EXTENDING HIGHER DIMENSIONAL QUASI-COCYCLES

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Abstract. Let G be a group admitting a non-elementary acylindrical action on a Gromov hyperbolic space (for example, a non-elementary relatively hyperbolic group, or the mapping class group of a closed hyperbolic surface, or $\mathrm{Out}(F_n)$ for $n \geq 2$). We prove that, in degree 3, the bounded cohomology of G with real coefficients is infinite-dimensional. Our proof is based on an extension to higher degrees of a recent result by Hull and Osin. Namely, we prove that, if H is a hyperbolically embedded subgroup of G and V is any $\mathbb{R}[G]$-module, then any n-quasi cocycle on H with values in V may be extended to G. Also, we show that our extensions detect the geometry of the embedding of hyperbolically embedded subgroups, in a suitable sense.

Bounded cohomology of discrete groups is very hard to compute. For example, as observed in [Mon06], there is not a single countable group G whose bounded cohomology (with trivial coefficients) is known in every degree, unless it is known to vanish in all positive degrees (this is the case, for example, of amenable groups). Even worse, no group G is known for which the supremum of the degrees n such that $H^n_b(G,\mathbb{R}) \neq 0$ is positive and finite.

In degree 2, bounded cohomology has been extensively studied via the analysis of *quasi-morphisms* (see e.g. [Bro81, EF97, Fuj98, BF02, Fuj00] for the case of trivial coefficients, and [HO13, BBF13] for more general coefficient modules). In this paper we exploit *quasi-cocycles*, which are the higher-dimensional analogue of quasi-morphisms, to prove non-vanishing results for bounded cohomology in degree 3. To this aim, we prove that quasi-cocycles may be extended from a hyperbolically embedded family of subgroups to the ambient group. We also discuss the geometric information carried by our extensions of quasi-cocycles, showing in particular that projections on hyperbolically embedded subgroups may be reconstructed from the extensions of suitably chosen quasi-cocycles.

Quasi-cocycles and bounded cohomology. Let G be a group, and V be a normed $\mathbb{R}[G]$-space, i.e. a normed real vector space endowed with an

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isometric left action of G. We denote by $C^n(G,V)$ the set of homogeneous n-cochains on G with values in V, and for every $\varphi \in C^n(G,V)$ we set

$$\|\varphi\|_\infty = \sup\{\|\varphi(g_0,\dots,g_n)\|_V \,|\, (g_0,\dots,g_n) \in G^{n+1}\} \ \in \ [0,\infty] \ .$$

We denote by $C^n_b(G,V) \subseteq C^n(G,V)$ the subspace of bounded cochains, and by $C^n(G,V)^G$, $C^n_b(G,V)^G$ the subspaces of invariant (bounded) cochains (see Section 1 for the precise definitions). The space of *n-quasi-cocycles* is defined as follows:

$$\mathrm{QZ}^n(G,V) = \{\varphi \in C^n(G,V) \,|\, \delta^n\varphi \in C^{n+1}_b(G,V)\} \ .$$

Roughly speaking, quasi-cocycles are those cochains whose differential is quasi-null. Just as in the case of quasi-morphisms, the *defect* of a quasi-cocycle $\varphi \in \mathrm{QZ}^n(G,V)$ is given by

$$D(\varphi) = \|\delta^n\varphi\|_\infty \ .$$

Any cochain which stays at bounded distance from a genuine cocycle is a quasi-cocycle. The existence of G-invariant quasi-cocycles that are not at bounded distance from any G-invariant cocycle is equivalent to the non-vanishing of the exact part $EH^{n+1}_b(G,V)$ of the bounded cohomology module $H^{n+1}_b(G,V)$ (see below for the definition of $EH^{n+1}_b(G,V)$), so quasi-cocycles are a useful tool in the study of bounded cohomology.

For technical reasons, it is convenient to consider the subspace of *alternating* quasi-cocycles, which is denoted by $\mathrm{QZ}^n_{\mathrm{alt}}(G,V)$. Hull and Osin recently proved that if G is a group and $\{H_\lambda\}_{\lambda\in\Lambda}$ is a hyperbolically embedded family of subgroups of G, then alternating 1-quasi-cocycles on the H_λ's may be extended to G [HO13]. In this paper we extend Hull and Osin's result to higher dimensions:

Theorem 1. *Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a hyperbolically embedded family of subgroups of G, and let V be a normed $\mathbb{R}[G]$-module. For every $n \geq 1$, there exists a linear map*

$$\Theta^n\colon \bigoplus_{\lambda\in\Lambda} \mathrm{QZ}^n_{\mathrm{alt}}(H_\lambda,V)^{H_\lambda} \to \mathrm{QZ}^n_{\mathrm{alt}}(G,V)^G$$

such that, for every $\varphi = (\varphi_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathrm{QZ}^n_{\mathrm{alt}}(H_\lambda,V)^{H_\lambda}$ and for every $\lambda \in \Lambda$, we have

$$\sup_{\overline{h}\in H_\lambda^{n+1}} \|\Theta^n(\varphi)(\overline{h}) - \varphi_\lambda(\overline{h})\|_V < \infty \ .$$

We refer the reader to Theorem 4.2 for a more general statement.

A natural question is whether possibly non-alternating quasi-cocycles could also be quasi-extended from the H_λ's to G. This is always true if $n = 1$ since 1-quasi-cocycles are at bounded distance from alternating ones (see Remark 1.1 for a brief discussion of this issue in higher degrees). However, it seems unlikely that our construction could be adapted to deal with the general case.

Our proof of Theorem 1 is based on the construction, which we carry out in Section 3, of the trace of a simplex on a coset, that we think of as a projection of an $(n+1)$-tuple in G^{n+1} to a given coset of a hyperbolically embedded subgroup. Since the projection on a coset of a hyperbolically embedded subgroup is a multi-valued function, the trace of a simplex is not a single simplex, but an average of simplices. In order to maximize the number of cancellations between traces of simplices and reduce the technical effort in the proof of the main theorem, we chose to work in the coned off graph $\widehat{G}$, that is obtained from a Cayley graph of G by adding an extra point for any coset. The metric properties of the coned graph $\widehat{G}$ allow us to prove that, for $n > 1$, given any n-simplex there is a set of at most $n(n+1)$ exceptional cosets such that the diameter of the trace of the simplex on any other coset is smaller than an universal constant. Our results here are very similar to analogous results proved by Hull and Osin in [HO13], and in fact our arguments were inspired by theirs (even if Hull and Osin's constructions take place in a slightly different context). Perhaps, it is worth mentioning that the exceptional cosets associated to a simplex also generalize the barycenter of the simplex as defined in [BBF$^+$14], where the case of amalgamated products is analyzed. Indeed, if $G = H * K$, then the family $\{H, K\}$ is hyperbolically embedded in G, and our construction provides a "quasification" of the strategy described in [BBF$^+$14].

Applications to bounded cohomology. For any group G, any normed $\mathbb{R}[G]$-module V and every $n \geq 0$, the inclusion of bounded cochains into ordinary cochains induces the *comparison map* $c^n\colon H_b^n(G,V) \to H^n(G,V)$. The kernel of c^n is the set of bounded cohomology classes whose representatives are exact, and it is denoted by $EH_b^n(G,V)$. If K is a subgroup of G, then the restriction of cochains on G to cochains on K induces the map $\mathrm{res}^\bullet\colon EH_b^\bullet(G,V) \to EH_b^\bullet(K,V)$. Building on Theorem 1, in Section 5 we prove the following result (see Proposition 5.1 for a slightly more general statement):

Corollary 2. *Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a hyperbolically embedded family of subgroups of G, and let V be a normed $\mathbb{R}[G]$-module. Fix $n \geq 2$, and denote by $\mathrm{res}_\lambda^n\colon H_b^n(G,V) \to H_b^n(H_\lambda,V)$ the restriction map. For every element $(\alpha_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} EH_b^n(H_\lambda,V)$, there exists $\alpha \in EH_b^n(G,V)$ such that*

$$\mathrm{res}_\lambda^n(\alpha) = \alpha_\lambda \quad \text{for every } \lambda \in \Lambda\ .$$

The norm $\|\cdot\|_\infty$ on $C_b^n(G,V)$ induces a seminorm on $H_b^n(G,V)$ that is usually referred to as *Gromov seminorm.* Let us now denote by $N_b^n(G,V)$ the subspace of $H_b^n(G,V)$ given by elements with vanishing seminorm, and let us set $\overline{H}_b^n(G,V) = H_b^n(G,V)/N_b^n(G,V)$, so $\overline{H}_b^n(G,V)$ is a Banach space. Following [Osi], we say that a group G is *acylindrically hyperbolic* if it admits an acylindrical action on a Gromov hyperbolic space. It is shown in [Osi] that being acylindrically hyperbolic is equivalent to containing a proper infinite hyperbolically embedded subgroup. Building on results from [DGO11], from Corollary 2 we deduce the following:

Corollary 3. *Let G be an acylindrically hyperbolic group. Then the dimension of both $\overline{H}^3_b(G,\mathbb{R})$ and $EH^3_b(G,\mathbb{R})$ is equal to the cardinality of the continuum. Therefore, the same is true also for $H^3_b(G,\mathbb{R})$.*

The class of acylindrically hyperbolic groups includes many examples of interest: non-elementary hyperbolic and relatively hyperbolic groups [DGO11], the mapping class group of the p-punctured closed orientable surface of genus g, provided that $3g+p\geq 6$ [DGO11, Theorem 2.18], $\mathrm{Out}(F_n)$ for $n\geq 2$ [DGO11, Theorem 2.20], groups acting geometrically on a proper CAT(0) space with a rank one isometry [Sis11] and [DGO11, Theorem 2.22], and fundamental groups of several graphs of groups [MO].

Further results. One may wonder whether Corollary 2 holds with bounded cohomology instead of exact bounded cohomology. In fact, the map $\Theta^\bullet$ of Theorem 1 extends to a map between alternating cochains sending bounded cochains to bounded cochains (see Theorem 4.1), so one may wonder whether that $\Theta^\bullet$ could be used to extend possibly non-exact bounded coclasses. However, in general our map $\Theta^\bullet$ does not carry cocycles to cocycles, but only to quasi-cocycles, so it is not a chain map. In fact, in Section 6 we prove the following:

Proposition 4. *For every $n\geq 2$, there exists a pair (G,H) such that G is relatively hyperbolic with respect to H (in particular, H is hyperbolically embedded in G), and the restriction $H^n_b(G,\mathbb{R})\to H^n_b(H,\mathbb{R})$ is not surjective.*

Even worse, $\Theta^\bullet$ does not induce a well-defined map on exact bounded cohomology in general (see Proposition 6.5 for an explicit example). In order to obtain a positive result in this direction, we need to make some further assumptions on the ordinary cohomology of the subgroups H_λ (see Proposition 5.2).

However, the fact that $\Theta^\bullet$ does not induce a well-defined map on bounded cohomology may be exploited to prove non-vanishing results. Namely, it may happen that a genuine (unbounded) real n-cocycle on a hyperbolically embedded subgroup H of G may be extended to a quasi-cocycle on G whose differential defines a non-trivial class in $H^{n+1}_b(G,\mathbb{R})$. For example, we can prove the following result (see Corollary 6.4):

Proposition 5. *Let H be an amenable hyperbolically embedded subgroup of the group G, let $n\geq 1$, and suppose that the inclusion $H\to G$ induces a non-injective map $H_n(H,\mathbb{R})\to H_n(G,\mathbb{R})$. Then $H^{n+1}_b(G,\mathbb{R})\neq 0$.*

As a consequence of (a variation of) Proposition 5, and building on a construction by McReynolds, Reid and Stover [MRS13], in Proposition 6.6 we show that, for every $n\geq 3$ and $2\leq k\leq n$, there exist infinitely many commensurability classes of cusped hyperbolic n-manifolds M such that $H^k_b(M,\mathbb{R})\neq 0$.

A natural question is whether, given n, it is possible to find a hyperbolically embedded finite family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of a group G so that the direct sum of the restriction map $\oplus\,\mathrm{res}^n_\lambda\colon EH^n_b(G,V)\to\bigoplus EH^n_b(H_\lambda,V)$ is an isomorphism (this map is surjective by Corollary 2). In dimension 3 this is never the case, due to the following:

Proposition 6. *Let G be a finitely generated group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite hyperbolically embedded family of subgroups of G, and let V be a normed $\mathbb{R}[G]$-module. Then the kernel of the restriction map $\oplus\,\mathrm{res}^3_\lambda\colon EH^3_b(G,V)\to\bigoplus EH^3_b(H_\lambda,V)$ is infinite-dimensional.*

We obtain Proposition 6 as a consequence of a refinement of a result of Dahmani, Guirardel and Osin [DGO11, Theorem 6.14] that might be of independent interest. Recall that a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups of G is *non-degenerate* if there is some λ so that H_λ is a proper, infinite subgroup of G. If $\{H_\lambda\}_{\lambda\in\Lambda}$ is a hyperbolically embedded family of subgroups of G, then each H_λ is hyperbolically embedded in G (see the first sentence of Remark 2.5). In particular, a consequence of [DGO11, Theorem 6.14] is that, if a group G contains a non-degenerate hyperbolically embedded family of subgroups, then G contains a maximal finite normal subgroup, which will be denoted by $K(G)$.

Theorem 7. *Let X be a (possibly infinite) generating system of the group G and let the non-degenerate family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ be hyperbolically embedded in (G,X). Then for each $n\geq 1$ there exists a copy F of the free group on n generators inside G so that $\{H_\lambda\}_{\lambda\in\Lambda}\cup\{F\times K(G)\}$ is hyperbolically embedded in (G,X).*

Since the proof of Theorem 7 uses techniques somehow different from the rest of the paper, and is heavily based on results of [DGO11], whereas the rest of the paper is almost self-contained, we decided to include the proof of Theorem 7 in an appendix, rather than in the main body of the paper.

Quasi-cocycles and projections. Any hyperbolically embedded family $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups of a group G comes along with a family of G-equivariant projections $\pi_B : G\to B$ for every coset B of a subgroup H_λ, satisfying certain axioms first introduced by Bestvina, Bromberg and Fujiwara in [BBF10]. It can be shown that the family of projections itself captures the fact that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G. Theorem 2.11, reported below, makes this statement precise, combining results in the literature. The BBF axioms are defined in Section 2.

Theorem 8. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite family of finitely generated subgroups of the finitely generated group G, and let $\mathcal{B}$ be the set of the (labelled) cosets of the H_λ's.*

(1) Suppose that it is possible to assign, for each pair of cosets $B_1,B_2\in\mathcal{B}$, a subset $\widetilde{\pi}_{B_1}(B_2)\subseteq B_1$ in an equivariant way (i.e. in such a

way that $\widetilde{\pi}_{gB_1}(gB_2) = g\widetilde{\pi}_{B_1}(B_2)$) and so that the BBF axioms are satisfied. Then $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G.

(2) Suppose $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X). Then the family of projections $\{\pi_B\}_{B\in\mathcal{B}}$ as in Definition 2.6 satisfies the BBF axioms.

Our extension of quasi-cocycles is only based on the good properties of the family of projections π_B and already contains all the information necessary to reconstruct the projections themselves. In fact, in Section 7 we make the following statement precise:

Informal Statement. Let H be a finitely generated group. Then, there exist a coefficient module V and a cocycle $c \in C^2_{\mathrm{alt}}(H, V)$ such that the following holds. Whenever H is hyperbolically embedded in G, the projections on the cosets of H may be recovered from the extension $\Theta^2(c)$, which, therefore, detects the geometry of the embedding of H in G.

So, by exploiting projections (and Theorem 8) we are able to "close the circle" and get back from our cocycle extensions to the fact H is hyperbolically embedded in G. We hope that in the future this will lead to a complete characterization of hyperbolically embedded subgroups in terms of bounded cohomology.

We emphasize that our argument does rely on c being a cocycle of dimension greater than 1, and the authors are not aware of ways to reconstruct projections using quasi-morphisms.

1. Basic facts about bounded cohomology

Let us recall some basic definitions about bounded cohomology of groups. Let G be a group, and V be a normed $\mathbb{R}[G]$-space. The set of n-cochains on G with values in V is given by

$$C^n(G, V) = \left\{\varphi\colon G^{n+1} \to V\right\} \ .$$

The vector space $C^n(G, V)$ is endowed with a left action of G defined by $(g \cdot \varphi)(g_0, \ldots, g_n) = g \cdot (\varphi(g^{-1}g_0, \ldots, g^{-1}g_n))$. We have defined in the introduction the submodule $C^n_b(G, V) \subseteq C^n(G, V)$ of bounded cochains. The action of G on $C^n(G, V)$ preserves $C^n_b(G, V)$, so $C^n_b(G, V)$ is a normed $\mathbb{R}[G]$-module. The differential

$$\delta^n\colon C^n(G, V) \to C^{n+1}(G, V)\,,$$
$$\delta^n\varphi(g_0, \ldots, g_{n+1}) = \textstyle\sum_{j=0}^{n+1}(-1)^j\varphi(g_0, \ldots, \widehat{g}_j, \ldots, g_{n+1})$$

restricts to a map $C^n_b(G, V) \to C^{n+1}_b(G, V)$, which will still be denoted by δ^n. If W is a (normed) $\mathbb{R}[G]$-module, then we denote by W^G the subspace of G-invariant elements of W. The differential δ^n sends invariant cochains to invariant cochains, thus endowing $C^\bullet(G, V)^G$ and $C^\bullet_b(G, V)^G$ with the structure of chain complexes. The cohomology (resp. bounded cohomology) of G with coefficients in V is the cohomology of the complex $C^\bullet(G, V)^G$ (resp. $C^\bullet_b(G, V)^G$).

Let us denote by $\mathfrak{S}_{n+1}$ the group of permutations of $\{0,\dots,n\}$. A cochain $\varphi\in C^n(G,V)$ is *alternating* if

$$\varphi(g_{\sigma(0)},\dots,g_{\sigma(n)})=\mathrm{sgn}(\sigma)\cdot\varphi(g_0,\dots,g_n)$$

for every $\sigma\in\mathfrak{S}_{n+1}$. Both the differential and the G-action preserve alternating cochains, that hence give a subcomplex $C^\bullet_{\mathrm{alt}}(G,V)^G$ of $C^\bullet(G,V)^G$, respectively $C^\bullet_{b,\mathrm{alt}}(G,V)^G$ of $C^\bullet_b(G,V)^G$. The space of alternating quasi-cocycles $\mathrm{QZ}^n_{\mathrm{alt}}(G,V)$ is just the intersection of $\mathrm{QZ}^n(G,V)$ with $C^n_{\mathrm{alt}}(G,V)$.

For every $n\geq 0$ we denote by $C_n(G)$ the real vector space with basis G^{n+1}. Elements of G^{n+1} are called *n-simplices*, and we say that an n-simplex $\overline{g}=(g_0,\dots,g_n)$ is supported in a subset $S\subseteq G$ if all its vertices lie in S, i.e. if $g_j\in S$ for every $j=0,\dots,n$. The subspace of $C_n(G)$ generated by simplices supported in S is denoted by $C_n(S)$. We also put on $C_n(G)$ the ℓ^1-norm defined by

$$\left\|\sum_{\overline{g}\in G^{n+1}}a_{\overline{g}}\overline{g}\right\|_1=\sum_{\overline{g}\in G^{n+1}}|a_{\overline{g}}|\ .$$

If $\overline{g}=(g_0,\dots,g_n)\in C_n(G)$, we denote by $\partial_j\overline{g}=(g_0,\dots,\widehat{g_j},\dots,g_n)\in C_{n-1}(G)$ the j-th face of $\overline{g}$, and we set $\partial\overline{g}=\sum_{j=0}^n(-1)^j\partial_j\overline{g}$.

Degenerate chains. If $S\subseteq G$ is any subset, then we may define an alternating linear operator $\mathrm{alt}_n\colon C_n(S)\to C_n(S)$ by setting, for every $\overline{s}=(s_0,\dots,s_n)\in S^{n+1}$,

$$\mathrm{alt}_n(\overline{s})=\frac{1}{(n+1)!}\sum_{\sigma\in\mathfrak{S}_{n+1}}\mathrm{sgn}(\sigma)(s_{\sigma(0)},\dots,s_{\sigma(n)})\ .$$

A chain $c\in C_n(S)$ is *degenerate* if $\mathrm{alt}_n(c)=0$. If K is a group and W is an $\mathbb{R}[K]$-module, then it is immediate to check that a cochain $\varphi\in C^n(K,W)$ is alternating if and only if it vanishes on degenerate chains in $C_n(K)$. If $\varphi\in C^n(S,V)$ is any cochain, then we may alternate it by setting

$$\mathrm{alt}^n(\varphi)(\overline{s})=\varphi(\mathrm{alt}_n(\overline{s}))$$

for every $\overline{s}\in S^{n+1}$.

In every degree, the G-equivariant chain map $\mathrm{alt}^n\colon C^n(G,V)\to C^n(G,V)$ provides a linear projection onto the subcomplex of alternating cochains, and $\mathrm{alt}^\bullet$ is G-equivariantly homotopic to the identity (see e.g. [FM11, Appendix B]).

Moreover, $\mathrm{alt}^\bullet$ restricts to a G-equivariant chain map $\mathrm{alt}^\bullet_b\colon C^\bullet_b(G,V)\to C^\bullet_{b,\mathrm{alt}}(G,V)$, and for every $n\in\mathbb{N}$ the map alt^n_b provides a norm non-increasing projection onto $C^n_{b,\mathrm{alt}}(G,V)$. The homotopy between $\mathrm{alt}^\bullet$ and the identity of $C^\bullet(G,V)$ may be chosen in such a way that it restricts to a homotopy between $\mathrm{alt}^\bullet_b$ and the identity of $C^\bullet_b(G,V)$, which is bounded in every degree. As a consequence, the bounded cohomology of G with coefficients in V may be computed as the cohomology of the complex $C^\bullet_{b,\mathrm{alt}}(G,V)$.

Observe that, if $\varphi \in C^n(G,V)$ is a quasi-cocycle, then

$$\|\delta^n \operatorname{alt}^n(\varphi)\|_\infty = \|\operatorname{alt}^n(\delta^n(\varphi))\|_\infty \leq \|\delta^n(\varphi)\|_\infty < +\infty ,$$

so alt^n projects $\mathrm{QZ}^n(G,V)$ onto $\mathrm{QZ}^n_{\mathrm{alt}}(G,V)$.

Remark 1.1. It is well-known that a quasi-cocycle $\varphi \in \mathrm{QZ}^1(G,V)$ is at bounded distance from the alternating quasi-cocycle $\operatorname{alt}^1(\varphi) \in \mathrm{QZ}^1_{\mathrm{alt}}(G,V)$. In fact, let $T^\bullet$ be a chain homotopy between $\operatorname{alt}^\bullet$ and the identity of $C^\bullet(G,V)$ which preserves boundedness of cochains. Then $T^1(\varphi)$, being a G-equivariant 0-cochain, is bounded (if the action of G is trivial, then it is even constant), so

$$\operatorname{alt}^1(\varphi) - \varphi = T^2(\delta^1\varphi) - \delta^0(T^1\varphi)$$

is itself bounded. On the contrary, if $\varphi \in \mathrm{QZ}^n(G,V)$, $n \geq 2$, then the cochain $T^{n+1}(\delta^n\varphi)$ is still bounded, while in general $\delta^{n-1}(T^n\varphi)$ (whence $\operatorname{alt}^n(\varphi) - \varphi$) can be unbounded.

Let us consider for example the group $\mathbb{Z}^2 = \langle a, b\rangle$ and the 1-cocyles $\alpha, \beta \in Z^1(\mathbb{Z}^2, \mathbb{R}) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{R})$ corresponding to the homomorphisms α', β' such that $\alpha'(a) = \beta'(b) = 1$, $\alpha'(b) = \beta'(a) = 0$. It is readily seen that, for every $n \in \mathbb{Z}$, $(\alpha \cup \beta)(1, b^n, a^n) = 0$, while $(\alpha \cup \beta)(a^n, 1, b^n) = -n^2$. This implies that the 2-cocycle $\alpha \cup \beta$ does not lie at bounded distance from any alternating cochain in $C^2_{\mathrm{alt}}(\mathbb{Z}^2, \mathbb{R})$.

2. Projections and hyperbolic embeddings

Let G be a group, and let us fix a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups of G. A (possibly infinite) subset $X \subseteq G$ is a *relative generating set* if $X \cup \bigcup_{\lambda\in\Lambda} H_\lambda$ generates G.

Definition 2.1 ([DGO11])**.** Let X be a relative generating set for G and let $\mathcal{H}$ denote the disjoint union $\mathcal{H} = \sqcup_{\lambda\in\Lambda} H_\lambda \backslash \{e\}$. We denote by $\mathrm{Cay}(G, X \sqcup \mathcal{H})$ the Cayley graph of G with respect to the alphabet $X \sqcup \mathcal{H}$. Notice that some letters in $X \sqcup \mathcal{H}$ may represent the same element of G, in which case $\mathrm{Cay}(G, X \sqcup \mathcal{H})$ has multiple edges corresponding to these letters. We label each edge of $\mathrm{Cay}(G, X \sqcup \mathcal{H})$ by the corresponding letter in $X \sqcup \mathcal{H}$.

For every $\lambda \in \Lambda$, we define the *relative metric* $d_\lambda : H_\lambda \times H_\lambda \to [0, \infty]$ by letting $d_\lambda(g, h)$ be the length of the shortest path in $\mathrm{Cay}(G, X \sqcup \mathcal{H})$ that connects g to h and has no edge that connects vertices of H_λ and is labelled by an element of $H_\lambda \backslash \{1\}$.

The family $\{H_\lambda\}_{\lambda\in\Lambda}$ is *hyperbolically embedded* in (G, X) if the Cayley graph $\mathrm{Cay}(G, X \sqcup \mathcal{H})$ is hyperbolic and, for every $\lambda \in \Lambda$, the metric space (H_λ, d_λ) is locally finite.

In general, one says that $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G if it is hyperbolically embedded in (G, X) for some relative generating set $X \subseteq G$. In this case we write $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ or $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ when we want to emphasize the choice of X.

Let us fix once and for all a subset $X \subseteq G$ so that $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X). Throughout the whole paper, any coset will be understood to be a *left* coset. We denote by $\mathcal{B}$ the set of cosets of the subgroups H_λ, $\lambda \in \Lambda$. More precisely, we let $\mathcal{B}$ be the *disjoint* union if the B_λ's, where $\mathcal{B}_\lambda$ is the set of cosets of H_λ for every $\lambda \in \Lambda$. We label every element of $\mathcal{B}_\lambda$ by the index $\lambda \in \Lambda$. Notice that, if there are repetitions among the H_λ's, then some cosets appear with repetitions (but with distinct labels) in $\mathcal{B}$.

Remark 2.2. Let $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ and fix an index $\lambda \in \Lambda$. Using the fact that the relative metric d_λ is locally finite it is easy to prove that, if $H_\lambda = H_{\lambda'}$ for some $\lambda' \neq \lambda$, then H_λ is finite and the set of indices $\lambda'' \in \Lambda$ such that $H_{\lambda''} = H_\lambda$ is finite. Therefore, at least for what concerns the main results of our paper, we could safely restrict our attention to the case when $H_\lambda \neq H_{\lambda'}$ for every $\lambda' \neq \lambda$.

We now define the object we will work with throughout the paper.

Definition 2.3. We denote by $(\widehat{G}, \widehat{d})$ the metric graph obtained by adding to $\mathrm{Cay}(G, X)$ a vertex $c(B)$ for each $B \in \mathcal{B}$ and edges $[c(B), h]$ of length $1/4$ for every $c(B)$ and $h \in B$.

Our $\widehat{G}$ is very similar to Farb's coned-off graph [Far98], but using $\widehat{G}$ rather than the coned-off graph or $\mathrm{Cay}(G, X \cup \mathcal{H})$ will allow us to streamline a few arguments. Hopefully, $\widehat{G}$ will turn out to be more convenient in other contexts as well.

If a geodesic γ of $\widehat{G}$ contains the vertex $c(B)$, then we denote by $\mathrm{in}_\gamma(B)$ and $\mathrm{out}_\gamma(B)$ respectively the last point of $\gamma \cap B$ preceding $c(B)$ and the first point of $\gamma \cap B$ following $c(B)$ along γ. If γ starts (resp. ends) at $c(B)$, then $\mathrm{out}_\gamma(B)$ (resp. $\mathrm{in}_\gamma(B)$) is not defined.

Remark 2.4. Suppose that the geodesic γ of $\widehat{G}$ intersects the coset B in at least two points p, q. Then γ contains $c(B)$, and $\gamma \cap B = \{\mathrm{in}_\gamma(B), \mathrm{out}_\gamma(B)\} = \{p, q\}$.

If $B \in \mathcal{B}$ is labelled by $\lambda \in \Lambda$, we endow B with the relative metric d_B obtained by translating d_λ. If $S \subseteq B$ is a subset of some coset $B \in \mathcal{B}$, then we denote by $\mathrm{diam}_B(S)$ the diameter of S with respect to d_B.

Remark 2.5. Let $\lambda_0 \in \Lambda$ be fixed. Since $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$, we have $H_{\lambda_0} \hookrightarrow_h (G, X \cup \mathcal{H}')$, where $\mathcal{H}' = \bigcup_{\omega\neq\lambda_0} H_\omega$. Therefore, [DGO11, Corollary 4.32] implies that, if G is finitely generated, then each H_λ is finitely generated. If, in addition, the family $\{H_\lambda\}_{\lambda\in\Lambda}$ is finite, then by [DGO11, Corollary 4.27] we can add to X the union of finite generating sets of the H_λ's without altering the fact that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$. Then [DGO11, Lemma 4.11-(b)] implies that the relative metric d_B is bi-Lipschitz equivalent to a word metric on B.

Hence, for the purposes of our paper, we could replace d_B with a more familiar word metric whenever we deal with finite families of hyperbolically embedded subgroups of a finitely generated group.

Projections on cosets. Projections, as defined below, will play a crucial role in this paper.

Definition 2.6. For every coset $B \in \mathcal{B}$ and every vertex x of $\widehat{G}$, the projection of x onto B is the set

$$\pi_B(x) = \{p \in B \mid \widehat{d}(x,p) = \widehat{d}(x,B)\} \ .$$

If $S \subseteq \widehat{G}$ is any subset, then we set $\pi_B(S) = \bigcup_{x\in S} \pi_B(x)$.

Remark 2.7. Of course $\pi_B(c(B)) = B$, while if $x \neq c(B)$ we have

$$\pi_B(x) = \{p \in B \mid p = \mathrm{in}_\gamma(B)\, ,\ \gamma \text{ geodesic joining } x \text{ to } c(B)\} \ .$$

An important result about projections is described in Lemma 2.8, which says that if two points project far away on a coset B then any geodesic connecting them contains $c(B)$. Similar properties are also true for other notions of projections in a relatively hyperbolic space, as discussed in [Sis13]. Also, the following lemma has strong connections with the bounded coset penetration property for Farb's coned-off graph [Far98].

Lemma 2.8. *There exists $D \geq 1$ with the following property. For $x, y \in \widehat{G}$ and a coset B, if $\mathrm{diam}_B(\pi_B(x) \cup \pi_B(y)) \geq D$ then all geodesics from x to y contain $c(B)$.*

Proof. Let as above $\mathcal{H} = \bigsqcup H_\lambda \backslash \{1\}$ and set $\Gamma = \mathrm{Cay}(G, X \sqcup \mathcal{H})$. Let $\widehat{\gamma}_1$, $\widehat{\gamma}_2$ be geodesics in $\widehat{G}$ from x to any $p \in \pi_B(x)$ and from y to any $q \in \pi_B(y)$ respectively. Notice that $\widehat{\gamma}_1 \cap B$ and $\widehat{\gamma}_2 \cap B$ each consists of a single point. We can form paths γ_i in Γ replacing all subpaths of $\widehat{\gamma}_i$ consisting of two edges intersecting at $c(B')$, for some coset B', with an edge in Γ (and possibly removing the first edge of $\widehat{\gamma}_i$ if x and/or y are in $\widehat{G}$ but not in G). Consider now a geodesic $\widehat{\gamma}$ from x to y, and construct a path γ in Γ similarly (and possibly add an edge at the beginning/end of γ to make sure that the endpoints of γ coincide with the starting points of γ_1, γ_2). Finally, if λ is the label of B, let e be the edge in Γ labelled by an element of H_λ connecting the endpoint of γ_1 to the endpoint of γ_2.

It is not hard to see that (the unit speed parametrizations of) γ, γ_i are, say, (2,2)-quasi-geodesics in Γ. For example, one can argue as follows. Given a geodesic α in Γ, we can replace each edge of α labelled by a letter from $\mathcal{H}$ by a path of length $1/2$ in $\widehat{G}$. This implies that $d_{\widehat{G}}(g,h) \leq d_\Gamma(g,h)$ for each $g, h \in G$. Now, whenever g, h are on, say, γ, and $\gamma|_{g,h}, \widehat{\gamma}|_{g,h}$ denote the subpaths of $\gamma, \widehat{\gamma}$ with endpoints g, h, we have

$$l(\gamma|_{g,h}) \leq 2l(\widehat{\gamma}|_{g,h}) = 2d_{\widehat{G}}(g,h) \leq 2d_\Gamma(g,h),$$

which easily implies that γ is a (2,2)-quasi-geodesic (we used in the equality that $\widehat{\gamma}$ is a geodesic). We proved that it is a (2,2)-quasi-geodesic rather than a (2,0)-quasi-geodesic because we showed the above inequality for $g, h \in G$ only. In fact, this estimate can be improved but we will not need to.

The paths $\gamma_1, e, \gamma_2, \gamma$ form a (2,2)-quasi-geodesic quadrangle in Γ, which is a hyperbolic metric space. Hence, there exists C depending on the hyperbolicity constant only so that any point on one side of the quadrangle is contained in the C-neighborhood of the union of the other three sides. Assume now that $\widehat{\gamma}$ does not contain $c(B)$ and hence that γ does not contain any edge connecting points in B. Under this assumption we now construct a cycle c whose length is bounded in terms of the hyperbolicity constant of Γ and so that the only edge contained in c that connects points in B is e. Such cycle is either a quadrangle, a pentagon or a hexagon formed by e, subpaths of γ_1, γ_2, possibly a subpath of γ and one or two paths of length bounded in terms of the hyperbolicity constant of Γ. The idea is illustrated in Figure 1.

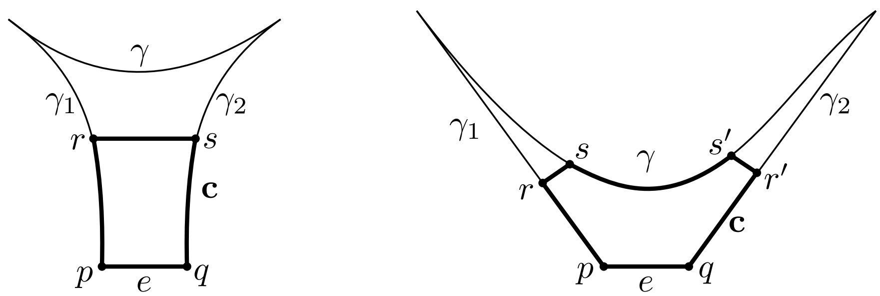

Figure 1

Consider a point r on γ_1 at distance $10C+10$ from the final point p of γ_1, or let r be the starting point of γ_1 if such point does not exist. We know that r is C-close to a point s on either γ_2 or γ (and we set $r = s$ if r is the starting point of γ_1). In the first case we let c be a quadrangle with vertices p, r, s, q as in the left part of Figure 1. The geodesic from r to s cannot contain any edge with both endpoints in B because (either it is trivial or) its length is at most C and one of its endpoints is at distance $10C + 10$ from a point in B. If instead s is on γ, we pick r' along γ_2 similarly to r. If r' is C-close to a point s' in γ_1 we form a quadrangle as above. Otherwise r' is C-close to a point $s' \in \gamma$ and we let c be a hexagon with vertices p, r, s, s', r', q as on the right part of Figure 1. Again, it is not difficult to show that e is the only edge of c that connects points in B.

Observe now that the cycle c has length bounded by $50C + 10$, and its only component labelled by an element of $H_\lambda \setminus \{1\}$ is the edge e. Therefore, by definition of the relative metric d_B, we get that $d_B(p, q) < 50C + 10$. This holds for all $p \in \pi_B(x)$ and $q \in \pi_B(y)$, and C only depends on G and X, so we are done. □

Let now B, B' be distinct cosets, and take points $x, y \in B'$ with $x \neq y$. The geodesic $[x, c(B')] \cup [c(B'), y]$ does not contain $c(B)$, so the previous lemma implies that $\operatorname{diam}_B(\pi_B(x) \cup \pi_B(y)) < D$. As a consequence, we easily get the following:

Lemma 2.9. *If the cosets B, B' are distinct, then* $\operatorname{diam}_B \pi_B(B') < D$. *In particular,* $\operatorname{diam}_B(\pi_B(x)) < D$ *for every* $x \in \widehat{G} \setminus \{c(B)\}$, *and* $\operatorname{diam}_B(B \cap B') < D$, $\operatorname{diam}_{B'}(B \cap B') < D$ *for any pair of distinct cosets* B, B'.

We also have the following:

Lemma 2.10. *Take $v_0, v_1 \in G$. Then the set*

$$\{B \in \mathcal{B} \mid \operatorname{diam}_B(\pi_B(v_0) \cup \pi_B(v_1)) \geq D\}$$

is finite.

Proof. By Lemma 2.8, if $d_B(\pi_B(v_0), \pi_B(v_1)) \geq D$ then any geodesic in $\widehat{G}$ joining v_0 with v_1 contains $c(B)$, as well as a subgeodesic of length $1/2$ centered at $c(B)$. Such subgeodesics can intersect at most at their endpoints, so the set of cosets described in the statement can contain at most $2\widehat{d}(v_0, v_1)$ elements. □

The BBF axioms. The projections on hyperbolically embedded subgroups satisfy certain axioms introduced by Bestvina, Bromberg and Fujiwara in [BBF10], which we will refer to as the *BBF axioms.* In order to simplify the statement of the theorem below, we restrict ourselves to the specific case we are interested in, namely cosets of subgroups of a given group. As opposed to the rest of the Section we restrict here to the case in which the group G is finitely generated and the family $\{H_\lambda\}_{\lambda \in \Lambda}$ is finite. We already pointed out that in this case each H_λ is finitely generated.

Let, as above, $\mathcal{B}$ be the collection of the (labelled) cosets of the H_λ's in G. We fix a finite system of generators $\mathcal{S}_\lambda$ of H_λ for each $\lambda \in \Lambda$, and if $B \in \mathcal{B}$ is labelled by λ we denote by $\mathcal{C}(B)$ a copy of the Cayley graph $\operatorname{Cay}(H_\lambda, \mathcal{S}_\lambda)$. For each B let $\widetilde{\pi}_B : \mathcal{B} \backslash \{B\} \to \mathcal{P}(\mathcal{C}(B))$ be a function (where $\mathcal{P}(\mathcal{C}(B))$ is the collection of all subsets of $\mathcal{C}(B)$). Define

$$d_Y(X, Z) = \operatorname{diam}_{\mathcal{C}(Y)}(\widetilde{\pi}_Y(X) \cup \widetilde{\pi}_Y(Z)).$$

Here the diameter is considered with respect to the word metric. We will say that the family of projections $\{\widetilde{\pi}_Y\}_{Y \in \mathcal{B}}$ satisfies the BBF axioms if the following holds. There exists $\xi < \infty$ so that, using the enumeration in [BBF10, Sections 2.1, 3.1]:

(0) $\operatorname{diam}_{\mathcal{C}(Y)}(\widetilde{\pi}_Y(X)) < \xi$ for all distinct $X, Y \in \mathcal{B}$,
(3) for all distinct $X, Y, Z \in \mathcal{B}$ we have $\min\{d_Y(X, Z), d_Z(X, Y)\} \leq \xi$,
(4) $\{Y : d_Y(X, Z) \geq \xi\}$ is a finite set for each $X, Z \in \mathcal{B}$.

Combining results in the literature, one can obtain the following theorem, which roughly speaking says that the family of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ is hyperbolically embedded in G if and only if one can define projections on the cosets of H_λ satisfying the BBF axioms.

Theorem 2.11. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite family of finitely generated subgroups of the finitely generated group G, and let $\mathcal{B}$ be the set of the (labelled) cosets of the H_λ's.*

(1) Suppose that it is possible to assign, for each pair of cosets $Y_1, Y_2 \in \mathcal{B}$, a subset $\widetilde{\pi}_{Y_1}(Y_2) \subseteq Y_1$ in an equivariant way (i.e. in such a way that $\widetilde{\pi}_{gY_1}(gY_2) = g\widetilde{\pi}_{Y_1}(Y_2)$) and so that the BBF axioms are satisfied. Then $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G.

(2) Suppose $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X). Then the family of projections $\{\pi_Y\}_{Y\in\mathcal{B}}$ as in Definition 2.6 satisfies the BBF axioms.

Proof. (1) The set of projections satisfying the BBF axioms can be used to construct a certain metric space out of $\{\mathcal{C}(B)\}_{B\in\mathcal{B}}$. We briefly overview the construction for the sake of completeness. The details can be found in [BBF10, Section 3.1].

First, the authors of [BBF10] define, using the functions d_Y, a certain graph $\mathcal{P}_K(\mathcal{B})$ with vertex set $\mathcal{B}$. We will not need the precise definition. Then, they construct the path metric space $\mathcal{C}(\mathcal{B})$ consisting of the union of all $\mathcal{C}(B)$'s and edges of length 1 connecting all points in $\widetilde{\pi}_X(Z)$ to all points in $\widetilde{\pi}_Z(X)$ whenever X, Z are connected by an edge in $\mathcal{P}_K(\mathcal{B})$.

As it turns out, $\mathcal{C}(\mathcal{B})$ is hyperbolic relative to $\{\mathcal{C}(B)\}_{B\in\mathcal{B}}$ [Sis12, Theorem 6.2] (even more, it is quasi-tree-graded [Hum12]). Moreover, the construction of $\mathcal{C}(\mathcal{B})$ is natural in the sense that G acts on $\mathcal{C}(\mathcal{B})$ by isometries. The action is such that for each $g \in G$ we have $g(\mathcal{C}(Y)) = \mathcal{C}(gY)$, and H_λ acts on $\mathcal{C}(H_\lambda)$ by left translations.

In particular, G acts coboundedly on $\mathcal{C}(\mathcal{B})$ in such a way that $\mathcal{C}(\mathcal{B})$ is hyperbolic relative to the orbits of the cosets of the H_λ's which coincide, for an appropriate choice of basepoints, with the copies of the $\mathcal{C}(B)$'s contained in $\mathcal{C}(\mathcal{B})$. Also, each H_λ acts properly. Using the characterization of being hyperbolically embedded given in [Sis12, Theorem 6.4] (see also [DGO11, Theorem 4.42]) in terms of actions on a relatively hyperbolic space, we can now conclude that $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G.

(2) Recall from Remark 2.5 that, for every $B \in \mathcal{B}$, the relative metric d_B and the word metric $d_{\mathcal{C}(B)}$ are bi-Lipschitz equivalent. Therefore, Axioms (0) and (4) follow respectively from Lemma 2.9 and 2.10. Let us now show Axiom (3) (cfr. [Sis11, Lemma 2.5]). Let X, Y, Z be distinct and suppose that $d_Y(X, Z) > \xi$ (for ξ large enough). We have to show that $d_Z(X, Y) \leq \xi$. Pick $x \in X$ and observe that Lemma 2.8 implies that any geodesic in $\widehat{G}$ from x to Z contains $c(Y)$. In particular, $\pi_Z(x)$ is contained in $\pi_Z(c(Y))$, and the conclusion easily follows (keeping into account Axiom (0)). □

Remark 2.12. Fix the notation of part (2) of the theorem. Since G is finitely generated, by [DGO11, Corollary 4.27] we may assume that $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X), where X is a (possibly infinite) set of generators of G. By [Sis12, Theorem 6.4], $\Gamma = \mathrm{Cay}(G, X)$ is (metrically) hyperbolic relative to the cosets of the H_λ's. It is observed in [MS13, Lemma

4.3] that the BBF axioms are satisfied in this setting when the π_Y's are defined as the closest point projections with respect to the metric of Γ. Hence, part (2) of the theorem also holds for this other set of projections. On the other hand, it could be shown using techniques from [Sis13] that projections as in Definition 2.6 and closest point projections in Γ coarsely coincide (but we will not need this).

3. The trace of a simplex on a coset

Throughout this section, we fix a group G with a hyperbolically embedded family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. We also denote by D the constant provided by Lemma 2.8.

For every $B \in \mathcal{B}$, if $\overline{g} = (g_0, \ldots, g_n) \in \widehat{G}^{n+1}$, then we set

$$\mathrm{diam}_B(\pi_B(\overline{g})) = \mathrm{diam}_B(\pi_B(g_0) \cup \ldots \cup \pi_B(g_n)) \ .$$

In particular, if $g_i \in B$ for every i, then $\mathrm{diam}_B(\overline{g}) = \mathrm{diam}_B(\{g_0, \ldots, g_n\})$.

We begin the section with a definition that is a version, suited to our context, of [HO13, Definition 3.1,3.6].

Definition 3.1. Let $B \in \mathcal{B}$ and let v_0, v_1 be vertices of $\widehat{G}$. We say that $v_0, v_1 \in \widehat{G}$ are *separated* by B if $\mathrm{diam}_B(\pi_B(v_0) \cup \pi_B(v_1)) \geq D$, and we denote by $\mathcal{S}(v_0, v_1)$ the set of cosets that separate v_0 from v_1. Let $\overline{g} = (g_0, \ldots, g_n) \in \widehat{G}^{n+1}$. A coset B is *relevant* for $\overline{g}$ if $\mathrm{diam}_B(\pi_B(\overline{g})) \geq 2D$, and we denote by $\mathcal{R}(\overline{g})$ the set of all relevant cosets for $\overline{g}$.

As the name suggests, if a coset B separates v_0 from v_1, then by Lemma 2.8 every geodesic joining v_0 to v_1 must contain $c(B)$ and intersect B unless $v_0 = v_1 = c(B)$. Moreover, we obviously have $\mathcal{R}(v_0, v_1) \subseteq \mathcal{S}(v_0, v_1)$.

For every pair of vertices v_0, v_1 of $\widehat{G}$, we are going to endow $\mathcal{S}(v_0, v_1)$ with a total ordering $<$ (so $\mathcal{R}(v_0, v_1)$ will be endowed with a total ordering as well).

Fix vertices v_0, v_1 of $\widehat{G}$, let B_0, B_1 be cosets in $\mathcal{S}(v_0, v_1)$, and take any geodesic γ starting at v_0 and ending at v_1. By Lemma 2.8 we know that γ must pass through $c(B_i)$, $i = 1, 2$, so $\widehat{d}(v_0, B_i) = \widehat{d}(v_0, \mathrm{in}_\gamma(B_i))$. In particular, we have that either $\widehat{d}(v_0, B_0) < \widehat{d}(v_0, B_1)$ (and along every geodesic starting at v_0 and ending at v_1 the point $c(B_1)$ follows $c(B_0)$), or $\widehat{d}(v_0, B_1) < \widehat{d}(v_0, B_0)$. We stipulate that $B_0 < B_1$ in $\mathcal{S}(v_0, v_1)$ in the first case, while $B_1 < B_0$ in the second case. It follows from the very definitions that $\mathcal{S}(v_0, v_1) = \mathcal{S}(v_1, v_0)$ as (unordered) sets. However, $B_0 < B_1$ in $\mathcal{S}(v_0, v_1)$ if and only if $B_1 < B_0$ in $\mathcal{S}(v_1, v_0)$.

Lemma 3.2. *Take $v_0, v_1 \in \widehat{G}$ and $\overline{g} \in \widehat{G}^{n+1}$. Then the sets $\mathcal{S}(v_0, v_1)$ and $\mathcal{R}(\overline{g})$ are finite.*

Proof. The first statement is just a restatement of Lemma 2.10, while the second one follows from the fact that, if $\overline{g} = (g_0, \ldots, g_n)$, then $\mathcal{R}(\overline{g}) = \bigcup_{i\neq j} \mathcal{R}(g_i, g_j) \subseteq \bigcup_{i\neq j} \mathcal{S}(g_i, g_j)$. □

Lemma 3.3. *Take points $v_0, v_1 \in \widehat{G}$ and cosets B_0, B_1 in $\mathcal{S}(v_0, v_1)$ such that $B_0 < B_1$. Then $\pi_{B_1}(v_0) = \pi_{B_1}(c(B_0))$ and $\pi_{B_0}(c(B_1)) = \pi_{B_0}(v_1)$.*

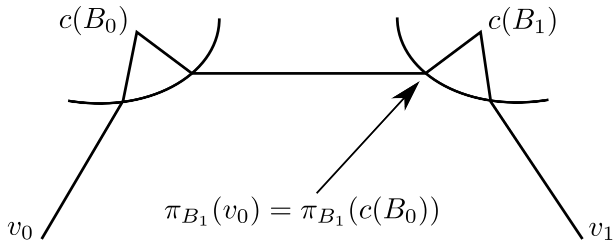

Figure 2

Proof. By symmetry, it is sufficient to show that $\pi_{B_1}(v_0) = \pi_{B_1}(c(B_0))$. Take a geodesic γ joining v_0 to v_1 (see Figure 2). Then $c(B_1)$ follows $c(B_0)$ along γ, so $\widehat{d}(v_0, c(B_1)) = \widehat{d}(v_0, c(B_0)) + \widehat{d}(c(B_0), c(B_1))$, and the concatenation of a geodesic between v_0 and $c(B_0)$ with any geodesic between $c(B_0)$ and $c(B_1)$ is itself a geodesic. This implies that $\pi_{B_1}(c(B_0)) \subseteq \pi_{B_1}(v_0)$. In order to conclude it is sufficient to show that every geodesic joining v_0 with $c(B_1)$ must contain $c(B_0)$. Suppose by contradiction that the geodesic γ joins v_0 to $c(B_1)$ and avoids $c(B_0)$. Since $B_0 < B_1$, there exists a geodesic γ' joining $c(B_1)$ to v_1 and avoiding $c(B_0)$. Since every geodesic joining v_0 to v_1 passes through $c(B_1)$, we have $\widehat{d}(v_0, v_1) = \widehat{d}(v_0, c(B_1)) + \widehat{d}(c(B_1), v_1)$, so the concatenation $\gamma * \gamma'$ is itself a geodesic. But $\gamma * \gamma'$ joins v_0 to v_1 without passing through $c(B_0)$, a contradiction. □

Proposition 3.4. *Let g_0, g_1, g_2 be elements of $\widehat{G}$. Then there exist at most two cosets $B \in \mathcal{R}(g_0, g_1)$ such that $\pi_B(g_2) \neq \pi_B(g_0)$ and $\pi_B(g_2) \neq \pi_B(g_1)$.*

Proof. Let us enumerate the elements $B_1, \ldots, B_k$ of $\mathcal{R}(g_0, g_1)$ in such a way that $B_i < B_{i+1}$. We set

$$\Omega = \{i \in \{1, \ldots, k\} \mid \operatorname{diam}_{B_i}(\pi_{B_i}(g_1), \pi_{B_i}(g_2)) \leq D\} \ .$$

Moreover, we set $i_0 = \max \Omega$ if $\Omega \neq \emptyset$, and $i_0 = 0$ otherwise. In the following arguments we will use the obvious fact that, if A_1, A_2, A_3 are non-empty subsets of a metric space, then

$$\operatorname{diam}(A_1 \cup A_2) \leq \operatorname{diam}(A_1 \cup A_3) + \operatorname{diam}(A_2 \cup A_3) \ .$$

In order to conclude, it is sufficient to prove Claims (1) and (2) below.

Claim 1. $\pi_{B_i}(g_2) = \pi_{B_i}(g_1)$ for every $i < i_0$.

We may suppose that $i_0 > 1$, otherwise the statement is empty. Since $i < i_0$, Lemma 3.3 implies that $\pi_{B_{i_0}}(c(B_i)) = \pi_{B_{i_0}}(g_0)$, so

$$(1) \quad \operatorname{diam}_{B_{i_0}}(\pi_{B_{i_0}}(c(B_i)), \pi_{B_{i_0}}(g_1)) = \operatorname{diam}_{B_{i_0}}(\pi_{B_{i_0}}(g_0), \pi_{B_{i_0}}(g_1)) \geq 2D \ ,$$

where the last inequality is due to the fact that $B_{i_0} \in \mathcal{R}(g_0, g_1)$. But $i_0 \in \Omega$, so $\operatorname{diam}_{B_{i_0}}(\pi_{B_{i_0}}(g_1), \pi_{B_{i_0}}(g_2)) \leq D$. Together with (1), this implies that

$\mathrm{diam}_{B_{i_0}}(\pi_{B_{i_0}}(c(B_i)), \pi_{B_{i_0}}(g_2)) \geq D$, i.e. $B_{i_0} \in \mathcal{S}(c(B_i), g_2)$. Of course also B_i belongs to $\mathcal{S}(c(B_i), g_2)$, and $B_i < B_{i_0}$ in $\mathcal{S}(c(B_i), g_2)$, so Lemma 3.3 implies that $\pi_{B_i}(g_2) = \pi_{B_i}(c(B_{i_0})) = \pi_{B_i}(g_1)$, where the last equality is due to the fact that $B_i < B_{i_0}$ in $\mathcal{R}(g_0, g_1)$.

Claim 2. $\pi_{B_i}(g_2) = \pi_{B_i}(g_0)$ for every $i > i_0 + 1$.

We set $i_1 = i_0 + 1$ for convenience. We may suppose that $i_1 < k$, otherwise the statement is empty. Since $i > i_1$, Lemma 3.3 implies that $\pi_{B_{i_1}}(c(B_i)) = \pi_{B_{i_1}}(g_1)$, so

$$\mathrm{diam}_{B_{i_1}}(\pi_{B_{i_1}}(g_2), \pi_{B_{i_1}}(c(B_i))) = \mathrm{diam}_{B_{i_1}}(\pi_{B_{i_1}}(g_2), \pi_{B_{i_1}}(g_1)) > D \ ,$$

where the last inequality is due to the fact that $i_1 \notin \Omega$. So $B_{i_1} \in \mathcal{S}(g_2, c(B_i))$. Of course also B_i belongs to $\mathcal{S}(g_2, c(B_i))$, and $B_{i_1} < B_i$ in $\mathcal{S}(g_2, c(B_i))$, so Lemma 3.3 implies that $\pi_{B_i}(g_2) = \pi_{B_i}(c(B_{i_1})) = \pi_{B_i}(g_0)$, where the last equality is due to the fact that $B_i > B_{i_1}$ in $\mathcal{R}(g_0, g_1)$. □

The trace of a simplex. Let us now come back to our original extension problem. In order to extend a cochain defined on H_λ to a cochain defined on the whole of G we need to be able to project a simplex with vertices in G onto a simplex (or, at least, onto a chain) supported in H_λ (or, more in general, on a coset of H_λ). To this aim we give the following definition.

Definition 3.5. Let $\overline{g} = (g_0, \ldots, g_n) \in G^{n+1}$ be any simplex, and fix a coset B. Then we define the *trace* $\mathrm{tr}_n^B(\overline{g}) \in C_n(B)$ of $\overline{g}$ on B by setting $\mathrm{tr}_n^B(\overline{g}) = 0$ if $B \notin \mathcal{R}(\overline{g})$ and

$$\mathrm{tr}_n^B(\overline{g}) = \frac{1}{\prod_{j=0}^n |\pi_B(g_j)|} \sum_{h_j \in \pi_B(g_j)} (h_0, \ldots, h_n)$$

if $B \in \mathcal{R}(\overline{g})$. In other words, the trace of $\overline{g}$ on B is either null, if B is not relevant for $\overline{g}$, or the average of the simplices obtained by projecting $\overline{g}$ onto B. The map tr_n^B uniquely extends to a linear map $\mathrm{tr}_n^B \colon C_n(G) \to C_n(B)$. By construction, this map is norm non-increasing.

The strategy to extend to the whole of G a cochain φ_λ defined on H_λ is clear: for every $\overline{g} \in G^{n+1}$, we just add up the sum of the values of φ_λ on the traces of $\overline{g}$ on the cosets of H_λ. In order to check that this procedure indeed takes quasi-cocycles to quasi-cocycles we need to prove that trace operators do not behave too wildly with respect to taking coboundaries. This boils down to showing that the trace operator defined on chains is “almost” a chain map, in a sense that is specified in Proposition 3.9. We warn the reader that there is no hope to replace traces with genuine chain maps: in fact, if this were possible, then, at least in the case when $\{H_\lambda\}_{\lambda\in\Lambda} = \{H\}$ consists of a single subgroup, it would be easy to prove that the restriction map $H_b^n(G, \mathbb{R}) \to H_b^n(H, \mathbb{R})$ is surjective. However, as anticipated in the introduction, this is not true in general (see Proposition 6.2).

Recall from Section 1 that a chain $c \in C_n(G)$ is *degenerate* if $\mathrm{alt}_n(c) = 0$.

Lemma 3.6. *Let $n \geq 2$ and take $\overline{g} \in G^{n+1}$. Then there exist at most $n(n+1)$ cosets $B \in \mathcal{B}$ such that $\mathrm{tr}_n^B(\overline{g})$ is* not *degenerate.*

Proof. Set $\overline{g} = (g_0, \ldots, g_n)$, and suppose that $B \in \mathcal{B}$ is such that $\mathrm{tr}_n^B(\overline{g})$ is not degenerate. Of course $B \in \mathcal{R}(\overline{g})$, so there exist $i, j \in \{0, \ldots, n\}$ such that $B \in \mathcal{R}(g_i, g_j)$ (in particular, $i \neq j$). Observe now that, if there exists $k \in \{0, \ldots, n\}$, $k \notin \{i, j\}$ such that $\pi_B(g_k) = \pi_B(g_i)$ or $\pi_B(g_k) = \pi_B(g_j)$, then $\mathrm{tr}_n^B(\overline{g})$ is degenerate. So the conclusion follows from Proposition 3.4: the number of cosets such that $\mathrm{tr}_n^B(\overline{g})$ is *not* degenerate is at most twice the number of pairs (i, j) of distinct elements of $\{0, \ldots, n\}$. □

Definition 3.7. Let $\overline{g} = (g_0, \ldots, g_n) \in G^{n+1}$. We say that $\overline{g}$ is *small* if there exists a coset $B \in \mathcal{B}$ such that $\overline{g}$ is supported in B and $\mathrm{diam}_B(\overline{g}) < 2D$. A chain $c \in C_n(G)$ is small if it is a linear combination of small simplices.

Lemma 3.8. *(1) Take $\overline{g} \in B^{n+1}$ for some $B \in \mathcal{B}$. Then $\overline{g}$ is small if and only if $\mathrm{diam}_B(\overline{g}) < 2D$.*
(2) For every $\lambda \in \Lambda$, the set of small n-simplices supported in H_λ is H_λ-invariant.
(3) The number of H_λ-orbits of small n-simplices supported in H_λ is finite.
(4) Take $\overline{g} \in B^{n+1}$ for some $B \in \mathcal{B}$. Then $\mathcal{R}(\overline{g}) = \emptyset$ if $\overline{g}$ is small, and $\mathcal{R}(\overline{g}) = \{B\}$ otherwise.

Proof. (1): If $\overline{g} \subseteq B^{n+1}$ is small, then there exists $B' \in \mathcal{B}$ such that $\overline{g} \subseteq (B')^{n+1}$ and $\mathrm{diam}_{B'}(\overline{g}) < 2D$. If $B = B'$ we are done. Otherwise $B \neq B'$ and $\overline{g}$ is supported in the intersection $B \cap B'$, so $\mathrm{diam}_B(\overline{g}) < D < 2D$ by Lemma 2.9. The converse implication is obvious.

Since the metric d_{H_λ} on H_λ is locally finite, points (2) and (3) immediately follow from (1). Finally, let $\overline{g} \in B^{n+1}$. By Lemma 2.9, if $B' \in \mathcal{B} \setminus \{B\}$, then $\mathrm{diam}_{B'}(\pi_{B'}(B)) < D$, so $B' \notin \mathcal{R}(\overline{g})$. Moreover, point (1) implies that $B \in \mathcal{R}(\overline{g})$ if and only if $\overline{g}$ is not small. This concludes the proof of the lemma. □

The following result shows that the trace operators commute with the boundary operator, up to small chains.

Proposition 3.9. *Fix $\overline{g} \in G^{n+1}$, $n \geq 2$. Then the chain*

$$\partial\, \mathrm{tr}_n^B(\overline{g}) - \mathrm{tr}_{n-1}^B(\partial \overline{g})$$

is small for every $B \in \mathcal{B}$.

Proof. If B is not relevant for $\overline{g}$, then it is not relevant for any face of $\overline{g}$, so $\partial\, \mathrm{tr}_n^B(\overline{g}) = \mathrm{tr}_{n-1}^B(\partial \overline{g}) = 0$. So, let $B \in \mathcal{R}(\overline{g})$. The equality $\partial(\mathrm{tr}_n^B(\overline{g})) = \mathrm{tr}_{n-1}^B(\partial \overline{g})$ may fail only when there exist some cosets in $\mathcal{R}(\overline{g})$ which are not relevant for some face of $\overline{g}$. More precisely, an easy computation shows that

$$\partial(\mathrm{tr}_n^B(\overline{g})) - \mathrm{tr}_{n-1}^B(\partial \overline{g}) = \sum_{i \in \Omega} (-1)^i c_i \ ,$$

where $\Omega = \{i \in \{0, \ldots, n\} \mid B \notin \mathcal{R}(\partial_i \overline{g})\}$ and

$$c_i = \frac{1}{\prod_{j \neq i} |\pi_B(g_j)|} \sum_{\substack{h_l \in \pi_B(g_l) \\ l \neq i}} (h_0, \ldots, h_n) \ ,$$

so we are left to show that c_i is small for every $i \in \Omega$. However, if $i \in \Omega$, then $B \notin \mathcal{R}(g_j, g_k)$ for every $j, k \in \{0, \ldots, n\}$ such that i, j, k are pairwise disjoint. In other words, for any such j, k we have $\operatorname{diam}_B(\pi_B(g_j), \pi_B(g_k)) < 2D$. This implies in turn that c_i is small, hence the conclusion. □

4. Proof of Theorem 1

Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a hyperbolically embedded family of subgroups of the group G. Moreover, let V be a normed $\mathbb{R}[G]$-module, and for every $\lambda \in \Lambda$ let U_λ be an $\mathbb{R}[H_\lambda]$-submodule of V. This section is devoted to the proof of Theorem 4.2, which specializes to Theorem 1 in the case when $U_\lambda = V$ for every $\lambda \in \Lambda$. In fact, we will deduce Theorem 4.2 from Theorem 4.1 below, which deals with extensions of alternating cochains that need not be quasi-cocycles.

We first fix some notation. For every $\varphi = (\varphi_\lambda)_{\lambda \in \Lambda} \in \oplus_{\lambda \in \Lambda} C^n_{\text{alt}}(H_\lambda, U_\lambda)^{H_\lambda}$ we denote by $\delta^n \varphi$ the element $\delta^n \varphi = (\delta^n \varphi_\lambda)_{\lambda \in \Lambda} \in \oplus_{\lambda \in \Lambda} C^{n+1}_{\text{alt}}(H_\lambda, U_\lambda)^{H_\lambda}$. We set

$$K(\varphi) = \max\{\|\varphi_\lambda(\overline{s})\|_{U_\lambda},\ \lambda \in \Lambda,\ \overline{s} \subseteq H_\lambda^{n+1} \text{ small}\} \ .$$

Since $\varphi_\lambda = 0$ for all but a finite number of indices, and the number of H_λ-orbits of small simplices in H_λ^{n+1} is finite (see Lemma 3.8), the value $K(\varphi)$ is well-defined and finite. We also set

$$\|\varphi\|_\infty = \max_{\lambda \in \Lambda} \|\varphi_\lambda\|_\infty \in [0, \infty] \,, \quad \|\delta^n \varphi\|_\infty = \max_{\lambda \in \Lambda} \|\delta^n \varphi_\lambda\|_\infty \in [0, \infty] \ .$$

In particular, $\|\delta^n \varphi\|_\infty < \infty$ if and only if every φ_λ is a quasi-cocycle. If this is the case, then we define the defect $D(\varphi)$ of φ by setting

$$D(\varphi) = \|\delta^n \varphi\|_\infty = \max_{\lambda \in \Lambda} D(\varphi_\lambda) \,.$$

Theorem 4.1. *For every $n \geq 1$, there exists a linear map*

$$\Theta^n \colon \bigoplus_{\lambda \in \Lambda} C^n_{\text{alt}}(H_\lambda, U_\lambda)^{H_\lambda} \to C^n_{\text{alt}}(G, V)^G$$

such that, for every $\varphi = (\varphi_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} C^n_{\text{alt}}(H_\lambda, U_\lambda)^{H_\lambda}$ and for every $\lambda \in \Lambda$, the following conditions hold:

(1) $\Theta^n(\varphi)(H_\lambda^{n+1}) \subseteq U_\lambda$;

(2) $\sup_{\overline{h} \in H_\lambda^{n+1}} \|\Theta^n(\varphi)(\overline{h}) - \varphi_\lambda(\overline{h})\|_{U_\lambda} \leq K(\varphi)$;

(3) If $n \geq 2$ then $\|\Theta^n(\varphi)\|_\infty \leq n(n+1) \cdot \|\varphi\|_\infty$;

(4) $\|\delta^n \Theta^n(\varphi) - \Theta^{n+1}(\delta^n \varphi)\|_\infty \leq 2(n+1)(n+2) K(\varphi)$.

Proof. For every coset $B \in \mathcal{B}$ we define a cochain $\varphi_B \in C^n_{\text{alt}}(B, V)$ as follows: if $\bar{b} = (b_0, \ldots, b_n) \in B^{n+1}$, then

$$\varphi_B(\bar{b}) = b_0 \cdot \varphi_\lambda(1, b_0^{-1}b_1, \ldots, b_0^{-1}b_n) \ ,$$

where $\lambda \in \Lambda$ is the label of B and $g \in G$ is such that $B = gH_\lambda$ (the fact that φ_B is indeed alternating is easily checked). Then, we define a new cochain $\varphi'_B \in C^n_{\text{alt}}(B, V)$ by setting

$$\varphi'_B(\bar{b}) = \begin{cases} 0 & \text{if } \bar{b} \text{ is small} \\ \varphi_B(\bar{b}) & \text{otherwise.} \end{cases}$$

The new cochain φ'_B stays at bounded distance from φ_B. More precisely, it follows from the definitions that

$$\|\varphi'_B - \varphi_B\|_\infty \leq K(\varphi)$$

for every $B \in \mathcal{B}$. If $B = H_\lambda$ for some $\lambda \in \Lambda$, then we set $\varphi'_\lambda = \varphi'_{H_\lambda}$, thus getting that $\|\varphi'_\lambda - \varphi_\lambda\|_\infty \leq K(\varphi)$.

We are now ready to define the element $\Phi = \Theta((\varphi_\lambda)_{\lambda\in\Lambda}) \in C^n_{\text{alt}}(G, V)^G$ as follows:

$$\Phi(\overline{g}) = \sum_{B\in\mathcal{B}} \varphi'_B(\mathrm{tr}^B_n(\overline{g})) \ . \tag{2}$$

Recall that $\mathrm{tr}^B_n(\overline{g}) = 0$ whenever $B \notin \mathcal{R}(\overline{g})$, so by Lemma 3.2 the sum on the right-hand side of (2) is finite. It is easy to check that Φ is alternating. Moreover, the H_λ-invariance of each φ_λ and the G-invariance of the set of small simplices readily imply that Φ is indeed G-invariant.

In order to show that conditions (1) and (2) are satisfied it is sufficient to show that the restriction of Φ to H_λ coincides with φ'_λ. So, suppose that $\overline{g} \in G^{n+1}$ is supported in H_λ. If $\overline{g}$ is small, then by Lemma 3.8–(4) we have $\varphi'_\lambda(\overline{g}) = \Phi(\overline{g}) = 0$. On the other hand, if $\overline{g}$ is not small, then $\mathcal{R}(\overline{g}) = \{H_\lambda\}$ again by Lemma 3.8–(4). Moreover, we obviously have $\mathrm{tr}^{H_\lambda}_n(\overline{g}) = \overline{g}$, so again $\Phi(\overline{g}) = \varphi'_\lambda(\overline{g})$.

Let us now suppose that each φ_λ is bounded, and observe that for every $B \in \mathcal{B}$ we have $\|\varphi'_B\|_\infty \leq \|\varphi_B\|_\infty \leq \|\varphi\|_\infty$. We fix an element $\overline{g} \in G^{n+1}$. Since $\|\mathrm{tr}^B_n(\overline{g}))\|_1 \leq 1$, for every $B \in \mathcal{B}$ we have $\|\varphi'_B(\mathrm{tr}^B_n(\overline{g}))\|_V \leq \|\varphi\|_\infty$. Moreover, since φ'_B is alternating, by Lemma 3.6 there are at most $n(n+1)$ cosets $B \in \mathcal{B}$ such that $\varphi'_B(\mathrm{tr}^B_n(\overline{g})) \neq 0$, so

$$\|\Phi(\overline{g})\|_V = \left\| \sum_{B\in\mathcal{B}} \varphi'_B(\mathrm{tr}^B_n(\overline{g})) \right\| \leq n(n+1)\|\varphi\|_\infty \ .$$

This proves condition (3).

Let us now concentrate our attention on condition (4). In order to compare $\Theta^{n+1}(\delta^n\varphi)$ with $\delta^n\Theta^n(\varphi)$ we first observe that $(\delta^n\varphi_B)'$ does *not* coincide in general with $\delta^n\varphi'_B$. In fact, let us fix an $(n+1)$-simplex $\bar{b} \in B^{n+2}$. If $\bar{b}$ is small, then also every face of $\overline{g}$ is small, and this readily implies that

$(\delta^n\varphi_B)'(\overline{b}) = \delta^n\varphi'_B(\overline{b}) = 0$. On the other hand, suppose that $\overline{b}$ is not small, and set

$$\Omega = \{i \in \{0, \ldots, n+1\} \mid \partial_i \overline{b} \text{ is small}\} \ .$$

Since $\overline{b}$ is not small, there exist distinct vertices b_{i_0}, b_{i_1} of $\overline{b}$ such that $d_B(b_{i_0}, b_{i_1}) \geq 2D$. This implies that $\partial_i \overline{b}$ is not small for every $i \notin \{i_0, i_1\}$, so $|\Omega| \leq 2$, and

$$\left\|((\delta^n\varphi_B)' - \delta^n\varphi'_B)(\overline{b})\right\|_V = \left\|\sum_{i\in\Omega} \varphi_B(\partial_i \overline{b})\right\|_V \leq 2K(\varphi) \ .$$

We have thus proved that, for every $B \in \mathcal{B}$, we have

$$\left\|(\delta^n\varphi_B)' - \delta^n\varphi'_B\right\| \leq 2K(\varphi) \ . \tag{3}$$

Let us now take any simplex $\overline{g} \in G^{n+2}$. Since φ'_B vanishes on small chains supported in B, Proposition 3.9 implies that $\varphi'_B(\mathrm{tr}^B_n(\partial\overline{g})) = \varphi'_B(\partial\, \mathrm{tr}^B_{n+1}(\overline{g}))$ for every $B \in \mathcal{B}$, so

$$\begin{aligned}\delta^n\Theta^n(\varphi)(\overline{g}) = \Theta^n(\varphi)(\partial\overline{g}) &= \sum_{B\in\mathcal{B}} \varphi'_B(\mathrm{tr}^B_n(\partial\overline{g})) = \sum_{B\in\mathcal{B}} \varphi'_B(\partial\, \mathrm{tr}^B_{n+1}(\overline{g})) \\ &= \sum_{B\in\mathcal{B}} \delta^n\varphi'_B(\mathrm{tr}^B_{n+1}(\overline{g})) \ .\end{aligned}$$

On the other hand, we have

$$\Theta^{n+1}(\delta^n\varphi)(\overline{g}) = \sum_{B\in\mathcal{B}} (\delta^n\varphi_B)'(\mathrm{tr}^B_{n+1}(\overline{g})) \ ,$$

so

$$\Theta^{n+1}(\delta^n\varphi)(\overline{g}) - \delta^n\Theta^n(\varphi)(\overline{g}) = \sum_{B\in\mathcal{B}} ((\delta^n\varphi_B)' - \delta^n\varphi'_B)(\mathrm{tr}^B_{n+1}(\overline{g})) \ . \tag{4}$$

Being alternating, the cochain $(\delta^n\varphi_B)' - \delta^n\varphi'_B$ vanishes on degenerate chains supported in B. On the other hand, recall from Lemma 3.6 that $\mathrm{tr}^B_{n+1}(\overline{g})$ is not degenerate on at most $(n+1)(n+2)$ cosets $B \in \mathcal{B}$. Therefore, since $\|\,\mathrm{tr}^B_{n+1}(\overline{g})\|_1 \leq 1$ for every $B \in \mathcal{B}$, from equation (4) and inequality (3) we get

$$\left\|\delta^n\Theta^n(\varphi)(\overline{g}) - \Theta^{n+1}(\delta^n\varphi)(\overline{g})\right\| \leq 2(n+1)(n+2)K(\varphi) \ .$$

This proves condition (4), and concludes the proof of the Theorem.

□

By considering the restriction to quasi-cocycles of the map Θ^n constructed in the previous theorem, we obtain the following result, which in turn implies Theorem 1:

Theorem 4.2. *For every $n \geq 1$, there exists a linear map*

$$\Theta^n \colon \bigoplus_{\lambda\in\Lambda} \mathrm{QZ}^n_{\mathrm{alt}}(H_\lambda, U_\lambda)^{H_\lambda} \to \mathrm{QZ}^n_{\mathrm{alt}}(G, V)^G$$

such that, for every $\varphi = (\varphi_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathrm{QZ}^n_{\mathrm{alt}}(H_\lambda, U_\lambda)^{H_\lambda}$ *and for every* $\lambda \in \Lambda$*, we have* $\Theta^n(\varphi)(H_\lambda) \subseteq U_\lambda$ *and*

$$\sup_{\overline{h}\in H_\lambda^{n+1}} \|\Theta^n(\varphi)(\overline{h}) - \varphi_\lambda(\overline{h})\|_{U_\lambda} \leq K(\varphi) \ ,$$

$$D(\Theta^n(\varphi)) \leq (n+1)(n+2)(D(\varphi) + 2K(\varphi)) \ .$$

Proof. We are only left to prove the estimate on the defect of $\Theta^n(\varphi)$ (which implies that Θ^n takes indeed quasi-cocycles into quasi-cocycles). However, by Theorem 4.1 we have

$$\begin{aligned} D(\Theta^n(\varphi)) = \|\delta^n(\Theta^n(\varphi))\|_\infty &\leq \|\delta^n(\Theta^n(\varphi)) - \Theta^{n+1}(\delta^n\varphi)\|_\infty + \|\Theta^{n+1}(\delta^n\varphi)\|_\infty \\ &\leq 2(n+1)(n+2)K(\varphi) + (n+1)(n+2)\|\delta^n\varphi\|_\infty \\ &= (n+1)(n+2)(2K(\varphi) + D(\varphi)) \ . \end{aligned}$$

□

5. Applications to bounded cohomology

This section is devoted to some applications of Theorem 1 to bounded cohomology. Throughout the section, we fix a hyperbolically embedded family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of a group G. We also fix a normed $\mathbb{R}[G]$-space V and, for every $\lambda \in \Lambda$, an H_λ-invariant submodule U_λ of V. The inclusion $C^n_b(H_\lambda, U_\lambda) \to C^n_b(H_\lambda, V)$ induces a map $i^n_\lambda \colon H^n_b(H_\lambda, U_\lambda) \to H^n_b(H_\lambda, V)$, which restricts to a map $EH^n_b(H_\lambda, U_\lambda) \to EH^n_b(H_\lambda, V)$. Finally, for every $\lambda \in \Lambda$ we denote by $\mathrm{res}^n_\lambda \colon H^n_b(G, V) \to H^n_b(H_\lambda, V)$ the restriction map.

The following result provides a generalization of Corollary 2.

Proposition 5.1. *Let* $n \geq 2$*. For every element* $(\alpha_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} EH^n_b(H_\lambda, U_\lambda)$*, there exists* $\alpha \in EH^n_b(G, V)$ *such that* $\mathrm{res}^n_\lambda(\alpha) = i^n_\lambda(\alpha_\lambda)$ *for every* $\lambda \in \Lambda$*.*

Proof. Recall that bounded cohomology can be computed from the complex of alternating bounded cochains, so, for every $\lambda \in \Lambda$, we may choose an alternating representative $a_\lambda \in C^n_{b,\mathrm{alt}}(H_\lambda, U_\lambda)^{H_\lambda}$ of α_λ. Since $\alpha_\lambda \in EH^n_b(H_\lambda, U_\lambda)^{H_\lambda}$, we have $a_\lambda = \delta^{n-1}\varphi_\lambda$ for some $\varphi_\lambda \in \mathrm{QZ}^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$. We have $\delta^{n-1}\,\mathrm{alt}^{n-1}(\varphi_\lambda) = \mathrm{alt}^n(\delta^{n-1}\varphi_\lambda) = a_\lambda$, so, up to replacing φ_λ with $\mathrm{alt}^{n-1}(\varphi_\lambda)$, we may suppose that φ_λ is alternating for every $\lambda \in \Lambda$.

We now consider the quasi-cocycle $\Phi = \Theta^{n-1}(\oplus_{\lambda\in\Lambda}\varphi_\lambda) \in \mathrm{QZ}^{n-1}_{\mathrm{alt}}(G, V)^G$, where Θ^{n-1} is the map described in Theorem 4.2, and we set $\alpha = [\delta^{n-1}\Phi] \in EH^n_b(G, V)$. In order to conclude it is sufficient to observe that, by Theorem 4.2, $\Phi|_{H^n_\lambda}$ and φ_λ differ by a bounded cochain for every $\lambda \in \Lambda$. □

The following result sharpens Proposition 5.1 under additional assumptions.

Proposition 5.2. *Let us assume that* $H^{n-1}(H_\lambda, U_\lambda) = 0$ *for every* $\lambda \in \Lambda$*. Then there exists a map*

$$\iota^n \colon \bigoplus_{\lambda\in\Lambda} EH^n_b(H_\lambda, U_\lambda) \to EH^n(G, V)$$

such that, for every $\alpha = (\alpha_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} EH_b^n(H_\lambda, U_\lambda)$,

$$\mathrm{res}_\lambda^n(\iota^n(\alpha)) = i_\lambda^n(\alpha_\lambda) \quad \text{for every } \lambda \in \Lambda \ .$$

Proof. The definition of ι^n has already been described in the proof of Proposition 5.1. Namely, once an element $\alpha = (\alpha_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} EH_b^n(H_\lambda, U_\lambda)$ is given, for every $\lambda \in \Lambda$ we choose an alternating representative $a_\lambda \in C_{b,\mathrm{alt}}^n(H_\lambda, U_\lambda)^{H_\lambda}$ of α_λ, and an alternating quasi-cocycle $\varphi_\lambda \in \mathrm{QZ}_{\mathrm{alt}}^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$ such that $a_\lambda = \delta^{n-1}\varphi_\lambda$. Of course we may suppose that $\varphi_\lambda = 0$ and $a_\lambda = 0$ for all but a finite number of indices, so $a = (a_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} C_{b,\mathrm{alt}}^n(H_\lambda, U_\lambda)^{H_\lambda}$, $\varphi = (\varphi_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} C_{b,\mathrm{alt}}^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$, and we can set

$$\iota^n(\alpha) = [\delta^{n-1}\Theta^{n-1}(\varphi)] \ .$$

In order to prove that this definition of ι^n is well-posed, we need to show that, if a_λ represents the null element of $EH_b^n(H_\lambda, U_\lambda)$ for every λ, then $[\delta^{n-1}\Theta^{n-1}(\varphi)] = 0$ in $EH_b^n(G, V)$. So, let us suppose that $a_\lambda = \delta^{n-1}b_\lambda$ for some $b_\lambda \in C_b^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$, and set as usual $b = (b_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} C_b^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$. Up to replacing b_λ with $\mathrm{alt}_b^{n-1}(b_\lambda)$, we may suppose that $b_\lambda \in C_{b,\mathrm{alt}}^{n-1}(H_\lambda, U_\lambda)^{H_\lambda}$. By construction we have $\delta^{n-1}(\varphi_\lambda - b_\lambda) = 0$, so our assumption that $H^{n-1}(H_\lambda, U_\lambda) = 0$ implies that $\varphi = b + \delta^{n-2}c$ for some $c \in \oplus_{\lambda\in\Lambda} C_{\mathrm{alt}}^{n-2}(H_\lambda, U_\lambda)$ (as usual, we may suppose that c_λ is alternating).

Therefore, we have

$$\begin{aligned}\delta^{n-1}\Theta^{n-1}(\varphi) &= \delta^{n-1}\Theta^{n-1}(b) + \delta^{n-1}\Theta^{n-1}(\delta^{n-2}c) \\ &= \delta^{n-1}\Theta^{n-1}(b) + \delta^{n-1}(\Theta^{n-1}(\delta^{n-2}c) - \delta^{n-2}\Theta^{n-2}(c)) \ .\end{aligned}$$

By Theorem 4.1, the right-hand side of this equality is the coboundary of a bounded cochain, and this concludes the proof. □

We will see in Proposition 6.5 that, if we drop the assumption that $H^{n-1}(H_\lambda, V) = 0$ for every $\lambda \in \Lambda$, then the construction just described does not yield a well-defined map on exact bounded cohomology.

The previous results may be exploited to deduce the non-vanishing of $EH_b^n(G, V)$ from the non-vanishing of $EH_b^n(H_\lambda, U_\lambda)$ for some $\lambda \in \Lambda$. For every group K and every normed K-module W we denote by $\overline{EH}_b^n(K, W)$ the quotient of $EH_b^n(K, W)$ by the subspace of its elements with vanishing seminorm.

Proposition 5.3. *Suppose that, for every λ, there exists an H_λ-equivariant norm non-increasing retraction $V \to U_\lambda$. Then*

$$\dim EH_b^n(G, V) \geq \dim\left(\oplus_{\lambda\in\Lambda} EH_b^n(H_\lambda, U_\lambda)\right) \ ,$$
$$\dim \overline{EH}_b^n(G, V) \geq \dim\left(\oplus_{\lambda\in\Lambda} \overline{EH}_b^n(H_\lambda, U_\lambda)\right) \ .$$

Proof. By Proposition 5.1, the map $\oplus_{\lambda\in\Lambda} \mathrm{res}_\lambda^n$ establishes a bounded epimorphism from $EH_b^n(G, V)$ to

$$(\oplus_{\lambda\in\Lambda} i_\lambda^n)\ (\oplus_{\lambda\in\Lambda} EH_b^n(H_\lambda, U_\lambda)) \ \subseteq\ \oplus_{\lambda\in\Lambda} EH_b^n(H_\lambda, V) \ .$$

Therefore, in order to conclude it is sufficient to observe that the existence of an H_λ-equivariant retraction $V \to U_\lambda$ ensures that the map $i^n_\lambda \colon EH^n_b(H_\lambda, U_\lambda) \to EH^n_b(H_\lambda, V)$ is an isometric embedding. □

In order to obtain concrete non-vanishing results we exploit the following fundamental result about acylindrically hyperbolic groups.

Theorem 5.4 (Theorem 2.23 of [DGO11])**.** *Let G be an acylindrically hyperbolic group. Then there exists a hyperbolically embedded subgroup H of G such that H is isomorphic to $F_2 \times K$, where K is finite.*

Putting together Proposition 5.3 and Theorem 5.4 we may reduce the non-vanishing of the exact bounded cohomology of an acylindrically hyperbolic group to the non-vanishing of the cohomology of free non-abelian groups. As an application of this strategy we provide a proof of Corollary 3 stated in the introduction, which we recall here for convenience:

Corollary 5.5. *Let G be an acylindrically hyperbolic group. Then the dimension of $\overline{EH}^3_b(G, \mathbb{R})$ is equal to the cardinality of the continuum.*

Proof. From Proposition 5.3 and Theorem 5.4 we deduce that

$$\dim \overline{EH}^3_b(G, \mathbb{R}) \geq \dim \overline{EH}^3_b(F_2 \times K, \mathbb{R}) = \dim \overline{H}^3_b(F_2 \times K, \mathbb{R}) \ ,$$

where K is a finite group. But $F_2 \times K$ surjects onto F_2 with amenable kernel, so $H^3_b(F_2 \times K, \mathbb{R})$ is isometrically isomorphic to $H^3_b(F_2, \mathbb{R})$, and the conclusion follows from the fact that $\dim \overline{H}^3_b(F_2, \mathbb{R})$ is equal to the cardinality of the continuum [Som97]. □

Monod and Shalom showed the importance of bounded cohomology with coefficients in $\ell^2(G)$ in the study of rigidity of G [NM03, NM04, MS06], and proposed the condition $H^2_b(G, \ell^2(G)) \neq 0$ as a cohomological definition of negative curvature for groups. More in general, bounded cohomology with coefficients in $\ell^p(G)$, $1 \leq p < \infty$ has been widely studied as a powerful tool to prove (super)rigidity results (see e.g. [Ham08] and [CFI]). However, little is known in this context about degrees higher than two. The following result shows that the non-vanishing of $H^n_b(G, \ell^p(G))$ may be reduced to the non-vanishing of $H^n_b(F_2, \ell^p(F_2))$ for a wide class of groups. However, as far as the authors know, for no degree $n \geq 3$ it is known whether $H^n_b(F_2, \ell^p(F_2))$ vanishes or not.

Corollary 5.6. *Let G be an acylindrically hyperbolic group. Then, for every $p \in [1, \infty)$, $n \geq 2$, we have*

$$\dim EH^n_b(G, \ell^p(G)) \geq \dim H^n_b(F_2, \ell^p(F_2)) \ .$$

Proof. By Theorem 5.4 there exists a hyperbolically embedded subgroup H of G which is isomorphic to a product $F_2 \times K$, where K is a finite group. We identify $\ell^p(H)$ with the H-submodule of $\ell^p(G)$ given by those functions that vanish outside H. Then, it is immediate to realize that $\ell^p(G)$ admits an H-equivariant norm non-increasing retraction onto $\ell^p(H)$. By Proposition 5.3, this implies that

$$\dim EH_b^n(G, \ell^p(G)) \geq \dim EH_b^n(H, \ell^p(H)) \ .$$

The conclusion follows from the fact that, since F_2 is a retract of $H \cong F_2 \times K$ and $\ell^p(H)$ admits an F_2-equivariant norm non-increasing retraction onto $\ell^p(F_2)$, there exist isometric embeddings

$$H_b^n(F_2, \ell^p(F_2)) = EH_b^n(F_2, \ell^p(F_2)) \to EH_b^n(F_2, \ell^p(H)) \to EH_b^n(H, \ell^p(H)) \ .$$

□

Corollary 5.5 can be combined with Theorem A.1 that we will prove in the appendix to show the non-injectivity of the restriction map we announced in the introduction, whose statement we recall here for the sake of completeness:

Proposition 5.7. *For any non-degenerate hyperbolically embedded finite family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of a finitely generated group G and for any normed $\mathbb{R}[G]$-module V, the kernel of the restriction map $\oplus_{\lambda\in\Lambda} \mathrm{res}^3_\lambda \colon H_b^3(G, V) \to \bigoplus_{\lambda\in\Lambda} H_b^3(H_\lambda, V)$ is infinite-dimensional.*

Proof. Of course, it is sufficient to show that the kernel of the restriction map $\oplus_{\lambda\in\Lambda} \mathrm{res}^3_\lambda \colon EH_b^3(G, V) \to \bigoplus_{\lambda\in\Lambda} EH_b^3(H_\lambda, V)$ is infinite-dimensional.

Since G is finitely generated we can assume that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X) where X is a generating system for G (see [DGO11, Corollary 4.27]). In particular Theorem A.1 implies that there exists a free group on two generators F such that the family $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{F \times K(G)\}$ is hyperbolically embedded in (G, X). Also observe that, since Λ is finite, the direct sum of restrictions define maps

$$\widehat{\eta} \colon EH_b^3(G, V) \to \bigoplus_{\lambda\in\Lambda} EH_b^3(H_\lambda, V) \oplus EH_b^3(F \times K(G)) \,,$$
$$\eta \colon EH_b^3(G, V) \to \bigoplus_{\lambda\in\Lambda} EH_b^3(H_\lambda, V).$$

These maps fit in the following commutative diagram

$$\begin{array}{ccc} EH_b^3(G, V) & \xrightarrow{\ \eta\ } & \bigoplus_{\lambda\in\Lambda} EH_b^3(H_\lambda, V) \\ & \searrow^{\widehat{\eta}} & \uparrow \\ & & \bigoplus_{\lambda\in\Lambda} EH_b^3(H_\lambda, V) \oplus H_b^3(F \times K(G)) \end{array}$$

where the vertical arrow is the projection onto the first summand.

But the module $EH_b^3(F \times K(G))$ is infinite dimensional, and the map $\widehat{\eta}$ is surjective by Proposition 5.1, so the conclusion follows. □

6. Examples and counterexamples

In this section we prove Propositions 4 and 5, and we provide examples showing that, in general, the map $\Theta^\bullet$ constructed in Theorem 4.1 does not induce a well-defined map on exact bounded cohomology. Throughout the whole section we will exploit the well-known fact that, if X is an aspherical manifold, then the ordinary and the bounded cohomology of X are canonically isomorphic to the ones of $\pi_1(X)$.

We begin with the following:

Lemma 6.1. *Let M be a compact orientable $(n+1)$-dimensional manifold with connected boundary, and suppose that the following conditions hold:*

- *M and ∂M are aspherical;*
- *the inclusion $\partial M \to M$ induces an injective map on fundamental groups;*
- *$\pi_1(\partial M)$ is Gromov hyperbolic.*

Then the restriction map

$$\mathrm{res}_b^n \colon H_b^n(\pi_1(M), \mathbb{R}) \to H_b^n(\pi_1(\partial M), \mathbb{R})$$

is not surjective.

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} H_b^n(\pi_1(M), \mathbb{R}) & \xrightarrow{\mathrm{res}_b^n} & H_b^n(\pi_1(\partial M), \mathbb{R}) \\ \Big\downarrow c_M^n & & \Big\downarrow c_{\partial M}^n \\ H^n(\pi_1(M), \mathbb{R}) & \xrightarrow{\mathrm{res}^n} & H^n(\pi_1(\partial M), \mathbb{R}) \end{array},$$

where vertical arrows represent comparison maps. Since any cycle in $Z^n(\partial M, \mathbb{R})$ bounds in M, by the Universal Coefficient Theorem the restriction of any element in $H^n(M, \mathbb{R})$ to $H^n(\partial M, \mathbb{R})$ is null. Since restrictions commute with the canonical isomorphisms $H^n(M, \mathbb{R}) \cong H^n(\pi_1(M), \mathbb{R})$, $H^n(\partial M, \mathbb{R}) \cong H^n(\pi_1(\partial M), \mathbb{R})$, this implies that res^n is the zero map. But $H^n(\pi_1(M), \mathbb{R}) \cong H^n(\partial M, \mathbb{R}) \cong \mathbb{R} \neq 0$, so the composition $c_{\partial M}^n \circ \mathrm{res}_b^n$ cannot be surjective. Now the main result of [Min01] implies that $c_{\partial M}^n$ is an epimorphism, so we can conclude that res_b^n cannot be surjective. □

We are now ready to prove Proposition 4 from the introduction:

Proposition 6.2. *For every $n \geq 2$, there exists a pair (G, H) such that G is relatively hyperbolic with respect to H (in particular, H is hyperbolically embedded in G), and the restriction $H_b^n(G, \mathbb{R}) \to H_b^n(H, \mathbb{R})$ is not surjective.*

Proof. By [LR01], for every $n \geq 2$ there exist examples of compact orientable $(n+1)$-dimensional hyperbolic manifolds with connected geodesic boundary. Let M^{n+1} be one such example, and let us set $G = \pi_1(M)$, $H = \pi_1(\partial M)$. It is well-known that G is relatively hyperbolic with respect to H. Moreover, the manifold M^{n+1} satisfies all the conditions described in Lemma 6.1, so the restriction $H_b^n(G, \mathbb{R}) \to H_b^n(H, \mathbb{R})$ is not surjective. □

We now provide examples where the map $\Theta^\bullet$ defined in Theorem 4.1 does not induce a well-defined map in bounded cohomology.

Proposition 6.3. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of the group G, and denote by $j_n^\lambda\colon H_n(H_\lambda,\mathbb{R})\to H_n(G,\mathbb{R})$ the map induced by the inclusion $H_\lambda\to G$. Suppose that the following conditions hold:*

- *H_λ is amenable for every $\lambda\in\Lambda$;*
- *The map $\oplus_{\lambda\in\Lambda}j_n^\lambda\colon\ \oplus_{\lambda\in\Lambda}H_n(H_\lambda,\mathbb{R})\to H_n(G,\mathbb{R})$ is not injective.*

Then there exists a collection $\varphi=(\varphi_\lambda)_{\lambda\in\Lambda}\in\oplus_{\lambda\in\Lambda}Z^n(H_\lambda,\mathbb{R})^{H_\lambda}$ of genuine cocycles such that, if $\Phi\in\mathrm{QZ}^n(G,\mathbb{R})^G$ is any quasi-cocycle such that $\Phi|_{H_\lambda}$ stays at uniformly bounded distance from φ_λ for every $\lambda\in\Lambda$, then $[\delta^n\Phi]\neq 0$ in $H_b^{n+1}(G,\mathbb{R})$.

Proof. Let $(w_\lambda)_{\lambda\in\Lambda}$ be a non-null element of $\ker\left(\oplus_{\lambda\in\Lambda}j_n^\lambda\right)$. Since $w_\lambda\neq 0$ for some $\lambda\in\Lambda$, by the Universal Coefficient Theorem we may choose cocycles $\varphi_\lambda\in Z^n(H_\lambda,\mathbb{R})^{H_\lambda}$ in such a way that $\varphi_\lambda=0$ for all but a finite number of indices and, if z_λ is any representative of w_λ, then $\sum_{\lambda\in\Lambda}\varphi_\lambda(z_\lambda)=1$.

Suppose now that Φ is as in the statement. Then for every $\lambda\in\Lambda$ there exists $b_\lambda\in C_b^n(H_\lambda,\mathbb{R})^{H_\lambda}$ such that $\Phi|_{H_\lambda}=\varphi_\lambda+b_\lambda$. We set $M=\sup_{\lambda\in\Lambda}\|b_\lambda\|_\infty$.

Let us assume by contradiction that $[\delta^n\Phi]$ vanishes in $H_b^{n+1}(G,\mathbb{R})$. This implies that $\Phi=\psi+b$, where $\psi\in Z^n(G,\mathbb{R})$ and $b\in C_b^n(G,\mathbb{R})$. Since each H_λ is amenable, the ℓ^1-seminorm on $H_n(H_\lambda,\mathbb{R})$ vanishes (see e.g. [MM85]), so for every $\lambda\in\Lambda$ we can choose a representative $z_\lambda\in Z_n(H_\lambda,\mathbb{R})$ of w_λ such that $\sum_{\lambda\in\Lambda}\|z_\lambda\|_1<(M+\|b\|_\infty)^{-1}$.

Let us set $z=\sum_{\lambda\in\Lambda}z_\lambda$. Since $\sum_{\lambda\in\Lambda}i_n^\lambda(w_\lambda)=0$, we have $\psi(z)=0$, and

$$|\Phi(z)|=|b(z)|\leq\|b\|_\infty\|z\|_1<\frac{\|b\|_\infty}{M+\|b\|_\infty}\ . \tag{5}$$

On the other hand, we have

$$\Phi(z)=\sum_{\lambda\in\Lambda}\Phi(z_\lambda)=\sum_{\lambda\in\Lambda}\varphi_\lambda(z_\lambda)+\sum_{\lambda\in\Lambda}b_\lambda(z_\lambda)=1+\sum_{\lambda\in\Lambda}b_\lambda(z_\lambda)\ ,$$

so

$$|\Phi(z)|\geq 1-\left|\sum_{\lambda\in\Lambda}b_\lambda(z_\lambda)\right|\geq 1-M\sum_{\lambda\in\Lambda}\|z_\lambda\|_1>1-\frac{M}{M+\|b\|_\infty}\ .$$

This contradicts inequality (5), and concludes the proof. □

Together with our main result on extensions of quasi-cocycles, Proposition 6.3 readily implies the following:

Corollary 6.4. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite hyperbolically embedded family of subgroups of the group G, and denote by $j_n^\lambda\colon H_n(H_\lambda,\mathbb{R})\to H_n(G,\mathbb{R})$ the map induced by the inclusion $H_\lambda\to G$. Suppose that the following conditions hold:*

- *H_λ is amenable for every $\lambda\in\Lambda$;*
- *The map $\oplus_{\lambda\in\Lambda}j_n^\lambda\colon\ \oplus_{\lambda\in\Lambda}H_n(H_\lambda,\mathbb{R})\to H_n(G,\mathbb{R})$ is not injective.*

Then $H_b^{n+1}(G,\mathbb{R}) \neq 0$.

The following proposition provides concrete examples for the phenomenon described in Proposition 6.3.

Proposition 6.5. *For every* $n \geq 1$*, there exist a group* G *relatively hyperbolic with respect to the finite family of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$ *such that the following holds. There exists a collection* $\varphi = (\varphi_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} Z^n_{\mathrm{alt}}(H_\lambda,\mathbb{R})$ *of genuine cocycles such that, if* $\Phi = \Theta^n(\varphi) \in \mathrm{QZ}^n_{\mathrm{alt}}(G,\mathbb{R})$ *is the quasi-cocycle constructed in Theorem 4.1, then* $[\delta^n\Phi] \neq 0$ *in* $H_b^{n+1}(G,\mathbb{R})$*.*

Proof. Let M be an orientable complete finite-volume non-compact hyperbolic $(n+1)$-manifold, and let Λ be the set of cusps of M. We set $G = \pi_1(M)$, and we denote by H_λ the subgroup of G corresponding to the cusp of M indexed by λ. It is well-known that Λ is finite, that each cusp is π_1-injective in M, and that G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$, so in order to conclude it is sufficient to show that $(G,\{H_\lambda\}_{\lambda\in\Lambda})$ satisfies the hypotheses of Proposition 6.3.

Each H_λ is the fundamental group of a compact Euclidean n-manifold, so it is virtually abelian, hence amenable. Moreover, if $C_1,\ldots,C_k$ are the cusps of M, then the map $\oplus_{i=1}^k H_n(C_i,\mathbb{R}) \to H_n(M,\mathbb{R})$ is not injective. Since the spaces M and C_i, $i = 1,\ldots,k$, are all aspherical, this implies in turn that the map $\oplus_{\lambda\in\Lambda} j_n^\lambda \colon \oplus_{\lambda\in\Lambda} H_n(H_\lambda,\mathbb{R}) \to H_n(G,\mathbb{R})$ is not injective. □

We can also use manifolds constructed by [MRS13] to show the following.

Proposition 6.6. *For every* $d \geq 4$ *and* $2 \leq k \leq d$ *there exist infinitely many commensurability classes of cusped orientable hyperbolic* d*-manifolds* M *such that* $H_b^k(\pi_1(M),\mathbb{R})$ *is non-trivial.*

Proof. Let us first suppose that $2 < k < d$. The authors of [MRS13] construct infinitely many commensurability classes of cusped orientable hyperbolic d-manifolds M containing a properly embedded totally geodesic submanifold N of dimension k with the following property. Denoting by $\mathcal{E}$ the cusp cross-sections of M and by $\mathcal{F}$ the cusp cross-sections of N, we have that:

(1) the homomorphism $H_{k-1}(\mathcal{F}) \to H_{k-1}(\mathcal{E})$ induced by the inclusion $N \to M$ is an injection.
(2) the homomorphism $H_{k-1}(\mathcal{F}) \to H_{k-1}(M)$ induced by the inclusion $N \to M$ is *not* an injection.

Denoting by $\{H_\lambda\}_{\lambda\in\Lambda}$ the set of all fundamental groups of the cusps of M and $G = \pi_1(M)$, we then see that the conditions of Corollary 6.4 are satisfied for $n = k-1$. In particular, $H_b^k(\pi_1(M),\mathbb{R}) \neq 0$, as required. Let us now consider the cases $k = 2, d$. Since $\pi_1(M)$ is relatively hyperbolic we have $H_b^2(\pi_1(M),\mathbb{R}) \neq 0$. Moreover, it is well-known that the straightened volume form on M (i.e. the d-dimensional cochain obtained by integrating the volume form on straight simplices) defines a non-trivial coclass in $H_b^d(M,\mathbb{R}) \cong H_b^d(\pi_1(M),\mathbb{R})$, and this concludes the proof. □

As mentioned at the end of the proof of the previous proposition, for every cusped orientable hyperbolic n-manifold M, the straightened volume form defines a non-trivial volume coclass $\omega_M \in H^n_b(\pi_1(M), \mathbb{R})$. We pose here the following:

Question 6.7. Let $\overline{M}$ denote the compact manifold with boundary obtained by truncating the cusps of M along horospherical sections of the cusps, denote by H_λ, $\lambda \in \Lambda$, the subgroups of $\pi_1(M)$ corresponding to the fundamental groups of the boundary components of $\overline{M}$, and let α be the element of $\oplus_{\lambda\in\Lambda} C^{n-1}(H_\lambda, \mathbb{R})$ corresponding to the (unbounded) Euclidean volume form on $\partial\overline{M}$. Is it true that $[\delta^{n-1}\Theta^{n-1}(\alpha)] = \omega_M$ in $H^n_b(\pi_1(M), \mathbb{R})$?

7. Extension of cocycles and projections

The purpose of this section is to give a precise meaning to the Informal Statement from the introduction, which says that, whenever H is a hyperbolically embedded subgroup of a finitely generated group G, our extension to G of a specific cocycle on H encodes the geometry of the embedding of H in G. In fact, we will construct a cocycle whose extension to G will allow us to define a family of projections $\pi_{gH} : G \to gH$ that satisfy the BBF axioms by evaluating the extension on suitable tuples, see Proposition 7.3 and Proposition 7.5. Recall from Theorem 2.11 that this is enough information to know that H is hyperbolically embedded in G.

The cocycle we will construct lies in $Z^2(H, \ell^2(E_H))$, where E_H is the set of edges of a Cayley graph of H, and is in fact an unbounded modification of the cocycle studied by Monod, Mineyev and Shalom in [MMS04]. In the case when $H \cong \mathbb{Z}^n$, $n \geq 2$, we can replace this cocycle by an n-cocycle with trivial coefficients: namely, the volume cocycle in $Z^n(\mathbb{Z}^n, \mathbb{R})$.

7.1. Construction for arbitrary groups.

Let us fix a hyperbolically embedded subgroup H of the group G. In order to avoid trivialities, we assume that H is proper and infinite. Moreover, throughout the section we assume that G is finitely generated. As a consequence, H is also finitely generated, and we can choose a symmetric finite generating set $\mathcal{S}_H$ of H contained in a symmetric finite generating set $\mathcal{S}_G$ of G. In this way, the Cayley graph $\mathrm{Cay}(H, \mathcal{S}_H)$ is naturally a subgraph of $\mathrm{Cay}(G, \mathcal{S}_G)$. By [DGO11, Corollary 4.27], we can fix a choice of $X \subseteq G$ so that $H \hookrightarrow_h (G, X)$ and X contains S_H. As usual, we denote by d_H the relative metric on H (see Definition 2.1). As observed in Remark 2.5, the metric d_H is bi-Lipschitz equivalent to the word metric $d_{\mathcal{S}_H}$ of $\mathrm{Cay}(H, \mathcal{S}_H)$.

We will denote by E_G (resp. E_H) the set of oriented edges of $\mathrm{Cay}(G, \mathcal{S}_G)$ (resp. of $\mathrm{Cay}(H, \mathcal{S}_H)$) and we will consider the normed G-module $\ell^2(E_G) = \ell^2(E_G, \mathbb{R})$ together with its H-submodule $\ell^2(E_H) = \ell^2(E_H, \mathbb{R})$ (where we identify an element $f \in \ell^2(E_H)$ with the function in $\ell^2(E_G)$ which coincides with f on E_H and vanishes elsewhere). We will also consider the bounded operator

$$\Psi_H \colon \ \ell^2(E_H) \ \to \ \ell^2(H)$$

that associates to any function f in $\ell^2(E_H)$ the function $\Psi_H(f) \in \ell^2(H)$ defined by

$$\Psi_H(f)(g) = \sum_{t(e)=v} f(e) - \sum_{o(e)=v} f(e)$$

where t (resp. o) is the function $E_H \to H$ associating to the edge e its final point (resp. its starting point). In the very same way, one can define the bounded operator

$$\Psi_G \colon \ \ell^2(E_G) \ \to \ \ell^2(G) \ .$$

We fix elements h_n of H such that $d_{\mathcal{S}_H}(1, h_n) = n^3$ (such elements exist since H is infinite).

Lemma 7.1. *For any point z in H there exists a constant c such that*

(1) $n^3 - c \leq d_{\mathcal{S}_H}(h_n, z) \leq n^3 + c$
(2) $d_{\mathcal{S}_H}(h_{n+1}, z) - d_{\mathcal{S}_H}(h_n, z) \geq 3n^2 - 2c$.

Proof. Set $c = d_{\mathcal{S}_H}(1, z)$. Then (1) follows from the triangle inequality, and (2) follows from (1) and (another) triangle inequality. $\square$

Let us now consider the cochain $c_H \in C^1_{\mathrm{alt}}(H, \ell^2(E_H))^H$ defined by

$$c_H(l_0, l_1) = \frac{d_{\mathcal{S}_H}(l_0, l_1)}{2\#[l_0, l_1]} \sum_{\gamma \in [l_0, l_1]} \chi_\gamma - \chi_{\overline{\gamma}}.$$

Here we denote by $[l_0, l_1]$ the set of geodesic paths in $\mathrm{Cay}(H, \mathcal{S}_H)$ with endpoints l_0, l_1, if γ is a geodesic path we denote by $\overline{\gamma}$ the geodesic with the opposite orientation, and, given a geodesic path γ, we denote by χ_γ the function in $\ell^2(E_H)$ that takes value 1 on the oriented edges that are contained in γ and is null everywhere else.

The element $\delta^1 c_H$ is an unbounded cocycle in $C^2_{\mathrm{alt}}(H, \ell^2(E_H))^H$ that separates the points of H in the sense specified by the following lemma:

Lemma 7.2. *For every $z \in H$ we have*

$$\lim_{n \to \infty} \Psi_H(\delta^1 c_H(h_n, h_{n+1}, z))(z) = \infty \ .$$

Moreover z, h_n, h_{n+1} are the only elements of H on which $\Psi_H(\delta^1 c_H(h_n, h_{n+1}, z))$ can be nonzero.

Proof. Let h, h', y be elements of H. It follows from the definition of c_H that $\Psi_H(c_H(h, h'))(y) \neq 0$ only if y belongs to a geodesic between h and h'. Moreover, if $y \neq h$, $y \neq h'$, then any such geodesic has an edge pointing to y and an edge exiting from y, so its contribution to $\Psi_H(c_H(h, h'))(y)$ is null. This implies the second assertion of the Lemma: $\Psi_H(\delta^1 c_H(h_n, h_{n+1}, z))(x) = 0$ if $x \notin \{z, h_n, h_{n+1}\}$.

On the contrary, it is immediate to check that

$$\Psi_H(c_H(h, h'))(h) = -d_{\mathcal{S}_H}(h, h') \,, \quad \Psi_H(c_H(h, h'))(h') = d_{\mathcal{S}_H}(h, h') \ .$$

This justifies the first assertion in the statement: if n is sufficiently large, then $z \neq h_n$ and $z \neq h_{n+1}$, so

$$\Psi_H(\delta^1 c_H(h_n, h_{n+1}, z))(z) = d_{\mathcal{S}_H}(h_{n+1}, z) - d_{\mathcal{S}_H}(h_n, z) ,$$

and the conclusion follows from Lemma 7.1. □

Let now C_G denote the quasi-cocycle $\Theta^2(\delta^1 c_H) \in C^2_{\text{alt}}(G, \ell^2(E_G))^G$ that extends the cocycle $\delta^1 c_H$, where Θ^2 is the extension operator provided by Theorem 4.1. The quasi-cocycle C_G allows us to reconstruct the projections π_{gH}: indeed let us define, for any coset gH of H and for any point y in $\widehat{G}$, the projection $\overline{\pi}_{gH}(y)$ to be the set of points $\overline{z} \in gH$ such that

$$\lim_{n\to\infty} \Psi_G(C_G(gh_n, gh_{n+1}, y))(\overline{z}) = \infty .$$

Proposition 7.3. *The projections $\overline{\pi}_{gH}$ are well defined and coincide with π_{gH} as in Definition 2.6. In particular, they satisfy the BBF axioms.*

Proof. Fix $y \in G$, and let us consider the simplex $\overline{s}_n(y) = (gh_n, gh_{n+1}, y)$. We first prove that, if n is large enough, then the only coset in $\mathcal{R}(\overline{s}_n(y))$ on which the trace of $\overline{s}_n(y)$ is not degenerate is gH. In fact, if n is large then $d_{gH}(gh_n, gh_{n+1}) > 2D$, $d_{gH}(gh_n, \pi_{gH}(y)) > 2D$, and also $d_{gH}(gh_{n+1}, \pi_{gH}(y)) > 2D$ (here D is as usual the constant provided by Lemma 2.8): indeed $d_{gH}(gh_n, gh_{n+1}) = d_H(h_n, h_{n+1})$ and d_H is bi-Lipschitz equivalent to the word metric $d_{\mathcal{S}_H}$.

This implies that gH is in $\mathcal{R}(\overline{s}_n(y))$. Moreover, every geodesic joining y with gh_n or with gh_{n+1} must contain $c(gH)$. This readily implies that, if $B \in \mathcal{R}(\overline{s}_n(y)) \setminus \{gH\}$, then $\pi_B(gh_n) = \pi_B(gh_{n+1}) = \pi_B(c(gH))$. Therefore, for any such B the trace $\text{tr}_2^B(\overline{s}_n(y))$ is degenerate.

Therefore, as a consequence of the definition of Θ^2, if n is large enough, then the function $C_G(gh_n, gh_{n+1}, y)$ is supported on $g \cdot E_H$ and

$$C_G(gh_n, gh_{n+1}, y) = g \cdot \left(\frac{1}{|\pi_{gH}(y)|} \sum_{gh \in \pi_{gH}(y)} \delta^1 c_H(h_n, h_{n+1}, h) \right) .$$

In particular it follows from Lemma 7.2 that, as n tends to infinity, the quantity

$$\Psi_G(C_G(gh_n, gh_{n+1}, y))(\overline{z}) = \frac{1}{|\pi_{gH}(y)|} \sum_{gh \in \pi_{gH}(y)} \Psi_H(\delta^1 c_H(h_n, h_{n+1}, h))(g^{-1}\overline{z})$$

tends to infinity if and only if $\overline{z}$ is in $\pi_{gH}(y)$. In particular, π_{gH} and $\overline{\pi}_{gH}$ coincide. □

As stated in the introduction, the construction described in Proposition 7.3 is based on the use of higher degree quasi-cocycles: in order to be able to recover the projection of a point z on H, we make use of (a sequence of) two auxiliary extra points, that in our construction are provided by the sequence of pairs (h_n, h_{n+1}). Moreover, here we have used the general formulation of our extension theorem: the cocycle c_H takes values in a proper submodule of the coefficient module of its extension to G.

The case $\mathbb{Z}^n \hookrightarrow_h G$, with $n > 1$. In the case when G admits $\mathbb{Z}^n$ as a hyperbolically embedded subgroup, we are able to reconstruct the projections on the cosets of $\mathbb{Z}^n$ from the extension of a certain alternating n-cocycle with *real* coefficients.

Let us consider the inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ and, for any $(n+1)$-tuple $\overline{z} = (z_0, \dots, z_n)$ of elements in $\mathbb{Z}^n$, let us denote by $\Delta(\overline{z})$ the affine simplex with vertices $(z_0, \dots, z_n)$. Let us moreover define the cocycle $\alpha \in C^n(\mathbb{Z}^n, \mathbb{R})$ by prescribing that the value of $\alpha(\overline{z})$ is the signed Euclidean volume of the simplex $\Delta(\overline{z})$. For every integer m, we will denote by y_0^m the point me_1 and by y_i^m the point $me_1 + me_i$, where $e_1, \dots, e_n$ are the natural generators of $\mathbb{Z}^n$.

For $z \in G$ (in particular z might be an element of $\mathbb{Z}^n$) denote by $\overline{s}_i^m(z)$ the simplex $(y_0^m, \dots, y_{i-1}^m, z, y_{i+1}^m, \dots, y_n^m) \in G^{n+1}$.

Lemma 7.4. *The cocycle $\alpha \in C^n(\mathbb{Z}^n, \mathbb{R})$ is alternating. Moreover, for any $z \in \mathbb{Z}^n$ and for any m in $\mathbb{N}$, the i-th coordinate of z can be computed for $i > 1$ as*

$$\frac{n!}{m^{n-1}} \alpha(\overline{s}_i^m(z))$$

and for $i = 1$ as

$$m + \frac{n!}{m^{n-1}} \alpha(\overline{s}_1^m(z)).$$

Proof. The signed area of the Euclidean simplex with vertices $(y_0, \dots, y_n)$ can be computed using the determinant of the matrix whose columns are the coordinates of $y_j - y_0$. In particular for $i = 1$ we get

$$\alpha(\overline{s}_1^m(z)) = \frac{1}{n!} \det \begin{pmatrix} z_1 - m & 0 & \\ z_2 & m & \\ \vdots & & \ddots \end{pmatrix} = \frac{m^{n-1}}{n!}(z_1 - m).$$

Analogously one gets $\alpha(\overline{s}_i^m(z)) = \frac{m^{n-1}}{n!} z_i$. □

Let now G be a group with $\mathbb{Z}^n \hookrightarrow_h G$, and let $A \in \mathrm{QZ}_{\mathrm{alt}}(G, \mathbb{R})$ be the quasi-cocycle obtained by setting $A = \Theta^n(\alpha)$, where Θ^n is the map described in Theorem 4.1.

Let us define, for any coset $g\mathbb{Z}^n$ and any point $z \in G$, the projection $\overline{\pi}_{g\mathbb{Z}^n}(z)$ to be the point

$$\overline{\pi}_{g\mathbb{Z}^n}(z) = g \lim_{m \to \infty} \left(\left\lfloor m + \frac{n! A(\overline{s}_1^m(g^{-1}z))}{m^{n-1}} \right\rfloor e_1 + \sum_{i=2}^{n} \left\lfloor \frac{n! A(\overline{s}_i^m(g^{-1}z))}{m^{n-1}} \right\rfloor e_i \right).$$

Proposition 7.5. *The projections $\overline{\pi}_{g\mathbb{Z}^n}$ are well defined and coincide up to bounded error with $\pi_{g\mathbb{Z}^n}$ as in Definition 2.6. In particular, they satisfy the BBF axioms.*

Proof. We will show that the expression in the definition of $\overline{\pi}_{g\mathbb{Z}^n}(z)$ is eventually constant in m, in particular $\overline{\pi}_{g\mathbb{Z}^n}$ is well defined. In showing this, we will prove that $\overline{\pi}_{g\mathbb{Z}^n}$ is at bounded distance from $\pi_{g\mathbb{Z}^n}$, hence concluding the proof.

Let D be the constant given by Lemma 2.8. For every i let us consider the simplex $\overline{s}_i^m(g^{-1}z)$. We claim that, if m is large enough, then $\mathbb{Z}^n$ is the only coset in $\mathcal{R}(\overline{s}_i^m(g^{-1}z))$ on which the trace of $\overline{s}_i^m(g^{-1}z)$ is not degenerate.

In fact, for every sufficiently large m, we have that $d_{\mathbb{Z}^n}(y_k^m, \pi_{\mathbb{Z}^n}(g^{-1}z)) > D$ for every $k = 0, \dots, n$. Therefore, Lemma 2.8 implies that any geodesic with endpoints $g^{-1}z$ and y_k^m contains the point $c(\mathbb{Z}^n)$. Assume by contradiction that there exists a coset B in $\mathcal{R}(\overline{s}_i^m(g^{-1}z))$ different from $\mathbb{Z}^n$ on which the trace $\mathrm{tr}_n^B(\overline{s}_i^m(g^{-1}z))$ is not degenerate. Then B belongs to $\mathcal{S}(g^{-1}z, y_k^m)$ for some $k \neq i$, and hence also to $\mathcal{S}(g^{-1}z, c(\mathbb{Z}^n))$. But this readily implies that, for every $j \neq i$, the projection $\pi_B(y_j^m)$ coincides with $\pi_B(c(\mathbb{Z}^n))$. This implies that the trace $\mathrm{tr}_n^B(\overline{s}_i^m(g^{-1}z))$ is degenerate, which is a contradiction.

Since $\mathbb{Z}^n$ is the only coset in $\mathcal{R}(\overline{s}_i^m(g^{-1}z))$ and $\overline{s}_i^m(g^{-1}z)$ is not small, by definition of Θ^n we have that $A(\overline{s}_1^m(g^{-1}z)) = \mathrm{tr}_n^{\mathbb{Z}^n}(\overline{s}_1^m(g^{-1}z))$. Therefore, if we denote by h_i the i-th coordinate of $h \in \mathbb{Z}^n$, then by Lemma 7.4 we get

$$A(\overline{s}_1^m(g^{-1}z)) = \frac{1}{|\pi_{\mathbb{Z}^n}(g^{-1}z)|} \sum_{h\in\pi_{\mathbb{Z}^n}(g^{-1}z)} \frac{m^{n-1}}{n!}(h_1 - m)$$

and

$$A(\overline{s}_i^m(g^{-1}z)) = \alpha(\mathrm{tr}_n^{\mathbb{Z}^n}(\overline{s}_i^m(g^{-1}z))) = \frac{1}{|\pi_{\mathbb{Z}^n}(g^{-1}z)|} \sum_{h\in\pi_{\mathbb{Z}^n}(g^{-1}z)} \frac{m^{n-1}}{n!} h_i$$

if $i > 1$. This implies that the expression defining $\overline{\pi}_{g\mathbb{Z}^n}$ is eventually constant in m, hence the limit is well defined.

Since the projection of $g^{-1}z$ to the coset $\mathbb{Z}^n$ has diameter at most D, and a consequence of what we just proved is that the expression for $\overline{\pi}_{g\mathbb{Z}^n}$ is a convex combination of the points in $\pi_{\mathbb{Z}^n}(g^{-1}z)$, we get that the point $\overline{\pi}_{g\mathbb{Z}^n}(z) = g\overline{\pi}_{\mathbb{Z}^n}(g^{-1}z)$ has distance at most D from the set $\pi_{g\mathbb{Z}^n}(z) = g\pi_{\mathbb{Z}^n}(g^{-1}z)$. □

Appendix A. On virtually free hyperbolically embedded subgroups

In most cases, the "natural" hyperbolically embedded subgroups one finds in a given group are virtually cyclic (for example this is the case for mapping class groups). However, many applications are based on the existence of virtually free non-abelian hyperbolically embedded subgroups. A crucial part of [DGO11] is therefore devoted to show that virtually free hyperbolically embedded subgroups can be constructed starting from a non-trivial hyperbolically embedded family. Theorem A.1 provides a strengthened version of this construction, that we used in the proof of Proposition 6.

Let G be a group containing a non-degenerate hyperbolically embedded family of subgroups. As observed before the statement of Theorem 7, G contains a maximal finite normal subgroup, which will be denoted $K(G)$.

Theorem A.1 ([DGO11], Theorem 6.14+ϵ)**.** *Let X be a (possibly infinite) generating system of the group G and let the non-degenerate family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ be hyperbolically embedded in (G,X). Then for each $n\geq 1$ there exists a copy F of the free group on n generators inside G so that $\{H_\lambda\}_{\lambda\in\Lambda}\cup\{F\times K(G)\}$ is hyperbolically embedded in (G,X).*

The two improvements on [DGO11, Theorem 6.14] that we make are that

(1) [DGO11, Theorem 6.14] guarantees the existence of a virtually free hyperbolically embedded subgroup, but it does not allow to keep the H_λ's in the hyperbolically embedded family, and
(2) [DGO11, Theorem 6.14] does not explicitly describe some $Y\subseteq G$ so that the virtually free subgroup is hyperbolically embedded in (G,Y).

The improvement that we actually need is the first one. However, it is worth pointing out that keeping track of the generating set is also part of [Hul13, Theorem 3.15] and [AMS13, Theorem 3.9], and in those papers such control turns out to be useful.

Along the way, we will show (a slightly more general form of) the following fact that might be of independent interest. An analogous statement in the setting of relatively hyperbolic groups was shown in [Osi06]. As usual $\{H_\lambda\}_{\lambda\in\Lambda}$ will be a non-degenerate hyperbolically embedded family of the group G, and we will denote by $\mathcal{H}$ the set $\sqcup_{\lambda\in\Lambda}(H_\lambda\backslash\{1\})$. For a definition of quasi-convexity and geometric separation see Subsection A.1.

Proposition A.2. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be hyperbolically embedded in (G,X), for X a generating system of the group G. Suppose that $\{E_i\}_{i\in I}$ is a finite family of finitely generated subgroups of G satisfying the following properties:*

(1) Each E_i is quasi-convex as a subset of $\Gamma=\mathrm{Cay}(G,X\sqcup\mathcal{H})$.
(2) The metric of Γ restricted to E_i is proper (i.e. balls of finite radius are finite).
(3) The family of all cosets of the E_i's, regarded as a family of (labelled) subsets of Γ, is geometrically separated.

Then $\{H_\lambda\}_{\lambda\in\Lambda}\cup\{E_i\}_{i\in I}\hookrightarrow_h(G,X)$.

A.1. **Preliminary facts.** In this subsection we collect a few facts that will be needed for the proof of Theorem A.1.

The following characterizations of hyperbolically embedded subgroups and relative hyperbolicity turn out to be convenient for the proof and, in the authors' opinion, they allow to provide a clear, worth-being-presented picture of why Theorem A.1 is true.

The reader unfamiliar with asymptotic cones and ultralimits is referred to [Dru02]. The following heuristic should however be enough to understand at least the ideas behind the proofs:

(1) an asymptotic cone of the metric space (X, d) is a "limit", in some suitable sense, of rescaled copies $(X, d/n)$ of X.
(2) the ultralimit of the sequence of subsets (A_i) of X in an asymptotic cone of X consists of all limit points of sequences (x_i) with $x_i \in A_i$.

Definition A.3 ([DS05])**.** Let Y be a geodesic metric space and $\mathcal{Q}$ a collection of closed connected subsets of Y. Then Y is *tree-graded* with respect to $\mathcal{Q}$ if

(T_1) for any distinct $P, Q \in \mathcal{Q}$ we have $|P \cap Q| \leq 1$, and
(T_2) any simple geodesic triangle is contained in some $P \in \mathcal{Q}$.

A simple geodesic triangle is a geodesic triangle with the property that distinct edges only intersect at their common endpoint.

Let X be a geodesic metric space and $\mathcal{P}$ a collection of subsets. Then X is *hyperbolic relative to* $\mathcal{P}$ if every asymptotic cone of X is tree-graded with respect to all (non-empty) ultralimits of sequences of elements of $\mathcal{P}$, and two such ultralimits coincide and contain at least two points only if the corresponding sequences coincide almost everywhere with respect to the ultrafilter chosen to construct the asymptotic cone.

(In [DS05] the latter condition is not explicitly stated but ultralimits of sequences that are not almost-everywhere equal are regarded as distinct sets throughout the paper.)

Theorem A.4 ([Sis12])**.** *The family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in (G, X), where X is a generating system for G, if and only if the Cayley graph* $\mathrm{Cay}(G, X)$ *is hyperbolic relative to the collection of all cosets of the H_λ's and the restriction of the metric of* $\mathrm{Cay}(G, X)$ *to each H_λ is proper (i.e. balls of finite radius are finite).*

Following [DGO11], we say that the collection of subsets $\mathcal{A}$ of a given metric space is *geometrically separated* if for each $D \geq 0$ there exists $K_D \geq 0$ so that for any distinct $A, B \in \mathcal{A}$ we have $\mathrm{diam}(N_D(A) \cap B) \leq K_D$.

The following variation of the definition of geometric separation will be useful:

Definition A.5. The collection of subsets $\mathcal{A}$ of a given metric space is K-*linearly geometrically separated* (K-LGS) if for any distinct $A, B \in \mathcal{A}$ and any $D \geq 1$ we have $\mathrm{diam}(N_D(A) \cap B) \leq KD$.

The following fact is a straightforward consequence of the definition.

Lemma A.6. *If the collection of subsets $\mathcal{A}$ of a metric space X is K-LGS then in any asymptotic cone of X distinct ultralimits of elements of $\mathcal{A}$ intersect in at most one point.*

Proof. Suppose that the conclusion is not true. Then there exist an ultrafilter ω, a sequence of scaling factors (r_i), sequences of sets (A_i^j) $(j = 1, 2)$ with $A_i^1 \neq A_i^2$ ω-a.e. and sequences of points $(x_{k,i})$ $(k = 1, 2)$ with the following properties:.

(1) $x_{k,i} \in A_i^1$.
(2) ω-lim $d(x_{k,i}, A_i^2)/r_i = 0$ for $k = 1, 2$.
(3) ω-lim $d(x_{1,i}, x_{2,i})/r_i = \epsilon > 0$.

In particular, for ω-a.e. i, we have $d(x_{1,i}, x_{2,i}) > \epsilon r_i/2$ and $x_{k,i} \in A_i^1 \cap N_{\epsilon r_i/(2K)}(A_i^2)$, in contradiction with the definition of K-LGS. □

Passing from geometric separation to linear geometric separation will be easy in our context due to the following (folklore) fact. We say that a subset S of a hyperbolic metric space is *quasi-convex* if there exists $\sigma \geq 0$ so that all geodesics connecting pairs of points on S are contained in the σ-neighborhood of S. A family of subsets is *uniformly quasi-convex* if all subsets in the family are quasi-convex with the same constant σ.

Lemma A.7. *Suppose that $\mathcal{A}$ is a collection of uniformly quasi-convex subsets of the hyperbolic metric space X. Then $\mathcal{A}$ is geometrically separated if and only if it is K-LGS for some $K \geq 0$.*

Proof. Clearly, we only need to show that geometric separation implies linear geometric separation.

Suppose that X is δ-hyperbolic, that every $A \in \mathcal{A}$ is σ-quasi-convex and that the diameter of the intersection of the $(\sigma+2\delta)$-neighborhoods of distinct elements of $\mathcal{A}$ is a most κ. Fix any $D \geq 1$ and suppose by contradiction that there exist $x, y \in N_D(A) \cap B$ with $d(x, y) \geq 2(D + 2\delta + 2) + \kappa + 1$ for some distinct $A, B \in \mathcal{A}$. Let $x_1, y_1 \in A$ be so that $d(x, x_1), d(y, y_1) \leq D + 1$. For each $a, b \in X$, denote by $[a, b]$ any choice of geodesic between a and b. Consider the point p (resp. q) along $[x, y]$ at distance $D+2\delta+2$ from x (resp. y). Notice that $d(p, q) \geq \kappa + 1$. We claim that $p, q \in N_\sigma(B) \cap N_{\sigma+2\delta}(A)$, in contradiction with the definition of κ.

The fact that p, q are within distance σ from B is just a consequence of quasi-convexity. Notice that p is within distance 2δ from $[x, x_1] \cup [x_1, y_1] \cup [y, y_1]$. We have

$$d(p, [x, x_1] \cup [y, y_1]) \geq \min\{d(p, x), d(p, y)\} - \max\{d(x, x_1), d(y, y_1)\} > 2\delta$$

so p must be within distance 2δ from $[x_1, y_1]$, which in turn is contained in the σ-neighborhood of A (see Figure 3). A similar argument also works for q, and this completes the proof. □

We can now prove the following:

Proposition A.8. *Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be hyperbolically embedded in (G, X), for X a generating system of the group G, and suppose that $\mathcal{E} \subseteq G$ is such that $\Gamma = \mathrm{Cay}(G, X \sqcup \mathcal{H} \sqcup \mathcal{E})$ is hyperbolic (e.g. $\mathcal{E} = \emptyset$). Suppose that $\{E_i\}_{i \in I}$ is a finite family of finitely generated subgroups of G satisfying the following properties:*

(1) Each E_i is quasi-convex as a subset of Γ.
(2) The metric of Γ restricted to E_i is proper.

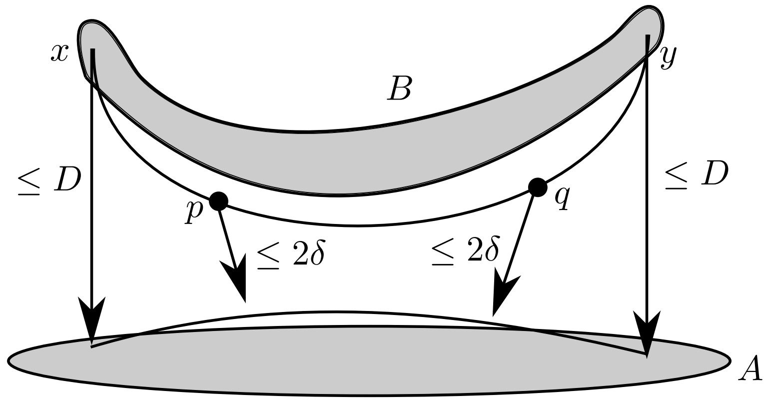

Figure 3

(3) The family of all cosets of the E_i's, regarded as a family of subsets of Γ, is geometrically separated.

Then $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_i\}_{i\in I} \hookrightarrow_h (G, X)$.

Proof. Let us denote by d_Γ the obvious metric on Γ. First of all, we remark that we can replace (2) with the condition

(2′) The metric of Γ restricted to E_i is quasi-isometric to a word metric on E_i.

In fact, thanks to quasi-convexity we can choose $D > 0$ such that the D-neighborhood N_i of E_i in Γ contains all the geodesics joining points of E_i. Then, the inclusion of E_i (endowed with the restriction of d_Γ) into N_i (endowed with the path metric induced from d_Γ) is an isometric embedding. As E_i acts properly and coboundedly on the path metric space N_i, the Milnor-Svarc Lemma tells us that the inclusion of E_i (now endowed with a word metric) into N_i is a quasi-isometry, so that the desired conclusion follows.

We need to show that, given an asymptotic cone of $\mathrm{Cay}(G, X)$, the ultralimits of cosets of the H_λ's and the E_i's are connected and they satisfy properties (T_1) and (T_2) of Definition A.3 (ultralimits of sequences of sets are always closed).

First of all, observe that property (T_2) holds just because it holds for the ultralimits of the H_λ's already.

Ultralimits of the E_i's are connected because they are bi-Lipschitz copies of an asymptotic cone of one of the E_i's: this is a consequence of (2′) and the fact that the inclusion of $\mathrm{Cay}(G, X)$ in Γ is 1-Lipschitz, which together imply that the restriction of the metric of $\mathrm{Cay}(G, X)$ to E_i is quasi-isometric to a word metric.

Also, conditions (3), (1) and Lemma A.7 imply that there exists K so that the family $\mathcal{B}$ of the cosets of the E_i's is K-LGS in Γ. For any subset $B \subseteq G$, let us denote by $N_D(B)$ (resp. $N^\Gamma_D(B)$) the D-neighborhood of B in G with respect to the word metric of $\mathrm{Cay}(G, X)$ (resp. the metric d_Γ). Now, if for some distinct $B_1, B_2 \in \mathcal{B}$ and $D \geq 1$ we have $x, y \in B_1 \cap N_D(B_2)$, then we also have $x, y \in B_1 \cap N_D^\Gamma(B_2)$ because the inclusion of $\mathrm{Cay}(G, X)$ in Γ is 1-Lipschitz. But then we get $d_\Gamma(x, y) \leq KD$. As this holds for any such pair x, y, in view of $(2')$ it is easy to deduce that $\mathcal{B}$ is K'-LGS in $\mathrm{Cay}(G, X)$ for some suitable K' (as

$$d_\Gamma(x, y) \leq d_{\mathrm{Cay}(G,X)}(x, y) \leq C_1 d^w_{B_1}(x, y) \leq C_2 d_\Gamma(x, y),$$

where $d^w_{B_1}$ is any word metric on B_1 and C_i are suitable constants).

In particular, distinct ultralimits of cosets of the E_i's intersect in at most one point by Lemma A.6. It is also easy to check that an ultralimit of cosets of the E_i's cannot intersect in more than one point an ultralimit of cosets of the H_λ: this is because a coset of an H_λ has diameter 1 in Γ so that, by (2), linear geometric separation holds in Γ and hence in $\mathrm{Cay}(G, X)$ as well.

So, keeping into account that property (T_1) holds for ultralimits of cosets of the H_λ's, we now showed that property (T_1) holds for ultralimits of cosets of the H_λ's and the E_i's, and the proof is complete. □

A.2. **Proof of Theorem A.1.** We can now show how to adapt the proof of [DGO11, Theorem 6.14] to prove Theorem A.1, whose notation we fix from now on.

Proceeding as in [DGO11], we consider the action of G on $\Gamma = \mathrm{Cay}(G, X \sqcup \mathcal{H})$. By [DGO11, Lemma 6.17], there exist elements h_1, h_2 in G so that

(1) each h_i acts hyperbolically on Γ,
(2) each h_i is contained in a maximal elementary subgroup $E(h_i)$,
(3) the family of the cosets of the $E(h_i)$'s is geometrically separated when such cosets are regarded as subsets of Γ,
(4) $E(h_1) \cap E(h_2) = K(G)$.

We need to modify the next step in the proof from [DGO11] a bit. In fact, [DGO11, Lemma 6.18] states that for any n there exists $Y \subseteq G$ and elementary subgroups $\{E_i\}_{i=1,\dots,n}$ of G such that $\{E_i\}_{i=1,\dots,n} \hookrightarrow_h (G, Y)$ and each E_i is of the form $\langle g_i \rangle \times K(G)$. We need to replace this result with the following:

Lemma A.9. *For each integer $n \geq 1$ there exist elementary subgroups $\{E_i\}_{i=1,\dots,n}$ of G such that $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_i\}_{i=1,\dots,n}$ is hyperbolically embedded in (G, X) and each E_i is of the form $\langle g_i \rangle \times K(G)$.*

Proof. The proof of Lemma 6.18 in [DGO11] starts with choosing $Y \subseteq G$ so that $\{E(h_1), E(h_2)\}$ is hyperbolically embedded in (G, Y). Instead of this, we invoke Proposition A.8 and get that $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E(h_1), E(h_2)\}$ is hyperbolically embedded in (G, X).

The rest of the proof of [DGO11, Lemma 6.18] takes place in $\mathrm{Cay}(G, Y \sqcup \mathcal{E})$, for $\mathcal{E} = (E(h_1) \cup E(h_2)) \setminus \{1\}$. However, the arguments apply to $\Gamma = \mathrm{Cay}(G, X \sqcup \mathcal{H} \sqcup \mathcal{E})$ as well, so that we can find pairwise non-commensurable elements g_i of G acting hyperbolically on Γ, each contained in a maximal elementary subgroup $E(g_i)$ which has the form $\langle g_i \rangle \times K(G)$. We can then invoke again Proposition A.8 to add the $E(g_i)$'s to the list of hyperbolically embedded subgroups.

Finally, the $E(h_i)$'s can be removed from the list, because the ultralimits of their cosets are bi-Lipschitz copies of $\mathbb{R}$, so they do not contain simple geodesic triangles and hence do not affect property (T_2) (while property (T_1) is preserved under passing to smaller collections of subsets). □

We can now use the argument after [DGO11, Lemma 6.18] with $Y = X$ and $\mathcal{E} = \mathcal{H} \cup \bigsqcup_{i=1}^n (E_i \backslash \{1\})$ and construct a subgroup $H = F \times K(G) < G$, with F free of rank n. The authors of [DGO11] then check that the hypotheses of [DGO11, Theorem 4.42] hold for F and the action of G on $\Gamma_{\mathcal{E}} = \mathrm{Cay}(G, X \sqcup \mathcal{E})$, namely they show that

(1) F is quasi-convex as a subset of $\Gamma_{\mathcal{E}}$,
(2) F acts properly on $\Gamma_{\mathcal{E}}$,
(3) the cosets of F are geometrically separated as subsets of $\Gamma_{\mathcal{E}}$.

Hence, Proposition A.8 allows us to add H to the list of hyperbolically embedded subgroups. □

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