

SECOND ORDER OPTIMALITY CONDITIONS FOR DIFFERENTIABLE MULTIOBJECTIVE PROBLEMS (*)

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Communicated by Franco GIANNESI

Abstract. – A second order optimality condition for multiobjective optimization with a set constraint is developed; this condition is expressed as the impossibility of nonhomogeneous linear systems. When the constraint is given in terms of inequalities and equalities, it can be turned into a John type multipliers rule, using a nonhomogeneous Motzkin Theorem of the Alternative. Adding weak second order regularity assumptions, Karush, Kuhn–Tucker type conditions are therefore deduced.

Keywords: Second order necessary optimality conditions, descent directions, second order contingent set, Abadie and Guignard type conditions.

1. INTRODUCTION

The study of optimality conditions is one of the main topics of Optimization Theory. For multiobjective programming, some of the first interesting results have been developed in the middle seventies [12]; since then, many papers appeared, dealing with first order necessary optimality conditions both for differentiable problems [12–14, 16, 20–22, 25, 28] and nondifferentiable ones [6, 11, 17]. When the problem satisfies suitable convexity assumptions, these conditions turn out to be also sufficient (see [14, 16, 19, 21] and the references therein). However, in the general case there may be feasible points, which satisfy the first order conditions but are not optimal solutions. In order to drop them, additional optimality conditions, involving second order derivatives of the given functions, can be developed. A few results in this direction have been presented in some recent papers [2, 23]. This paper aims to deepen this type of analysis, providing more general results.

(*) Received December 18, 1999.

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First, we investigate differentiable multiobjective problems, where the constraint is given in set form. By linearizing techniques, we obtain necessary conditions in terms of the impossibility of nonhomogeneous linear systems, involving the Jacobians and the Hessians of the objective functions and the second order contingent set [3, 26] of the feasible region. We stress that these systems depend upon the choice of a common descent direction for the objective functions. Moreover, we show that the gap between first order conditions for single and multi-objective problems exploited in [25] holds also for second order conditions.

Then, we apply our results to the case where the feasible region is expressed by both inequality and equality constraints. This can be done, exploiting the connections between the second order contingent set of the feasible region and the second order derivatives of the constraining functions.

By means of theorems of the alternative, we are therefore able to deduce a John type multipliers rule, involving both the Jacobians and the Hessians of the objective and constraining functions. Since the multipliers are not fixed but they depend upon the chosen descent direction, this rule extends to the multiobjective case the results of [4].

In the last section, we analyse some conditions, which guarantee the existence of nonzero multipliers corresponding to the objective functions; following the approach developed in [10] for scalar problems, we consider a constraint qualification, which is weaker than those used in [2, 23], and we show that the Guignard type constraint qualification is useless without convexity assumptions; on the contrary, we introduce a Guignard type condition, which involves also the objective functions and needs no convexity assumptions to achieve the goal.

2. PRELIMINARIES

In this section, we introduce some notations and definitions which are used throughout the paper. Let $\mathbb{R}^\ell$ be the ℓ -dimensional Euclidean space; $\mathbb{R}_+^\ell := \{x \in \mathbb{R}^\ell : x_i \geq 0, i = 1, \dots, \ell\}$ is the positive orthant.

For each set $A \subseteq \mathbb{R}^\ell$, $\text{int } A$, $\text{cl } A$, and $\text{conv } A$ denote the topological interior, the topological closure and the convex hull of A , respectively. Given any two vectors $x, y \in \mathbb{R}^\ell$ we use the following notation

$$x \leq_{\text{int } \mathbb{R}_+^\ell} y \iff y - x \in \text{int } \mathbb{R}_+^\ell.$$

In order to obtain multipliers rules, we need the following nonhomogeneous form of the Motzkin Theorem of the Alternative [15].

LEMMA 2.1: (*Nonhomogeneous Motzkin Theorem*). Let $a_j, b_i, c_i \in \mathbb{R}^n$ and $\alpha_j, \beta_i, \gamma_i \in \mathbb{R}$ with $j \in J \neq \emptyset$, $i \in I^-$ and $i \in I^o$ finite index sets. Then, the system (in the unknown $x \in \mathbb{R}^n$)

$$\begin{cases} a_j x + \alpha_j < 0, & j \in J \\ b_i x + \beta_i \leq 0, & i \in I^- \\ c_i x + \gamma_i = 0, & i \in I^o \end{cases} \quad (1)$$

is impossible if and only if the system (in the unknowns $\theta_j, \lambda_i, \mu_i$)

$$\begin{cases} \sum_{j \in J} \theta_j a_j + \sum_{i \in I^-} \lambda_i b_i + \sum_{i \in I^o} \mu_i c_i = 0 \\ \sum_{j \in J} \theta_j \alpha_j + \sum_{i \in I^-} \lambda_i \beta_i + \sum_{i \in I^o} \mu_i \gamma_i = \theta_0 \\ \theta_0 + \sum_{j \in J} \theta_j > 0 \\ \theta_j \geq 0, j \in J \cup \{0\}, \lambda_i \geq 0, i \in I^- \end{cases} \quad (2)$$

is possible. Moreover, if the rows of (1) are linearly independent, then all the solutions of (2) have $\theta_0 > 0$.

Proof: Since the impossibility of (1) is equivalent to that of the homogeneous linear system

$$\begin{cases} a_j x + \alpha_j z < 0, & j \in J \\ 0x - z < 0, \\ b_i x + \beta_i z \leq 0, & i \in I^- \\ c_i x + \gamma_i z = 0, & i \in I^o, \end{cases}$$

we achieve the result by applying Motzkin theorem of the Alternative (see [15]). $\square$

Remark 2.1: From Lemma 2.1 we deduce that the impossibility of (1) implies the possibility of the system

$$\left\{ \begin{array}{l} \sum_{j \in J} \theta_j a_j + \sum_{i \in I^-} \lambda_i b_i + \sum_{i \in I^o} \mu_i c_i = 0 \\ \sum_{j \in J} \theta_j \alpha_j + \sum_{i \in I^-} \lambda_i \beta_i + \sum_{i \in I^o} \mu_i \gamma_i \geq 0 \\ \sum_{j \in J} \theta_j + \sum_{i \in I^-} \lambda_i + \sum_{i \in I^o} \mu_i^2 > 0 \\ \theta_j \geq 0, j \in J, \lambda_i \geq 0, i \in I^-. \end{array} \right. \quad (3)$$

Let $\varphi = (\varphi_1, \dots, \varphi_\ell) : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$ be a twice differentiable function, $J := \{1, \dots, \ell\}$ be the index set³ which identifies the components of φ , and let $\bar{x} \in \mathbb{R}^n$ be given. We set:

$$J(\bar{x}) := \{j \in J : \varphi_j(\bar{x}) = 0\}.$$

By means of these notations, we introduce the following definitions:

- the *set of the descent directions* for φ at $\bar{x}$ is

$$D_{\leq}(\varphi, \bar{x}) := \{d \in \mathbb{R}^n : \nabla \varphi_j(\bar{x})d \leq 0, \forall j \in J(\bar{x})\};$$

- the *set of the attainable directions* for φ at $\bar{x}$ is

$$D_{=}(\varphi, \bar{x}) := \{d \in \mathbb{R}^n : \nabla \varphi_j(\bar{x})d = 0, \forall j \in J(\bar{x})\}.$$

For each given direction $d \in \mathbb{R}^n$, we set

$$J(\bar{x}, d) := \{j \in J(\bar{x}) : \nabla \varphi_j(\bar{x})d = 0\}.$$

Therefore

- the *set of the second order strict descent directions* for φ at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ is

$$D_{<}^2(\varphi, \bar{x}, d) := \{w \in \mathbb{R}^n : \nabla \varphi_j(\bar{x})w + \nabla^2 \varphi_j(\bar{x})(d, d) < 0, \forall j \in J(\bar{x}, d)\};$$

³ For the sake of simplicity, we use the same notation as in Lemma 2.1.

- the set of the second order descent directions for φ at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ is

$$D_{\leq}^2(\varphi, \bar{x}, d) := \{w \in \mathbb{R}^n : \nabla \varphi_j(\bar{x})w + \nabla^2 \varphi_j(\bar{x})(d, d) \leq 0, \forall j \in J(\bar{x}, d)\};$$

- the set of the second order attainable directions for φ at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ is

$$D_{=}^2(\varphi, \bar{x}, d) := \{w \in \mathbb{R}^n : \nabla \varphi_j(\bar{x})w + \nabla^2 \varphi_j(\bar{x})(d, d) = 0, \forall j \in J(\bar{x}, d)\}.$$

Let $f = (f_1, \dots, f_\ell) : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$ be a twice differentiable function, $J := \{1, \dots, \ell\}$ and $X \subseteq \mathbb{R}^n$. Consider the following multiobjective problem

$$\min_{\text{int } \mathbb{R}_+^\ell} f(x) \quad \text{subject to} \quad x \in X \quad (4)$$

where $\min_{\text{int } \mathbb{R}_+^\ell}$ marks vector minimum with respect to the cone $\text{int } \mathbb{R}_+^\ell$: $\bar{x} \in X$ is said to be a *local vector minimum point* of (4) iff there exists a neighbourhood N of $\bar{x}$, such that no $x \in X \cap N$ satisfies $f(x) \leq_{\text{int } \mathbb{R}_+^\ell} f(\bar{x})$. It is a widespread tradition to call these minima *weak vector Pareto* ones; the term “weak”, which is here mathematically misleading, comes from the following situation: the solutions of (4) are also solutions of the problem where vector minima are considered with respect to the larger cone $\mathbb{R}_+^\ell \setminus \{0\}$; this last problem is actually a different one from (4), since it has a different “ordering” cone. For a more detailed discussion, see [7].

For the sake of simplicity, we will write $D_{\leq}(f, \bar{x})$ instead of $D_{\leq}(f - f(\bar{x}), \bar{x})$ and analogously for the other sets.

DEFINITION 2.1: Let $X \subseteq \mathbb{R}^n$. The second order contingent set of X at $\bar{x} \in \text{cl } X$ in the direction $d \in \mathbb{R}^n$ is:

$$T^2(X, \bar{x}, d) := \{w \in \mathbb{R}^n : \exists \{w_n\} \rightarrow w, \\ \exists \{t_n\} \downarrow 0 \text{ s.t. } \bar{x} + t_n d + 2^{-1} t_n^2 w_n \in X\}.$$

The second order contingent set represents an extension of the Bouligand tangent cone $T(X, \bar{x})$ and it preserves many properties of such a cone: for instance, it is closed and it is isotone, i.e. if $X_1 \subseteq X_2$ and $\bar{x} \in \text{cl } X_1$ then

$$T^2(X_1, \bar{x}, d) \subseteq T^2(X_2, \bar{x}, d).$$

Moreover, we observe that

$$T^2(X, \bar{x}, 0) = T(X, \bar{x}).$$

It is easy to show that

$$d \notin T(X, \bar{x}) \implies T^2(X, \bar{x}, d) = \emptyset. \quad (5)$$

For more properties of this type of approximation, see [3] and references therein.

3. NECESSARY OPTIMALITY CONDITIONS WITH A SET CONSTRAINT

By linearizing techniques with second order accuracy, we achieve the following result.

THEOREM 3.1: *If $\bar{x} \in X$ is a local vector minimum point of (4), then for each descent direction $d \in D_{\leq}(f, \bar{x})$ the system*

$$\nabla f_j(\bar{x})w + \nabla^2 f_j(\bar{x})(d, d) < 0, \quad j \in J(\bar{x}, d) \quad (6)$$

has no solution $w \in T^2(X, \bar{x}, d)$.

Proof: *Ab absurdo*, let $d \in D_{\leq}(f, \bar{x})$ be given and $w \in T^2(X, \bar{x}, d)$ be a solution to (6). By the definition of second order contingent set, there exist $\{t_n\} \downarrow 0$ and $\{w_n\} \rightarrow w$ such that

$$x_n := \bar{x} + t_n d + 2^{-1} t_n^2 w_n \in X, \quad \forall n \in \mathbb{N}.$$

Since f_j is twice differentiable, we have

$$\begin{aligned} f_j(x_n) - f_j(\bar{x}) &= t_n [\nabla f_j(\bar{x})(d + 2^{-1} t_n w_n) \\ &\quad + 2^{-1} t_n \nabla^2 f_j(\bar{x})(d + 2^{-1} t_n w_n, d + 2^{-1} t_n w_n) + t_n \varepsilon_n] \end{aligned}$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Let us consider the following two cases.

- If $j \notin J(\bar{x}, d)$, we have

$$\nabla f_j(\bar{x})(d + 2^{-1} t_n w_n) < 0,$$

and therefore for all n large enough we have

$$f_j(x_n) < f_j(\bar{x}).$$

- If $j \in J(\bar{x}, d)$, since

$$\begin{aligned} \lim_{n \rightarrow \infty} \nabla f_j(\bar{x})w_n + \nabla^2 f_j(\bar{x})(d + 2^{-1} t_n w_n, d + 2^{-1} t_n w_n) + \varepsilon_n \\ = \nabla f_j(\bar{x})w + \nabla^2 f_j(\bar{x})(d, d) < 0, \end{aligned}$$

for all n large enough we have

$$f_j(x_n) - f_j(\bar{x}) = 2^{-1}t_n^2[\nabla f_j(\bar{x})w_n + \nabla^2 f_j(\bar{x})(d + 2^{-1}t_n w_n, d + 2^{-1}t_n w_n) + \varepsilon_n] < 0.$$

Therefore, $\exists \bar{n} \in \mathbb{N}$, such that

$$f(x_n) \leq_{\text{int } \mathbb{R}_+^{\ell}} f(\bar{x}), \quad \forall n \geq \bar{n},$$

which contradicts the assumption. $\square$

We note that, when $\ell = 1$, Theorem 3.1 collapses to the second order necessary optimality condition given in [18] (see also [26]). In particular, choosing $d = 0$, we deduce the following result.

COROLLARY 3.1: *If $\bar{x} \in X$ is a local vector minimum point of (4), then the system*

$$\nabla f_j(\bar{x})w < 0, \quad j \in J, \quad (7)$$

has no solution $w \in T(X, \bar{x})$.

As observed in [25], the inconsistency of (7) does not hold when we replace the set $T(X, \bar{x})$ with the set $\text{cl conv } T(X, \bar{x})$. The same happens also with the second order contingent set as the following example shows.

Example 3.1: Consider the multiobjective problem (4) with

$$f(x_1, x_2) := (-x_1^2 - x_2, x_2 - x_1^2),$$

and the feasible region given by

$$X := \{x \in \mathbb{R}^2 : x_1^4 \leq x_2^2 \leq 4x_1^4\}.$$

Easy calculations show that $\bar{x} = (0, 0)$ is a vector minimum point. Moreover, given the direction $d = (1, 0) \in D_{\leq}(f, \bar{x})$, the second order contingent set is

$$T^2(X, \bar{x}, d) = \{w \in \mathbb{R}^2 : 2 \leq |w_2| \leq 4\}.$$

System (6) becomes $|w_2| < 2$. Thus, it has no solution $w \in T^2(X, \bar{x}, d)$ as Theorem 3.1 claims, but it has a solution $w \in \text{cl conv } T^2(X, \bar{x}, d)$ since

$$\text{cl conv } T^2(X, \bar{x}, d) = \{w \in \mathbb{R}^2 : |w_2| \leq 4\}.$$

In order to obtain the impossibility of system (6) with $w \in \text{clconv } T^2(X, \bar{x}, d)$, we can assume the following form of generalized convexity for problem (4).

DEFINITION 3.1: *Given a set $X \subseteq \mathbb{R}^n$, a function $f : X \rightarrow \mathbb{R}^\ell$ is said to be subconvexlike on X iff for any $x_1, x_2 \in X$, any $t \in]0, 1[$ and any $\lambda \geq_{\text{int } \mathbb{R}_+^\ell} 0$, there exists $x_3 \in X$, such that*

$$\lambda + tf(x_1) + (1-t)f(x_2) - f(x_3) \geq_{\text{int } \mathbb{R}_+^\ell} 0.$$

THEOREM 3.2: *Suppose that f is subconvexlike on X . If $\bar{x} \in X$ is a local vector minimum point of (4), then for each descent direction $d \in D_{\leq}(f, \bar{x})$ system (6) has no solution $w \in \text{cl conv } T^2(X, \bar{x}, d)$.*

Proof: *Ab absurdo*, suppose there exist $\bar{d} \in D_{\leq}(f, \bar{x})$ and $\bar{w} \in \text{clconv } T^2(X, \bar{x}, \bar{d})$ such that (6) holds. From (5) it follows that $\bar{d} \in T(X, \bar{x})$. By Theorem 3.1 in [24], there exists a nonzero $\theta \in \mathbb{R}_+^\ell$ such that $\bar{x}$ is a local minimum point of the scalar problem

$$\min \varphi(x) := \sum_{j \in J} \theta_j f_j(x) \quad \text{subject to} \quad x \in X.$$

By Corollary 3.1 in [18], we have

$$\nabla \varphi(\bar{x})d \geq 0, \quad \forall d \in T(X, \bar{x}), \quad (8)$$

and, if $\nabla \varphi(\bar{x})d = 0$, also

$$\nabla \varphi(\bar{x})w + \nabla^2 \varphi(\bar{x})(d, d) \geq 0, \quad \forall w \in \text{clconv } T^2(X, \bar{x}, d). \quad (9)$$

Choosing $d = \bar{d}$, (8) implies that $\theta_j = 0$ for all $j \notin J(\bar{x}, \bar{d})$; then $\bar{w}$ contradicts (9). $\square$

4. JOHN TYPE NECESSARY OPTIMALITY CONDITIONS

In this section we consider the feasible set X defined by inequality and equality constraints. Let $g = (g_1, \dots, g_p) : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $h = (h_1, \dots, h_q) : \mathbb{R}^n \rightarrow \mathbb{R}^q$ be twice differentiable functions and let $I^- := \{1, \dots, p\}$ and $I^o := \{1, \dots, q\}$ be the corresponding index sets. From now onwards, we suppose that the feasible region of problem (4) is given by

$$X := X_g \cap X_h$$

where

$$X_g := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I^-\},$$

$$X_h := \{x \in \mathbb{R}^n : h_i(x) = 0, i \in I^o\}.$$

Following the notations given in Section 2, let us introduce:

- the set of descent directions at $\bar{x}$ for problem (4), namely

$$D(\bar{x}) := D_{\leq}(f, \bar{x}) \cap D_{\leq}(g, \bar{x}) \cap D_{=}(h, \bar{x});$$

- the weak second order linearizing set of X at $\bar{x}$ in the direction $d \in \mathbb{R}^n$, namely

$$L_{<}^2(\bar{x}, d) := D_{<}^2(g, \bar{x}, d) \cap D_{=}^2(h, \bar{x}, d);$$

- the second order linearizing set of X at $\bar{x}$ in the direction $d \in \mathbb{R}^n$, namely

$$L_{\leq}^2(\bar{x}, d) := D_{\leq}^2(g, \bar{x}, d) \cap D_{=}^2(h, \bar{x}, d).$$

In order to achieve second order necessary optimality conditions for problem (4), we state the following result which connects the second order linearizing sets of X and the second order contingent set of X .

LEMMA 4.1: Let $\bar{x} \in X$ and $d \in D_{\leq}(g, \bar{x}) \cap D_{=}(h, \bar{x})$ be a given direction. Then

$$T^2(X, \bar{x}, d) \subseteq L_{\leq}^2(\bar{x}, d). \quad (10)$$

If $\{\nabla h_i(\bar{x})\}_{i \in I^o}$ are linearly independent, then

$$L_{<}^2(\bar{x}, d) \subseteq T^2(X, \bar{x}, d). \quad (11)$$

Proof: Given $w \in T^2(X, \bar{x}, d)$, there exist $\{t_n\} \downarrow 0$ and $\{w_n\} \rightarrow w$, such that

$$x_n = \bar{x} + t_n d + 2^{-1} t_n^2 w_n \in X, \quad \forall n \in \mathbb{N}.$$

Since g_i and h_i are twice differentiable, we have

$$\begin{aligned} 0 \geq g_i(x_n) &= g_i(\bar{x}) + t_n \nabla g_i(\bar{x})(d + 2^{-1} t_n w_n) \\ &\quad + 2^{-1} t_n^2 \nabla^2 g_i(\bar{x})(d + 2^{-1} t_n w_n, d + 2^{-1} t_n w_n) + t_n^2 \varepsilon_n, \quad \forall i \in I^- \end{aligned} \quad (12)$$

$$\begin{aligned} 0 = h_i(x_n) &= h_i(\bar{x}) + t_n \nabla h_i(\bar{x})(d + 2^{-1} t_n w_n) \\ &\quad + 2^{-1} t_n^2 \nabla^2 h_i(\bar{x})(d + 2^{-1} t_n w_n, d + 2^{-1} t_n w_n) + t_n^2 \varepsilon_n, \quad \forall i \in I^o. \end{aligned} \quad (13)$$

Therefore, for each $i \in I^-(x, d)$, dividing (12) by $2^{-1}t_n^2$, we have

$$0 \geq \nabla g_i(\bar{x})w_n + \nabla^2 g_i(\bar{x})(d + 2^{-1}t_n w_n, d + 2^{-1}t_n w_n) + 2\varepsilon_n,$$

and considering the limit as $n \rightarrow \infty$, we get $w \in D_{\leq}^2(g, \bar{x}, d)$. For each $i \in I^o$, dividing (13) by t_n and considering the limit we get $\nabla h_i(\bar{x})d = 0$. Now, we can divide by $2^{-1}t_n$ to obtain

$$0 = \nabla h_i(\bar{x})w_n + \nabla^2 h_i(\bar{x})(d + 2^{-1}t_n w_n, d + 2^{-1}t_n w_n) + 2\varepsilon_n;$$

considering the limit as $n \rightarrow \infty$, we get $w \in D_{\leq}^2(h, \bar{x}, d)$. Thus, (10) follows.

Let us prove the second part. Theorem 3.5 in [27] implies that $D_{\leq}^2(h, \bar{x}, d) = T^2(X_h, \bar{x}, d)$; therefore, for each $w \in L_{<}^2(\bar{x}, d)$, there exist $\{t_n\} \downarrow 0$ and $\{w_n\} \rightarrow w$ such that

$$x_n := \bar{x} + t_n d + 2^{-1}t_n^2 w_n \in X_h, \quad \forall n \in \mathbb{N}.$$

It will be enough to show that $x_n \in X_g$ for all n large enough.

- For each $i \notin I(\bar{x})$, the continuity of g_i implies that $g_i(x_n) < 0$.
- For each $i \in I(\bar{x}) \setminus I^-(\bar{x}, d)$ we have

$$g_i(x_n) = t_n[\nabla g_i(\bar{x})(d + 2^{-1}t_n w_n) + \varepsilon_n] < 0.$$

- For each $i \in I^-(\bar{x}, d)$ we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \nabla g_i(\bar{x})w_n + \nabla^2 g_i(\bar{x})(d + 2^{-1}t_n w_n, d + 2^{-1}t_n w_n) \\ = \nabla g_i(\bar{x})w + \nabla^2 g_i(\bar{x})(d, d) < 0, \end{aligned}$$

and thus

$$g_i(x_n) = 2^{-1}t_n^2[\nabla g_i(\bar{x})w_n + \nabla^2 g_i(\bar{x})(d + 2^{-1}t_n w_n, d + 2^{-1}t_n w_n) + \varepsilon_n] < 0.$$

Therefore, (11) is satisfied. $\square$

Now, it is immediate to deduce the following second-order optimality condition.

THEOREM 4.1: *Suppose that $\{\nabla h_i(\bar{x})\}_{i \in I^o}$ are linearly independent. If $\bar{x} \in X$ is a local vector minimum point of (4), then for each descent direction $d \in D(\bar{x})$ the system (in the unknown $w \in \mathbb{R}^n$)*

$$\begin{cases} \nabla f_j(\bar{x})w + \nabla^2 f_j(\bar{x})(d, d) < 0, & j \in J(\bar{x}, d), \\ \nabla g_i(\bar{x})w + \nabla^2 g_i(\bar{x})(d, d) < 0, & i \in I^-(\bar{x}, d), \\ \nabla h_i(\bar{x})w + \nabla^2 h_i(\bar{x})(d, d) = 0, & i \in I^o \end{cases} \quad (14)$$

is impossible.

Proof: It follows immediately from Theorem 3.1 and Lemma 4.1. $\square$

Applying Lemma 2.1, we can deduce the following John type multipliers rule from Theorem 4.1.

THEOREM 4.2: *If $\bar{x} \in X$ is a local vector minimum point of (4), then for each descent direction $d \in D(\bar{x})$ there exist $\theta \in \mathbb{R}_+^l$, $\lambda \in \mathbb{R}_+^p$, and $\mu \in \mathbb{R}^q$ not all zero such that*

$$(i) \quad \sum_{j \in J} \theta_j \nabla f_j(\bar{x}) + \sum_{i \in I^-} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in I^0} \mu_i \nabla h_i(\bar{x}) = 0;$$

$$(ii) \quad \left(\sum_{j \in J} \theta_j \nabla^2 f_j(\bar{x}) + \sum_{i \in I^-} \lambda_i \nabla^2 g_i(\bar{x}) + \sum_{i \in I^0} \mu_i \nabla^2 h_i(\bar{x}) \right) (d, d) \geq 0;$$

$$(iii) \quad \lambda_i g_i(\bar{x}) = 0, \text{ for each } i \in I^-;$$

$$(iv) \quad \theta_j \nabla f_j(\bar{x})d = 0, \text{ for each } j \in J, \text{ and } \lambda_i \nabla g_i(\bar{x})d = 0, \text{ for each } i \in I^-.$$

Proof: If $\{\nabla h_i(\bar{x})\}_{i \in I^0}$ are linearly dependent, let us choose $\theta_j = \lambda_i = 0$ and μ_i not all zero such that

$$\sum_{i \in I^0} \mu_i \nabla h_i(\bar{x}) = 0.$$

Conditions (i), (iii), and (iv) are trivially satisfied; if (ii) does not hold, it is enough to change μ_i with $-\mu_i$. Otherwise, the theorem follows immediately from Lemma 2.1 applied to system (14), just setting $\theta_j = \lambda_i = 0$ for all $j \notin J(\bar{x}, d)$ and $i \notin I^-(\bar{x}, d)$. $\square$

It is worth noting that Theorem 4.1 and Theorem 4.2 embrace also the first order optimality conditions (see, for instance [9]), which can be obtained just considering the particular direction $d = 0$.

5. NECESSARY OPTIMALITY CONDITION WITH CONSTRAINT QUALIFICATIONS

The multipliers rule in Theorem 4.2 does not guarantee that at least one multiplier corresponding to the objective functions is nonzero; obviously, when they are all zero, the objective functions do not play any role in the

optimality condition. To overcome this drawback, some further assumptions on the problem have to be introduced.

DEFINITION 5.1: Given a feasible point $\bar{x} \in \mathbb{R}^n$,

- the Abadie second order constraint qualification (ASOCQ) holds at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ iff

$$L_{\leq}^2(\bar{x}, d) = T^2(X, \bar{x}, d);$$

- the Guignard second order constraint qualification (GSOCQ) holds at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ iff

$$L_{\leq}^2(\bar{x}, d) \subseteq \text{cl conv } T^2(X, \bar{x}, d).$$

Observe that (ASOCQ) and (GSOCQ) collapse to the well known Abadie constraint qualification and Guignard constraint qualification, choosing $d = 0$. Obviously, if (ASOCQ) holds, then also (GSOCQ) is satisfied; the converse does not hold as well known for $d = 0$. The following result can be trivially deduced from Theorem 3.1.

THEOREM 5.1: Suppose that (ASOCQ) holds at $\bar{x} \in X$ in the descent direction $d \in D(\bar{x})$. If $\bar{x}$ is a local vector minimum point of (4), then the system (in the unknown $w \in \mathbb{R}^n$)

$$\begin{cases} \nabla f_j(\bar{x})w + \nabla^2 f_j(\bar{x})(d, d) < 0, & j \in J(\bar{x}, d), \\ \nabla g_i(\bar{x})w + \nabla^2 g_i(\bar{x})(d, d) \leq 0, & i \in I^-(\bar{x}, d), \\ \nabla h_i(\bar{x})w + \nabla^2 h_i(\bar{x})(d, d) = 0, & i \in I^o \end{cases} \quad (15)$$

is impossible.

A similar result was presented in [23] and improved in [2] where the Authors do not require the assumption of weak convex inclusion as it has been done in [23]. In our result we further relax the constraint qualification considered in Theorem 3.3 of [2]. In fact, in that theorem it is required that (ASOCQ) holds in every descent direction. The following example shows a case in which Theorem 5.1 can be applied even if (ASOCQ) holds only in some descent directions.

Example 5.1: Consider the multiobjective problem with

$$\begin{aligned} f(x_1, x_2, x_3) &:= (x_1, x_2, x_3 - x_1^2 - x_2^2), \\ g(x_1, x_2, x_3) &:= x_1 x_2 x_3 - x_3, \\ h(x_1, x_2, x_3) &:= x_1 x_2. \end{aligned}$$

It is easy to show that $\bar{x} = (0, 0, 0)$ is a vector minimum point. The set of the descent directions is

$$D(\bar{x}) = \{d \in \mathbb{R}^3 : d_1 \leq 0, d_2 \leq 0, d_3 = 0\}.$$

Calculations show that, for such directions d , we have

$$L_{\leq}^2(\bar{x}, d) = \begin{cases} \emptyset, & \text{if } d_1 d_2 < 0, \\ \{w \in \mathbb{R}^3 : w_3 \geq 0\}, & \text{if } d_1 d_2 = 0, \end{cases}$$

and

$$T^2(X, \bar{x}, d) = \begin{cases} \emptyset, & \text{if } d_1 d_2 < 0, \\ \{w \in \mathbb{R}^3 : w_i = 0, w_3 \geq 0\}, & \text{if } d_i = 0 \text{ with } i = 1, 2. \end{cases}$$

Thus (ASOCQ) holds only in the descent directions d with $d_1 d_2 < 0$ and so Theorem 3.1 in [2] cannot be applied; indeed, for the other nonzero descent directions d system (15) admits the solution $w = (-1, -1, 0)$.

Now, we can deduce the following Karush, Kuhn–Tucker type multipliers rule from Theorem 5.1.

THEOREM 5.2: *Suppose that (ASOCQ) holds at $\bar{x} \in X$ in the descent direction $d \in D(\bar{x})$. If $\bar{x}$ is a local vector minimum point of (4), then there exist $\theta \in \mathbb{R}_+^\ell$, $\lambda \in \mathbb{R}_+^p$, and $\mu \in \mathbb{R}^q$ with $\theta \neq 0$ such that (i–iv) of Theorem 4.2 hold.*

Proof: It follows immediately from Lemma 2.1 applied to system (15) just setting $\theta_j = 0$ and $\lambda_i = 0$ for all $j \notin J(\bar{x}, d)$ and $i \notin I^-(\bar{x}, d)$. $\square$

This result improves the second order necessary optimality conditions presented in some papers [2, 23]. Moreover, choosing $d = 0$ as descent direction, we obtain the result presented in [22]. By means of Theorem 3.2 we deduce immediately the following result.

THEOREM 5.3: *Suppose that (GSOCQ) holds at $\bar{x} \in X$ in the descent direction $d \in D(\bar{x})$ and that f is subconvexlike on X . If $\bar{x}$ is a local vector minimum point of (4), then there exist $\theta \in \mathbb{R}_+^\ell$, $\lambda \in \mathbb{R}_+^p$, and $\mu \in \mathbb{R}^q$ with $\theta \neq 0$ such that (i–iv) of Theorem 4.2 hold.*

Proof: It follows immediately from Lemma 2.1 and Theorem 3.2. $\square$

In the previous theorems, the constraint qualifications imply that the (vector) multiplier θ is nonzero; now, we want to investigate conditions which ensure that such a vector has many (possibly all) positive components.

For this reason, we introduce the *set of the second order descent directions at $\bar{x}$ for problem (4) with respect to $d \in \mathbb{R}^n$* , namely

$$D^2(\bar{x}, d) := D_{\leq}^2(f, \bar{x}, d) \cap D_{\leq}^2(g, \bar{x}, d) \cap D_{\leq}^2(h, \bar{x}, d).$$

Observe that $D^2(\bar{x}, 0) = D(\bar{x})$.

DEFINITION 5.2: *Given a feasible point $\bar{x} \in X$, the Guignard second order regularity condition (GSORC) holds at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ iff*

$$D^2(\bar{x}, d) \subseteq \bigcap_{s=1}^{\ell} \text{cl conv } T^2(X^s, \bar{x}, d),$$

where

$$X^s := \{x \in X : f_j(x) - f_j(\bar{x}) < 0, \forall j \in J \setminus \{s\}\} \cup \{\bar{x}\}.$$

Now, we are able to prove the following result.

THEOREM 5.4: *Suppose that (GSORC) holds at $\bar{x} \in X$ in the descent direction $d \in D(\bar{x})$. If $\bar{x}$ is a local vector minimum point of (4), then for each $s \in J(\bar{x}, d)$ the linear system (in the unknown $w \in \mathbb{R}^n$)*

$$\begin{cases} \nabla f_s(\bar{x})w + \nabla^2 f_s(\bar{x})(d, d) < 0, \\ \nabla f_j(\bar{x})w + \nabla^2 f_j(\bar{x})(d, d) \leq 0, & j \in J(\bar{x}, d) \setminus \{s\}, \\ \nabla g_i(\bar{x})w + \nabla^2 g_i(\bar{x})(d, d) \leq 0, & i \in I^-(\bar{x}, d), \\ \nabla h_i(\bar{x})w + \nabla^2 h_i(\bar{x})(d, d) = 0, & i \in I^o, \end{cases}$$

is impossible.

Proof: *Ab absurdo*, suppose that w solves the system for some $s \in J(\bar{x}, d)$. Thus, (GSORC) implies that $w \in \text{cl conv } T^2(X^s, \bar{x}, d)$. From the definition of local vector minimum point, it follows immediately that $\bar{x}$ is also a local minimum point of the scalar problem

$$\min f_s(x) \quad \text{subject to} \quad x \in X^s.$$

By Corollary 3.1 in [18], we get the contradiction $\nabla f_s(\bar{x})w + \nabla^2 f_s(\bar{x})(d, d) \geq 0$. $\square$

Theorem 5.4 leads to the following Karush, Kuhn–Tucker type multipliers rule.

THEOREM 5.5: *Suppose that (GSORC) holds at $\bar{x} \in X$ in the descent direction $d \in D(\bar{x})$. If $\bar{x}$ is a local vector minimum point of (4), then there exist $\theta \in \mathbb{R}_+^\ell$, $\lambda \in \mathbb{R}_+^p$, and $\mu \in \mathbb{R}^q$ with $\theta_j > 0$ for all $j \in J(\bar{x}, d)$ such that $(i - iv)$ of Theorem 4.2 hold.*

Proof: It follows immediately applying Lemma 2.1 to each linear system of Theorem 5.4 (setting $\theta_j = 0$ and $\lambda_i = 0$ for all $j \notin J(\bar{x}, d)$ and $i \notin I^-(\bar{x}, d)$) and summing up the resulting multipliers. $\square$

Throughout the whole paper, the impossibility of linear systems has been turned into multipliers rules using Lemma 2.1. Equivalently, following the image space approach techniques (see [7] and the references therein), this can be done by separation arguments; the multipliers will be just the gradients of separating hyperplanes. Thus, it is possible to analyse regularity conditions exploiting the tools developed in [5].

6. CONCLUDING REMARKS

We studied second-order necessary optimality conditions for multiobjective problems where the ordering cone is the interior of the positive orthant. It is clear that the conditions we developed are necessary also for the multiobjective problem where the ordering cone is the positive orthant. Is it possible to obtain necessary conditions for this problem that are not necessary for the one we studied? Other interesting question deals with the general case, where the ordering is given by any convex cone with nonempty interior. In fact, the peculiar structure of the positive orthant allows to use the single components of the objective function in order to identify the descent directions; in the general case, they seem not to be suitable for this aim. So, how is it possible to extend the results of this paper to the case of any ordering cone?

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