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Generating the efficient frontier of a class of bicriteria generalized fractional programming

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Abstract In this paper a particular class of bicriteria maximization problems over a compact polyhedron is considered. The first component of the objective function is the ratio of powers of affine functions and the second one is linear. Several theoretical properties are provided, such as the pseudoconcavity of the first criterium of the objective function, the connectedness and compactness of both the efficient frontier and the set of efficient points The obtained results allows us to propose a new simplex-like solution method for generating the whole efficient frontier; to better clarify the use of the suggested algorithm several examples are described and the results of a computational test are presented.

Keywords Bicriteria programming · Generalized fractional programming · Pseudoconcavity

Mathematics Subject Classification (2000) 90C29 · 90C31 · 90C32

1 Introduction

Over the last decades, great attention has been devoted to bicriteria fractional programming from both a theoretical and an algorithmic point of view. The great interest for this class of problems is justified also by the numerous applications in different areas such as management science, finance and transportation theory (see for all the surveys [3, 14] and see [13] for an applicative view).

In this paper a particular class of bicriteria maximization problems is studied; more precisely the first component of the objective function is the ratio

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of powers of affine functions, the second component is a linear function, while the feasible region is a compact polyhedron.

Looking for the efficient points of this class of fractional problems deserves a great attention because of the particular feature of the first objective; in fact as this class of functions may admit maximum points which are not vertices (see [4]), efficient points can be either a vertex, or be part of an edge or even lie in the interior of the feasible polyhedron. The algorithm proposed in this paper looks for the efficient points in a simplex-like way. The feasible region defined by nonnegative variables and equality constraints allows to implement the simplex-like method by using a tabular approach; the simplex-like table allows to move from vertex to vertex by means of pivot operations and to recognize the optimality of the points by means of "reduced costs". This kind of approach has been introduced in [1] and then profitably used in other bicriteria fractional programming literature (see for example [11]).To better grasp how the algorithm works, some detailed examples are given and the procedure is described step by step. Furthermore a computational test shows how the performance of the algorithm changes as the dimension of the treated problems varies.

The paper is organized as follows: in Section 2 and 3 some theoretical results are stated with the aim of obtaining suitable conditions for algorithmically determine the efficient points. In particular, the set of efficient points is characterized by means of suitable parametric subproblems. The stated theoretical properties are related both to a general bicriteria problem (Section 2) and to the analyzed class of bicriteria fractional programs (Section 3). In Section 4 the simplex-like algorithm is proposed, while the numerical examples and the computational test results are described in Section 5.

2 Preliminary results for a general bicriteria problem

The aim of this section is to provide some new general results which will be useful in the rest of the paper in order to propose an algorithm which generates the efficient frontier of a particular class of bicriteria problems. In this light, let us consider the following general bicriteria problem:

$$\begin{cases} \max \;\; f(x) = (f_1(x), f_2(x)) \\ \qquad\qquad x \in X \end{cases} \tag{1}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous functions and $X \subset \mathbb{R}^n$ is a compact polyhedron. For the sake of completeness, let us first recall the definitions of efficient point and of efficient frontier.

Definition 1 Let us consider Problem (1).

- a point $x^0 \in X$ is said to be an efficient solution for problem P if there exists no $\bar{x} \in X$ such that $f(\bar{x}) \in f(x^0) + \mathbb{R}^2_+ \setminus \{0\}$, that is to say that $f(X) \cap \{f(x^0) + \mathbb{R}^2_+\} = \{f(x^0)\}$;

– let us denote by X_E the set of all efficient points for problem P. The efficient frontier is the set of the values of all efficient solutions, that is $f(X_E) \subset f(X)$.

As it is well known, the efficient points of a bicriteria problem are strictly related to the maximum points of a suitable family of parametric scalar problems (see for all [7]). With this aim, the following notations are introduced:

$$\xi_{\max} = \max_{x \in X} \{f_1(x)\} \qquad \xi_{\min} = \max_{x \in X, f_2(x) = \theta_{max}} \{f_1(x)\}$$
$$\theta_{\max} = \max_{x \in X} \{f_2(x)\} \qquad \theta_{\min} = \max_{x \in X, f_1(x) = \xi_{\max}} \{f_2(x)\}$$

It can be easily proved that $f_1(X_E) \subseteq [\xi_{\min}, \xi_{\max}]$ and $f_2(X_E) \subseteq [\theta_{\min}, \theta_{\max}]$. Some further notations, where $\theta \in [\theta_{\min}, \theta_{\max}]$, are needed:

$$S_\theta = \{x \in X, f_2(x) \geq \theta\} \qquad \bar{S}_\theta = \{x \in X, f_2(x) = \theta\}$$
$$R_\theta = \arg\max_{x \in S_\theta} \{f_1(x)\} = \arg\max_{x \in X, f_2(x) \geq \theta} \{f_1(x)\}$$
$$\bar{R}_\theta = \arg\max_{x \in \bar{S}_\theta} \{f_1(x)\} = \arg\max_{x \in X, f_2(x) = \theta} \{f_1(x)\}$$
$$\xi(\theta) = \max_{x \in \bar{S}_\theta} \{f_1(x)\} = \max_{x \in X, f_2(x) = \theta} \{f_1(x)\} = f_1(\bar{R}_\theta)$$

Notice that the continuity of f_1 and f_2, together with the compactness of X, imply that for all $\theta \in [\theta_{\min}, \theta_{\max}]$, the sets S_θ, $\bar{S}_\theta$, R_θ and $\bar{R}_\theta$ are nonempty and compact. Moreover $\xi : \theta \mapsto \max_{x \in \bar{S}_\theta} \{f_1(x)\}$ is a function such that $\xi(\theta_{\min}) = \xi_{\max}$ and $\xi(\theta_{\max}) = \xi_{\min}$.

In the literature efficient points are usually studied in relationship with the elements in $R(\theta)$ (see for all [3]). Actually, the elements in $\bar{R}(\theta)$ are to be used for the algorithmic purposes of this paper. For this very reason the following results are stated.

Theorem 1 *Let us consider Problem (1). It is*

$$X_E \subseteq \bigcup_{\theta \in [\theta_{\min}, \theta_{\max}]} R_\theta \qquad \textit{and} \qquad X_E \subseteq \bigcup_{\theta \in [\theta_{\min}, \theta_{\max}]} \bar{R}_\theta$$

Proof We are going to prove that for any efficient point x^*, it is $x^* \in R_{\theta^*}$ and $x^* \in \bar{R}_{\theta^*}$ where $\theta^* = f_2(x^*) \in [\theta_{min}, \theta_{max}]$.

First notice that by definition $x^* \in \bar{S}_{\theta^*} \subseteq S_{\theta^*}$. Assume by contradiction that $x^* \notin R_{\theta^*}$ [$x^* \notin \bar{R}_{\theta^*}$]. This yields that $\exists \bar{x} \in X$ such that $f_2(\bar{x}) \geq \theta^* = f_2(x^*)$ [$f_2(\bar{x}) = \theta^* = f_2(x^*)$] and $f_1(\bar{x}) > f_1(x^*)$. As a consequence $f(\bar{x}) \in f(x^*) + \mathbb{R}^2_+ \setminus \{0\}$ so that $x^* \notin X_E$ which is a contradiction.

The reverse inclusion relations between the family of the sets $\{R(\theta)\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$ ($\{\bar{R}(\theta)\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$) and X_E does not in general hold. The following theorem provides a sufficient condition for a complete characterization of the efficient set in terms of both $\{R(\theta)\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$ and $\{\bar{R}(\theta)\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$

Theorem 2 . *Let us consider Problem (1) and let* $\theta \in [\theta_{min}, \theta_{\max}]$.
If $R_\theta \subseteq \bar{R}_\theta$ *then* $R_\theta = \bar{R}_\theta \subseteq X_E$.

Proof Let us first prove that $R_\theta \subseteq \bar{R}_\theta$ implies $R_\theta = \bar{R}_\theta$. Assume by contradiction that $\exists x^0 \in \bar{R}_\theta$ such that $x^0 \notin R_\theta$. Condition $x^0 \in \bar{R}_\theta \subseteq \bar{S}_\theta \subseteq S_\theta$ means that:

$$f_1(x^0) \geq f_1(x) \ \forall x \in X \text{ such that } f_2(x) = \theta = f_2(x^0) \tag{2}$$

while $x^0 \notin R_\theta$ with $x^0 \in \bar{S}_\theta \subseteq S_\theta$ implies:

$$\exists x^* \in X \text{ such that } f_2(x^*) \geq \theta = f_2(x^0) \text{ and } f_1(x^*) > f_1(x^0) \tag{3}$$

These two last equations yield $f_2(x^*) > \theta = f_2(x^0)$ so that:

$$\exists \bar{x} \in R_\theta \text{ such that } f_2(\bar{x}) > \theta = f_2(x^0) \text{ and } f_1(\bar{x}) > f_1(x^0) \tag{4}$$

This implies that $\bar{x} \notin \bar{R}_\theta$ which contradicts the hypothesis $R_\theta \subseteq \bar{R}_\theta$.

Finally, let us prove that $R_\theta \subseteq X_E$. Assume by contradiction that $\exists x^0 \in R_\theta$ such that $x^0 \notin X_E$. Condition $x^0 \in R_\theta \subseteq \bar{R}_\theta \subseteq \bar{S}_\theta \subseteq S_\theta$ means that:

$$f_1(x^0) \geq f_1(x) \ \forall x \in X \text{ such that } f_2(x) \geq \theta = f_2(x^0) \tag{5}$$

while $x^0 \notin X_E$ implies:

$$\exists x^* \in X \text{ such that } f(x^*) \in f(x^0) + \mathbb{R}^2_+ \setminus \{0\} \tag{6}$$

These two last equations yield $x^* \in S_\theta$ and $f_1(x^*) = f_1(x^0)$. As a consequence, $f_2(x^*) > f_2(x^0)$ and hence $x^* \in R_\theta$ but $x^* \notin \bar{R}_\theta$, which contradicts the hypothesis $R_\theta \subseteq \bar{R}_\theta$.

The following result presents a class of functions for which condition $R_\theta \subseteq \bar{R}_\theta \ \forall \theta \in [\theta_{\min}, \theta_{\max}]$ is verified and hence provides a general assumption allowing to study efficient points by means of the elements in $\bar{R}(\theta)$.

Theorem 3 *Let us consider Problem (1) and assume that function* f_1 *verifies the following property:*

- *each local maximum point of* f_1 *is also a global maximum.*

Then, $R_\theta = \bar{R}_\theta \ \forall \theta \in [\theta_{min}, \theta_{\max}]$ *so that*

$$X_E = \bigcup_{\theta \in [\theta_{\min}, \theta_{\max}]} R_\theta = \bigcup_{\theta \in [\theta_{\min}, \theta_{\max}]} \bar{R}_\theta \tag{7}$$

and

$$f(X_E) = \left\{ (\xi(\theta), \theta) \in \mathbb{R}^2 \text{ such that } \theta \in [\theta_{\min}, \theta_{\max}] \right\} \tag{8}$$

Proof We are going to prove that $R_\theta \subseteq \bar{R}_\theta\ \forall\theta \in [\theta_{\min}, \theta_{\max}]$, so that the whole result will follow from Theorems 1 and 2. In the case $\theta = \theta_{\max}$ just notice that by means of the definitions it is $R_{\theta_{\max}} = \bar{R}_{\theta_{\max}}$. Consider now the case $\theta_{\min} \le \theta < \theta_{\max}$ and assume by contradiction that $\exists x^0 \in R_\theta$ such that $x^0 \notin \bar{R}_\theta$, so that:

$$f_2(x^0) > \theta \text{ and } f_1(x^0) \ge f_1(x)\ \forall x \in X \text{ such that } f_2(x) \ge \theta \tag{9}$$

By means of the continuity of f_2 there exists a neighbourhood of x_0, namely I_{x^0}, such that $f_2(x) > \theta\ \forall x \in X \cap I_{x^0}$. Due to (9) it is $f_1(x^0) \ge f_1(x)\ \forall x \in X \cap I_{x^0}$, that is to say that x^0 is a local maximum point of f_1. By means of the assumptions x^0 is then a global maximum point of f_1 and hence $f_1(x^0) = \xi_{\max}$. It then results $f_2(x^0) > \theta \ge \theta_{min} = \max_{x \in X, f_1(x) = \xi_{\max}} \{f_2(x)\} \ge f_2(x^0)$ which is a contradiction.

The class of functions for which the local maximum points are also global, includes the class of semistrictly quasiconcave functions. By assuming the semistrictly quasiconcavity of f_1 and the quasiconcavity of f_2 we get the following result.

Theorem 4 *Let us consider Problem (1), and assume that function f_1 is semistrictly quasiconcave while function f_2 is quasiconcave.*
Then, the set $R_\theta = \bar{R}_\theta$ is convex for all $\theta \in [\theta_{\min}, \theta_{\max}]$.

Proof First recall that a local maximum point of a semistrictly quasiconcave function is also a global one, that the set of maximum points of a continuous semistrictly quasiconcave function is convex, and that the upper level sets of a quasiconcave function are convex ones. Hence, the sets S_θ are convex for all $\theta \in [\theta_{\min}, \theta_{\max}]$ and the sets of maximum points $R_\theta \subset S_\theta$ of function f_1 are convex. The result then follows from Theorem 3.

Under the assumption "f_1 has no local maximum point different from the global ones", function ξ and the sets X_E and $f(X_E)$ exhibit properties which can be usefully exploited from an alogorithmical point of view (see Section 3 and Section 4). With this regards we present the following theorem which extends Theorem 2.2 in [10].

Theorem 5 *Let us consider Problem (1) and assume that function f_1 verifies the following property:*

- *each local maximum point of f_1 is also a global maximum.*

Then, function ξ results to be strictly decreasing and continuous.

Proof Let us first prove that $\xi(\theta)$ is strictly decreasing. With this aim, let θ_1 and θ_2, with $\theta_{\min} \le \theta_1 < \theta_2 \le \theta_{\max}$, be arbitrary values. For Theorems 2 and 3 it is $R_{\theta_1} = \bar{R}_{\theta_1} \subseteq X_E$ and $R_{\theta_2} = \bar{R}_{\theta_2} \subseteq X_E$. Being $S_{\theta_1} \subset S_{\theta_2}$ it then yields:

$$\xi(\theta_1) = f_1(\bar{R}_{\theta_1}) = f_1(R_{\theta_1}) \ge f_1(R_{\theta_2}) = f_1(\bar{R}_{\theta_2}) = \xi(\theta_2)$$

Let $x_1 \in \bar{R}_{\theta_1} \subseteq X_E$, $x_2 \in \bar{R}_{\theta_2} \subseteq X_E$ and assume by contradiction that $\xi(\theta_1) = f_1(x_1) = f_1(x_2) = \xi(\theta_2)$; by definition of $\bar{R}_\theta$ it results $f_2(x_1) = \theta_1 < \theta_2 = f_2(x_2)$ so that $f(x_2) \in f(x_1) + \mathbb{R}^2_+ \setminus \{0\}$, and this is a contradiction since x_1 is an efficient point. As a consequence, it is $\xi(\theta_1) > \xi(\theta_2)$ which implies the strict decreaseness of function $\xi(\theta)$.
Let us now prove that ξ is continuous. Since ξ is monotone then the limit $\lim_{\theta\to\hat{\theta}} \xi(\theta)$ exists for all $\hat{\theta} \in [\theta_{\min}, \theta_{\max}]$. Let $x_\theta \in \bar{R}_\theta$ so that $\xi(\theta) = f_1(x_\theta)$. The assumptions guarantee that $R_\theta = \bar{R}_\theta \subseteq X_E$, since X_E is a compact set it then yields $\lim_{\theta\to\hat{\theta}} x_\theta = x^* \in X_E$ and for the continuity of function f_2 it is $f_2(x^*) = \lim_{\theta\to\hat{\theta}} f_2(x_\theta) = \lim_{\theta\to\hat{\theta}} \theta = \hat{\theta}$. Consequently, it is $x^* \in \bar{R}_{\hat{\theta}}$. The continuity of f_1 then yields $\lim_{\theta\to\hat{\theta}} \xi(\theta) = \lim_{\theta\to\hat{\theta}} f_1(x_\theta) = f_1(x^*) = f_1(\bar{R}_{\hat{\theta}}) = \xi(\hat{\theta})$ so that $\xi(\theta)$ is continuous too.

Theorem 6 *Let us consider Problem (1) and assume that function f_1 verifies the following property:*

- *each local maximum point of f_1 is also a global maximum.*

Then, the set of efficient points X_E and the efficient frontier $f(X_E)$ are both compact and connected.

Proof Let us first prove the compactness of X_E. The set X_E is bounded since $X_E \subseteq X$ and X is bounded. Assume now by contradiction that X_E is not closed, that is to say that there exists a sequence $\{x_k\} \subset X_E$ such that: $\lim_{k\to+\infty} x_k = \hat{x}$ with $\hat{x} \in X$, $\hat{x} \notin X_E$. Notice also that for the continuity of f_1 and f_2 it is:

$$\lim_{k\to+\infty} f_1(x_k) = f_1(\hat{x}) \quad \text{and} \quad \lim_{k\to+\infty} f_2(x_k) = f_2(\hat{x})$$

Since $f_i(X_E) \subseteq [\xi_{\min}, \xi_{\max}]$ with $i = 1, 2$, it is $f_2(x_k) \in [\theta_{\min}, \theta_{\max}]$ for all k so that, for the compactness of the interval $[\theta_{\min}, \theta_{\max}]$ and by means of a suitable subsequence if needed, it results:

$$\lim_{k\to+\infty} f_2(x_k) = f_2(\hat{x}) \in [\theta_{\min}, \theta_{\max}]$$

Let $\hat{\theta} = f_2(\hat{x})$; since $\hat{x} \notin X_E$ there exists $x^* \in R_{\hat{\theta}}$ such that $f(x^*) \in f(\hat{x}) + \mathbb{R}^2_+ \setminus \{0\}$. Due to Theorems 2 and 3 it results $R_{\hat{\theta}} = \bar{R}_{\hat{\theta}} \subseteq X_E$, hence $f_2(x^*) = \hat{\theta} = f_2(\hat{x})$ which implies $f_1(x^*) > f_1(\hat{x})$.
In the case $\hat{\theta} = \theta_{\max}$ for the continuity of f_1 there exists a value $\tilde{k}$ great enough such that for all $k > \tilde{k}$ it is: $f_1(x^*) > f_1(x_k)$ and $f_2(x^*) = \hat{\theta} = \theta_{\max} \geq f_2(x_k)$ and this is a contradiction since $\{x_k\} \subset X_E$.
In the case $\hat{\theta} < \theta_{\max}$ there exists $\bar{x} \in X \cap I_{x^*}$, where I_{x^*} is a suitable neighborhood of x^*, such that: $f_1(x^*) > f_1(\bar{x}) > f_1(\hat{x})$and $f_2(\bar{x}) > \hat{\theta} = f_2(x^*) = f_2(\hat{x})$ For the continuity of f_1 and f_2 there exists a value $\tilde{k}$ great enough such that for

all $k > \tilde{k}$ it is: $f_1(\bar{x}) > f_1(x_k)$ and $f_2(\bar{x}) > f_2(x_k)$ and this is a contradiction since $\{x_k\} \subset X_E$.
The compactness of $f(X_E)$ follows from the continuity of f being $f(X_E)$ the image of a compact set.
Let us now prove the connectedness of $f(X_E)$. From Theorem 3, the set $f(X_E)$ is nothing but the graph of function ξ in the domain $[\theta_{\min}, \theta_{\max}]$. The connectedness of such a graph follows directly from the continuity of $\xi(\theta)$ proved in Theorem 5 and the connectedness of the interval $[\theta_{\min}, \theta_{\max}]$.
Since $f(X_E)$ is connected and f is continuous then X_E is connected too.

Notice that in [12] closedness is proved for a componentwise strictly quasiconcave function while in [6] connectedness is obtained by considering semistrictly quasiconcave functions. The connectedness of the efficient frontier is studied also in [9] and in [5] by using a different approach based on the image space.

3 A generalized fractional bicriteria problem: statement and theoretical properties

In what follows we are going to consider the following problem:

$$P : \begin{cases} \max \ f(x) = (f_1(x), f_2(x)) \\ x \in X = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\} \end{cases} \tag{10}$$

where:

- $f_1(x) = \dfrac{\left(c^T x + c_0\right)^\alpha}{(d^T x + d_0)^\beta}$ with $0 < \alpha < \beta$;
- $f_2(x) = a^T x$;
- $f : D \to \mathbb{R}^2$, with $D = \{x \in \mathbb{R}^n : d^T x + d_0 > 0\}$, f differentiable on D;
- $c^T x + c_0 > 0 \ \forall x \in X$ and $d^T x + d_0 > 0 \ \forall x \in X$;
- $A \in \mathbb{R}^{m \times n}$ with rank$[A] = m < n$, $b \in \mathbb{R}^m$, $c, d, a \in \mathbb{R}^n$, $c \neq 0$, $d \neq 0$, $a \neq 0$, c and d are linearly independent, $c_0, d_0 \in \mathbb{R}$ and $X \neq \emptyset$ is a compact polyhedron.

We first present several theoretical properties of Problem P which play a key role in generating the efficient frontier (see Section 4). With this aim, let us write the gradient and the Hessian matrix of f_1

$$\begin{aligned} \nabla f_1(x) &= \frac{(c^T x + c_0)^{\alpha - 1}}{(d^T x + d_0)^{\beta + 1}} \left((d^T x + d_0)\alpha c - (c^T x + c_0)\beta d\right) \\ &= f_1(x) \left(\frac{\alpha c}{c^T x + c_0} - \frac{\beta d}{d^T x + d_0} \right) \end{aligned} \tag{11}$$

$$\begin{aligned} H_1(x) = \frac{(c^T x + c_0)^{\alpha - 2}}{(d^T x + d_0)^{\beta + 2}} &\Big(\left(d^T x + d_0\right)^2 \left(\alpha^2 - \alpha\right) cc^T - \\ &\alpha\beta \left(c^T x + c_0\right) \left(d^T x + d_0\right) \left(cd^T + dc^T\right) + \left(c^T x + c_0\right)^2 \left(\beta^2 + \beta\right) dd^T \Big) \end{aligned} \tag{12}$$

Referring to the properties of function f_1, we get the following results.

Theorem 7 *Given function f_1 in Problem P, the following statements hold:*
i) f_1 is pseudoconcave on X;
ii) f_1 has no critical points;
iii) f_1 admits a maximum point which is either a vertex or a point belonging to an edge;
iv) x^0 is a maximum point of f_1 on X if and only if for every feasible direction u it is $\nabla f_1(x)^T u \leq 0$;
v) x^0 is a maximum point of f_1 on X if and only it is a maximum point on X of $f_1^*(x) = \dfrac{\left(c^T x + c_0\right)}{(d^T x + d_0)^{\frac{\beta}{\alpha}}}$.

Proof i) Function f_1 can be seen as a special case of function f studied in [4]. Since $\alpha < \beta$ and $c^T x + c_0 > 0$ for every $x \in X$, the result follows from Theorem 4 in [4].
ii) Follows from (11) and from rank$[c, d] = 2$.
iii) Since X is compact, the result follows directly from Theorem 1 in [4].
iv) Follows from the convexity of X and the pseudoconcavity of f_1.
v) Follows directly from the monotonicity of the power function $y^{\frac{1}{\alpha}}$ over the positive domain given by $c^T x + c_0 > 0$ and $d^T x + d_0 > 0$ for every $x \in X$.

The following theorem provides a complete characterization for the existence of alternative optimal solutions of f_1 over X.

Theorem 8 *Let $x^0 \in X$ be an optimal solution of $\max\limits_{x \in X} f_1(x)$. The following conditions are equivalent:*

i) $\exists \bar{x} \in X$, $\bar{x} \neq x^0$, which is an optimal solution of $\max\limits_{x \in X} f_1(x)$;
ii) $\exists \bar{x} \in X$, $\bar{x} \neq x^0$, such that $d^T x^0 = d^T \bar{x}$ and $c^T x^0 = c^T \bar{x}$;
iii) there exists a feasible direction $u \in \mathbb{R}^n$ such that $d^T u = c^T u = 0$.

Proof i) $\Rightarrow$ ii) Let $u = \bar{x} - x^0$ and consider the segment connecting x^0 and $\bar{x}$, $s = \left\{x \in \mathbb{R}^n \ : \ x = x^0 + tu, \ t \in [0,1]\right\}$. Being $f_1(x)$ a pseudoconcave function and being X a convex set, the segment s results to be a feasible set of maximum points. Hence, the restriction $f_1(x^0 + tu)$ is constant so that its first order derivative is zero, that is to say that for all $t \in [0,1]$ it results:

$$0 = \nabla f_1(x^0 + tu)^T u = f_1(x^0 + tu) \left[\frac{\alpha c^T u}{c^T (x^0 + tu) + c_0} - \frac{\beta d^T u}{d^T (x^0 + tu) + d_0} \right]$$

Being $f_1(x) > 0 \ \forall x \in X$ it yields for all $t \in [0,1]$ that:

$$\alpha c^T u (d^T x^0 + t d^T u + d_0) = \beta d^T u (c^T x^0 + t c^T u + c_0)$$

which yields:

$$\alpha c^T u (d^T x^0 + d_0) - \beta d^T u (c^T x^0 + c_0) + t c^T u d^T u (\alpha - \beta) = 0 \quad \forall t \in [0,1]$$

Being $0 < \alpha < \beta$ it then follows: $\alpha c^T u (d^T x^0 + d_0) = \beta d^T u (c^T x^0 + c_0)$ and $c^T u d^T u = 0$. Let us now prove that $d^T u = 0$; with this aim assume by contradiction that $d^T u \neq 0$ which yields $c^T u = 0$; being $\beta > 0$ it follows $c^T x^0 + c_0 = 0$

which is a contradiction since $x^0 \in X$ and $c^T x + c_0 > 0 \; \forall x \in X$. By using the same lines it can be proved that $c^T u = 0$ too. The result then follows being $u = \bar{x} - x^0$.

ii) $\Rightarrow$ *iii)* Just consider the direction $u = \bar{x} - x^0$ observing that its feasibility follows from the convexity of X.

iii) $\Rightarrow$ *i)* Consider the segment $s = \{x \in \mathbb{R}^n \;:\; x = x^0 + tu, \; t \in [0, \epsilon]\}$ which results to be a subset of X for a suitable $\epsilon > 0$ due to the feasibility of direction u. Assumption $d^T u = c^T u = 0$ yields $f_1(x) = f_1(x^0)$ for all $x \in s$ so that all the points in s are optimal solutions of $\max_{x \in X} f_1(x)$. The result then follows since $s \neq \{x^0\}$.

Moving from function f_1 to the bicriteria function f, we first characterize the efficient points on its open domain D and then we will focus our attention on the efficient points belonging to the polyhedron X.

Lemma 1 *Consider function f and let $\bar{x} \in D = \{x \in \mathbb{R}^n : d^T x + d_0 > 0\}$. The following conditions are equivalent:*

i) *$\bar{x}$ is an efficient point for f over D;*
ii) *$\exists \eta_c, \eta_d \in \mathbb{R}$, $\eta_c < 0$ and $\eta_d > 0$, such that $a = \eta_c c + \eta_d d$, moreover $\bar{x}$ lies on the hyperplane $\frac{\eta_c}{\alpha}(c^T x + c_0) + \frac{\eta_d}{\beta}(d^T x + d_0) = 0$.*

Proof Let us preliminary observe that from i) of Theorem 7, f is componentwise pseudoconcave. Therefore, $\bar{x}$ is an efficient point for f over D if and only if $\exists (\mu_1, \mu_2) \in \mathbb{R}^2 \setminus \{(0)\}$ such that $\mu_1 \nabla f_1(\bar{x}) + \mu_2 \nabla f_2(\bar{x}) = 0$ (see for example Theorem 4.23 in [2]). From ii) of Theorem 7 and $a \neq 0$, we get $\nabla f_1(\bar{x}) \neq 0$ and $\nabla f_2(\bar{x}) \neq 0$ so that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Consequently, $\nabla f_2(\bar{x}) = -\frac{\mu_2}{\mu_1} \nabla f_1(\bar{x})$.
i) $\Rightarrow$ *ii)* Assumption $\nabla f_2(\bar{x}) = \lambda \nabla f_1(\bar{x})$ implies:

$$a = \lambda f_1(\bar{x}) \left[\frac{\alpha c}{c^T \bar{x} + c_0} - \frac{\beta d}{d^T \bar{x} + d_0} \right] = \left(\frac{\lambda f_1(\bar{x}) \alpha}{c^T \bar{x} + c_0} \right) c + \left(\frac{-\lambda f_1(\bar{x}) \beta}{d^T \bar{x} + d_0} \right) d$$

that is to say that a is a linear combination of c and d, hence $\exists \eta_c, \eta_d \in \mathbb{R}$ such that $a = \eta_c c + \eta_d d$. It then follows:

$$\eta_c = \frac{\lambda f_1(\bar{x}) \alpha}{c^T \bar{x} + c_0} \quad , \quad \eta_d = \frac{-\lambda f_1(\bar{x}) \beta}{d^T \bar{x} + d_0} \tag{13}$$

that is to say $\dfrac{\eta_c}{\alpha}(c^T \bar{x} + c_0) = \lambda f_1(\bar{x})$, $\dfrac{\eta_d}{\beta}(d^T \bar{x} + d_0) = -\lambda f_1(\bar{x})$ which yields $\frac{\eta_c}{\alpha}(c^T \bar{x} + c_0) + \frac{\eta_d}{\beta}(d^T \bar{x} + d_0) = 0$. Finally, notice that from (13) and from the assumptions of problem P it follows $\eta_c < 0$ and $\eta_d > 0$.

ii) $\Rightarrow$ *i)* Since $\bar{x}$ lies on the hyperplane $\frac{\eta_c}{\alpha}(c^T x + c_0) + \frac{\eta_d}{\beta}(d^T x + d_0) = 0$ and being $f_1(x) > 0 \; \forall x \in X$ it results $\dfrac{\eta_c}{\alpha f_1(\bar{x})}(c^T \bar{x} + c_0) + \dfrac{\eta_d}{\beta f_1(\bar{x})}(d^T \bar{x} + d_0) = 0$, so that the following value $\lambda < 0$ can be defined:

$$\lambda = \frac{\eta_c}{\alpha f_1(\bar{x})}(c^T \bar{x} + c_0) = -\frac{\eta_d}{\beta f_1(\bar{x})}(d^T \bar{x} + d_0)$$

It then results $\eta_c = \dfrac{\lambda f_1(\bar{x})\alpha}{c^T\bar{x}+c_0}$ and $\eta_d = \frac{-\lambda f_1(\bar{x})\beta}{d^T\bar{x}+d_0}$, so that from $a = \eta_c c + \eta_d d$ it follows $a = \lambda f_1(\bar{x})\left[\frac{\alpha c}{c^T\bar{x}+c_0} - \frac{\beta d}{d^T\bar{x}+d_0}\right]$ and hence $\nabla f_2(\bar{x}) = \lambda \nabla f_1(\bar{x})$.

According to the previous Lemma, establishing whether f admits interior efficient point is just a matter of verifying if a is a "suitable" linear combination of c and d. Whenever this condition is verified we can immediately identify the whole set of efficient points on D. The following theorem holds.

Theorem 9 *Consider function f of P and let D be its open domain. The following conditions are equivalent:*

i) $\exists \eta_c, \eta_d \in \mathbb{R}$, $\eta_c < 0$ *and* $\eta_d > 0$, *such that* $a = \eta_c c + \eta_d d$;

ii) all the points $x \in D \cap \pi$, $D \cap \pi \neq \emptyset$, *are efficient points for* f *over* D, *where* π *is the hyperplane defined as follows:*

$$\pi = \left\{ x \in \mathbb{R}^n : \frac{\eta_c}{\alpha}(c^T x + c_0) + \frac{\eta_d}{\beta}(d^T x + d_0) = 0 \right\} \tag{14}$$

with $\eta_c, \eta_d \in \mathbb{R}$, $\eta_c < 0$ *and* $\eta_d > 0$, *be such that* $a = \eta_c c + \eta_d d$.

Proof *i)* $\Rightarrow$ *ii)* Let $\eta_c, \eta_d \in \mathbb{R}$, $\eta_c < 0$ and $\eta_d > 0$, be such that $a = \eta_c c + \eta_d d$. Being $0 < \alpha < \beta$ it is possible to define the hyperplane π. Notice that π is nonempty and that it is not parallel to the hyperplane $\{x \in \mathbb{R}^n : d^T x + d_0 = 0\}$, as a consequence $D \cap \pi \neq \emptyset$. The result then follows from Lemma 1 which guarantees the efficiency of all the points $\bar{x} \in D \cap \pi$.

ii) $\Rightarrow$ *i)* Follows from Lemma 1.

Let us now consider function f on the polyhedron X. From Theorem 9, f admits interior efficient points if and only if the intersection between the set X and the hyperplane π defined in (14) is non-empty. Moreover, from iii) of Theorem 7 and the linearity of a, f clearly admits efficient boundary points. To characterize X_E, let us consider the following scalar problem

$$P_\theta : \max_{x \in X} \frac{c^T x + c_0}{(d^T x + d_0)^{\frac{\beta}{\alpha}}}, x \in \{x \in X : a^T x = \theta\} \tag{15}$$

Using the notation introduced in Section 2, let $\bar{R}_\theta$ be the set of maximum points for Problem P_θ.

Recalling that f_1 is pseudoconcave and hence all its local maxima are global ones, the set of efficient points and the set of efficient frontier are connected and compact; furthermore Theorem 3 can be specified as follows.

Theorem 10 *Consider Problem P; let X_E be the set of efficient points and let* $M = \max_{x \in X} \dfrac{c^T x + c_0}{(d^T x + d_0)^{\frac{\beta}{\alpha}}}$.

i) It is $X_E = \bigcup_{\theta \in [\theta_{\min}, \theta_{\max}]} \bar{R}_\theta$ *and*

$$f(X_E) = \left\{ (\xi(\theta), \theta) \in \mathbb{R}^2 \text{ such that } \theta \in [\theta_{\min}, \theta_{\max}] \right\}$$

where $\theta_{\min} = \max a^T x, x \in X \cap \left\{ \frac{c^T x + c_0}{(d^T x + d_0)^{\frac{\beta}{\alpha}}} = M \right\}$, $\theta_{\max} = \max_{x \in X} a^T x$.

4 Sequential method

Taking into account Theorem 10, we are able to propose a simplex-like algorithm for generating the set of efficient points X_E and the set of efficient frontier $f(X_E)$. The suggested procedure starts with the maximization of the linear function f_2 over the feasible region X and then it parametrically visits the sets S_θ as θ varies from $\theta_{\max}$ down to $\theta_{\min}$. By means of a suitable post-optimality simplex-like analysis, the set $\bar{R}_\theta$ and $\xi(\theta)$ are determined. In each iteration, new vertex/edge of the polyhedron X is reached and the optimality conditions for efficiency are maintained by means of suitable behavior of "directional derivatives". Therefore, step by step, $X_E = \bigcup_{\theta\in[\theta_{\min},\theta_{\max}]} \bar{R}_\theta$ and $f(X_E) = \{(\xi(\theta), \theta),\ \theta \in [\theta_{\min}, \theta_{\max}]\}$ are obtained. This approach has been introduced for bicriteria fractional programming in [1] and then extended for a wider class of problems in [11]. For a critical discussion of this approach and a comparison with other algorithms for bicriteria linear fractional programming, the interested reader can see [3] and references therein (see also [8]). Going into detail, let x^* be an optimal vertex of the problem $\max_{x\in X} f_2(x) = \max_{x\in X} a^T x$. It is obvious that x^* is an efficient point; if x^* is also the maximum for the fractional function f_1 on X, we have found the ideal maximum point and no further analysis is needed. Beyond this very uncommon case, the efficient frontier is constructed by separately analyzing the interior of X and its boundary. Taking into account iii) and iv) of Theorem 9, we first establish whether a is a linear combination of c and d. If this is the case, the interior efficient points are those in the intersection between the hyperplane π in (14) and the set X. Regarding the boundary efficient points which do not belong to the hyperplane π, they are found starting from a solution of the following parametric maximization problem

$$P_\theta : \begin{cases} \max f_1^*(x) = \dfrac{\left(c^T x + c_0\right)}{(d^T x + d_0)^{\frac{\beta}{\alpha}}} \\ s.t. \quad x \in X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\} \subset D, \\ \qquad a^T x = a^T x^* - \theta \end{cases}$$

by setting $\theta = 0 = \theta_0$; if x^* is the unique solution of $\max_{x\in X} f_2(x)$, by construction, it is an optimal solution of problem P_{θ_0}. If this is not the case, we solve Problem P_{θ_0} by using the procedure presented in [4]. With respect to the initial problem P, P_θ has the additional constraint $a^T x = a^T x^* - \theta = \theta_{\max} - \theta$, and hence it has an additional slack variable x_{n+1}. Let $x(\theta_0)$ be the optimal solution of P_{θ_0} and let us suppose that $x(\theta_0)$ is a vertex (for the case $x(\theta_0)$ is not a vertex see Remark 1); the iterative part of the procedure begins with a basis associated with $x(\theta_0)$ such that:

1) the slack variable x_{n+1} of the parametric constraint is a non-basic variable;
2) the directional derivatives of f_1 associated with all the non-basic variables, but x_{n+1}, are non-positive;
3) the directional derivatives of f_1 associated with x_{n+1} is non-negative. The

existence of such a basis follows from iv) of Theorem 7 and from the fact that $x(\theta_0)$ is binding to the parametric constraint. Using a very standard notation, let us define B_0 and N_0 as the sets of indices associated with the basic and the non-basic variables respectively; accordingly we can partition the vectors $c = [c_{B_0}, c_{N_0}]^T$, $d = [d_{B_0}, d_{N_0}]^T$, $a = [a_{B_0}, a_{N_0}]^T$ and the matrix $A = [A_{B_0}, A_{N_0}]$. As θ increases, we find the basic solution $x(\theta) = (x_{B_0}(\theta), 0) = (x_{B_0} + \theta u_{B_0}, 0)$; function $f_1^*(x_{B_0}(\theta), 0)$ and $\nabla f_1^*((x_{B_0}(\theta), 0)$ can be expressed as follows

$$f_1^*(x_{B_0}(\theta), 0) = \frac{c_{B_0}^T x_{B_0} + \theta c_{B_0}^T u_{B_0} + c_0}{\left(d_{B_0}^T x_{B_0} + \theta d_{B_0}^T u_{B_0} + d_0\right)^{\frac{\beta}{\alpha}}}$$

Setting $p = \frac{\beta}{\alpha}$, $\bar{c}_0(\theta) = c_{B_0}^T x_{B_0} + \theta c_{B_0}^T u_{B_0} + c_0$, $\bar{d}_0(\theta) = d_{B_0}^T x_{B_0} + \theta d_{B_0}^T u_{B_0} + d_0$, we get

$$\nabla f_1^*((x_{B_0}(\theta), 0) = \frac{f_1^*(x_{B_0}(\theta), 0)}{\bar{d}_0(\theta)} \left(\bar{d}_0(\theta) c^T - p \bar{c}_0(\theta) d^T\right)$$

$(x_{B_0}(\theta), 0)$ is optimal for Problem $P(\theta)$, if and only if the following conditions are satisfied:

i) $(x_{B_0}(\theta), 0)$ is feasible, that is $(x_{B_0} + \theta u_{B_0}, 0) \geq 0$; therefore a feasibility interval $[0, \theta_{\text{feas}}]$ is obtained.

ii) the directional derivatives with respect to every non-basic variable $j \neq n+1$ is non-positive (optimality with respect to f_1), that is

$$\gamma_j(\theta) = \nabla f_1(x_{B_0}(\theta), 0)^T e^j = \bar{d}_0(\theta) \bar{c}_{N_j} - p \bar{c}_0(\theta) \bar{d}_{N_j} \leq 0$$

where $e^j = \left(e_i^j\right)_{i=1}^n$ with $e_i^j = 0$ if $i \neq j$ and $e_i^j = 1$ if $i = j$, $\bar{c}$ and $\bar{d}$ are the updated values of c and d with respect to the basis (B_0, N_0). This condition defines the so called stability interval $[0, \theta_{\text{opt}}]$.

iii) the directional derivative with respect to the last non-basic variable $n+1$ is non-negative, that is

$$\gamma_{n+1}(\theta) = \nabla f_1(x_{B_0}(\theta), 0)^T e^{n+1} = \bar{d}_0(\theta) \bar{c}_{N_j} - p \bar{c}_0(\theta) \bar{d}_{N_j} \geq 0.$$

Actually, as θ increases, a lower value of f_2 has to be compensated by an higher value of f_1. According to this third condition, θ must belong to the following interval $[0, \theta_{f_2}]$

Setting $\theta_k = \theta_0$ and by using the usual notation B_k, N_k as the sets of indices associated with the basic and the non-basic variables, we are able to present the iterative part of the procedure.
- *Iterative part of the procedure*
Taking into account conditions i), ii) and iii), we determine

$$\theta_{k+1} = \min\{\theta_{\text{feas}}, \theta_{\text{opt}}, \theta_{f_2}\}.$$

The following exhaustive cases may occur:

Case a) $\theta_{k+1} = \theta_{\text{feas}}$: correspondingly to θ_{feas} we get a new efficient vertex of the feasible region and therefore, all the points between $x(\theta_k)$ and $x(\theta_{k+1})$ are efficient points for problem P. The efficient frontier contains the values of f along the line segment $[x(\theta_k), x(\theta_{k+1})]$. The procedure goes on by setting $\theta = \theta + \theta_{k+1}$; as for $\theta > 0$ the feasibility is lost, we restore it by applying (if possible) an iteration of the dual simplex algorithm and the new basis solution $x(\theta_{k+1})$ is determined. New values for θ_{feas}, θ_{opt} and θ_{f_2} are computed. Then, we set $k = k + 1$ and we go back at the beginning of the iterative part. If no dual iteration is available, the procedure stops $\theta_{k+1} = \theta_{\min}$ and $x(\theta_{k+1})$ is the last efficient point for P.

Case b) $\theta_{k+1} = \theta_{\text{opt}}$: in this case all the points between the vertex $x(\theta_k)$ and $x(\theta_{k+1})$ are efficient and $x(\theta_{k+1})$ lies on an edge of X. We update the value $\theta = \theta + \theta_{k+1}$ and we compute the direction derivatives $\gamma_j(\theta)$ for every non-basic variable j. Corresponding to the new $x(\theta_k)$, we compute again $\hat{\theta} = \min[\theta_{\text{feas}}, \theta_{\text{opt}}, \theta_{\text{f2}}]$. Therefore, one of the following three exhaustive subcases may occur:

- b.1) $\hat{\theta} = \theta_{\text{opt}}$: it means that $\gamma_j^*(0) = 0$ for at least one $j^* \in N_k$ and there exists a path of efficient points starting from $x(\theta_{k+1})$, along the relative interior of a face (in $\mathbb{R}^2$, in the interior of X). The value of non-basic variable j^* can be increased as follows $x_{N_j^*}(\theta) = \frac{\gamma_{j^*(\theta)}}{(p-1)\bar{c}_{N_j}\bar{d}_{N_j}}$. Consequently the values of the basic variables are updated, that is $x_{B_k}(\theta) = x_{B_k}(\theta) - A_{B_k}^{-1} A_{N_k} x_{N_j^*}$. Recalling that $x_{B_k}(\theta) \geq 0$, θ is increased up to the greatest value $\tilde{\theta}$ which guarantees the non-negativity of the basic variables. From a geometrical point of view, we move from $x(\theta_{k+1})$ along a feasible direction belonging to the hyperplane π (or to the relative interior of the boundary) and we get to the edge where $x_{B_k}(\tilde{\theta})$ lies. Moreover, all the points of the segment $[x(\theta_{k+1}), x(\tilde{\theta})]$ are efficient. At this stage, we update the value $\theta = \theta + \tilde{\theta}$. As for $\theta > 0$ the feasibility is lost, we restore it by applying (if possible) an iteration of the dual simplex algorithm and the new basic solution $x_{B_{k+1}}(\theta)$ is determined. We set $k = k + 1$ and we go back at the beginning of the iterative part. If no dual iteration is available, the procedure stops $\tilde{\theta} = \theta_{\min}$ and $x(\tilde{\theta})$ is the last efficient point for P.
- b.2) $\hat{\theta} = \theta_{\text{feas}}$ all the points between $x(\theta_{k+1})$ and $x(\hat{\theta})$ are efficient points for problem P. The efficient frontier contains the values of f along the line segment $[x(\theta_{k+1}), x(\hat{\theta})]$. Then, the procedure follows the same steps of Case a).
- b.3) $\hat{\theta} = \theta_{\text{f}_2}$. In this case the algorithm stops since a lower level of θ would correspond to a lower value for both f_1 and f_2. The value $\theta_{\min}$ is reached.

Case c) $\theta_{k+1} = \theta_{f2}$: as in case b3), the algorithm stops.

- *End of the iterative part of the procedure*

Remark 1 If the problem $\max_{x \in X} f_2$ has alternative solutions, it may happen that the optimal solution $x(\theta_0)$ of initial maximization problem P_{θ_0} is not a vertex.

In this case, if $x(\theta_0)$ is also the maximum point of f_1 on X, then we get the ideal maximum point of problem P and no further analysis is needed. If on the contrary $x(\theta_0)$ is not the ideal maximum point, to start the suggested procedure, we need to recover an efficient vertex. With this regards, let us observe that the set of efficient points is connected and hence the directional derivative, evaluated at $x(\theta_0)$, with respect to at least one non-basic variable is zero. Therefore, either $x(\theta_0)$ belongs to the hyperplane of efficient points π or $x(\theta_0)$ is an element of a path of efficient points which is contained in the relative interior of the boundary of S. To recover an optimal basic solution to start with, we observe that $x(\theta_0)$ is an optimal vertex of the following auxiliary problem

$$\hat{P}_\theta : \begin{cases} \max f_1^*(x) = \dfrac{\left(c^T x + c_0\right)}{(d^T x + d_0)^{\frac{\beta}{\alpha}}} \\ s.t. \quad x \in X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\} \subset D, \\ \quad a^T x = a^T x^* - \theta \\ \quad d^T x = d^T x(\theta_0) \end{cases}$$

Therefore, we have a basic solution to start, and the algorithm can continue; of course, in each step, the feasibility interval of θ is determined without taking into account the last constraint.

Remark 2 The correctness of the algorithm is given by the theoretical results of the previous sections where it is proved how to determine and recognize the efficient points. The convergence of this simplex-like algorithm follows noticing that at the end of each iteration either a new vertex is reached or an edge is left, taking into account that the region X is parametrically visited from $\theta_{\max}$ down to $\theta_{\min}$, so that an edge or a vertex cannot be visited twice, and finally recalling that a compact polyhedron has a finite number of vertices and edges. Clearly, since this is a simplex-like algorithm, it has the same complexity of simplex method, that is to say that in the worst case all of the vertices have to be visited.

5 Numerical examples and computational test

To clarify how the suggested procedure works, in this section three examples are provided. In the first one, both interior and boundary efficient points are determined, while in the second one, the "efficiency path" completely lies on the boundary. The last one describes a particular case of the procedure, namely Case b.2) which has interesting geometric meaning.

Example 1 Consider the problem P

$$P : \begin{cases} \max f(x) = \left(f_1(x) = \dfrac{(x_1 + x_2 + 3x_3 + 4)}{(x_1 + 2x_2 + x_3 + 1)^2}, f_2(x) = 3x_1 + 7x_2 + x_3 \right) \\ x \in X = \{x \in \mathbb{R}^3 : 0 \leq x_i \leq 4, i = 1, 2, 3\} \end{cases} \tag{16}$$

The maximum value of f_2 on X is 44 and it is attained at the vertex $(4,4,4)$, so that $\theta_{\max} = 44$. As the maximum point of f_1 is $(0,0,0)$, $(4,4,4)$ is not an ideal point and we start with the procedure. Let us preliminary observe that $a = -c + 4d$ and $\pi = \{x \in \mathbb{R}^3 : 2x_1 + 6x_2 - 2x_3 - 4 = 0\} \cap X \neq \emptyset$, so that there exist interior efficient points. In Figure 1 the grey area is the feasible region and the yellow one is the intersection between the plane π and X. Let us analyze the boundary of X; consider the parametric problem

$$P(\theta) : \begin{cases} \max \dfrac{x_1 + x_2 + 3x_3 + 4}{(x_1 + 2x_2 + x_3 + 1)^2} \\ x \in X \cap 3x_1 + 7x_2 + x_3 = 44 - \theta \end{cases}$$

whose tabular representation is the following

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
c	$-24+\theta/7$	0	0	0	$-4/7$	0	$-20/7$	$-1/7$
d	$-17+2\theta/7$	0	0	0	$-1/7$	0	$-5/7$	$-2/7$
	4	1	0	0	1	0	0	0
$x \in S$	$4-\theta/7$	0	1	0	$-3/7$	0	$-1/7$	$1/7$
	4	0	0	1	0	0	1	0
$f_2(x) = \theta$	$\theta/7$	0	0	0	$3/7$	1	$1/7$	$-1/7$

where x_4, x_5 and x_6 are the slack variables associated with the constraints defining the feasible region and x_7 is the slack variable associated with the parametric constraint.
We set $\theta_0 = 0$; looking at the feasibility we get $\theta \in [0,28]$ so that $\theta_{\text{feas}} = 28$. The directional derivatives associated with the non-basic variables are $\gamma_4(\theta) = -\frac{20}{7} + \frac{6}{49}\theta$, $\gamma_6(\theta) = -\frac{100}{7} + \frac{30}{49}\theta$ and $\gamma_7(\theta) = \frac{79}{7} - \frac{2}{49}\theta$. Therefore $\theta_{\text{opt}} = 70/3$ and $\theta_{f_2} = 553/2$, so that $\theta_1 = \min\{\theta_{\text{feas}}, \theta_{\text{opt}}, \theta_{f_2}\} = 70/3$. (We are in case b)). By substituting $\theta = 70/3$ we obtain $(4, 2/3, 4)$; all the points belonging to the segment $[(4,4,4)\,(4,2/3,4)]$ are efficient and the value of f is $\left(f_1\left(4, 4 - \frac{1}{7}\theta, 4\right), f_2\left(4, 4 - \frac{1}{7}\theta, 4\right)\right)$ with $\theta \in [0, 70/3]$. In Figure 1 the segment is colored in blue and in Figure 2 the corresponding values of (f_1, f_2) are represented in blue too. By updating θ as $\theta = \theta + \frac{70}{3}$ we get the following tabular form

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
c	$-62/3+1/7\theta$	0	0	0	$-4/7$	0	-20/7	$-1/7$
d	$-31/3+2/7\,\theta$	0	0	0	$-1/7$	0	$-5/7$	$-2/7$
	4	1	0	0	1	0	0	0
$x \in S$	$2/3-1/7\theta$	0	1	0	-3/7	0	$-1/7$	$1/7$
	4	0	0	1	0	0	1	0
$f_2(x) = \theta$	$1/7\theta+10/3$	0	0	0	$3/7$	1	$1/7$	$-1/7$

The new directional derivatives are $\gamma_4(\theta) = \frac{6}{49}\theta$, $\gamma_6(\theta) = \frac{30}{49}\theta$ and $\gamma_7(\theta) = \frac{31}{3} - \frac{2}{49}\theta$. As both γ_4 and γ_6 are zero for $\theta = 0$, we can either increase the value of x_4 or the value of x_6. By choosing $x_4 = \frac{3}{2}\theta$ we move along the feasible direction $\left[4 - \frac{3}{2}\theta, \frac{2}{3} + \frac{\theta}{2}, 4, -\frac{\theta}{2} + \frac{10}{3}\right]^T$; the feasibility condition imposes $\theta \in [0, 8/3]$ and hence we set $\theta_2 = 8/3$. For $\theta = 8/3$ we get the efficient point $(0,2,4)$; the points belonging to the segment $[(4, 2/3, 4), (0, 2, 4)]$ are in the relative interior of the boundary of X and they are efficient. In Figure 1 the segment is colored in green and in Figure 2 the corresponding values of (f_1, f_2) are represented in green too. We update the value of θ, $\theta = \theta + \frac{8}{3}$; as $\theta > 0$ the feasibility of x_1 is lost and hence we perform an iteration of the dual simplex algorithm to restore it. The new tabular form associated with $(0, 2, 4)$ is the following:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
c	$-18+\theta/7$	4/7	0	0	0	0	−20/7	−1/7
d	$-9+2/7\,\theta$	1/7	0	0	0	0	−5/7	−2/7
	4	1	0	0	1	0	0	0
$x \in S$	$2-1/7\theta$	3/7	1	0	0	0	−1/7	1/7
	4	0	0	1	0	0	1	0
$f_2(x)=\theta$	$\theta/7+2$	−3/7	0	0	0	1	1/7	−1/7

The new directional derivatives are $\gamma_1(\theta) = -\frac{6}{49}\theta$; $\gamma_6(\theta) = \frac{30}{49}\theta$ and $\gamma_7(\theta) = 9-\frac{2}{49}\theta$. The non-basic variable x_6 is increased as follows $x_6 = \frac{3}{10}\theta$ and we move along the direction $\left[4, 2-\frac{1}{10}\theta, 4-\frac{3}{10}\theta, 10\theta+2\right]^T$. The feasibility condition requires $\theta \in [0, 40/3]$, so that we move up $\theta = 40/3$ which corresponds to $(0, 2/3, 0)$. In Figure 1 the segment of efficient points $[(0, 2, 4), (0, 2/3, 0)]$ is the red one as well as their corresponding values in Figure 2. We set $\theta_3 = 40/3$, $\theta = \theta + 40/3$ and again we perform an iteration of the dual algorithm to restore the feasibility; the tabular form associated with $(0, 2/3, 0)$ is the following:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
c	$-14/3+\theta/7$	4/7	0	20/7	0	0	0	−1/7
d	$-7/3+2/7\,\theta$	1/7	0	5/7	0	0	0	−2/7
	4	1	0	0	1	0	0	0
$x \in S$	$2/3-1/7\theta$	3/7	1	1/7	0	0	0	1/7
	4	0	0	1	0	0	1	0
$f_2(x)=\theta$	$1/7\theta+10/3$	−3/7	0	−1/7	0	1	0	−1/7

Regarding feasibility we get $\theta \in [0, 14/3]$, so that $\theta_{feas} = 14/3$. The directional derivatives are $\gamma_1(\theta) = -\frac{6}{49}\theta$, $\gamma_3(\theta) = -\frac{30}{49}\theta$ and $\gamma_7(\theta) = \frac{7}{3} - \frac{2}{49}\theta$, so that $\theta_{\text{opt}} = +\infty$ and $\theta_{f2} = 3436$. Consequently $\theta_3 = 14/3$; the value θ_4 corresponds to the vertex $(0, 0, 0)$ and the points belonging to the segment $[(0, 2/3, 0), (0, 0, 0)]$ are efficient (it is brown colored in Figures 1 and 2). The tabular form associated with $(0, 0, 0)$ is the following

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
c	$-4+\theta/7$	4/7	0	20/7	0	0	0	−1/7
d	$-1+2/7\theta$	1/7	0	5/7	0	0	0	−2/7
	4	1	0	0	1	0	0	0
$x \in S$	$-1/7\theta$	3/7	1	1/7	0	0	0	1/7
	4	0	0	1	0	0	1	0
$f_2(x)=\theta$	$4+\theta/7$	−3/7	0	−1/7	0	1	0	−1/7

No dual iteration is possible, $\theta_{\min}$ has been reached and the procedure stops.

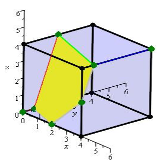

Fig. 1 Decision space

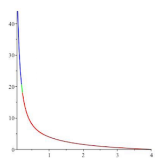

Fig. 2 Image space

Example 2 Consider the problem P

$$P:\begin{cases}\max f(x)=\left(f_1(x)=\dfrac{\left(x_1+2x_2+\frac{2}{3}x_3+3\right)^3}{(x_1+3x_2+x_3+12)^5}, f_2(x)=-2x_1+x_2+x_3\right)\\ x\in X=\{x\in\mathbb{R}^3:0\le x_i\le 5, i=1,2,3\}\end{cases}\tag{17}$$

The maximum value of the linear function is 16 and it is attained at $(0,5,5)$ which is not an ideal maximum point. It is $\text{rank}[a,c,d]=3$; from Lemma 1 and Theorem 9 there exist no interior efficient point. The vertices of the generated efficient path are the following: $(0,5,5)$, $(0,4/3,5)$, $(5/3,0,5)$, $(3,0,5)$, $(39/8,0,0)$, and $(5,0,0)$. Figures 3 and 4 represent the solution of the problem in the decision and in the image space respectively.

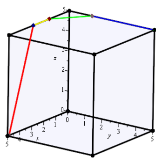

Fig. 3 Decision space

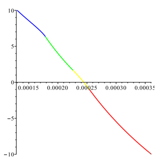

Fig. 4 Image space

Example 3 Consider the problem P

$$P:\begin{cases}\max f(x)=\left(f_1(x)=\dfrac{(9x_1-10x_2-10x_3+31)^2}{(9x_1+8x_2+8x_3+1)^5}, f_2(x)=x_1+5x_2+5x_3\right)\\ x\in X=\{x\in\mathbb{R}^3:0\le x_1\le 9, 0\le x_2\le 1, 0\le x_3\le 2\}\end{cases}\tag{18}$$

The maximum value of the linear function is 24 and it is attained at $(9,1,2)$. Since $\text{rank}[a,c,d]=3$, there exist interior efficient points belonging to the plane $-\frac{5\,x}{12}+\frac{91\,y}{54}+\frac{91\,z}{54}=\frac{125}{36}$. The procedure determines the following path of efficient points: $(9,1,2)$, $(19/5,1,2)$, $(0,11/182,2)$, $(0,0,2)$, $(0,0,0)$. Moreover, with respect to the efficient point $(0,11/182,2)$, the directional derivative of a non-basic variable is 0 and the corresponding value $\bar{c}_{N_j}\bar{d}_{N_j}$ is zero, that is Case b.2) occurs. From a geometrical point of view, this implies that all the points belonging to the face $x=0$ and lying under the line $(0,11/182+t,2-t), t\in[0,1]$ are efficient (see Figure 5). The efficient frontier in the image space is represented in Figure 6.

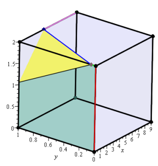

Fig. 5 Decision space

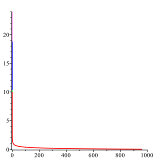

Fig. 6 Image space

The proposed algorithm has been fully implemented with the software MATLAB 9.2 R2016b on a macOS computer having 32 Gb RAM and an Intel Core i7 quad core processor at 4 GHz. Within the procedures, the linear problems have been solved by using the Gurobi 7.0.2 engine. Various instances have been randomly generated and solved, with a grand total of 1060000 problems solved. Referring to the parameter specifications, 8 different pairs of α and β have been conceived. Matrix $A \in \mathbb{R}^{m\times n}$ and vectors $a, c, d \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ have been randomly generated with components in the interval [-10,10] by using the "randi()" MATLAB function (integer numbers generated with uniform distribution). In compliance with the assumptions of Problem P, the values c_0, d_0 have been chosen in order to guarantee the non-negativity of $c^T x + c_0$ and the positivity of $d^T x + d_0$, over the feasible region. The results of the computational test are described in the following two tables. Table 1 provides the average number of vertices of the generated efficient paths and Table 2 collects the average spent times (given by the "tic" and "toc" MATLAB commands). In this light, notice that "$m \times n$" represents the dimension of matrix A in the considered problems, "num" is the number of randomly generated problems solved for the corresponding dimension $m \times n$. Moreover, the pairs of α and β provide the parameters chosen for the objectives.

Table 1 Computational results - Average number of vertices of the generated efficient paths

$m \times n$	num	$\alpha = 1$ $\beta = 2$	$\alpha = 1$ $\beta = 3$	$\alpha = 3$ $\beta = 5$	$\alpha = 5$ $\beta = 6$	$\alpha = 1/3$ $\beta = 1/2$	$\alpha = 1/3$ $\beta = 3/4$	$\alpha = 1/4$ $\beta = 3/2$	$\alpha = 1/4$ $\beta = 1/3$
30×40	50000	8.11	7.65	8.35	8.94	8.53	7.95	7.13	8.73
60×80	30000	12.23	11.40	12.66	13.62	12.93	11.97	10.56	13.31
90×120	20000	15.87	14.86	16.51	17.84	16.91	15.61	13.70	17.45
120×150	12000	19.18	17.91	19.84	21.53	20.30	18.79	16.61	20.74
150×200	8000	21.68	20.48	22.42	24.27	23.05	21.40	18.82	23.28
180×240	5000	23.84	22.33	24.65	26.42	25.18	23.44	20.42	25.89
210×280	3000	25.68	24.20	26.51	28.47	26.97	25.11	22.65	27.21
240×320	2000	27.21	25.55	27.29	29.45	28.32	26.23	23.95	28.91
270×360	1500	28.50	27.49	29.08	31.04	29.85	27.82	25.58	29.98
300×400	1000	29.93	28.16	31.56	32.07	30.75	29.05	27.65	31.69

Table 2 Computational Results - Average elapsed times (secs)

$m \times n$	num	$\alpha = 1$ $\beta = 2$	$\alpha = 1$ $\beta = 3$	$\alpha = 3$ $\beta = 5$	$\alpha = 5$ $\beta = 6$	$\alpha = 1/3$ $\beta = 1/2$	$\alpha = 1/3$ $\beta = 3/4$	$\alpha = 1/4$ $\beta = 3/2$	$\alpha = 1/4$ $\beta = 1/3$
30×40	50000	0.010	0.010	0.010	0.011	0.010	0.010	0.010	0.010
60×80	30000	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033
90×120	20000	0.104	0.105	0.103	0.102	0.103	0.105	0.107	0.103
120×150	12000	0.217	0.220	0.215	0.213	0.215	0.219	0.227	0.212
150×200	8000	0.440	0.451	0.435	0.429	0.435	0.446	0.459	0.426
180×240	5000	0.796	0.816	0.789	0.769	0.782	0.807	0.826	0.782
210×280	3000	1.072	1.099	1.062	1.035	1.045	1.076	1.140	1.02
240×320	2000	1.607	1.632	1.539	1.524	1.556	1.586	1.693	1.532
270×360	1500	2.786	2.933	2.734	2.673	2.733	2.786	2.982	2.689
300×400	1000	3.925	3.993	4.021	3.709	3.799	3.904	4.355	3.775

The obtained results point out on the behavior of the solution algorithm with respect to the problem dimension. In particular, it is worth noticing that:

- the algorithm manages real parameters α and β which can be non-integers;
- it is possible to solve quite large problems in a reasonable time;
- the average number of the vertices of the generated efficient path increases with the dimension of the problem;
- the time needed to solve the instances increases exponentially with the dimension of the problem.

6 Conclusions

In this paper an algorithm to generate the efficient frontier of a class of bicriteria problems is proposed. With this aim, some new theoretical results are stated in order to recognize the efficient points. Numerical examples and a computational test are also provided in order to clarify the use and the performance of the algorithm.

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