A unified matrix approach to the representation of Appell polynomials

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In this paper we propose a unified approach to matrix representations of different types of Appell polynomials. This approach is based on the creation matrix - a special matrix which has only the natural numbers as entries and is closely related to the well known Pascal matrix. By this means we stress the arithmetical origins of Appell polynomials. The approach also allows to derive, in a simplified way, the properties of Appell polynomials by using only matrix operations.

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1. Introduction

In the last years, the interest in Appell polynomials and their applications in different fields has significantly increased. As recent applications of Appell polynomials in fields like probability theory and statistics we mention [6] and [28]. Generalized Appell polynomials as tools for approximating 3D-mappings were introduced for the first time in [22] in combination with Clifford analysis methods. As another new example we mention representation theoretic results like those of [9] and [20], that gave evidence to the central role of Appell polynomials as orthogonal polynomials. Representation theory is also the tool for their application in quantum physics as explained in [31]. Starting from Appell polynomials, but in the general framework of noncommutative Clifford algebras, one can find more traditionally motivated operational approaches to generalize Laguerre, Gould-Hopper and Chebyshev polynomials in the recent papers [10–12].

At the same time other authors were concerned with finding new characterizations of Appell polynomials themselves through new approaches. To quote some of them we mention, for instance, the novel approach developed in [29], which makes use of the generalized Pascal functional matrices introduced in [30], and also a new characterization based on a determinantal definition proposed in [15] and recently applied in [17] to Sheffer sequences too. Both of them have allowed to derive some properties of Appell polynomials by employing only linear algebra tools and to generalize some classical Appell polynomials.

Some decades ago, an intensive study of sequences of Appell polynomials was also realized by Roman and Rota in the context of the *Umbral Calculus* based on the well

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known functional approach [26, 27]. In the introduction to [26] they mentioned as traditional approaches for describing number sequences three different types: by recursion, by generating functions and by transform methods. All of them are also applicable for describing polynomial sequences and have been widely used more recently in almost all investigations on this field.

In this paper we concentrate on another unifying tool for studying polynomial sequences, namely the representation of Appell polynomials in matrix form. This was possible through the consideration of a related system of first order ordinary differential equations with certain initial conditions. It seems to be rather remarkable that the obtained matrix structure revealed how one and the same simple constant matrix H defined by [1]

$$(H)_{ij} = \begin{cases} i, & i = j+1 \\ 0, & \text{otherwise}, \qquad i,j = 0,1,\ldots,m. \end{cases} \tag{1}$$

determines the structure of all Appell sequences. Needless to emphasize the simplicity of the structure of H: it is a subdiagonal matrix which contains as nonzero entries only the sequence of natural numbers. In the context of this paper H is called *creation matrix* because it is the matrix from which all the special types of Appell polynomials can be created. With other words, our approach sheds light on the most fundamental arithmetical origins of the class of Appell polynomials.

At the same time the present paper can be considered as a far reaching generalization of the papers [2–5] where the authors concentrated their attention on the Pascal matrix and highlighted its connections with matrix representations of other special polynomials like, for instance, Bernoulli and Bernstein polynomials. The common Appell polynomial background was not a subject of their concern.

The paper is organized as follows. In Section 2 the concept of Appell polynomials and the generating functions of some particular cases are recalled. Besides the most common examples like Euler, Bernoulli, Hermite and Laguerre polynomials, also Legendre and Chebyshev polynomials, both of the first and of the second kind, are referred, the latter, however, in a modified Appell sequence form suggested in a remark by Carlson [14]. Section 3 explains how to obtain a matrix representation of all referred polynomials. Therefore we rely also on a so called *transfer matrix* obtained in terms of the creation matrix. In addition, we show that our matrix approach is also applicable for representing polynomial sequences that do not fulfill all requirements of Appell sequences as, for instance, Genocchi polynomials. In Section 4 we prove some general properties of Appell polynomials by employing only matrix operations and showing through short proofs the efficiency of our matrix approach. Finally, in Section 5 some conclusions are presented.

2. Classical equivalent characterizations of Appell polynomials

In 1880 Appell introduced in [7] sequences of polynomials $\{p_n(x)\}_{n\geq 0}$ satisfying the relation

$$\frac{\mathrm{d}}{\mathrm{d}x} p_n(x) = n\, p_{n-1}(x), \qquad n = 1,2,\ldots, \tag{2}$$

[1]Notice that, in general, H could be considered as an infinite dimensional square matrix corresponding to the considered infinite sequence of Appell polynomials. But due to the aim of looking for the representation of a polynomial of a certain fixed degree m, we reduce our consideration to the case of a square matrix of order $m+1$.

in which

$$p_0(x) = c_0, \qquad c_0 \in \mathbb{R} \setminus \{0\}. \tag{3}$$

Remark 2.1 Some authors use another definition for $\{p_n(x)\}_{n\geq 0}$, where the factor n on the right hand side of (2) is omitted. However, considering that the prototype of such sequences is the monomial power basis, i.e., $\{x^n\}_{n\geq 0}$, we prefer to consider Appell's original definition.

From (2)-(3) it can easily be checked that Appell polynomials are of the following form:

$$\begin{aligned} p_0(x) &= c_0 \\ p_1(x) &= c_1 + c_0\, x \\ p_2(x) &= c_2 + 2\, c_1\, x + c_0\, x^2 \\ &\vdots \end{aligned} \tag{4}$$

It is evident that the condition imposed in (3) on c_0 implies that all polynomials $\{p_n(x)\}_{n\geq 0}$ are exactly of degree n. The compact form of all $p_n(x)$ follows immediately:

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k}\, x^k, \quad n = 0, 1, \ldots, \qquad c_0 \neq 0. \tag{5}$$

In addition, if $f(t)$ is a formal power series of the form

$$f(t) = \sum_{n=0}^{+\infty} c_n \frac{t^n}{n!}, \qquad f(0) \neq 0, \tag{6}$$

then Appell sequences can also be defined by means of their *exponential generating function* $G(x,t)$ given by

$$G(x,t) \equiv f(t)\, e^{xt} = \sum_{n=0}^{+\infty} p_n(x) \frac{t^n}{n!}, \tag{7}$$

cf. [7]. By appropriately choosing $f(t)$, many of the classical polynomials can be derived. In particular, we get [2]

- the *monomials* $\{x^n\}_{n\geq 0}$ when

$$f(t) = 1;$$

- the *Bernoulli polynomials* $\{B_n(x)\}_{n\geq 0}$ when

$$f(t) = \frac{t}{e^t - 1} = \left(\frac{\sum_{n=0}^{+\infty} \frac{t^n}{n!} - 1}{t} \right)^{-1} = \left(E_{1,2}(t)\right)^{-1},$$

[2]The expression of the formal power series $f(t)$ for Bernoulli and Euler polynomials can be found in [25], the one of the monic Hermite polynomials in [8] and that of Laguerre polynomials in [18], cf. also [23].

where, by denoting with $\Gamma(\cdot)$ the Euler's Gamma function,

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha, \beta > 0, \tag{8}$$

is the two-parameter Mittag-Leffler function [23];

- the *Euler polynomials* $\{E_n(x)\}_{n\geq 0}$ when
$$f(t) = \frac{2}{e^t + 1};$$
- the *monic Hermite polynomials* $\{\widehat{H}_n(x)\}_{n\geq 0} \equiv \{2^{-n} H_n(x)\}_{n\geq 0}$, with $H_n(x)$ the classical Hermite polynomials, when
$$f(t) = e^{-\frac{t^2}{4}} = E_{1,1}\left(-\frac{t^2}{4}\right);$$
- the *'modified' generalized Laguerre polynomials* $\{(-1)^n \, n! \, L_n^{(\alpha-n)}(x)\}_{n\geq 0}$, $\alpha > -1$, when
$$f(t) = (1-t)^\alpha.$$

Some other families of special polynomials do not seem to be Appell sets since they are usually defined by a different type of generating function. However, Carlson in [14] highlights that sometimes it is possible to transform a given polynomial sequence into one of Appell type by suitable changes of variables. This is the case, for example, of the *Legendre polynomials* $\{P_n(x)\}_{n\geq 0}$ defined on the interval $(-1, 1)$ if as generating function is chosen [3]

$$J_0\left(t\,(1-x^2)^{\frac{1}{2}}\right) e^{xt} = \sum_{n=0}^{+\infty} P_n(x) \frac{t^n}{n!},$$

where $J_0(y) = \sum_{n=0}^{+\infty} (-y^2)^n / (2^{2n} \, (n!)^2)$ denotes the Bessel function of the first kind and index 0. However, setting

$$x = \frac{z}{(z^2+1)^{\frac{1}{2}}}, \qquad t = \tau (z^2+1)^{\frac{1}{2}}, \tag{9}$$

we can obtain a generating function of the form (7) for the *'modified' Legendre polynomials* as follows:

$$J_0(\tau)\, e^{z\tau} = \sum_{n=0}^{+\infty} P_n\left(\frac{z}{(z^2+1)^{\frac{1}{2}}}\right) (z^2+1)^{\frac{n}{2}} \frac{\tau^n}{n!}.$$

Similarly, it can be verified that also the Chebyshev polynomials, both of the first and of the second kind, can be treated as Appell sequences. In fact, the *Chebyshev polynomials*

[3] The generating functions of P_n, T_n, U_n here reported can be found in [25].

of the first kind $\{T_n(x)\}_{n\geq 0}$ have as generating function

$$\cosh\left(t\,(x^2-1)^{\frac{1}{2}}\right)\,e^{xt} = \sum_{n=0}^{+\infty} T_n(x)\,\frac{t^n}{n!}, \qquad x\in(-1,1), \tag{10}$$

and the *Chebyshev polynomials of the second kind* $\{U_n(x)\}_{n\geq 0}$ have as generating function

$$\frac{\sinh\left(t\,(x^2-1)^{\frac{1}{2}}\right)}{(x^2-1)^{\frac{1}{2}}}\,e^{xt} = \sum_{n=0}^{+\infty} U_n(x)\,\frac{t^{n+1}}{(n+1)!}, \qquad x\in(-1,1). \tag{11}$$

Therefore, by using again the transformations (9) in (10) as well as in (11) we get

$$\cosh\left(i\,\tau\right)\,e^{z\tau} = \sum_{n=0}^{+\infty} T_n\left(\frac{z}{(z^2+1)^{\frac{1}{2}}}\right)\,(z^2+1)^{\frac{n}{2}}\,\frac{\tau^n}{n!},$$

respectively

$$\frac{\sinh\left(i\,\tau\right)}{i}\,e^{z\tau} = \sum_{n=0}^{+\infty} U_n\left(\frac{z}{(z^2+1)^{\frac{1}{2}}}\right)\,(z^2+1)^{\frac{n}{2}}\,\frac{\tau^{n+1}}{(n+1)!}.$$

Consequently, by taking into account the well known identities

$$\cosh(i\,\tau) = \frac{e^{i\,\tau}+e^{-i\,\tau}}{2} = \cos\tau, \qquad \frac{\sinh(i\,\tau)}{i} = \frac{e^{i\,\tau}-e^{-i\,\tau}}{2\,i} = \sin\tau,$$

it follows that the generating functions of 'modified' Chebyshev polynomials of the first and of the second kind are

$$\cos\tau\,e^{z\tau} = \sum_{n=0}^{+\infty} T_n\left(\frac{z}{(z^2+1)^{\frac{1}{2}}}\right)\,(z^2+1)^{\frac{n}{2}}\,\frac{\tau^n}{n!}$$

and

$$\operatorname{sinc}\tau\,e^{z\tau} = \sum_{n=0}^{+\infty} \frac{1}{n+1}\,U_n\left(\frac{z}{(z^2+1)^{\frac{1}{2}}}\right)\,(z^2+1)^{\frac{n}{2}}\,\frac{\tau^n}{n!},$$

respectively, where, as usual, $\operatorname{sinc}\tau = \sin\tau/\tau$.

Summarizing, the changes of variables given by (9) lead to the following Appell sequences:

- the *'modified' Legendre polynomials* $\{(z^2+1)^{n/2}P_n(z/(z^2+1)^{1/2})\}_{n\geq 0}$ when

$$f(\tau) = J_0\left(\tau\right),$$

  is the Bessel function of the first kind and index 0;

- the *'modified' Chebyshev polynomials of the first kind* $\{(z^2+1)^{n/2}T_n(z/(z^2+1)^{1/2})\}_{n\geq 0}$ when

$$f(\tau) = \cos\tau = E_{2,1}(-\tau^2); \tag{12}$$

- the *'modified' Chebyshev polynomials of the second kind* $\{(z^2+1)^{n/2}U_n(z/(z^2+1)^{1/2})/(n+1)\}_{n\geq 0}$ when

$$f(\tau) = \operatorname{sinc}\tau = E_{2,2}(-\tau^2). \tag{13}$$

We finish this section with some additional remarks about the matrix approach applied to polynomial sequences which are not Appell in the strong sense, that is which verify the relations in (2) but do not fulfill the condition imposed in (3) on c_0. This can be equivalently stated by means of the following requirements

$$\frac{\mathrm{d}}{\mathrm{d}x}p_n(x) = n\,p_{n-1}(x), \quad n = 0, 1, \ldots, \tag{14}$$

where the right-hand side is taken to be zero when $n = 0$. It is worth to note that, in such case, the degree of each polynomial $p_n(x)$ is less than or equal to n and, in particular, it is strictly less than n if $p_0(x) = 0$. The latter is the case of the Genocchi sequence $\{G_n(x)\}_{n\geq 0}$ whose generating function is [4]

$$f(t)e^{xt} \equiv \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{15}$$

Indeed, to quote only few of them,

$$G_0(x) = 0,\ G_1(x) = 1,\ G_2(x) = 2x-1,\ G_3(x) = 3x^2-3x$$

from which one deduces that the degree of $G_n(x)$ is $n-1$, for each $n \geq 1$. Another example, already mentioned by Carlson in [14], is the polynomial set

$$\{0, 1, 2x, 3x^2, \ldots\} \tag{16}$$

whose generating function is

$$f(t)e^{xt} \equiv t\,e^{xt} = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!}. \tag{17}$$

Furthermore, in the same paper, it has been proved in Theorem 1 that $\{p_n(x)\}_{n\geq 0}$ verifies (14) if and only if it satisfies a binomial theorem of the form

$$p_n(x+y) = \sum_{k=0}^{\infty}\binom{n}{k}p_k(x)y^{n-k}, \qquad n = 0, 1, \ldots. \tag{18}$$

This property is fulfilled by polynomials that cannot be considered Appell polynomials according to (2)-(3). This is the case of the Genocchi polynomials, for instance.

[4]The expression of the function $f(t)$ for the Genocchi polynomials can be found in [21].

3. Appell polynomials: the matrix approach

As mentioned in the Introduction, our unified matrix approach basically relies on the properties of the *creation matrix* (1) of order $m+1$. It is worth to observe that it is a nilpotent matrix of degree $m+1$, i.e.,

$$H^s = O, \qquad \text{for all } s \geq m+1. \tag{19}$$

This property is one of the essential ingredients for the unified matrix approach to Appell polynomials that now follows.

In order to handle the Appell sequence $\{p_n(x)\}_{n\geq 0}$ in a closed form we introduce

$$\mathbf{p}(x) = [p_0(x) \;\; p_1(x) \;\; \cdots \;\; p_m(x)]^T, \tag{20}$$

hereafter called *Appell vector.*

Due to (2), the application of the creation matrix (1) implies that the Appell vector satisfies the first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{p}(x) = H\,\mathbf{p}(x), \tag{21}$$

whose general solution is

$$\mathbf{p}(x) = e^{xH}\,\mathbf{p}(0) \equiv P(x)\,\mathbf{p}(0), \tag{22}$$

where $P(x)$ is a matrix defined by

$$(P(x))_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j \\ 0, & \text{otherwise}, \qquad i,j = 0,1,\ldots,m. \end{cases} \tag{23}$$

The matrix (23) is called *generalized Pascal matrix* because $P(1) \equiv P$ is the lower triangular *Pascal matrix* [2, 13] defined by

$$(P)_{ij} = \begin{cases} \binom{i}{j}, & i \geq j \\ 0, & \text{otherwise}, \qquad i,j = 0,1,\ldots,m. \end{cases} \tag{24}$$

Notice that $P(0) \equiv I$ is the identity matrix.

Consider now the vector of monomial powers

$$\xi(x) = [1 \;\; x \;\; \cdots \;\; x^m]^T \tag{25}$$

and the matrix M defined by

$$(M)_{ij} = \begin{cases} \binom{i}{j} c_{i-j}, \; i \geq j \\ 0, \qquad \text{otherwise}, \quad i,j = 0,1,\ldots,m, \qquad c_0 \neq 0. \end{cases} \tag{26}$$

Obviously, M is a nonsingular matrix and according to (4) we have

$$\mathbf{p}(x) = M\xi(x) \tag{27}$$

Such relation between the vectors $\xi(x)$ and $\mathbf{p}(x)$ motivates the following definition.

Definition 3.1 The matrix M given by (26) is called *transfer matrix* for the Appell vector (20).

Considering that $\xi(0) = [1\ \ 0\ \cdots\ 0]^T$, from (26) immediately follows that

$$\mathbf{p}(0) = M\xi(0) = [c_0\ \ c_1\ \cdots\ c_m]^T, \qquad c_0 \neq 0. \tag{28}$$

Therefore, as we should expect, from (22) and (27) follows that the different kinds of Appell polynomials are determined by the entries of $\mathbf{p}(0)$ or, equivalently, by the entries of the transfer matrix M. It is now only one step left to arrive to the formula which combines the properties of the creation matrix H with the explicit form of the transfer matrix M. The key result is the following.

THEOREM 3.2 *Let H be the creation matrix defined by (1). If $G(x,t) \equiv f(t)e^{xt}$ is the generating function of the Appell sequence $\{p_n(x)\}_{n\geq 0}$, then the transfer matrix M of order $m+1$ is the nonsingular matrix $f(H)$.*

Proof. Since (see (7))

$$G(x,t) = \sum_{n=0}^{+\infty} p_n(x)\frac{t^n}{n!},$$

setting $x = 0$, we obtain $f(t) = \sum_{n=0}^{+\infty} p_n(0)\, t^n/n!$. Substituting t by H and by taking into account formula (6) as well as (19) we get

$$f(H) = \sum_{n=0}^{m} c_n \frac{H^n}{n!}, \qquad c_0 \neq 0.$$

Furthermore, denoting by $\mathbf{e}_s, s = 0, 1, \ldots, m$, the standard unit basis vector in $\mathbb{R}^{m+1}$ and adopting the convention that $\mathbf{e}_s = \mathbf{0}$ whenever $s > m$ (null vector), then the entries of $f(H)$ are obtained by

$$\begin{aligned}(f(H))_{ij} &= \sum_{n=0}^{m} \frac{c_n}{n!}\mathbf{e}_i^T H^n \mathbf{e}_j = \sum_{n=0}^{m} \frac{c_n}{n!}(j+1)^{(n)}\mathbf{e}_i^T \mathbf{e}_{j+n} \\ &= \sum_{n=0}^{m} c_n \frac{(j+1)^{(n)}}{n!}\delta_{i,j+n},\end{aligned} \tag{29}$$

where $\delta_{i,j}$ is the Kronecker symbol and $(j+1)^{(n)} = (j+1)(j+2)\cdots(j+n)$ is the ascending factorial with $(j+1)^{(0)} := 1$. Thus, $(f(H))_{ij} = 0$ if $i < j$, and when $i = j+n$, i.e. $i \geq j$, then follows

$$(f(H))_{ij} = c_{i-j}\frac{i!}{(i-j)!j!} = \binom{i}{j} c_{i-j}$$

which completes the proof due to the definition of the transfer matrix (26). ■

We saw that the knowledge of the transfer matrix M is sufficient for a concrete matrix representation formula for any Appell sequence (see (27)). Taking into account the previous theorem, each transfer matrix is a suitable function of the creation matrix H.

We now specify for the particular cases referred in Section 2 the corresponding transfer matrices.

(i) For the monomials $\{x^n\}_{0\le n\le m}$,

$$M = I,$$

where I is the identity matrix of order $m+1$.

(ii) For the Bernoulli polynomials $\{B_n(x)\}_{0\le n\le m}$,

$$M = \left(\sum_{n=0}^{m} \frac{H^n}{(n+1)!}\right)^{-1}. \tag{30}$$

(iii) For the Euler polynomials $\{E_n(x)\}_{0\le n\le m}$,

$$M = 2(e^H + I)^{-1} = 2(P+I)^{-1}, \tag{31}$$

where P is the Pascal matrix of order $m+1$ (see (24)).

(iv) For the monic Hermite polynomials $\{\widehat{H}_n(x)\}_{0\le n\le m}$,

$$M = e^{-\frac{H^2}{4}} = \sum_{n=0}^{m} \frac{(-H^2)^n}{2^{2n} n!}. \tag{32}$$

In this case, introducing the diagonal matrix

$$D[\ell] = \mathrm{diag}[\ell^0, \ell^1, \ell^2, \dots, \ell^m], \qquad \ell \neq 0, \tag{33}$$

the vector of the classical Hermite polynomials $\mathbf{H}(x) = [H_0(x)\, H_1(x) \cdots H_m(x)]^T$, and recalling that $\widehat{H}_n(x) = 2^{-n} H_n(x)$, we get

$$(D[2])^{-1}\,\mathbf{H}(x) = e^{-\frac{H^2}{4}} \xi(x) \quad \Leftrightarrow \quad \mathbf{H}(x) = D[2] e^{-\frac{H^2}{4}} \xi(x),$$

that is $M = D[2]e^{-H^2/4}$ is the transfer matrix associated to the classical Hermite polynomials.

(v) For the 'modified' generalized Laguerre polynomials

$$\{(-1)^n n! L_n^{(\alpha-n)}(x)\}_{0\le n\le m}$$

we have

$$M = (I-H)^{\alpha}.$$

It is worth noting that the matrix point of view provides an easy way to relate $\{(-1)^n n! L_n^{(\alpha-n)}(x)\}_{0\le n\le m}$ with the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{0\le n\le m}$ [25]. In fact, introducing the vector

$$\mathbf{L}(x) = [L_0^{(\alpha)}(x)\;\; L_1^{(\alpha-1)}(x)\;\; \cdots\;\; L_m^{(\alpha-m)}(x)]^T$$

and the diagonal matrix (here the subscript f indicates 'factorials')

$$D_f = \mathrm{diag}[0!, 1!, 2!, \cdots, m!],$$

we obtain (see (33)) $D[-1]D_f\mathbf{L}(x) = (I - H)^{\alpha}\xi(x)$ or, equivalently,

$$\mathbf{L}(x) = (D_f)^{-1} D[-1](I - H)^{\alpha}\xi(x). \tag{34}$$

In addition, the recurrence relation reported in [25]

$$L_n^{(\alpha)}(x) = L_{n-1}^{(\alpha)}(x) + L_n^{(\alpha-1)}(x), \quad n > 0,$$

gives, successively,

$$\begin{aligned}
L_1^{(\alpha-1)}(x) &= L_1^{(\alpha)}(x) - L_0^{(\alpha)}(x) \\
L_2^{(\alpha-2)}(x) &= L_2^{(\alpha-1)}(x) - L_1^{(\alpha-1)}(x) = (L_2^{(\alpha)}(x) - L_1^{(\alpha)}(x)) - (L_1^{(\alpha)}(x) - L_0^{(\alpha)}(x)) \\
&= L_2^{(\alpha)}(x) - 2L_1^{(\alpha)} + L_0^{(\alpha)}(x) \\
&\vdots \\
L_m^{(\alpha-m)}(x) &= \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} L_n^{(\alpha)}(x).
\end{aligned}$$

Denoting by $\boldsymbol{\mathcal{L}}(x) = [L_0^{(\alpha)}(x) \;\; L_1^{(\alpha)}(x) \;\; \cdots \;\; L_m^{(\alpha)}(x)]^T$ the vector of the first $m+1$ generalized Laguerre polynomials and using (23) we have

$$\mathbf{L}(x) = P(-1)\boldsymbol{\mathcal{L}}(x).$$

Finally, from (34) it follows that

$$\boldsymbol{\mathcal{L}}(x) = P(D_f)^{-1}D[-1](I - H)^{\alpha}\xi(x).$$

In particular, when $\alpha = 0$ we get the ordinary *Laguerre polynomials*.

(vi) For the 'modified' Legendre polynomials $\{(z^2+1)^{\frac{n}{2}}P_n(z/(z^2+1)^{\frac{1}{2}})\}_{0\le n\le m}$ we get

$$M = J_0(H) = \sum_{n=0}^{m} \frac{(-H^2)^n}{2^{2n}(n!)^2}.$$

(vii) For the 'modified' Chebyshev polynomials of the first kind $\{(z^2+1)^{\frac{n}{2}}T_n(z/(z^2+1)^{\frac{1}{2}})\}_{0\le n\le m}$,

$$M = \cos H = \sum_{n=0}^{m} \frac{(-H^2)^n}{(2n)!}.$$

(viii) For the 'modified' Chebyshev polynomials of the second kind $\{(z^2+1)^{\frac{n}{2}}U_n(z/(z^2+$

$1)^{\frac{1}{2}})/(n+1)\}_{0\le n\le m}$,

$$M \equiv M_{\mathbf{U}} = \sum_{n=0}^{m} \frac{(-H^2)^n}{(2n+1)!}. \tag{35}$$

Of course, considering

$$z = \frac{x}{(1-x^2)^{\frac{1}{2}}}, \qquad x \in (-1,1),$$

we obtain the first $m+1$ elements of the classical sequences $\{P_n(x)\}_{n\ge 0}$, $\{T_n(x)\}_{n\ge 0}$ and $\{U_n(x)\}_{n\ge 0}$. Collecting these elements into the vectors $\mathbf{P}(x)$, $\mathbf{T}(x)$, and $\mathbf{U}(x)$, respectively, we get

$$\begin{aligned}
\mathbf{P}(x) &= D[(1-x^2)^{\frac{1}{2}}]\, J_0(H) D^{-1}[(1-x^2)^{\frac{1}{2}}]\, \xi(x),\\
\mathbf{T}(x) &= D[(1-x^2)^{\frac{1}{2}}]\, \cos H D^{-1}(1-x^2)^{\frac{1}{2}}]\, \xi(x),\\
\mathbf{U}(x) &= D_{m+1} D[(1-x^2)^{\frac{1}{2}}]\, M_{\mathbf{U}} D^{-1}[(1-x^2)^{\frac{1}{2}}]\, \xi(x),
\end{aligned}$$

where $D_{m+1} = \mathrm{diag}[1, 2, \ldots, m+1]$.

Remark 3.3 The matrix $M_{\mathbf{U}}$ given in (35) satisfies $HM_{\mathbf{U}} = \sin H$.

As a byproduct, our approach is also applicable for representing polynomial sequences that do not fulfill all requirements of Appell sequences. Here we shall focus only on the two examples just mentioned at the end of Section 2. Concerning the case of Genocchi polynomials, by virtue of (15) it is possible to get a corresponding transfer matrix $M = M_{\mathbf{G}}$ in the same way as for Appell sequences. The only one difference is that in this case M is singular ($c_0 = 0$). Actually, from Theorem 3.2 we get

$$M_{\mathbf{G}} = 2H(e^H + I)^{-1} = 2H(P+I)^{-1}.$$

In the case of the polynomial sequence (16), due to (17), the corresponding transfer matrix $M = M_{\mathbf{C}}$ is

$$M_{\mathbf{C}} = H.$$

4. Some properties of Appell polynomials

In order to establish several identities of Appell polynomials in a friendly and unified way, different from other approaches, we now use the transfer matrix and some properties of the generalized Pascal matrix. It is worth mentioning that most of them were derived in [15, 16] by using a determinantal approach, involving an upper Hessenberg matrix and elementary matrix calculus.

Lemma 4.1 *Let $P(x)$ be the generalized Pascal matrix and $\xi(x)$ the vector containing the ordinary monomials as previously defined. Then,*

$$\xi(x+y) = P(x)\,\xi(y), \qquad \forall\, x, y \in \mathbb{R}.$$

Proof. The result is a consequence of the binomial theorem. In fact, (see (23))

$$(\xi(x+y))_i = (x+y)^i = \sum_{k=0}^{i} \binom{i}{k} x^{i-k}\, y^k = (P(x)\,\xi(y))_i\,.$$

■

The next result (binomial identity, [cf. 16, p. 16]) is the matrix version of (18) and at the same time a generalization of the previous Lemma. It stresses once again how easy some proofs become using the matrix approach.

Proposition 4.2 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence and $P(x)$ the generalized Pascal matrix defined by (23). For the corresponding Appell vector we have*

$$\mathbf{p}(x+y) = P(x)\,\mathbf{p}(y), \qquad \forall\, x, y \in \mathbb{R}. \tag{36}$$

Proof. From (27) and Lemma 4.1 one has

$$\mathbf{p}(x+y) = M\xi(x+y) = MP(x)\,\xi(y) = P(x)M\xi(y) = P(x)\mathbf{p}(y)$$

since $P(x)$ and M commute being both functions of the creation matrix H. ■

Corollary 4.3 *([cf. 16, p. 18-19]) Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. Then, for any constant a and for all $x \in \mathbb{R}$, the Appell vector of the given sequence satisfies the following relations:*

(i) forward difference:

$$\mathbf{p}(x+1) - \mathbf{p}(x) = (P-I)\,\mathbf{p}(x);$$

(ii) multiplication theorem:

$$\mathbf{p}(ax) = P((a-1)\,x)\,\mathbf{p}(x), \tag{37}$$

$$\mathbf{p}(ax) = MD[a]\,\xi(x), \tag{38}$$

where $D[a]$ is defined by (33).

Proof. (*i*) The result follow from (36) with $y = 1$ and by recalling that $P(1) \equiv P$.
(*ii*) The relation (37) can be immediately deduced from (36) by putting $y = (a-1)x$. The formula (38) follows from (27) and by observing that $\xi(ax) = D[a]\,\xi(x)$. ■

It is worth noting that (37) generalizes, for all kinds of Appell polynomials, the well-known properties for the Bernoulli and the Euler polynomials (see, e.g., [1, 15, 24])

$$B_n(ax) = \sum_{i=0}^{n} \binom{n}{i} B_i(x)(a-1)^{n-i}x^{n-i}, \qquad E_n(ax) = \sum_{i=0}^{n} \binom{n}{i} E_i(x)(a-1)^{n-i}x^{n-i}.$$

In addition, there are some identities involving Appell polynomials which at a first glance are not equivalent. To provide an example we consider the following ones involving the Bernoulli polynomials:

$$B_n(1-x) = (-1)^n B_n(x), \qquad B_n(1) = (-1)^n B_n(0).$$

It is trivial to check that the relation on the left-hand side implies the one on the right, while it is not clear that the opposite implication also holds true. Similar arguments can be used about the following relations involving Euler polynomials:

$$E_n(1-x) = (-1)^n E_n(x), \qquad E_n(1) = (-1)^n E_n(0).$$

However, the referred equivalences (symmetry relations) are general properties which will become evident from the next theorem ([cf. 15, p. 1537]):

THEOREM 4.4 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. For the corresponding Appell vector the following equivalence holds*

$$(\,\mathbf{p}(h-x) = D[-1]\,\mathbf{p}(x),\, \forall\, h, x \in \mathbb{R}\,) \Leftrightarrow (\,\mathbf{p}(h) = D[-1]\,\mathbf{p}(0),\, \forall\, h \in \mathbb{R}\,), \tag{39}$$

where $D[-1]$ is defined by (33).

Proof. ($\Rightarrow$) This implication is trivial from the hypothesis with $x = 0$.
($\Leftarrow$) Using (36) and by observing that $P(-x) = D[-1]P(x)D[-1]$ (see (23)) we get

$$\mathbf{p}(h-x) = P(-x)\mathbf{p}(h) = D[-1]P(x)D[-1]D[-1]\,\mathbf{p}(0) = D[-1]P(x)\,\mathbf{p}(0) = D[-1]\,\mathbf{p}(x).$$

■

It is interesting to ask for the consequences of Theorem 4.4 for the coefficients of the Appell polynomials ([cf. 15, p. 1537-1538]).

COROLLARY 4.5 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. For the Appell vector we get*

$$(\,\mathbf{p}(-x) = D[-1]\,\mathbf{p}(x), \quad \forall\, x \in \mathbb{R}\,) \quad \Leftrightarrow \quad (\,c_{2n+1} = 0, \quad n = 0, 1, \ldots\,).$$

Proof. From Theorem 4.4, by fixing $h = 0$ one has that

$$(\,\mathbf{p}(-x) = D[-1]\,\mathbf{p}(x), \quad \forall\, x \in \mathbb{R}\,) \quad \Leftrightarrow \quad (\,\mathbf{p}(0) = D[-1]\,\mathbf{p}(0)\,).$$

The relation on the right of this equivalence together with the fact that $\mathbf{p}(0) = [c_0\ c_1\ \cdots\ c_m]^T$ (see (28)) lead to the desired result. ■

Remark 4.6 We notice that Hermite, Legendre and Chebyshev polynomials of the first and of the second kind verify the equivalence of the previous corollary. The reason for can be read off from their transfer matrices. It is due to the fact that their transfer matrices are expansions of even powers of H whose entries satisfy (see (29))

$$\mathbf{e}_i^T H^{2n} \mathbf{e}_j = (j+1)^{(2n)} \mathbf{e}_i^T \mathbf{e}_{j+2n} = 0, \qquad i - j \neq 2n.$$

Among the properties of Appell polynomials we now refer to the ones stated in Theorem 11 of [15, p. 1534], which were proved applying a determinantal approach. Here we propose an alternative proof based on our matrix approach and making use of the transfer matrix.

THEOREM 4.7 *Let $\{v_n(x)\}_{n\geq 0}$ and $\{u_n(x)\}_{n\geq 0}$ be two sequences of Appell polynomials and $\mathbf{v}(x)$ and $\mathbf{u}(x)$ their corresponding Appell vectors. Then,*

(i) for all $\lambda, \mu \in \mathbb{R}$, $\lambda\mathbf{v}(x) + \mu\mathbf{u}(x)$ is an Appell vector for the sequence $\{\lambda v_n(x) + \mu u_n(x)\}_{n\geq 0}$;

(*ii*) *replacing in* $v_n(x)$ *the powers* x^0, $x^1, \ldots, x^n$ *by* $u_0(x)$, $u_1(x), \ldots, u_n(x)$ *and denoting the resulting polynomial by* $w_n(x)$, *the vector* $\mathbf{w}(x) = [w_0(x)\, w_1(x)\, \ldots\, w_m(x)]^T$ *is the Appell vector of* $\{w_n(x)\}_{n\geq 0}$

Proof. Let M_v and M_u be the transfer matrices for $\mathbf{v}(x)$ and $\mathbf{u}(x)$, respectively, i.e.,

$$\mathbf{v}(x) = M_v \xi(x), \qquad \mathbf{u}(x) = M_u \xi(x).$$

Then,

(*i*) $\lambda \mathbf{v}(x) + \mu \mathbf{u}(x) = (\lambda M_v + \mu M_u)\, \xi(x)$,

(*ii*) $\mathbf{w}(x) = M_v \mathbf{u}(x) = M_v M_u \xi(x)$. [5]

To prove that $\lambda \mathbf{v}(x) + \mu \mathbf{u}(x)$ and $\mathbf{w}(x)$ are Appell vectors, we need to check if they satisfy the relation (21). The result follows immediately by recalling that M_v and M_u are both functions of H (see Theorem 3.2). Consequently, they commute with H as well any of their linear combination. ■

In particular, the last theorem allows us to obtain in a straightforward way some sequences of Appell polynomials recently introduced in [19, p. 9471]. In fact, the transfer matrix

(*i*) for the 2-iterated Bernoulli polynomials $\{B_n^{[2]}(x)\}_{0\leq n\leq m}$ is (see (30))

$$M = \left(\sum_{n=0}^{m} \frac{H^n}{(n+1)!} \right)^{-2};$$

(*ii*) for the 2-iterated Euler polynomials $\{E_n^{[2]}(x)\}_{0\leq n\leq m}$ is (see (31))

$$M = 4(e^H + I)^{-2} \equiv 4(P + I)^{-2};$$

(*iii*) for the Bernoulli-Euler polynomials $\{\,_E B_n(x)\}_{0\leq n\leq m}$ (or Euler-Bernoulli polynomials $\{\,_B E_n(x)\}_{0\leq n\leq m}$) is (see (30) and (31))

$$M = 2 \left((P + I) \sum_{n=0}^{m} \frac{H^n}{(n+1)!} \right)^{-1}.$$

Some other properties of Appell polynomials can be obtained by making explicitly use of the inverse of the transfer matrix, [cf. 16, p. 5]. To achieve them, we recall that (see Theorem 3.2 and (6))

$$M = \sum_{k=0}^{m} c_k \frac{H^k}{k!}, \qquad c_0 \neq 0.$$

Setting

$$M^{-1} = \sum_{k=0}^{m} \gamma_k \frac{H^k}{k!}, \tag{40}$$

[5]In fact, we recognize easily that the process of substitution described in the theorem in the same form as in [15] is the composition $w_n(x) := v_n(x) \circ u_n(x)$ realized by matrix multiplication. Our matrix approach makes this fact explicit.

we have

$$\gamma_0 = \frac{1}{c_0}, \qquad \gamma_k = -\frac{1}{c_0}\sum_{s=0}^{k-1}\binom{k}{s} c_{k-s}\gamma_s, \qquad k = 1, 2, \ldots, m. \tag{41}$$

In fact,

$$I = MM^{-1} = \left(\sum_{k=0}^{m} c_k \frac{H^k}{k!}\right)\left(\sum_{r=0}^{m} \gamma_r \frac{H^r}{r!}\right) = \sum_{n=0}^{m}\left(\sum_{k+r=n} c_k\gamma_r \frac{H^{k+r}}{k!r!}\right)$$
$$= \sum_{n=0}^{m}\left(\sum_{r=0}^{n} \frac{n!c_{n-r}\gamma_r}{(n-r)!r!}\right)\frac{H^n}{n!} = \sum_{n=0}^{m}\left(\sum_{r=0}^{n}\binom{n}{r} c_{n-r}\gamma_r\right)\frac{H^n}{n!}.$$

Consequently, (27) implies that

$$M^{-1}\mathbf{p}(x) = \xi(x)$$

or, equivalently,

$$\sum_{k=0}^{n}\binom{n}{k}\gamma_{n-k}\, p_k(x) = x^n, \qquad n = 0, 1, \ldots, m,$$

from which we deduce a *general recurrence relation* for Appell polynomials:

$$p_n(x) = \frac{1}{\gamma_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\gamma_{n-k}\, p_k(x)\right), \qquad n = 0, 1, \ldots.$$

By taking into account (40), it is an easy matter to notice from (30), (31) and (32), that ([cf. 15, p. 1539-1540])

- for the Bernoulli polynomials: $\gamma_k = 1/(k+1)$, $k = 0, 1, \ldots, m$;
- for the Euler polynomials: $\gamma_0 = 1$, $\gamma_k = 1/2$, $k = 1, \ldots, m$;
- for the monic Hermite polynomials:

$$\gamma_k = \begin{cases} \frac{1}{2^k}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}, \qquad k = 0, 1, \ldots, m.$$

Furthermore, for the *generalized Euler polynomials* introduced in [15, p. 1542] we have $\gamma_0 = 1$, $\gamma_k = \bar{\gamma} = const.$, $k = 1, \ldots, m$, which leads to

$$M^{-1} = I + \sum_{k=1}^{m}\bar{\gamma}\,\frac{H^k}{k!} = (1-\bar{\gamma})I + \sum_{k=0}^{m}\bar{\gamma}\,\frac{H^k}{k!} = (1-\bar{\gamma})\,I + \bar{\gamma}\,P.$$

Knowing the relationship between the coefficients of M and its inverse, we can prove the following result which relates the coefficients of an Appell polynomial with those of the general recurrence relation (cf. Corollary 4.5).

PROPOSITION 4.8 *([cf. 15, p. 1537]) The elements of the sets $\{c_n\}_{0\le n\le m}$ and $\{\gamma_n\}_{0\le n\le m}$ characterizing the transfer matrix M and its inverse, respectively, satisfy the following* *equivalences:*

$$c_{2j+1} = 0 \quad \Leftrightarrow \quad \gamma_{2j+1} = 0, \qquad j = 0, 1, \ldots, \frac{m-1}{2}.$$

Proof. We proceed by induction on j. If $j = 0$, the statement is verified directly using (41). Next, let us suppose that it is true for $j-1$ and we prove that it is true also for j. From (41)

$$\gamma_{2j+1} = -\frac{1}{c_0}\sum_{s=0}^{2j}\binom{2j+1}{s} c_{2j+1-s}\gamma_s = -\frac{1}{c_0}c_{2j+1}\gamma_0 +$$
$$-\frac{1}{c_0}\sum_{\substack{1\le s\le 2j-1\\ s\ odd}}\binom{2j+1}{s} c_{2j+1-s}\gamma_s - \frac{1}{c_0}\sum_{\substack{2\le s\le 2j\\ s\ even}} c_{2j+1-s}\binom{2j+1}{s}\gamma_s.$$

By using the induction hypothesis the last two sums vanish, and this completes the proof. ■

5. Conclusion

Being the central ingredients of the presented unified matrix approach to Appell polynomials, the roles of the creation matrix H as well as of the transfer matrix M are studied. Furthermore, the paper confirmed the effectiveness of the unified matrix representation by showing that some new types of recently introduced Appell polynomials can immediately be deduced. Finally, the special role of the transfer matrix is also stressed and advantageously used for deriving, in an easy and compact way, the relationship between the coefficients of Appell polynomials and their general recurrence relations. The paper also tries to call attention to the Appell polynomial nature of *modified* Legendre and Chebyshev polynomials. They are obtained from the classical ones (so far not studied in the context of Appell polynomials) by a straightforward substitution suggested in [14] and applying corresponding generating functions given in [25].

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