AN ELEMENTARY PROOF OF UNIQUENESS OF THE PARTICLE TRAJECTORIES FOR SOLUTIONS OF A CLASS OF SHEAR-THINNING NON-NEWTONIAN 2D FLUIDS

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ABSTRACT. We prove some regularity results for a class of two dimensional non-Newtonian fluids. By applying results from [Dashti and Robinson, *Nonlinearity*, 22 (2009), 735-746] we can then show uniqueness of particle trajectories.

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1. INTRODUCTION

In this paper we consider the following system of partial differential equations

$$u_t - \nu_0 \Delta u - \nu_1 \operatorname{div} S(\mathcal{D}u) + (u \cdot \nabla)\, u + \nabla \pi = f \quad \text{in} \quad [0,T] \times \Omega, \tag{1.1a}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad [0,T] \times \Omega, \tag{1.1b}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega, \tag{1.1c}$$

where Ω denotes either a two-dimensional bounded domain or the two dimensional flat torus, the vector field $u = (u_1, u_2)$ is the velocity, the scalar π is the kinematic pressure, the vector $f = (f_1, f_2)$ is the external body force, u_0 is the initial velocity, and ν_0, ν_1 are positive constants. We denote by

$$\mathcal{D}u := \frac{1}{2}(\nabla u + \nabla u^T) = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{for } i,j = 1,2,$$

the symmetric part of ∇u, the convective term is $(u \cdot \nabla)\, u := \sum_{k=1}^2 u_k \partial_k u$, and S denotes the extra stress tensor, defined by

$$S(\mathcal{D}u) := (\delta + |\mathcal{D}u|)^{p-2} \mathcal{D}u, \qquad p \in [1,2), \tag{1.2}$$

where δ is a non-negative constant. System (1.1) describes a shear-thinning homogeneous fluid and for an introduction to the mathematical theory see Málek, Rajagopal, and Růžička [19]. We mainly study the problem, endowed with homogeneous Dirichlet boundary conditions

$$u_{|\Gamma} = 0 \quad \text{where} \quad \Gamma = \partial\Omega, \tag{1.3}$$

but we give some remarks also on the periodic case.

The main goal of this paper is to study the problem of uniqueness for the *particle trajectories* (or characteristics), which are solutions of the following Cauchy problem

$$\begin{cases} \dfrac{dX(t)}{dt} = u(X(t),t) \qquad t \in [0,T], \\ X(0) = x \in \Omega, \end{cases} \tag{1.4}$$

where u is the fluid velocity in (1.1). For the 3D Navier-Stokes equations the problem of existence of particle trajectories and Lagrangian representation of the flow started with the work of Foias, Guillopé, and Temam [13], and related results

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of regularity in $\mathbb{R}^n$ are proved in Chemin and Lerner [6] by means of Littlewod-Paley decomposition. The question of uniqueness has been recently addressed by elementary tools and in a more general context in Robinson *et al.* [10, 21, 22] and it is strictly related with uniqueness for linear transport equations. We consider here the same problem, in the case of shear-thinning fluids, described by (1.1). To this end, we will study certain regularity properties of the solutions of (1.1), investigating when the velocity will verify the appropriate hypotheses for uniqueness results.

In particular, classical results concerning Lipschitz continuous fields u (which generally can be verified checking that ∇u is bounded in the space variables) are not easily applicable here, since such a regularity is very difficult to be proved, even in the two dimensional case, for (1.1). We recall that, *restricting to the two dimensional case*, some $C^{1,\gamma}$-results are obtained in Kaplický, Málek, and Stará [14, 15] in the stationary case. Early results in the time dependent case (but not up-to-the-boundary) are those by Seregin [23], while results in the space-periodic time-dependent case have been obtained in [16]. We observe that essentially all the above results require the extra-stress tensor S to be slightly smoother than that in (1.2). In particular, it is requested that the stress-tensor is replaced, for instance, by $S(\mathcal{D}u) = (\delta+|\mathcal{D}u|^2)^{\frac{p-2}{2}}\mathcal{D}u$. In any case we study the regularity up-to-the-boundary with non-smooth initial data and our results, proved in an elementary way, are original. Moreover, the difficulties appearing in the 3D case seem completely out of the current mathematical knowledge for such fluids, and this explains why we restrict to the two dimensional case.

Since we want to have *elementary* proofs (in order to possibly extend the results to the widest possible class of solutions and stress-tensors) we will work with the classical energy-type methods. Concerning uniqueness of particle trajectories, there have been some recent improvements, strictly related with the Osgood criterion and with Log-Lipschitz properties of Sobolev functions $W^{s+1,q}(\mathbb{R}^d)$ in the case of limiting Sobolev exponents such that $q = \frac{d}{s}$. In particular we will use the result below, proved in [10, Theorem 2.1].

Theorem 1.1. *Let Ω be either the whole space $\mathbb{R}^d$, $d \geq 2$, a periodic d-dimensional domain, or an open bounded subset of $\mathbb{R}^d$ with a sufficiently smooth boundary. Let assume that for some $p > 1$*

$$u \in L^p(0,T; W^{\frac{d-2}{2},2}(\Omega)) \quad \textit{and} \quad \sqrt{t}\, u \in L^2(0,T; W^{\frac{d+2}{2},2}(\Omega)),$$

with $u_{|\Gamma} = 0$, when Ω is a domain with boundary. Then, the Cauchy problem (1.4) *has a unique solution in* $[0,T]$.

The latter result shows that certain (slightly weaker than $C^{1,\gamma}$) results of Sobolev space-regularity can be used to obtain uniqueness for (1.4). On the other hand, the $W^{2,2}(\mathbb{R}^2)$ regularity for fluid with shear-dependent viscosity is another non-trivial task (while in 3D proving $u \in W^{5/2,2}(\mathbb{R}^3)$, seems at the moment out of sight). Some recent results (in the stationary case) for second-order space-derivatives appeared in [2, 4, 9] even if the square integrability of second order derivatives is not reached in general domains, or if certain limitations on the smallness of the force are not satisfied. For the non-stationary case, we recall the result in the space periodic setting (obtained uniformly in $\delta \geq 0$) from [5, 11].

We also point out that one of the main technical obstructions is represented by the pressure and the associated divergence-free constraint. In the case of the p-Laplacian systems, in fact, the recent results in Beirão da Veiga and Crispo [3] show that $u \in W^{2,q}(\Omega)$, for arbitrary q, if f is smooth, and under certain restrictions on the range of $p \in (1,2)$. These latter results are proved in the stationary case, they have no counterpart for the p-Stokes system, and most likely they can be adapted to the time-dependent case.

We point out that in the case of non-Newtonian fluids many features of the problem are critical: The type of boundary conditions, the range of p, and if the parameter δ is strictly larger than zero. We will discuss later on some of the technical issues of the problem and we will explain why we have to reduce to the 2D case with $\nu_0, \delta > 0$. We start by considering the easier case of the periodic setting where Ω is the flat 2D torus $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}$ and we will prove the following result.

Proposition 1.1. *Let $\nu_0 > 0$, $\delta \geq 0$, and $p \in (1, 2]$. Let be given $u_0 \in L^2(\mathbb{T}^2)$ such that $\operatorname{div} u_0 = 0$ and $f \in L^2(0, T; L^2(\mathbb{T}^2))$. Then, weak solutions to (1.1) satisfy $\sqrt{t}\, u \in L^2(0, T; W^{2,2}(\mathbb{T}^2))$ and hence problem (1.4) admits a unique solution.*

We emphasize that the assumption $\nu_0 > 0$ is crucial in our method. When $\nu_0 = 0$ it is possible to prove a regularity result that, although it is not useful to get an application of Theorem 1.1, seems interesting by itself. See Prop. 3.1, cf. Kost [17].

In the Dirichlet case the problem of regularity is more delicate. We will consider problem (1.1) in a domain with flat boundary. We first prove a regularity result, by using techniques similar to those used in [8] and formerly introduced, for the case $p > 2$, in [1]. With smooth data, we have the following result.

Proposition 1.2. *Let $\delta > 0$, $\nu_0 > 0$, $p \in \left[\frac{3}{2}, 2\right]$, $u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ with $\operatorname{div} u_0 = 0$, and $f \in W^{1,2}(0, T; L^2(\Omega))$. Then, weak solutions to Problem (1.1)-(1.3) satisfy*

$$\begin{aligned} \|u_t\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \pi\|_{L^2(0,T;L^2)} \\ + \|\nabla u_t\|_{L^2(0,T;L^2)} + \|D^2 u\|_{L^2(0,T;L^2)} \leq C, \end{aligned} \tag{1.5}$$

where C depends on p, δ, ν_0, ν_1, $\|f\|_{W^{1,2}(0,T;L^2)}$, $\|u_0\|_{2,2}$, T, and Ω.

Some hypotheses can be relaxed, since the time regularity is unnecessary for the proof of uniqueness of particle trajectories, but the arguments used to prove Proposition 1.2 will play a fundamental role to demonstrate our main uniqueness criterion for the problem (1.4). The main result of this paper reads as follows:

Theorem 1.2. *Let $\delta > 0$, $\nu_0 > 0$, $p \in \left[\frac{3}{2}, 2\right]$, $u_0 \in L^2(\Omega)$ with $\operatorname{div} u_0 = 0$, such that $(u_0 \cdot n)_{|\Gamma} = 0$, and $f \in L^2(0, T; L^2(\Omega))$. Then, weak solutions to (1.1)-(1.3) satisfy $\sqrt{t}\, u \in L^2(0, T; W^{2,2}(\Omega))$, and consequently (1.4) admits a unique solution.*

Plan of the paper. In Section 2 we introduce the notation and we give some preliminary results. In Section 3, we consider the space-periodic setting and we prove Proposition 1.1. Thereafter, in Section 4, we prove a preliminary space-time regularity result for the solutions of (1.1)-(1.3) and then we demonstrate Proposition 1.2. Finally, in Section 5, we give the proof of Theorem 1.2.

2. Preliminaries and basic results

Let us introduce the notation related especially to the problem (1.1) with Dirichlet boundary conditions. The needed assumptions or changes for the space periodic case are specified in Section 3.

Throughout the article, when Ω is a bounded domain with boundary, it will be a two dimensional cube $\Omega =]-1, 1[^2$ and we denote by Γ the two opposite sides in the x_2 direction

$$\Gamma := \{x = (x_1, x_2) \colon |x_1| < 1,\, x_2 = -1\} \cup \{x = (x_1, x_2) \colon |x_1| < 1,\, x_2 = 1\},$$

We use the following boundary conditions

$$
\begin{cases} u_{|\Gamma} = 0, \\ u \quad \text{is} \quad 2\text{-periodic w.r.t } x_1. \end{cases} \tag{2.1}
$$

Here, x_1 represents the tangential direction to Γ and this idealized setting of a "periodic strip" corresponds to the half-space, but without complications at infinity.

Given $q \geq 1$, by $L^q(\Omega)$, we indicate the usual Lebesgue space with norm $\|\cdot\|_q$. Moreover, by $W^{k,q}(\Omega)$, k a non-negative integer and q as before, we denote the usual Sobolev space with norm $\|\cdot\|_{k,q}$. We also denote by $W_0^{1,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$ and by $W^{-1,q'}(\Omega)$, $q' = q/(q-1)$, the dual of $W_0^{1,q}(\Omega)$ with norm $\|\cdot\|_{-1,q'}$. Let X be a real Banach space with norm $\|\cdot\|_X$. We will use the customary spaces $W^{k,q}(0,T;X)$, with norm denoted by $\|\cdot\|_{W^{k,q}(0,T;X)}$, recalling that $W^{0,q}(0,T;X) = L^q(0,T;X)$. We will also use the notation $\Omega_T := \Omega \times (0,T)$ and we will not distinguish between scalar and vector fields and the symbol $\langle \cdot\, , \cdot \rangle$ will indicate a duality pairing. Here and in the sequel, we denote by C positive constants that may assume different values, even in the same equation. We also define

$$
V_q := \big\{ v \in W^{1,q}(\Omega) : \ \nabla \cdot v = 0,\ v_{|\Gamma} = 0,\ v \text{ is 2-periodic w.r.t. } x_1 \big\},
$$

with dual space V_q'. Since the extra-stress tensor S is a function not of the gradient, but of the deformation tensor (in order to have frame invariant equations) we recall a Korn-type inequality, see [8]

Lemma 2.1. *There exists a positive constant $C = C(\Omega)$ such that*

$$
\|v\|_q + \|\nabla v\|_q \leq C \|\mathcal{D}v\|_q, \quad \textit{for each} \quad v \in V_q.
$$

We write $f \simeq g$, if there exist $c_0, c_1 > 0$ such that $c_0 f \leq g \leq c_1 f$. When considering the tensor $S(D) = (\delta + |D^{sym}|)^{p-2} D^{sym}$, introduced in (1.2) (where D is a second order tensor and D^{sym} its symmetric part) it can be easily checked that for any second order tensor C, the following relations are verified

$$
\sum_{i,j,k,l=1}^{2} \partial_{kl} S_{ij}(D) C_{ij} C_{kl} \geq (p-1)(\delta + |D^{sym}|)^{p-2} |C|^2, \tag{2.2a}
$$

$$
|\partial_{kl} S_{ij}(D)| \leq (3-p)(\delta + |D^{sym}|)^{p-2}. \tag{2.2b}
$$

The symbol $\partial_{kl} S_{ij}$ represents the partial derivative $\partial S_{ij}/\partial D_{kl}$ of the (i,j)-component of S with respect to the (k,l)-component of the underlying space of 2×2 matrices. Monotonicity and growth properties of S are characterized in the following standard lemma.

Lemma 2.2. *Assume that $p \in (1,\infty)$ and $\delta \in [0,\infty)$. Then, for all $A, B \in \mathbb{R}^{2\times 2}$ there holds*

$$
\begin{gathered}
\big(S(A) - S(B)\big) \cdot (A^{sym} - B^{sym}) \simeq (\delta + |B^{sym}| + |A^{sym}|)^{p-2} |A^{sym} - B^{sym}|^2, \\
|S(A) - S(B)| \simeq (\delta + |B^{sym}| + |A^{sym}|)^{p-2} |A^{sym} - B^{sym}|,
\end{gathered}
$$

where the constants c_0, $c_1 > 0$ depend only on p, and are independent of $\delta \geq 0$.

From the elementary inequality $a^p \leq a^2 b^{p-2} + b^p$, valid for all $0 \leq a$, $0 < b$, and $p \in [1,2]$, we get the relation

$$
\delta^{\frac{p}{2}} + t^{\frac{p}{2}} \simeq (\delta + t)^{\frac{p-2}{2}} t + \delta^{\frac{p}{2}}, \qquad \delta, t \ \geq 0 \tag{2.4}
$$

with constants depending only on p (see [5, Corollary 2.19]).

Since in the Dirichlet case we need to handle in a different way tangential and normal derivatives, we denote by D^2u the set of all the second-order partial derivatives of u. In addition, the symbol D^2_*u denotes all partial derivatives $\partial^2_{ik}u_j$, except for the derivative $\partial^2_{22}u_1$, namely

$$|D^2_*u|^2 := |\partial_{22}u_2|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(2,2)}}^{2} |\partial^2_{ik}u_j|^2.$$

We introduce the following quantities strictly related to the stress tensor S and coming naturally in the problem, when using the techniques introduced in [2, 11, 19]:

$$\mathcal{I}_1(u) := \int_\Omega (\delta + |\mathcal{D}u|)^{p-2}|\partial_1\mathcal{D}u|^2dx, \tag{2.5a}$$

$$\mathcal{I}(u) := \int_\Omega (\delta + |\mathcal{D}u|)^{p-2}|\nabla\mathcal{D}u|^2dx, \tag{2.5b}$$

$$\mathcal{J}(u) := \int_\Omega (\delta + |\mathcal{D}u|)^{p-2}|\mathcal{D}u_t|^2dx, \tag{2.5c}$$

where $\mathcal{I}$ is obtained by integration by parts when testing the extra stress-tensor S with $-\Delta u$ (and this is possible in the periodic-case); a multiple of $\mathcal{I}_1$ is obtained testing with $-\partial_{11}u$ and the calculations are possible in the flat domain; Finally a multiple of $\mathcal{J}$ is obtained testing with u_{tt} and calculations are valid also in the Dirichlet case, for a generic domain.

We will also use this classical result, see Nečas [20].

Lemma 2.3. *If it holds $\nabla g = \operatorname{div} G$, for some $G \in (L^q(\Omega))^{2\times 2}$, for $1 < q < +\infty$ then*

$$\left\| g - \int_\Omega g(x)\,dx \right\|_q \leq c\|G\|_q.$$

Let us recall the definition of *weak solution* to the Problem (1.1)-(2.1).

Definition 2.1. *Let $T > 0$ and assume that $f \in L^2(0,T;V_2')$. We say that u is a weak solution of problem* (1.1) *if:*

$$u \in L^2(0,T;V_2) \cap L^\infty(0,T;L^2(\Omega)), \tag{2.6a}$$

$$u_t \in L^2(0,T;V_2'), \tag{2.6b}$$

$$\begin{aligned}&\int_\Omega u(t)\,\varphi\,dx + \nu_0\int_0^t\int_\Omega \nabla u(s)\,\nabla\varphi\,dxds + \nu_1\int_0^t \big\langle S(\mathcal{D}u(s)),\mathcal{D}\varphi\big\rangle\,ds\\ &\quad -\int_0^t\int_\Omega (u(s)\cdot\nabla)\,\varphi\,u(s)\,dxds = \int_\Omega u_0\,\varphi\,dx + \int_0^t \langle f(s),\varphi\rangle\,ds \quad \forall\,\varphi\in V_2.\end{aligned} \tag{2.6c}$$

Due to the fact that $\nu_0 > 0$, the existence of weak solutions follows for all $p \geq 1$ in a standard way, and one has not to resort to very sophisticated tools as in Diening, Růžička, and Wolf [12]. We will come back later on, for the motivation on this assumption on ν_0. In particular, we do not have any further restriction on p and the proof follows the same lines of the classical work on monotone operators, as summarized in Lions [18]. The result below is part of the folklore associated with non-Newtonian fluids. We will give a sketch of the proof since some of the calculations will be used many times in the sequel.

Theorem 2.1. *Let be given $\nu_0, \nu_1 > 0$, $p \in [1,2]$, $u_0 \in L^2(\Omega)$ with $\operatorname{div} u_0 = 0$ and $(u_0\cdot n)_{|\Gamma} = 0$, and $f \in L^2(0,T,V_2')$. Then, there exists a unique solution u*

to (1.1)-(2.1) *satisfying* (2.6a)-(2.6c). *Moreover, the following estimates are verified*

$$\|u\|^2_{L^\infty(0,T;L^2)} + \nu_0\|\nabla u\|^2_{L^2(0,T;L^2)} \leq C$$
$$\|u_t\|^2_{L^2(0,T;V_2')} \leq C,$$

where $C = C(p, \delta, \nu_0, \nu_1, \|f\|_{L^2(0,T;V_2')}, \|u_0\|_2, T, \Omega)$.

Proof. We deduce the *a priori* estimates on which the existence of weak solutions to (1.1)-(2.1) is based. More properly, one should consider approximate Galerkin solutions defined as follows. Let $\{\omega^r\}$, with $r \in \mathbb{N}$, be the eigenfunctions of the Stokes operator and let $\{\lambda^r\}$ be the corresponding eigenvalues; we define $X_m := \text{span}\{\omega^1, \dots, \omega^m\}$ and P_m is the orthogonal projection operator over X_m. We will seek approximate functions $u^m(t,x) = \sum_{r=1}^m c_r^m(t)\omega^r(x)$ as solutions of the system of equations below, for all $1 \leq r \leq m$, $t \in [0,T]$

$$\int_\Omega \Big[u_t^m \omega^r + \nu_0 \nabla u^m \nabla \omega^r + \nu_1 S(\mathcal{D}u^m)\,\mathcal{D}\omega^r + (u^m \cdot \nabla)u^m\,\omega^r\Big]\,dx = \langle f, \omega^r\rangle,$$
$$u^m(0) = P^m u_0.$$

Taking the L^2-product of (1.1a) with u^m, using suitable integrations by parts and Young inequality we get

$$\frac{1}{2}\frac{d}{dt}\|u^m\|_2^2 + \nu_0\|\nabla u^m\|_2^2 + \frac{\nu_1}{2}\int_\Omega (\delta + |\mathcal{D}u^m|)^{p-2}|\mathcal{D}u^m|^2 dx$$
$$\leq \frac{\nu_0}{2}\|\nabla u^m\|_2^2 + \frac{1}{2\nu_0}\|f\|^2_{V_2'},$$

Using (2.4) and integrating in time we arrive at the following inequality

$$\|u^m(t)\|_2^2 + \nu_0\int_0^t \|\nabla u^m(s)\|_2^2\,ds + C\nu_1\int_0^t \|\mathcal{D}u^m(s)\|_p^p\,ds$$
$$\leq \|u_0\|_2^2 + \frac{1}{\nu_0}\int_0^t \|f(s)\|^2_{-1,2}\,ds + C(p)\nu_1\delta^p,$$

for a.e. $t \in [0,T]$. We estimate, by comparison, the time derivative. The only term which requires some care is the extra-stress tensor S. Since $p \leq 2$ we get

$$\int_0^T \langle S(\mathcal{D}u^m), \mathcal{D}\varphi\rangle\,ds \leq \|S(\mathcal{D}u^m)\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}$$
$$\leq \|\mathcal{D}u^m\|^{p-1}_{L^{2p-2}(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}$$
$$\leq C(T,\Omega)\|\nabla u^m\|^{p-1}_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}.$$

Whence, by standard calculations

$$\int_0^t \|u_t^m(s)\|^2_{-1,2}\,ds \leq C, \tag{2.7}$$

for a constant C depending on p, ν_0, ν_1, $\|f\|_{L^2(0,T;V_2')}$, $\|u_0\|_2$, T, and Ω. This proves that if u^m is a Galerkin approximate solution then, uniformly in $m \in \mathbb{N}$,

$$u^m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_2) \quad \text{and} \quad u_t^m \in L^2(0,T;V_2').$$

Note that we can extract sub-sequences converging weakly to some u in $L^2(0,T;V_2)$, weakly* in $L^\infty(0,T;L^2(\Omega))$ and, by Aubin-Lions theorem, strongly in $L^2(\Omega_T)$, and a.e. in Ω_T. We have enough regularity to pass to the limit in the convective term. Moreover, since $S(\mathcal{D}u^m)$ is bounded uniformly in $L^2(\Omega_T)$, it follows that $S(\mathcal{D}u^m) \rightharpoonup A$ for some A in $L^2(\Omega_T)$. (Observe that without the Laplacian term we would have only a bound in $L^{p'}(\Omega_T)$). We have now to check that $A = S(\mathcal{D}u)$. This is obtained with the monotonicity trick, see e.g. [18, §2-5.2]. By usual Sobolev embeddings (since we are in two dimensions) the function $t \mapsto \int_\Omega (u\cdot\nabla)\, u\, u\, dx \in L^1(0,T)$, hence we can write the energy equality between any couple $0 \le s_0 \le s \le T$

$$\frac{1}{2}\|u(s)\|_2^2 + \nu_0 \int_{s_0}^{s} \|\nabla u\|_2^2\, d\tau + \nu_1 \int_{s_0}^{s} \langle A, u\rangle\, d\tau = \frac{1}{2}\|u(s_0)\|_2^2 + \int_{s_0}^{s} (f,u)\, d\tau. \tag{2.8}$$

Defining for $\phi \in L^2(0,T;V_2)$ (a test function with the same regularity of u)

$$\mathcal{X}_s^m := \nu_1 \int_0^s \langle S(\mathcal{D}u^m) - S(\mathcal{D}\phi), \mathcal{D}u^m - \mathcal{D}\phi\rangle\, d\tau + \nu_0 \int_0^s \|\nabla u^m\|_2^2\, d\tau + \frac{1}{2}\|u^m(s)\|_2^2,$$

it follows, by using that S is monotone and by semi-continuity of the norm, that $\liminf_m \mathcal{X}_s^m \ge \nu_0 \int_0^s \|\nabla u\|_2^2\, d\tau + \frac{1}{2}\|u(s)\|_2^2$, and also that

$$\lim_m \mathcal{X}_s^m = \int_0^s (f,u) + \frac{1}{2}\|u_0\|_2^2 - \nu_1 \int_0^s \langle A, \mathcal{D}\phi\rangle\, d\tau - \nu_1 \int_0^s \langle S(\mathcal{D}\phi), \mathcal{D}u - \mathcal{D}\phi\rangle\, d\tau.$$

Hence, by using the equality (2.8) we get

$$\nu_1 \int_0^s \langle A - S(\mathcal{D}\phi), \mathcal{D}u - \mathcal{D}\phi\rangle\, d\tau \ge 0 \qquad a.e.\ s \in [0,T].$$

We fix $\phi = u - \lambda\,\psi$ for $\psi \in L^2(0,T;V_2)$ and $\lambda > 0$. Finally, letting $\lambda \to 0^+$ the thesis follows.

It is important to point out that the weak solution above constructed is unique. Let us suppose that we have two solutions u_1 and u_2 corresponding to the same data. We obtain the following inequality for $U := u_1 - u_2$ (This follows by using the usual interpolation inequalities as for the Navier-Stokes equations and since U is allowed as test function, see Constantin and Foias [7])

$$\|U(t)\|_2^2 + \nu_0 \int_0^t \|\nabla U(s)\|_2^2\, ds + \nu_1 \int_0^t \langle S(\mathcal{D}u_1) - S(\mathcal{D}u_2), \mathcal{D}u_1 - \mathcal{D}u_2\rangle\, ds \le \frac{C}{\nu_0} \int_0^t \|\nabla u_1(s)\|_2^2 \|U(s)\|_2^2\, ds.$$

Since S is monotone (cf. Lemma 2.2) the integral involving the extra stress-tensor is non-negative. Using the Gronwall lemma and the energy estimate one obtains that $U \equiv 0$. □

This latter result is very relevant since it allows to conclude that *all* the sequence $\{u^m\}$ converges to u. Moreover, if we have other *a priori* estimates on u^m, the extra-regularity is inherited by weak solutions directly. This will be used in the proof of Theorem 1.1. Observe also that, at moment, we do not have any information on the pressure, apart that there exists as a distribution, by using De Rham theorem.

3. The space-periodic case

In this section we are concerned with the space-periodic case, that is $\Omega = \mathbb{T}^2$. Each considered function w will satisfy $w(x + 2\pi e_i) = w(x)$, $i = 1,2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. We also require all functions to have vanishing mean value, to ensure the validity of the Poincaré inequality. We prove some regularity results and we will show why the hypothesis $\nu_0 > 0$ seems necessary in many arguments. We define $\mathcal{V}_{\text{per}}(\Omega)$ as the space of vector-valued functions on Ω that are smooth, divergence-free, and space periodic with zero mean value. For $1 < q < \infty$ and $k \in \mathbb{N}$, set

$$W^{k,q}_{\text{div}}(\Omega) := \big\{\text{closure of} \quad \mathcal{V}_{\text{per}}(\Omega) \quad \text{in} \quad W^{k,q}(\Omega)\big\},$$

endowed, with the usual norms.

In the space periodic setting many calculations are simpler since we can use $-\Delta u$ as test function (now formally but the procedure goes through the Galerkin approximation). Since in the 2D space-periodic case $\int_\Omega (u\cdot\nabla)\, u\Delta u\, dx = 0$ we get

$$\frac{d}{dt}\|\nabla u\|^2 + \nu_0\|\Delta u\|^2 + \nu_1\mathcal{I}(u) \leq C\|f\|^2, \tag{3.1}$$

hence, if we are able to construct such a solution (this is not trivial at all due to some technical issues when passing to the limit in $\int_0^T \mathcal{I}(u^m(s))\, ds$, for a fixed $T>0$) that and if $\nu_0 = 0$ we obtain as higher order estimate

$$\int_0^T \mathcal{I}(u)\, dt = \int_0^T \int_{\mathbb{T}^2} (\delta + |\mathcal{D}u|)^{p-2} |\nabla \mathcal{D}u|^2\, dxdt < +\infty.$$

We recall the following lemma, which is an adaption of [5, Lemma 4.4] to the two dimensional case.

Lemma 3.1. *Let $p \in (1,2]$, $\delta \in (0,\infty)$, and $\ell \in [1,2)$. Then, for all sufficiently smooth functions u with vanishing mean value over Ω, the following relations hold true*

$$\|u\|_{2,\ell}^p \leq c\big(\mathcal{I}(u) + \delta^p\big),$$

Hence, the information on the regularity in the space variable which we can extract from (3.1), in the case $\nu_0 = 0$, could be at most

$$u \in W^{2,\ell}(\mathbb{T}^2) \qquad \forall\, \ell < 2, \quad a.e.\ t\in[0,T].$$

This is not enough to employ Thm. 1.1 and explains the introduction of the hypothesis $\nu_0 > 0$.

Proof of Proposition 1.1. In the light of the above observations the proof follows as in the 2D Navier-Stokes equations, see[10]. We test the equations by $-t\,\Delta u^m$ and we have

$$\frac{d}{dt}(t\,\|\nabla u^m\|^2) + \nu_0\, t\,\|\Delta u^m\|^2 + \nu_1\, t\,\mathcal{I}(u^m) \leq C\, t\,\|f\|^2 + \|\nabla u\|^2.$$

Hence, no matter of the non-negative term coming from the extra-stress tensor, integrating in time over $[0,T]$ we have that $\sqrt{t}\, u^m \in L^2(0,T;W^{2,2}(\mathbb{T}^2))$. Due to uniqueness of the solution the whole sequence $\{u^m\}$ converges to u and by lower-semicontinuity of the norm we obtain that $\sqrt{t}\, u \in L^2(0,T;W^{2,2}(\mathbb{T}^2))$. □

For the sake of completeness, we recall that in the periodic 2D case, with $\nu_0 = 0$ it is possible to prove the following result of existence of regular solutions, see Kost [17], which is an adaption of those in [5] for the 3D case. (Observe that in absence of the Laplacian also the existence and uniqueness of weak solutions is more delicate and the limit process on Galerkin solutions requires some care). The following result, which is of interest by itself, is not enough for our purposes of studying uniqueness for solutions to (1.4).

Proposition 3.1. *Let be given $\delta \in [0,\delta_0]$, for some $\delta_0 > 0$, set $\nu_0 = 0$, $\nu_1 > 0$, and let $p \in (1,2]$. Given $T > 0$, assume that $f \in L^\infty(0,T;W^{1,2}(\mathbb{T}^2)) \cap W^{1,2}(0,T;L^2(\mathbb{T}^2))$. Let $u_0 \in W^{2,2}(\mathbb{T}^2)$ be such that $\operatorname{div} u_0 = 0$ and $\operatorname{div} S(\mathcal{D}u_0) \in L^2(\mathbb{T}^2)$. Then, there is a time $0 < T' \leq T$ (depending on the data of the problem) such that the system (1.1), has a strong solution u on $[0,T']$ satisfying, for $r \in (4/3,2)$,*

$$u \in L^q(0,T';W^{2,r}(\mathbb{T}^2)) \cap C(0,T';W^{1,q}(\mathbb{T}^2)), \qquad \forall\, q < \infty.$$

Remark 3.1. *One can obtain further regularity results for u_t and also for $\nabla\pi$ (the latter if $\delta > 0$).*

4. Space-time regularity in the Dirichlet case

In this section we consider the time evolution problem with Dirichlet boundary conditions and we prove a result of regularity for smooth data. Then, we will relax some of the assumptions to prove the main result of the paper. We start by showing a first regularity result for the time derivative of the solutions to the problem (1.1) with Dirichlet boundary conditions. We prove now some results by using as test functions first and second order time derivatives of the velocity. These are legal test functions, since if u is divergence-free and $u_{|\Gamma} = 0$, then $\frac{\partial^k u}{\partial t^k}$ shares the same two properties, for all $k \in \mathbb{N}$. In particular, the following result is valid in any smooth and bounded domain, while the hypothesis of flat boundary will be used for the $W^{2,2}(\Omega)$-regularity.

Lemma 4.1. *Let $p \in (1, 2]$, $\delta > 0$, $f \in W^{1,2}(0, T; L^2(\Omega))$, $u_0 \in W^{2,2}(\Omega) \cap V_2$, and let u be a weak solution of problems* (1.1)-(2.1)*. Then,*

$$\|u_t\|^2_{L^\infty(0,T;L^2)} + \|\nabla u\|^2_{L^\infty(0,T;L^2)} + \nu_0 \|\nabla u_t\|^2_{L^2(0,T;L^2)} + \nu_1 \|\mathcal{J}(u)\|_{L^1(0,T)} \leq C, \tag{4.1}$$

where the constant C depends on p, δ, ν_0, ν_1, $\|f\|_{W^{1,2}(0,T;L^2)}$, $\|u_0\|_{2,2}$, T, and Ω.

As in the previous result we only prove the a priori estimates. A complete proof can be obtained through a Galerkin approximation and for the reminder of this section we drop the superscript "m". We also define

$$M(t) := \int_0^t (\delta + s)^{p-2} s \, ds \geq 0, \quad \text{for} \quad t \geq 0.$$

Observe that $M(t) \simeq (\delta + t)^{p-2} t^2$ and also $(\delta + t)^{p-2} t^2 \leq t^p$, with $1 \leq p \leq 2$. This shows that

$$0 \leq \mathcal{M}(u) := \int_\Omega M(|\mathcal{D}u|) \, dx \leq C(p) \|\mathcal{D}u\|_p^p, \quad \text{with} \quad 1 \leq p \leq 2. \tag{4.2}$$

Proof of Lemma 4.1. First, we multiply (1.1a) by u_t and integrate by parts. We observe that taking the duality of $-\operatorname{div} S(\mathcal{D}u)$ against u_t, we get

$$-\big\langle \operatorname{div} S(\mathcal{D}u), u_t \big\rangle = \big\langle S(\mathcal{D}u), \mathcal{D}u_t \big\rangle = \frac{d}{dt} \mathcal{M}(u). \tag{4.3}$$

By suitable integrations (since $\operatorname{div} u_t = 0$) we obtain

$$\|u_t\|_2^2 + \frac{\nu_0}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \nu_1 \frac{d}{dt} \mathcal{M}(u) = \int_\Omega \big(f\, u_t - (u \cdot \nabla)\, u\, u_t\big)\, dx.$$

By using Hölder and Gagliardo-Nirenberg inequalities, with the boundedness of the kinetic energy, we get, for all $\varepsilon > 0$

$$\begin{aligned}\left| \int_\Omega (u \cdot \nabla)\, u\, u_t \, dx \right| &\leq \|u\|_4 \|\nabla u\|_2 \|u_t\|_4 \\ &\leq C \|u\|_2^{\frac12} \|\nabla u\|_2^{\frac12} \|\nabla u\|_2 \|u_t\|_2^{\frac12} \|\nabla u_t\|_2^{\frac12} \\ &\leq c_\varepsilon \big(\|\nabla u\|_2^2 + \|u_t\|_2^2 \|\nabla u\|_2^2\big) + \varepsilon \|\nabla u_t\|_2^2,\end{aligned}$$

Thus, we obtain the following differential inequality

$$\|u_t\|_2^2 + \frac{d}{dt}\big(\nu_0 \|\nabla u\|_2^2 + \nu_1 \mathcal{M}(u)\big) \leq c_\varepsilon \big(\|\nabla u\|_2^2 + \|u_t\|_2^2 \|\nabla u\|_2^2 + \|f\|_2^2\big) + \varepsilon \|\nabla u_t\|_2^2, \tag{4.4}$$

which we clearly cannot use directly, due to the lack of control for ∇u_t.

Remark 4.1. *Another path will be that of using improved estimates for ∇u to estimate the convective term, see the last section.*

We take now the time derivative of (1.1a), multiply by u_t and integrate by parts (recalling that $\int_\Omega (u\cdot\nabla)\,u_t\,u_t\,dx=0$) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\nu_0\|\nabla u_t\|^2+\nu_1\langle\partial_t\big(S(\mathcal{D}u)\big),\mathcal{D}u_t\rangle\le\left|\int_\Omega\Big((u_t\cdot\nabla)\,u\,u_t+f_t\,u_t\Big)\,dx\right|. \tag{4.5}$$

By (2.2a) the term involving S in (4.5) is non-negative being estimated from below by a multiple of $\mathcal{J}(u)\ge 0$. Let us focus on the right-hand side of (4.5). By using Hölder and interpolation inequality, and the energy estimate we get, for each $\eta>0$,

$$\begin{aligned}\left|\int_\Omega (u_t\cdot\nabla)\,u\,u_t\,dx\right|=\left|\int_\Omega (u_t\cdot\nabla)\,u_t\,u\,dx\right|&\le\|u_t\|_4\|\nabla u_t\|_2\|u\|_4\\&\le C\|u_t\|_2^{\frac12}\|\nabla u_t\|_2^{\frac32}\|u\|_2^{\frac12}\|\nabla u\|_2^{\frac12}\\&\le c_\eta\|\nabla u\|_2^2\|u_t\|_2^2+\eta\|\nabla u_t\|_2^2,\end{aligned}$$

hence, choosing $\eta>0$ small enough we get

$$\frac{d}{dt}\|u_t\|_2^2+\nu_0\|\nabla u_t\|_2^2+\nu_1\mathcal{J}(u)\le C\big(\|\nabla u\|_2^2\|u_t\|_2^2+\|f_t\|_2^2\big). \tag{4.6}$$

Summing up (4.4)-(4.6) and choosing $\varepsilon>0$ small enough we get finally

$$\begin{aligned}\frac{d}{dt}\Big(\|u_t\|_2^2+\nu_0\|\nabla u\|_2^2+\nu_1\mathcal{M}(u)\Big)&+\|u_t\|_2^2+\nu_0\|\nabla u_t\|_2^2+\nu_1\mathcal{J}(u)\\&\le C\Big(\|\nabla u\|_2^2\|u_t\|_2^2+\|\nabla u\|_2^2+\|f\|_2^2+\|f_t\|_2^2\Big).\end{aligned}$$

To integrate over $[0,T]$ we need to make sense to $\|u_t(0,\cdot)\|_2$. From the assumptions on the data, the fact that $\delta>0$, and $u^m(0)=P^m u_0$ we easily get (cf. [5, § 5]) that

$$\|u_t^m(0)\|_2\le c\big(\|u_0^m\|_{2,2}^2+\|f^m(0)\|_2\big).$$

Recall that we are working on the finite dimensional approximation u^m and taking the limit $m\to+\infty$. With Gronwall lemma and by using the fact that $\nabla u\in L^2(0,T;L^2(\Omega))$, we get for a.e $t\in[0,T]$

$$\begin{aligned}\|u_t(t)\|_2^2+\nu_0\|\nabla u(t)\|_2^2+\int_0^t\Big(\|u_t(s)\|_2^2+\nu_0\|\nabla u_t(s)\|_2^2+\mathcal{J}(u(s))\Big)\,ds&\\ \le C(\nu_0,\nu_1,\delta,T,\|f\|_{W^{1,2}(0,T;L^2)},\|u_0\|_{2,2},\Omega)&,\end{aligned}$$

hence the thesis. □

Remark 4.2. *The hypotheses on the external force can be slightly relaxed, but this is inessential in our treatment.*

We now prove Proposition 1.2. For the reader's convenience we split the proof into two parts. First, we perform a preliminary study of the system obtained removing the convective term $(u\cdot\nabla)\,u$ from (1.1a).

$$u_t-\nu_0\Delta u-\nu_1\,\mathrm{div}\,S(\mathcal{D}u)+\nabla\pi=f\quad\text{in}\quad[0,T]\times\Omega, \tag{4.7a}$$

$$\mathrm{div}\,u=0\quad\text{in}\quad[0,T]\times\Omega, \tag{4.7b}$$

$$u=0\quad\text{in}\quad[0,T]\times\Gamma, \tag{4.7c}$$

$$u(0)=u_0\quad\text{in}\quad\Omega, \tag{4.7d}$$

and focusing on the role of the nonlinear stress-tensor. The system (4.7) can be treated similarly to a steady state problem if we have good enough *a priori* estimates on u_t. We will then address the full problem (1.1)-(2.1), by adding suitable estimates for the convective term.

Lemma 4.2. *Let* $\nu_0>0$, $\delta>0$ *and* $p\in\big[\frac32,2\big]$. *Given* $T>0$, *assume that* $u_0\in W^{2,2}(\Omega)\cap V_2$ *and* $f\in W^{1,2}(0,T;L^2(\Omega))$. *Then, problem* (4.7)-(2.1) *admits a unique solution, such that* (1.5) *holds true.*

Proof. We adapt to the time-dependent case a technique with three intermediate steps taken from [2, 8]: In the first step we bound the tangential derivative of velocity and pressure; In the second step we estimate the normal derivative of the velocity field; In the last step we estimate the normal derivative of the pressure.

Again we merely prove the *a priori* estimates. Observe that for this simpler problem without convection, the same existence proved in Theorems 2.1 and regularity from Lemma 4.1 clearly hold true (this is particularly relevant for what concerns u_t).

Step 1. We first prove that the following estimates, concerning the tangential derivatives, hold true

$$\nu_0\|\nabla\partial_1 u\|^2 + \nu_0\|\partial^2_{22}u_2\|^2_{L^2(0,T;L^2)} + \|\partial_1\pi\|_{L^2(0,T;L^2)} \le C, \tag{4.8}$$

where C depends on p, δ, ν_0, ν_1, $\|f\|_{W^{1,2}(0,T;L^2)}$, $\|u_0\|_{2,2}$, T, and Ω.

We now use the particular features of the flat domain. Multiplying equation (4.7a) by $-\partial^2_{11}u$ and integrating by parts, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\partial_1 u\|_2^2 + \nu_0\|\nabla\partial_1 u\|_2^2 + (p-1)\nu_1\int_\Omega(\delta+|\mathcal{D}u|)^{p-2}|\partial_1\mathcal{D}u|^2\,dx \le \|f\|_2\,\|\partial^2_{11}u\|_2.$$

By applying Young inequality and using relation (2.5a), we get a.e. in $[0,T]$

$$\begin{aligned}\|\partial_1 u(t)\|_2^2 + \int_0^t \big(\nu_0\|\nabla\partial_1 u(s)\|_2^2 + \nu_1\mathcal{I}_1(u(s))\big)\,ds&\\ \le C\Big(\|\nabla u_0\|^2 + \frac{1}{\nu_0}\int_0^t\|f(s)\|_2^2\,ds\Big),&\end{aligned}\tag{4.9}$$

and, since $\operatorname{div} u = 0$, $\partial^2_{22}u_2 = -\partial^2_{21}u_1$ and the estimate on $\partial^2_{22}u_2$ follows.

Let us focus on the pressure term. Differentiating the equation (4.7a) with respect to the tangential direction x_1, one has that

$$\nabla\partial_1\pi = \nu_0\operatorname{div}\partial_1\nabla u + \nu_1\operatorname{div}\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big] - \partial_1 u_t + \partial_1 f \qquad a.e. \text{ in } \Omega_T.$$

We observe that $\partial_1 u_t = \operatorname{div}\begin{pmatrix}\partial_t u_1 & 0\\ \partial_t u_2 & 0\end{pmatrix}$ and $\partial_1 f = \operatorname{div}\begin{pmatrix} f_1 & 0\\ f_2 & 0\end{pmatrix}$. Hence to apply Lemma 2.3 to estimate $\partial_1\pi$, we only have to bound the term $\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big]$. A direct computation gives

$$\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big] = (\delta+|\mathcal{D}u|)^{p-2}\partial_1\mathcal{D}u + (p-2)(\delta+|\mathcal{D}u|)^{p-3}(\mathcal{D}u\cdot\partial_1\mathcal{D}u)\frac{\mathcal{D}u}{|\mathcal{D}u|},$$

and consequently

$$\big|\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big]\big| \le (3-p)(\delta+|\mathcal{D}u|)^{p-2}|\partial_1\mathcal{D}u| \qquad a.e. \text{ in } \Omega_T.$$

Therefore, by comparison $\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big] \in L^2(\Omega)$ and it follows that

$$\int_\Omega\big|\partial_1\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big]\big|^2\,dx \le c\,\delta^{p-2}\,\mathcal{I}_1(u) \qquad a.e.\ t\in[0,T].$$

By applying Lemma 2.3 we have that

$$\|\partial_1\pi\|_2^2 \le \|u_t\|_2^2 + \nu_0\|\partial_1\nabla u\|_2^2 + \nu_1\,C\,\delta^{p-2}\,\mathcal{I}_1(u) + \|f\|_2^2 \qquad a.e.\ t\in[0,T],$$

from which, integrating in time over $[0,T]$, using (4.9) and recalling the bounds previously proved on u_t, $\partial_1\nabla u$, and $\mathcal{I}_1$, then (4.8) follows.

Step 2. To bound $\|\partial^2_{22}u_1\|_{L^2(0,T;L^2)}$, we consider a narrower range of values for the parameter p. Under the same hypotheses as before, but for $p\in\big[\frac{3}{2},2\big)$, we have

$$\|\partial^2_{22}u_1\|_{L^2(0,T;L^2)} \le C,$$

where the constant C depends on p, δ, ν_0, ν_1, $\|f\|_{W^{1,2}(0,T;L^2)}$, $\|u_0\|_{2,2}$, T, and Ω.

We follow the main lines established in the proof of [8, Lemma 3.3]. By calculating $\partial_2\big[(\delta+|\mathcal{D}u|)^{p-2}\mathcal{D}u\big]$, the first equation in (4.7a) can be written as

$$\alpha_1\partial^2_{22}u_1=-F_1-f_1+\partial_t u_1+\partial_1\pi, \tag{4.10}$$

where

$$\alpha_1:=\nu_0+\frac{\nu_1}{2}(\delta+|\mathcal{D}u|)^{p-2}+\nu_1(p-2)\frac{(\delta+|\mathcal{D}u|)^{p-3}}{|\mathcal{D}u|}(\mathcal{D}u)_{12}(\mathcal{D}u)_{12},$$

and

$$\begin{aligned}F_1:=&\big[\nu_0+\nu_1(\delta+|\mathcal{D}u|)^{p-2}\big]\partial^2_{11}u_1+\frac{\nu_1}{2}(\delta+|\mathcal{D}u|)^{p-2}\partial^2_{12}u_2\\&+\nu_1(p-2)\frac{(\delta+|\mathcal{D}u|)^{p-3}}{|\mathcal{D}u|}\Big[\sum_{k,l=1}^{2}(\mathcal{D}u)_{kl}\partial_1(\mathcal{D}u)_{kl}(\mathcal{D}u)_{11}+\partial^2_{22}u_2(\mathcal{D}u)_{22}(\mathcal{D}u)_{12}\\&\qquad+\frac{\partial^2_{12}u_2}{2}(\mathcal{D}u)_{12}(\mathcal{D}u)_{12}\Big].\end{aligned}$$

By direct calculations it can be easily seen that

$$|F_1|\le C\Big[\nu_0+\nu_1\Big(p-\frac{3}{2}\Big)(\delta+|\mathcal{D}u|)^{p-2}\Big]|D^2_*u|\qquad a.e.\text{ in }\Omega_T$$

and by using that $p\ge\frac{3}{2}$ we get

$$\alpha_1\ge\Big[\nu_0+\nu_1\Big(p-\frac{3}{2}\Big)(\delta+|\mathcal{D}u|)^{p-2}\Big]\ge\nu_0>0.$$

Division of both sides of (4.10) by α_1 is then legitimate and we infer that

$$|\partial^2_{22}u_1|\le C\big(|D^2_*u|+\frac{1}{\nu_0}\big(|\partial_1\pi|+|\partial_t u_1|+|f_1|\big)\qquad a.e.\text{ in }\Omega_T.$$

Therefore, squaring and integrating over Ω_T we get

$$\int_0^T\|\partial^2_{22}u_1(s)\|_2^2\,ds\le\frac{C}{\nu_0}\int_0^T\Big(\|D^2_*u(s)\|_2^2+\|\partial_1\pi(s)\|_2^2+\|\partial_t u_1(s)\|_2^2+\|f_1(s)\|_2^2\Big)\,ds,$$

which, by the previous results is finite. This finally shows that $D^2u\in L^2(\Omega_T)$.

Step 3. The final step, which is not strictly required for the particle trajectories uniqueness, is the regularity of the normal derivative of pressure. Nevertheless, we include it for the sake of completeness. Under the same hypotheses we have

$$\|\partial_2\pi\|_{L^2(0,T;L^2)}\le C,$$

where the constant C depends on p, δ, ν_0, ν_1, $\|f\|_{W^{1,2}(0,T;L^2)}$, $\|u_0\|_{2,2}$, T, and Ω.

By using the second equation in (4.7a), one can write

$$|\partial_2\pi|\le c\Big(\nu_0+\nu_1(p-2)(\delta+|\mathcal{D}u|)^{p-2}\Big)|D^2u|+|\partial_2u_t|+|f_2|\qquad a.e.\text{ in }\Omega_T.$$

Hence, straightforward calculations lead to

$$\int_0^T\|\partial_2\pi(s)\|_2^2\,ds\le c\int_0^T\Big(\Big[\nu_0+\nu_1\delta^{2(p-2)}\Big]\|D^2u(s)\|_2^2+\|\partial_2u_t(s)\|_2^2+\|f(s)\|_2^2\Big)\,ds,$$

and the assertion follows as a consequence of the previous results. □

We finally prove the same regularity results also in presence of the convective term. We use a perturbation argument, treating $(u\cdot\nabla)\,u$ as a right-hand side in equation (1.1a).

Proof of Proposition 1.2. Here, we use the a priori estimates obtained for the problem (4.7) with external body force

$$\mathcal{F} := -(u\cdot\nabla)\,u + f.$$

In the derivation of estimates for u_t we used that $\|f\|_{W^{1,2}(0,T;L^2)}$, while in Lemma 4.2 the estimates depend essentially on the $L^2(\Omega_T)$-norm of the external force. Hence, by using Lemma 4.1 it is then sufficient to estimate $\|(u\cdot\nabla)\,u\|_{L^2(0,T;L^2)}$ in terms of second order derivatives of u, to follow the same calculations in Step 1–3 of the previous result.

By applying Hölder, Gagliardo-Nirenberg, and Young inequalities and the energy estimate, we get for each $\varepsilon>0$

$$\begin{aligned}\|(u\cdot\nabla)\,u\|_2 \le \|u\|_4\|\nabla u\|_4 &\le c\|u\|_2^{\frac12}\|\nabla u\|_2^{\frac12}\|\nabla u\|_2^{\frac12}\|D^2u\|_2^{\frac12}\\ &\le c_\varepsilon\|\nabla u\|_2^2+\varepsilon\|D^2u\|_2.\end{aligned}\tag{4.11}$$

By using the same calculations as in the previous proposition and the *a-priori* estimates in (4.1) –especially that $\nabla u\in L^\infty(0,T;L^2(\Omega))$– we have

$$\begin{aligned}\int_0^T\big(\|u\|_{2,2}^2+\|\pi\|_{1,2}^2\big)\,ds &\le C\int_0^T\big(\|f\|_2^2+\|u_t\|_2^2+\|(u\cdot\nabla)\,u\|_2^2\big)\,ds\\ &\le C(p,\delta,\nu_0,\nu_1,\|f\|_{W^{1,2}(0,T;L^2)},\|u_0\|_{2,2},T,\Omega,\varepsilon)+\varepsilon\int_0^T\|D^2u\|_2^2\,ds,\end{aligned}$$

and, by choosing $\varepsilon>0$ small enough, we end the proof. □

As a consequence of the above result we have full L^2-space-time regularity of the solution up to second-order space-derivatives, hence the uniqueness of particle trajectories. The result is not optimal in view of application to uniqueness of trajectories, in the sense that some of the hypotheses can be slightly relaxed. For instance $f_t\in L^2(\Omega_T)$ and $u_0\in W^{2,2}(\Omega)$ can be removed (at the price of less regularity on u_t) by following a slightly different path as we do in the next section.

5. Proof of Theorem 1.2

In this section we finally address the problem of the uniqueness of particle trajectories under "minimal" assumptions on the data. We will show how the previous regularity result, together with Theorem 1.1, allow us to prove Theorem 1.2.

Proof of Theorem 1.2. In the same way as in the proof of Lemma 4.2, we perform separately the *a priori* estimates for the normal and tangential derivative of the time-weighted $\sqrt{t}\,u^m$ (which we call $\sqrt{t}\,u$). In particular, here we do not use a lot of regularity on u_t, but we have to face with a non-smooth u_0. By adapting standard weighted estimates, we multiply the equation (1.1a) by $-t\,\partial_{11}^2u$. Integrating by parts, and with Young inequality we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\big(t\,\|\partial_1u\|_2^2\big)+\nu_0\,t\,\|\nabla\partial_1u\|_2^2+(p-1)\nu_1\,t\,\mathcal{I}_1(u)\\ &\qquad\le\frac{\nu_0}{2}t\,\|\partial_{11}^2u\|_2^2+\frac{C}{\nu_0}t\,\big(\|(u\cdot\nabla)u\|_2^2+\|f\|_2^2\big)+\|\partial_1u\|_2^2.\end{aligned}$$

Integrating in time and using the energy estimate to bound $\int_0^t\|\partial_1u\|^2\,ds$ it follows

$$\begin{aligned}&t\,\|\partial_1u(t)\|_2^2+\nu_0\int_0^t s\,\|\nabla\partial_1u(s)\|_2^2\,ds+\nu_1\int_0^t s\,\mathcal{I}_1(u(s))\,ds\\ &\qquad\le\frac{C}{\nu_0}\Big[\|u_0\|_2^2+\int_0^t s\,\big(\|(u(s)\cdot\nabla)\,u(s)\|_2^2+\|f(s)\|_2^2\big)\,ds\Big]\quad\text{a.e. in }[0,T].\end{aligned}\tag{5.1}$$

We take now the L^2-inner product of (1.1a) with $t\,u_t$. By suitable integrations by parts, and using (4.2)-(4.3) we reach

$$\begin{aligned} t\,\|u_t\|_2^2 + \frac{\nu_0}{2}\frac{d}{dt}\big(t\,\|\nabla u\|_2^2\big) + \nu_1\frac{d}{dt}\big(t\,\mathcal{M}(u)\big) \\ \leq t\left(\left|\int_\Omega (u\cdot\nabla)u\,u_t\,dx\right| + \left|\int_\Omega f\,u_t\,dx\right|\right) + \nu_0\,\|\nabla u\|_2^2 + C\,\nu_1\,\mathcal{M}(u) \\ \leq \frac{t}{4}\big(\|(u\cdot\nabla)\,u\|_2^2 + \|f\|_2^2\big) + \frac{t}{2}\|u_t\|^2 + \nu_0\|\nabla u\|_2^2 + C\,\nu_1\|\mathcal{D}u\|_p^p. \end{aligned}$$

Integrating this inequality in time, by appealing to the energy inequality and recalling that $\mathcal{M}(u)\geq 0$, it follows that for a.e. $t\in[0,T]$

$$\begin{aligned} \nu_0\,t\,\|\nabla u(t)\|_2^2 + \int_0^t s\,\|u_t(s)\|_2^2\,ds \\ \leq C\Big[\|u_0\|_2^2 + \int_0^t s\,\big(\|(u(s)\cdot\nabla)\,u(s)\|_2^2 + \|f(s)\|_2^2\big)\,ds\Big], \end{aligned} \tag{5.2}$$

where C depends on p, δ, ν_0, ν_1, T, and Ω.

Let us now focus on the normal derivatives of u. Arguing as in Step 2 of the proof of Lemma 4.2, and replacing f with $f+(u\cdot\nabla)u$, we infer that

$$|\partial_{22}^2 u_1| \leq C\Big(|D_*^2 u| + \frac{1}{2\nu_0}\Big[|\partial_1\pi| + |u_t| + \big|(u\cdot\nabla)u\big| + |f|_2^2\Big]\Big) \quad \text{a.e. in } \Omega_T.$$

Then, squaring, multiplying by t, and integrating over $(0,t)\times\Omega$, we find

$$\begin{aligned} \int_0^t s\,\|\partial_{22}^2 u_1(s)\|^2\,ds \\ \leq \frac{C}{\nu_0}\int_0^t s\,\big(\|D_*^2 u(s)\|_2^2 + \|\partial_1\pi(s)\|_2^2 + \|u_t(s)\|_2^2 + \|(u(s)\cdot\nabla)\,u(s)\|_2^2\big)\,ds, \end{aligned} \tag{5.3}$$

To control $\int_0^t s\,\|\partial_1\pi(s)\|_2^2\,ds$ we proceed again as in Step 2 of the proof of Lemma 4.2. Thus, for a.e. $t\in[0,T]$, the following inequality holds true

$$\begin{aligned} &\int_0^t s\,\|\partial_1\pi(s)\|_2^2\,ds \\ &\leq C\int_0^t s\,\big(\|u_t(s)\|_2^2 + \|\partial_1\nabla u(s)\|_2^2 + \delta^{p-2}\mathcal{I}_1(u)(s) + \|f(s)\|_2^2 + \|(u(s)\cdot\nabla)\,u(s)\|_2^2\big)\,ds \\ &\leq C\Big[\|u_0\|_2^2 + \int_0^t s\,\big(\|(u(s)\cdot\nabla)\,u(s)\|_2^2 + \|f(s)\|_2^2\big)\,ds\Big], \end{aligned}$$

where we have used relations (5.1) and (5.2). Once again we apply (5.1), so that relation (5.3) gives, for a.e. $t\in[0,T]$

$$\int_0^t s\,\|\partial_{22}^2 u_1(s)\|_2^2\,ds \leq C\Big[\|u_0\|_2^2 + \int_0^t s\,\big(\|(u(s)\cdot\nabla)\,u(s)\|_2^2 + \|f(s)\|_2^2\big)\,ds\Big],$$

where C depends on p, δ, ν_0, ν_1, T, and Ω. Summing up the above inequality with (5.1) and (5.2), we get for a.e. $t\in[0,T]$

$$\begin{aligned} t\,\|\nabla u(t)\|_2^2 + \nu_0\int_0^t s\,\|D^2 u(s)\|_2^2\,ds \\ \leq C\Big[\|u_0\|_2^2 + \int_0^t s\,\big(\|(u(s)\cdot\nabla)\,u(s)\|_2^2 + \|f(s)\|_2^2\big)\,ds\Big], \end{aligned}$$

whit C depending on p, δ, ν_0, ν_1, T, and Ω. The convective term can be estimated as in (4.11) and choosing $\varepsilon > 0$ small enough we get, for a.e. $t \in [0,T]$,

$$t\,\|\nabla u(t)\|_2^2 + \nu_0 \int_0^t s\,\|D^2 u(s)\|_2^2\,ds$$
$$\leq c_\varepsilon \int_0^t \big(s\,\|\nabla u(s)\|_2^2\big)\,\|\nabla u(s)\|_2^2\,ds + C(p,\delta,\nu_0,\nu_1,\|f\|_{L^2(0,T;L^2)},\|u_0\|_2,T,\Omega).$$

Hence, by using Gronwall inequality over $[\lambda, T]$ (for any $\lambda > 0$), letting $\lambda \to 0^+$, and by using the energy inequality we get

$$\int_0^T t\,\|D^2 u(t)\|_2^2\,dt \leq C(p,\delta,\nu_0,\nu_1,\|f\|_{L^2(0,T;L^2)},\|u_0\|_2,T,\Omega).$$

Then, the assertion follows by means of Theorem 1.1. □

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