \forall v \in g(K, y)$$

and so

$$\langle\lambda^*, v\rangle \leq 0, \quad \forall v \in \text{conv}g(K, y) - D(y),$$

which implies that

$$\langle\lambda^*, v\rangle \leq 0, \quad \forall v \in \text{cl}\,\text{cone}\,[\text{conv}g(K, y) - D(y)],$$

Since the generalized Slater condition (13) holds, from Remark 5.2 one has $\text{cl}\,\text{cone}\,[\text{conv}g(K, y) - D(y)] = \mathcal{Z}$ and thus it follows that $\lambda^* = 0$, a contradiction. This implies that (vii) holds. □

Theorem 5.4 *Assume that $\mathcal{Z}$ is a locally convex topological linear space. Let $y \in K$. Suppose that $\text{qri}D(y) \neq \emptyset$, $\text{cl}(C(y) - C(y)) = \mathcal{Y}, \text{cl}(D(y) - D(y)) = \mathcal{Z}$ and that the generalized Slater condition (13) holds. If there exist $\theta^* \in C(y)^*$ and $\lambda^* \in D(y)^*$, $(\theta^*, \lambda^*) \neq 0$, such that (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$, then y is a weak v.q.m.p. for (P).*

Proof Assume that (λ^*, y) is a saddle point for L_{y,θ^*} on $D(y)^* \times K$. By Proposition 5.2 it easily follows that the second inequality in (9) is equivalent to (10) and, in turn, to the condition

$$\langle\theta^*, u\rangle + \langle\lambda^*, v\rangle \leq 0, \quad \forall (u, v) \in \mathcal{K}_y.$$

Similarly to the proof in Proposition 5.4, we can show that $\theta^* \neq 0$. Since $\theta^* \neq 0$, from Theorem 5.3 we obtain

$$\langle\theta^*, u\rangle + \langle\lambda^*, v\rangle > 0, \quad \forall (u, v) \in (\text{qri}C(y)) \times D(y) = \mathcal{H}_y.$$

It follows that (3) holds and y is a weak v.q.m.p. for (P). □

Theorems 5.1, 5.2 and 5.4 generalize Theorem 6.15 in [27]. In case $C(y)$ and $D(y)$ are fixed cones for every $y \in X$ and the function g does not depend on y, then Theorem 5.4 is related to Theorem 3.2 in [17] stated for a set-valued opimization problem where the saddle point condition for the Lagrangian function is replaced by the equivalent characterization given in Proposition 5.2.

5.2 Applications to Vector Dynamic Variational Inequalities

Let $\mathscr{T} := [0,T]$ with $T > 0$ and $E_r := L^p(\mathscr{T}, \mathbb{R}^r)$, where $p > 0$ and $r \in \mathbb{N}$. The dual space of E_r, $L^q(\mathscr{T}, \mathbb{R}^r)$,, where $\frac{1}{q} + \frac{1}{p} = 1$, will be denoted by E_r^*. The canonical bilinear form on $E_r^* \times E_r$ is given by

$$\ll g, h \gg := \int_{\mathscr{T}} \langle g(t), h(t) \rangle dt, \quad \forall (g,h) \in E_r^* \times E_r.$$

Let $D_r := \{h \in E_r : h(t) \in \mathbb{R}^r_+, \text{ a.e. on } \mathscr{T}\}$. D_r is a closed convex cone in E_r with $D_r - D_r = E_r$. The polar of D_r is given by $D_r^* = \{V^* \in E_r^* : V^*(t) \in \mathbb{R}^r_+, \text{ a.e. on } \mathscr{T}\}$. It is known [9, 10] that $\text{qi}D_r = D_{r+} := \{g \in E_r : g(t) \in \text{int}\mathbb{R}^r_+, \text{ a.e. on } \mathscr{T}\}$.

Let $J := (J_1, ..., J_k)$, where $J_i \in E_r^*, \ i = 1, ..., k$ and $k \in \mathbb{N}$, and define

$$\begin{aligned} \ll J, h \gg_k &:= (\ll J_1, h \gg, \cdots, \ll J_k, h \gg) \\ &= \Big(\int_{\mathscr{T}} \langle J_1(t), h(t) \rangle dt, \cdots, \int_{\mathscr{T}} \langle J_k(t), h(t) \rangle dt \Big), \forall h \in E_r. \end{aligned}$$

In this section, without other specifications, let K be a nonempty convex subset of E_r, $T := (T_1, ..., T_k)$, where $T_i : E_r \to E_r^*$, $i = 1, ..., k$ and $C : E_r \to \mathbb{R}^k$ such that for each $H \in K$, $C(H)$ is a nonempty closed pointed convex cone in $\mathbb{R}^k$. Let $g : E_r \to E_z$ be $-D_z$-convex on X and $f : E_r \to E_w$ be affine, where $z, w \in \mathbb{N}$.

Let $\Omega := \{H \in K : g(H) \in D_z, f(H) = 0\}$, which is supposed to be nonempty. Since g is $-D_z$-convex on K and f is affine, Ω is convex. We consider the following vector dynamic variational inequality (for short, VDVI), with a variable ordering relation: find $\bar{H} \in \Omega$ such that

$$\ll T(\bar{H}), H - \bar{H} \gg_k \notin C(\bar{H}) \backslash \{0\}, \quad \forall H \in \Omega.$$

(VDVI) provides a generalization of time-dependent traffic equilibrium problem (see, e.g., [31–33]).

Let $\bar{H} \in \Omega$. Set

$$\begin{aligned} F(H; \bar{H}) &:= (\ll T(\bar{H}), H - \bar{H} \gg_k, g(H), f(H)), \\ \mathcal{H} &:= (C(\bar{H}) \backslash \{0\}) \times D_z \times \{0\}, \\ \mathcal{E} &:= \mathcal{K}_{\bar{H}} - \text{cl}\mathcal{H} = F(K; \bar{H}) - \text{cl}\mathcal{H} \\ &= (\ll T(\bar{H}), \cdot - \bar{H} \gg_k, g(\cdot), f(\cdot))(K) - [C(\bar{H}) \times D_z \times \{0\}]. \end{aligned}$$

For the sake of simplicity we avoid mentioning the dependence of the sets $\mathcal{E}$ and $\mathcal{H}$ on $\bar{H}$.

Since g is $-D_z$-convex on K and f is affine, then $\mathcal{E}$ is convex. We observe that (VDVI) is equivalent to (3) and from Proposition 3.1 one has the following characterization of the optimality condition of (VDVI) in the image space:

Proposition 5.5 *$\bar{H} \in \Omega$ is a solution of* (VDVI) *iff*

$$\mathcal{H} \cap \mathcal{K}_{\bar{H}} = \emptyset, \quad \text{or equivalently}, \quad \mathcal{H} \cap \mathcal{E} = \emptyset.$$

The following proposition plays an important role in investigating Lagrangian-type necessary and sufficient optimality conditions for (VDVI).

Theorem 5.5 *Let $\bar{H} \in \Omega$ be a solution of* (VDVI). *Then the following statements are true:*

(a) *If* $\mathrm{qri}\mathcal{E} \neq \emptyset$*, then* $\mathrm{qri}[(g, f)(K) - (D_z \times \{0\})] \neq \emptyset$*; Similarly, if* $\mathrm{qi}\mathcal{E} \neq \emptyset$*, then* $\mathrm{qi}[(g, f)(K) - (D_z \times \{0\})] \neq \emptyset$*;*

(b) $\mathrm{qri}(\mathrm{cl}\mathcal{H}) \neq \emptyset$.

Moreover, suppose $g : E_r \to E_z$ and $f : E_r \to E_w$ are affine with $g(x) := g_0(x) + U_0$ and $f(x) := f_0(x) + V_0$, where $g_0 : E_r \to E_z$ and $f_0 : E_r \to E_w$ are linear continuous operators, $U_0 \in E_z$ and $V_0 \in E_w$ are given. Then:

(c) *If* $\mathrm{qri}K \neq \emptyset$*, then* $\mathrm{qri}\mathcal{K}_{\bar{H}} \neq \emptyset$, $\mathrm{qri}\mathcal{E} \neq \emptyset$ *and* $\mathrm{qri}[(g, f)(K) - (D_z \times \{0\})] \neq \emptyset$*;*

(d) *If there exists a point $x_0 \in K$ such that* $(g(x_0)f(x_0)) \in \mathrm{qi}[(g, f)(K)]$ *and* $(g(x_0), f(x_0)) \in D_z \times \{0\}$*, or equivalently,* $\mathrm{qi}[(g, f)(K)] \cap (D_z \times \{0\}) \neq \emptyset$*, then* $(0, 0) \in \mathrm{qi}[(g, f)(K) - (D_z \times \{0\})]$.

Proof (a) The proof is similar to that in [34].
(b) Clearly, $\mathrm{cl}\mathcal{H} = C(\bar{H}) \times D_z \times \{0\}$. Since $C(\bar{H})$ is a nonempty closed pointed convex cone in $\mathbb{R}^k$, $\mathrm{qri}C(\bar{H}) = \mathrm{ri}C(\bar{H}) \neq \emptyset$. From [9, 10, 34], we have $\mathrm{qri}D_z = \mathrm{qi}D_z = D_{z+}$ and $\mathrm{qri}\{0\} = \{0\}$. As a consequence, (ii) in Lemma 2.1 allows that $\mathrm{qri}(\mathrm{cl}\mathcal{H}) = \mathrm{ri}C(\bar{H}) \times D_{z+} \times \{0\}$.
(c) Assume that $\mathrm{qri}K \neq \emptyset$. Since g_0 and f_0 are linear continuous operators, so is $H \mapsto (\ll T(\bar{H}), H \gg_k, g_0(H), f_0(H))$ and moreover $\mathcal{K}_{\bar{H}}$ is convex. From (iii) in Lemma 2.1 and Lemma 2.3 it follows that

$$\begin{aligned}
\mathrm{qri}\mathcal{K}_{\bar{H}} &= \mathrm{qri}[(\ll T(\bar{H}), \cdot - \bar{H} \gg_k, g(\cdot), f(\cdot))(K)] \\
&= \mathrm{qri}[(\ll T(\bar{H}), -\bar{H} \gg_k, U_0, V_0) + (\ll T(\bar{H}), \cdot \gg_k, g_0(\cdot), f_0(\cdot))(K)] \\
&= (\ll T(\bar{H}), -\bar{H} \gg_k, U_0, V_0) + \mathrm{qri}[(\ll T(\bar{H}), \cdot \gg_k, g_0(\cdot), f_0(\cdot))(K)] \\
&\supseteq (\ll T(\bar{H}), -\bar{H} \gg_k, U_0, V_0) + (\ll T(\bar{H}), \cdot \gg_k, g_0(\cdot), f_0(\cdot))(\mathrm{qri}K) \\
&= (\ll T(\bar{H}), \cdot - \bar{H} \gg_k, g(\cdot), f(\cdot))(\mathrm{qri}K) \neq \emptyset.
\end{aligned}$$

Since $\mathcal{K}_{\bar{H}}$ is convex and $\mathcal{E} = \mathcal{K}_{\bar{H}} - \mathrm{cl}\mathcal{H}$, it follows from (i) and (iv) in Lemma 2.1 and (b) that

$$\mathrm{qri}\mathcal{E} = \mathrm{qri}(\mathcal{K}_{\bar{H}} - \mathrm{cl}\mathcal{H}) \supseteq \mathrm{qri}\mathcal{K}_{\bar{H}} - \mathrm{qri}(\mathrm{cl}\mathcal{H}) \neq \emptyset$$

and therefore $\mathrm{qri}[(g, f)(K) - D_z \times \{0\}] \neq \emptyset$ in view of (a).
(d) Notice that the condition that there is $x_0 \in K$ such that $(g(x_0)f(x_0)) \in \mathrm{qi}[(g, f)(K)]$ and $(g(x_0)f(x_0)) \in (D_z \times \{0\})$ is equivalent to $\mathrm{qi}[(g, f)(K)] \cap (D_z \times$

$\{0\}) \neq \emptyset$. If $\mathrm{qi}[(g,f)(K)] \cap (D_z \times \{0\}) \neq \emptyset$, then $(0,0) \in \mathrm{qi}[(g,f)(K)] - (D_z \times \{0\})$. As a consequence, from (ix) in Lemma 2.1 one has

$$\mathrm{qi}[(g,f)(K) - (D_z \times \{0\})] \quad \supseteq \quad \mathrm{qi}[(g,f)(K)] - D_z \times \{0\} \ni (0,0).$$

The proof is complete. □

The following proposition illustrates that under suitable conditions, the assumption in (d) of Theorem 5.5 holds.

Proposition 5.6 *Suppose $g : E_r \to E_z$ and $f : E_r \to E_w$ are affine with $g(x) := g_0(x) + U_0$ and $f(x) := f_0(x) + V_0$, where $g_0 : E_r \to E_z$ and $f_0 : E_r \to E_w$ are linear continuous operators, $U_0 \in E_z$ and $V_0 \in E_w$ are given. Then:*

(i) *If $\mathrm{qri}K \neq \emptyset$, $\mathrm{qri}[(g_0,f_0)(K)] \neq \emptyset$ and $(g_0,f_0)(\mathrm{qri}K) \cap ((D_z - U_0) \times \{-V_0\}) \neq \emptyset$, then $\mathrm{qri}[(g,f)(K)] \cap (D_z \times \{0\}) \neq \emptyset$.;*

(ii) *If $\mathrm{qi}K \neq \emptyset$, $\mathrm{qi}[(g_0,f_0)(K)] \neq \emptyset$ and $(g_0,f_0)(\mathrm{qi}K) \cap ((D_z - U_0) \times \{-V_0\}) \neq \emptyset$, then $\mathrm{qi}[(g,f)(K)] \cap (D_z \times \{0\}) \neq \emptyset$.*

Proof We declare that $\mathrm{qri}[(g_0,f_0)(K)] + (U_0,V_0) = \mathrm{qri}[(g,f)(K)]$. In fact, from (iii) in Lemma 2.1 we have

$$\mathrm{qri}[(g_0,f_0)(K)] = \mathrm{qri}[(g,f)(K) - (U_0,V_0)] = \mathrm{qri}[(g,f)(K)] - (U_0,V_0).$$

This implies that $\mathrm{qri}[(g_0,f_0)(K)] + (U_0,V_0) = \mathrm{qri}[(g,f)(K)]$. If $\mathrm{qi}[(g_0,f_0)(K)] \neq \emptyset$, then from (ix) in Lemma 2.1 we have that $\mathrm{qi}[(g_0,f_0)(K)] + (U_0,V_0) = \mathrm{qi}[(g,f)(K)]$.

(i) Since g_0 and f_0 are linear continuous operators, so is $H \mapsto (g_0(H), f_0(H))$. If $\mathrm{qri}K \neq \emptyset$, $\mathrm{qri}[(g_0,f_0)(K)] \neq \emptyset$ and $(g_0,f_0)(\mathrm{qri}K) \cap ((D_z - U_0) \times \{-V_0\}) \neq \emptyset$, then from above we have $\mathrm{qri}[(g,f)(K)] = \mathrm{qri}[(g_0,f_0)(K)] + (U_0,V_0) \neq \emptyset$ and from Lemma 2.3 it follows that

$$\begin{aligned} \mathrm{qri}[(g,f)(K)] - (D_z \times \{0\}) &= \mathrm{qri}[(g_0,f_0)(K)] + (U_0,V_0) - (D_z \times \{0\}) \\ &\supseteq (g_0,f_0)(\mathrm{qri}K) + (U_0,V_0) - (D_z \times \{0\}) \\ &= (g_0,f_0)(\mathrm{qri}K) - (D_z - U_0) \times \{-V_0\} \ni (0,0). \end{aligned}$$

(ii) Assume that $\mathrm{qi}K \neq \emptyset$, $\mathrm{qi}[(g_0,f_0)(K)] \neq \emptyset$ and $(g_0,f_0)(\mathrm{qi}K) \cap ((D_z - U_0) \times \{-V_0\}) \neq \emptyset$. Notice that $\mathrm{qri}K = \mathrm{qi}K \neq \emptyset$. The rest of the proof is similar to that in (i). □

Next, we aim to prove that there exists $(\theta^*, U^*, V^*) \in N_{\mathcal{E}}(0,0,0)$ with $(\theta^*, U^*) \in (C(\bar{H})^* \backslash \{0\}) \times D_z^*$.

Theorem 5.6 *Let $\bar{H} \in \Omega$ be a solution of* (VDVI)*, let g be $-D_z$-convex on K and f be affine. If $(0,0,0) \notin \mathrm{qi}\mathcal{E}$ and*

$$(0,0) \in \mathrm{qi}[(g,f)(K) - (D_z \times \{0\})], \tag{14}$$

then $\mathcal{K}_{\bar{H}}$ and $\mathcal{H}$ admit a regular linear separation, i.e., there exists $(\theta^*, U^*, V^*) \in (C(\bar{H})^* \backslash \{0\}) \times {D_z}^* \times E_w^*$ such that

$$\langle \theta^*, \ll T(\bar{H}), H - \bar{H} \gg_k \rangle + \ll U^*, g(H) \gg + \ll V^*, f(H) \gg \leq 0, \quad \forall H \in K. \tag{15}$$

Proof From the hypotheses that g is $-D_z$-convex on K and f is affine it follows that $(g,f)(K)-(D_z\times\{0\})$ and $\mathcal{E}$ are convex sets (see Remark 5.1).

Since $\bar{H}\in\Omega$ is a solution of (VDVI), one has $(0,0,0)\in\mathcal{E}$ and since $(0,0,0)\notin$ qi$\mathcal{E}$, Theorem 4.3 and Remark 4.3 allow that $\mathcal{H}$ and $\mathcal{K}_{\bar{H}}$ are linearly separable, that is, there is $(\theta^*,U^*,V^*)\in\mathcal{H}^*\backslash\{(0,0,0)\}=(C(\bar{H})^*\times D_z^*\times E_w^*)\backslash\{(0,0,0)\}$ such that (15) holds. It follows that for any $H\in K,\alpha\in C(\bar{H})$ and $U\in D_z$,

$$\langle\theta^*,\ll T(\bar{H}),H-\bar{H}\gg_k-\alpha\rangle+\ll U^*,g(H)-U\gg+\ll V^*,f(H)\gg\leq 0. \quad (16)$$

Recalling that $(0,0,0)\in\mathcal{E}$, it follows from (16) that $(\theta^*,U^*,V^*)\in N_{\mathcal{E}}(0,0,0)$.

It remains to prove $\theta^*\neq 0$. Suppose to the contrary that $\theta^*=0$. Then from (16) it follows that

$$\ll U^*,g(H)-U\gg+\ll V^*,f(H)\gg\leq 0,\quad \forall H\in K,\forall U\in D_z,$$

or equivalently, $(U^*,V^*)\in N_{(g,f)(K)-(D_z\times\{0\})}(0,0)$. From (14) it follows that $N_{(g,f)(X)-D_z\times\{0\}}(0,0)=\{(0,0)\}$ and thus, $(U^*,V^*)=(0,0)$. Consequently, $(\theta^*,U^*,V^*)=(0,0,0)$, a contradiction. This completes the proof. □

Remark 5.3 From Theorem 5.6, one can see that the condition (14) guaranteed by qi$[(g,f)(K)]\cap(D_z\times\{0\})\neq\emptyset$ (see Theorem 5.5 (d)) plays an important role in deriving the regular linear separation between $\mathcal{K}_{\bar{H}}$ and $\mathcal{H}$.

Next proposition provides a further sufficient condition for (14).

Proposition 5.7 *Let g be $-D_z$-convex on K, f be affine and the following assumptions hold:*

$$\exists\bar{H}\in K\quad s.t.\quad g(\bar{H})\in \mathrm{qi}D_z=\mathrm{qri}D_z\quad and\ f(\bar{H})=0, \quad (17)$$

and $f(K)=E_w$. Then, (14) is fulfilled.

Proof Since g is $-D_z$-convex on K and f is affine, $(g,f)(K)-(D_z\times\{0\})$ is a convex set (see Remark 5.1).

Condition (17) implies that $(0,0)\in(g,f)(K)-(D_z\times\{0\})$. Ab absurdo, assume that (14) is not fulfilled. This is equivalent to say that

$$N_{(g,f)(K)-(D_z\times\{0\})}(0,0)\neq\{(0,0)\},$$

that is, there exists $(U^*,V^*)\in {E_z}^*\times E_w^*\backslash\{(0,0)\}$ such that

$$\ll U^*,g(H)-U\gg+\ll V^*,f(H)\gg\leq 0,\quad \forall H\in K,\forall U\in D_z, \quad (18)$$

Computing (18) for $H:=\bar{H}$ we obtain:

$$-\ll U^*,U\gg+\ll U^*,g(\bar{H})\gg\leq 0,\quad \forall U\in D_z. \quad (19)$$

We declare that $U^*\in {D_z}^*$. Suppose to the contrary that $U^*\notin {D_z}^*$. Then there is $\bar{U}\in D_z$ such that $\ll U^*,\bar{U}\gg<0$. Letting $U:=t\bar{U},t>0$, in (19) leads to

$$+\infty\leftarrow-\ll U^*,t\bar{U}\gg+\ll U^*,g(\bar{H})\gg\leq 0$$

as $t \to +\infty$, a contradiction. Setting $U := 0$ in (19) we obtain

$$\ll U^*, g(\bar{H}) \gg \leq 0.$$

Since $g(\bar{H}) \in \mathrm{qi}D_z$, by Theorem 5.3 it follows that $U^* = 0$: actually, if $U^* \in {D_z}^* \backslash \{0\}$ we should have $\ll U^*, g(\bar{H}) \gg > 0$. Therefore $U^* = 0$ and (18) becomes

$$\ll V^*, f(H) \gg \leq 0, \quad \forall H \in K.$$

This implies that $V^* = 0$, since $f(K) = E_w$. Thus, $(U^*, V^*) = (0, 0)$, a contradiction, which completes the proof. □

By the previous proposition and Theorem 5.6, we obtain the following result.

Corollary 5.1 *Assume that g is $-D_z$-convex on K, f is affine with $f(K) = E_w$, $(0,0,0) \notin \mathrm{qi}\mathcal{E}$ and condition* (17) *is fulfilled. If $\bar{H} \in \Omega$ is a solution of* (VDVI)*, then $\mathcal{K}_{\bar{H}}$ and $\mathcal{H}$ admit a regular linear separation.*

Proof It follows from Proposition 5.7 and Theorem 5.6. □

From the previous results it follows that the existence of a saddle point for the Lagrangian function $L_{\bar{H},\theta^*} : D_z^* \times E_w^* \times K \to \mathbb{R}$ associated with (VDVI) defined by

$$L_{\bar{H},\theta^*}(U, V, H) := \langle \theta^*, \ll T(\bar{H}), H - \bar{H} \gg_k \rangle + \ll U, g(H) \gg + \ll V, f(H) \gg$$

is a necessary optimality condition for (VDVI), where $\bar{H} \in K$ and $\theta^* \in \mathbb{R}^k$.

Corollary 5.2 *Let $\bar{H} \in \Omega$ be a solution of* (VDVI)*, let g be $-D_z$-convex on K and f be affine. If $(0,0,0) \notin \mathrm{qi}\mathcal{E}$ and* (14) *holds, then there exists $(\theta^*, U^*, V^*) \in (C(\bar{H})^* \backslash \{0\}) \times {D_z}^* \times E_w^*$ such that $(U^*, V^*, \bar{H})$ is a saddle point for $L_{\bar{H},\theta^*}$ on ${D_z}^* \times E_w^* \times K$, i.e.,*

$$L_{\bar{H},\theta^*}(U^*, V^*, H) \leq L_{\bar{H},\theta^*}(U^*, V^*, \bar{H}) \leq L_{\bar{H},\theta^*}(U, V, \bar{H}), \quad \forall (U, V, H) \in {D_z}^* \times E_w^* \times K. \tag{20}$$

Proof Since $\bar{H} \in \Omega$ and (14) holds, from Theorem 5.6 one has (15) is fulfilled and moreover $\ll U^*, g(\bar{H}) \gg \geq 0$. Putting $\bar{H}$ in (15) we get

$$\ll U^*, g(\bar{H}) \gg \leq 0,$$

which implies $\ll U^*, g(\bar{H}) \gg = 0$ and in turn $L_{\bar{H},\theta^*}(U^*, V^*, \bar{H}) = 0$, so that the first inequality in (20) is equivalent to (15). The second inequality in (20)

$$\ll U, g(\bar{H}) \gg + \ll V, f(\bar{H}) \gg \geq 0, \quad \forall (U, V) \in {D_z}^* \times E_w^*,$$

follows from the fact that $\bar{H} \in \Omega$. □

Finally, we remark that, similarly to (VDVI), it is possible to consider a weak vector dynamic variational inequality (for short, WVDVI): find $\bar{H} \in \Omega$ such that

$$\ll T(\bar{H}), H - \bar{H} \gg_k \notin \mathrm{int}C(\bar{H}), \quad \forall H \in \Omega,$$

where we assume that $\text{int}C(\bar{H}) \neq \emptyset$.

We note that the necessary optimality conditions obtained for (VDVI) in Theorem 5.6, Corollaries 5.1 and 5.2 also hold in case $\bar{H}$ is a solution of (WVDVI). Moreover, it can be shown the saddle point condition (20) is a sufficient optimality condition for (WVDVI) provided that $\theta^* \neq 0$, while the sufficient optimality conditions for (VDVI) can also be obtained under stronger regularity conditions (see, for example, [4, 33]).

Theorem 5.7 *Let g be $-D_z$-convex on K and f be affine. Assume that* (14) *holds and that there exists $(\theta^*, U^*, V^*) \in (C(\bar{H})^* \backslash \{0\}) \times {D_z}^* \times E_w^*$ such that $(U^*, V^*, \bar{H})$ is a saddle point for $L_{\bar{H},\theta^*}$ on ${D_z}^* \times E_w^* \times K$, i.e.,* (20) *holds. Then $\bar{H}$ is a solution of* (WVDVI).

6 Conclusions

We have investigated image convexity properties of a generalized system with infinite dimensional image by exploiting the quasi relative interior and we have obtained necessary and/or sufficient conditions for the impossibility of this generalized system. Moreover, we have applied these new results to investigate vector quasi optimization problems and vector dynamic variational inequalities.

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