

# ISOMETRIES OF NILPOTENT METRIC GROUPS

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ABSTRACT. We consider Lie groups equipped with arbitrary distances. We only assume that the distances are left-invariant and induce the manifold topology. For brevity, we call such objects metric Lie groups. Apart from Riemannian Lie groups, distinguished examples are sub-Riemannian Lie groups, homogeneous groups, and, in particular, Carnot groups equipped with Carnot–Carathéodory distances. We study the regularity of isometries, i.e., distance-preserving homeomorphisms. Our first result is the analyticity of such maps between metric Lie groups. The second result is that if two metric Lie groups are connected and nilpotent then every isometry between the groups is the composition of a left translation and an isomorphism. There are counterexamples if one does not assume the groups to be either connected or nilpotent. The first result is based on a solution of the Hilbert’s fifth problem by Montgomery and Zippin. The second result is proved, via the first result, reducing the problem to the Riemannian case, which was essentially solved by Wolf.

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## 1. INTRODUCTION

In this paper, with the term *metric Lie group* we mean a Lie group equipped with a left-invariant distance that induces the manifold topology. An *isometry* is a distance-preserving bijection. Hence, a priori it is only a homeomorphism. As a general fact we show the following regularity result.

**Theorem 1.1.** *Isometries between metric Lie groups are analytic maps.*

We say that a map between groups is *affine* if it is the composition of a left translation and a group homomorphism. For nilpotent groups we have the following stronger result.

**Theorem 1.2.** *Isometries between nilpotent connected metric Lie groups are affine.*

In particular we have that

- [1.2.i] two isometric nilpotent connected metric Lie groups are isomorphic;
- [1.2.ii] given a connected metric Lie group  $N$ , its isometry group  $\text{Isom}(N)$ , which always is a Lie group, is a semidirect product if  $N$  is nilpotent. Namely,

$$\text{Isom}(N) = N \rtimes \text{AutIsom}(N),$$

where  $N$  is seen inside  $\text{Isom}(N)$  as left translations and  $\text{AutIsom}(N)$  denotes the group of automorphisms of  $N$  that are isometries.

Moreover, with the above notation, we have

- [1.2.iii]  $N$  is a maximal connected nilpotent subgroup of  $\text{Isom}(N)$  and the Lie algebra of  $N$  is the nilradical of the Lie algebra of  $\text{Isom}(N)$ , see Section 3.2.

Theorem 1.2 is a generalization of previous results. On the one hand, in the case of nilpotent Lie groups equipped with left-invariant *Riemannian* distances the result is essentially known from the work of Wolf, see [Wol63, Wil82] and Remark 3.3. On the other hand, Theorem 1.2 has been shown in the case of Carnot groups equipped with Carnot–Carathéodory distances, see [Pan89, Ham90, Kis03, LO16]. In fact our strategy of proofs is built on both [Wol63] and [LO16].

Examples of groups not considered before are sub-Riemannian, and more generally sub-Finsler, groups that are not Carnot groups (i.e., the sub-Riemannian structure is not coming from the first layer of a stratification), together with their subgroups, and their snowflakes. Other examples are given by the Heisenberg group equipped with the Korányi gauge and, more generally, by any other homogeneous group (in the sense of Folland and Stein), i.e., a graded group equipped with a homogeneous norm, see more in [LN16, LR17].

We remark that both assumptions ‘connectedness’ and ‘nilpotency’ are necessary for Theorem 1.2 to hold. In this respect in Section 4 we provide some counterexamples.

The large-scale analogue of Theorem 1.2 is a challenging open problem that has raised a lot of attention since the papers of Pansu and Shalom [Pan89, Sha04]. What is expected is that if two finitely generated nilpotent groups are torsion-free, then every quasi-isometry between them induces an isomorphism between their Malcev completions. The quasi-isometric classification of locally compact groups is also a very active area, see the (quasi-)survey [Cor15].

We spend the rest of the introduction to explain the strategy of the proofs of the two theorems and the structure of the paper. To study isometries between two metric Lie groups, we first treat the case when the two groups are the same, i.e., they are isometric via a Lie group isomorphism. If  $M$  is a connected metric Lie group, we consider its isometry group  $G$ , that is, the set of self-isometries of  $M$  equipped with the composition rule and the compact-open topology. Hence, the group  $G$  acts continuously, transitively and by isometries on  $M$ . It is crucial that  $G$  is a locally compact group. This latter fact follows from Ascoli–Arzelà Theorem but it needs some argument since closed balls are not necessarily assumed to be compact. At this point, the theory of locally compact groups, [MZ74], provides a Lie group structure on  $G$  such that the action  $G \curvearrowright M$  is smooth, see Section 2.1.

Assume that  $M_1, M_2$  are metric Lie groups and  $F: M_1 \rightarrow M_2$  is an isometry. We consider the above-mentioned Lie group structures on the respective isometry groups  $G_1, G_2$ . The conjugation by  $F$  provides a map from  $G_1$  to  $G_2$  that is a continuous homomorphism between Lie groups, hence it is analytic. This observation will give the conclusion of the proof of Theorem 1.1, see Section 2.2.

An important consequence of Theorem 1.1 is that every isometry between metric Lie groups can be seen as a Riemannian isometry. Namely, for every map  $F: M_1 \rightarrow M_2$  as above there are Riemannian left-invariant structures  $g_1, g_2$  such that  $F: (M_1, g_1) \rightarrow (M_2, g_2)$  is a Riemannian isometry, see Proposition 2.4. Of a separate interest is the fact that the Riemannian structures can be chosen independently of  $F$ . Together with Wolf’s study of nilpotent Riemannian Lie groups, Theorem 1.2 and the other statements now follow.

We also show that if  $M$  is a group equipped with a left-invariant distance, then its isometries are affine if and only if its isometry group  $G$  splits as semi-direct product

$$G = M \rtimes \text{Stab}_1(G),$$

where  $\text{Stab}_1(G)$  is the set of isometries fixing the identity element 1 of  $M$ . We provide the simple proof in Lemma 3.2.

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## 2. REGULARITY OF ISOMETRIES

**2.1. Lie group structure of isometry groups.** The first aim of this section is to show that the isometry group of a metric Lie group is a Lie group. Such a fact is a consequence of the solution of the Hilbert's fifth problem by Montgomery–Zippin, together with the observation that the isometry group is locally compact. This latter property follows by Ascoli–Arzelà Theorem.

We stress that a metric Lie group  $(M, d)$  may not be boundedly compact. Namely, the closed balls  $\bar{B}_d(1_M, r) := \{p \in M : d(p, 1_M) \leq r\}$  with respect to  $d$  may not be compact. For example, this is the case for the distance  $\min\{d_E, 1\}$  on  $\mathbb{R}$ , where  $d_E$  denotes the Euclidean distance.

*Remark 2.1.* If  $(M, d)$  is a connected metric Lie group, then there exists a distance  $\rho$  such that  $(M, \rho)$  is a metric Lie group that is boundedly compact and  $\text{Isom}(M, d) \subseteq \text{Isom}(M, \rho)$ . Indeed, since the topology induced by  $d$  is the manifold topology, then there exists some  $r_0 > 0$  such that  $\bar{B}_d(1_M, r_0)$  is compact. Then we can consider the distance

$$\rho(p, q) := \inf \left\{ \sum_{i=1}^k d(p_{i-1}, p_i) : k \in \mathbb{N}, p_i \in M, p_0 = p, p_k = q, d(p_{i-1}, p_i) \leq r_0 \right\}.$$

Once can check that  $(M, \rho)$  is a metric Lie group, for all  $r > 0$  the set  $\bar{B}_\rho(1_M, r)$  is compact, and  $\text{Isom}(M, d) \subseteq \text{Isom}(M, \rho)$ .

Let us clarify now why the isometry group of a connected metric Lie group is locally compact, which was not justified in [LO16]. With the terminology of Remark 2.1 the stabilizer  $S$  of 1 in  $\text{Isom}(M, d)$  is a closed subgroup of the stabilizer  $S_\rho$  of 1 in  $\text{Isom}(M, \rho)$ . Furthermore, for any  $r > 0$  and  $f \in S_\rho$  we have that  $f(\bar{B}_\rho(1, r)) = \bar{B}_\rho(1, r)$ , which is compact. Hence, the maps from  $S$  restricted to  $\bar{B}(1, r)$  form an equi-uniformly continuous and pointwise precompact family. Ascoli–Arzelà Theorem implies that  $S_\rho$  is compact, being also closed in  $C^0(M, M)$ . Consequently,  $S$  is compact and because  $M$  is locally compact, then also  $\text{Isom}(G, d)$  is locally compact. At this point we are allowed to use the theory of locally compact groups after Gleason–Montgomery–Zippin [MZ74]. In fact, the argument in [LO16, Proposition 4.5] concludes the proof of the following result.

**Proposition 2.2.** *Let  $M$  be a metric Lie group with isometry group  $G$ . Assume that  $M$  is connected.*

- (1) *The stabilizers of the action  $G \curvearrowright M$  are compact.*
- (2) *The topological group  $G$  is a Lie group (finite dimensional and with finitely many connected components) acting analytically on  $M$ .*

*Remark 2.3.* The assumption of  $M$  being connected in Proposition 2.2 is necessary. Indeed, one can take as a counterexample the group  $\mathbb{Z}$  with the discrete distance.

**2.2. Proof of smoothness.** With the use of Proposition 2.2, we give the proof of the analyticity of isometries (Theorem 1.1). We remark that in the Riemannian setting the classical result of Myers and Steenrod gives smoothness of isometries, see [MS39], and more generally [CL16]. However, the following proof is different in spirit and, nonetheless, it will imply (see Proposition 2.4) that such metric isometries are Riemannian isometries for some Riemannian structures.

*Proof of Theorem 1.1.* Let  $F: M_1 \rightarrow M_2$  be an isometry between metric Lie groups. Without loss of generality we may assume that  $F(1_{M_1}) = 1_{M_2}$  and that both  $M_1$  and  $M_2$  are connected, since left translations are analytic isometries and connected components of identity elements are open. By Proposition 2.2, for  $i \in \{1, 2\}$ , the space  $G_i := \text{Isom}(M_i)$  is a Lie group. The map  $C_F: G_1 \rightarrow G_2$  defined as  $I \mapsto F \circ I \circ F^{-1}$  is a group isomorphism that is continuous, see [Are46, Theorem 4]. Hence, the map  $C_F$  is analytic, see [Hel01, p. 117, Theorem 2.6].

Consider also the inclusion  $\iota: M_1 \rightarrow G_1$ ,  $m \mapsto L_m$ , which is analytic being a continuous homomorphism, and the orbit map  $\sigma: G_2 \rightarrow M_2$ ,  $I \mapsto I(1_{M_2})$ , which is analytic since the action is analytic (Proposition 2.2). We deduce that  $\sigma \circ C_F \circ \iota$  is analytic. We claim that this map is  $F$ . Indeed, for any  $m \in M_1$  it holds

$$(\sigma \circ C_F \circ \iota)(m) = \sigma(F \circ L_m \circ F^{-1}) = (F \circ L_m \circ F^{-1})(1_{M_2}) = F(m). \quad \square$$

**2.3. Isometries as Riemannian isometries.** We show next that isometries between metric Lie groups are actually Riemannian isometries for some left-invariant structures. Let us point out that when  $M$  is a Lie group and  $g$  is a left-invariant Riemannian metric tensor on  $g$ , then one has an induced Riemannian distance  $d_g$  and, by the theorem of Myers and Steenrod [MS39], the group  $\text{Isom}(M, d_g)$  of distance-preserving bijections coincides with the group  $\text{Isom}(M, g)$  of tensor-preserving diffeomorphisms. In what follows we shall write  $(M, g)$  to denote the metric Lie group  $(M, d_g)$ .

**Proposition 2.4.** *If  $(M_1, d_1)$  and  $(M_2, d_2)$  are connected metric Lie groups, then there exists left-invariant Riemannian metrics  $g_1$  and  $g_2$  on  $M_1$  and  $M_2$ , respectively, such that  $\text{Isom}(M_i, d_i) \subseteq \text{Isom}(M_i, g_i)$  for  $i \in \{1, 2\}$  and for all isometries  $F: (M_1, d_1) \rightarrow (M_2, d_2)$  the map  $F: (M_1, g_1) \rightarrow (M_2, g_2)$  is a Riemannian isometry.*

Let us first deal with the case  $(M_1, d_1) = (M_2, d_2)$ .

**Lemma 2.5.** *If  $(M, d)$  is a connected metric Lie group, then there is a Riemannian metric  $g$  such that  $\text{Isom}(M, d) \subseteq \text{Isom}(M, g)$ .*

*Proof of Lemma 2.5.* Fix a scalar product  $\langle \langle \cdot, \cdot \rangle \rangle$  on the tangent space  $T_1M$  at the identity 1 of  $M$ . From Proposition 2.2, the stabilizer  $S$  of 1 in  $\text{Isom}(M, d)$  is compact

and acts smoothly on  $M$ . Let  $\mu_S$  be the probability Haar measure on  $S$ . Consider for  $v, w \in T_1M$

$$\langle v, w \rangle := \int_S \langle \langle dFv, dFw \rangle \rangle d\mu_S(F).$$

Then  $\langle \cdot, \cdot \rangle$  defines an  $S$ -invariant scalar product on  $T_1M$ , and one can take  $g$  as the left-invariant Riemannian metric that coincides with  $\langle \cdot, \cdot \rangle$  at the identity.  $\square$

*Proof of Proposition 2.4.* By Lemma 2.5 let  $g_2$  be a Riemannian metric on  $M_2$  with

$$(2.6) \quad \text{Isom}(M_2, d_2) \subseteq \text{Isom}(M_2, g_2).$$

Fix  $F: (M_1, d_1) \rightarrow (M_2, d_2)$  an isometry. By Theorem 1.1 the map  $F$  is smooth, and we may define a Riemannian metric on  $M_1$  by  $g_1 := F^*g_2$ . There are two things to check: a)  $\text{Isom}(M_1, d_1) \subseteq \text{Isom}(M_1, g_1)$ , which in particular gives that  $g_1$  is left-invariant and b) every isometry  $H: (M_1, d_1) \rightarrow (M_2, d_2)$  is an isometry of Riemannian manifolds.

For part a, since by construction  $F$  is also a Riemannian isometry, the map  $I \mapsto F \circ I \circ F^{-1}$  is a bijection between  $\text{Isom}(M_1, d_1)$  and  $\text{Isom}(M_2, d_2)$  and between  $\text{Isom}(M_1, g_1)$  and  $\text{Isom}(M_2, g_2)$ . Therefore the inclusion (2.6) implies the inclusion  $\text{Isom}(M_1, d_1) \subseteq \text{Isom}(M_1, g_1)$ .

For part b, since  $H \circ F^{-1} \in \text{Isom}(M_2, d_2) \subseteq \text{Isom}(M_2, g_2)$ , then  $(H \circ F^{-1})^*g_2 = g_2$ . Consequently, we get  $H^*g_2 = F^*(H \circ F^{-1})^*g_2 = g_1$ .  $\square$

### 3. AFFINE DECOMPOSITION

**3.1. Preliminary lemmas.** Given a group  $M$  we denote by  $M^L$  the group of left translations on  $M$ . The following two results make sense in the settings of groups equipped with left-invariant distances. We call such groups *metric groups*.

**Lemma 3.1.** *Let  $M_1$  and  $M_2$  be metric groups. Suppose  $F: M_1 \rightarrow M_2$  is an isometry and  $F \circ M_1^L \circ F^{-1} = M_2^L$ . Then  $F$  is affine.*

*Proof.* Up to precomposing with a translation, we assume that  $F(1_{M_1}) = 1_{M_2}$ . So we want to prove that  $F$  is an isomorphism. The map  $C_F: \text{Isom}(M_1) \rightarrow \text{Isom}(M_2)$ ,  $I \mapsto F \circ I \circ F^{-1}$ , is an isomorphism and by assumption it gives an isomorphism between  $M_1^L$  and  $M_2^L$ . We claim that  $F$  is the same isomorphism when identifying  $M_i$  with  $M_i^L$ . Namely, we want to show that for all  $m \in M_1$  we have  $L_{F(m)} = C_F(L_m)$ . By assumption for all  $m_1 \in M_1$  exists  $m_2 \in M_2$  such that  $L_{m_2} = C_F(L_{m_1})$ . Evaluating at  $1_{M_2}$ , we get

$$m_2 = L_{m_2}(1_{M_2}) = C_F(L_{m_1})(1_{M_2}) = F(L_{m_1}(F^{-1}(1_{M_2}))) = F(m_1). \quad \square$$

With the next result we clarify that the condition of self-isometries being affine is equivalent to left translations being a normal subgroup of the group of isometries. Equivalently, we have a semi-direct product decomposition of the isometry group. Namely, given a metric group  $M$  and denoting by  $G$  the isometry group and by

$\text{Stab}_1(G)$  the stabilizer of the identity element, we have that  $M$  has affine isometries if and only if  $G = M^L \rtimes \text{Stab}_1(G)$ . We denote by  $\text{Aff}(M)$  the group of affine maps from  $M$  to  $M$  and by  $\text{Aut}(M)$  the group of automorphisms of  $M$ .

**Lemma 3.2.** *Let  $M$  be a metric group with isometry group  $G$ . Then the following are equivalent:*

- (a)  $M^L \triangleleft G$ , i.e.,  $F \circ M^L \circ F^{-1} = M^L$ , for all  $F \in G$ ;
- (b)  $G < \text{Aff}(M)$ ;
- (c)  $\text{Stab}_1(G) < \text{Aut}(M)$ ;
- (d)  $G = M^L \rtimes \text{Stab}_1(G)$ ;
- (e)  $G = M^L \rtimes (G \cap \text{Aut}(M))$ .

*Proof.* Property (a) implies (b) by Lemma 3.1. Regarding the fact that (b) implies (a), consider a map  $F \in G$ , which we know is of the form  $F = \tau \circ \Phi$  with  $\tau \in M^L$  and  $\Phi \in \text{Aut}(M)$ . For all  $p \in M$  we get

$$\begin{aligned} F \circ L_p \circ F^{-1} &= (\tau \circ \Phi) \circ L_p \circ (\tau \circ \Phi)^{-1} \\ &= \tau \circ \Phi \circ L_p \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \Phi \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \tau^{-1} \in M^L, \end{aligned}$$

which gives  $M^L \triangleleft G$ .

The equivalence of (b) with (c) is trivial. The equivalence of (a) with (d) follows from the facts  $M^L \cdot \text{Stab}_1(G) = G$  and  $M^L \cap \text{Stab}_1(G) = \{\text{id}\}$ . Finally, (e) implies (d), and (e) is implied by (d) together with (c).  $\square$

*Remark 3.3.* As said in the introduction, Theorem 1.2 is essentially due to Wolf in the Riemannian setting. Indeed, in [Wol63, p. 278, Theorem 4.2] he proved the semi-direct product decomposition of the isometry group of a Riemannian nilpotent Lie group, which is equivalent to self-isometries being affine, as in the lemma above. To conclude that an isometry  $F: N_1 \rightarrow N_2$  between Riemannian nilpotent Lie groups is affine one considers the self-isometry of the product  $N_1 \times N_2$  given by  $(n, m) \mapsto (F^{-1}(m), F(n))$ . Also, one can check that the proof of [Wil82, Theorem 3] gives the same result.

**3.2. Theorem 1.2 from Proposition 2.4.** For every Riemannian nilpotent Lie group Wolf proved a characterization of the group inside its isometry group. In fact, he described the nilpotent group as the nilradical of its isometry group. We shall give the same characterization in the general setting. We introduce some terminology inspired by [Wol63, Wil82, GW88].

**Definition 3.4** (Nilradical condition). Let  $\mathfrak{g}$  be a Lie algebra. The *nilradical* of  $\mathfrak{g}$ , denoted by  $\text{nil}(\mathfrak{g})$ , is the largest nilpotent ideal of  $\mathfrak{g}$ . We say that a connected metric Lie group  $N$  with isometry group  $\text{Isom}(N)$  satisfies *the nilradical condition* if it holds

$$(3.5) \quad \text{Lie}(N^L) = \text{nil}(\text{Lie}(\text{Isom}(N))).$$

Clearly, a metric Lie group  $N$  can satisfy the nilradical condition only if it is nilpotent. The nilradical of a Lie algebra  $\mathfrak{g}$  can also be defined as the sum of all nilpotent ideals of  $\mathfrak{g}$ , see [HN12, Definition 5.2.10].

*Remark 3.6.* The nilradical condition is satisfied by Riemannian nilpotent Lie groups, where the distance is induced by a left-invariant metric tensor. Such a result was proved by Wolf [Wol63, p. 278, Theorem 4.2], see also [Wil82, p. 341 Theorem 2]. Actually, Wolf proved the stronger statement that such a group  $N$  is a maximal connected nilpotent subgroup inside  $\text{Isom}(N)$ , which implies the nilradical condition since  $N^L \triangleleft \text{Isom}(N)$ . Clearly, there may be several maximal connected nilpotent subgroups inside  $\text{Isom}(N)$ .

The nilradical condition is an algebraic characterization of the Lie algebra of a nilpotent metric Lie group inside the Lie algebra of its isometry group. Hence, by Lemma 3.1 it is clear that if two connected metric Lie groups  $N_1$  and  $N_2$  satisfy the nilradical condition (3.5), then any isometry  $F: N_1 \rightarrow N_2$  is affine. Indeed, the map  $I \mapsto F \circ I \circ F^{-1}$  induces a Lie algebra isomorphism between  $\text{Lie}(\text{Isom}(N_1))$  and  $\text{Lie}(\text{Isom}(N_2))$ , and therefore, since the exponential map is surjective, one concludes that the map sends  $N_1^L$  to  $N_2^L$ .

We also mention that the work of Wolf, together with the work of Gordon and Wilson, is one of the initial steps in the study of (Riemannian) nilmanifolds, solvmanifolds, and homogeneous Ricci solitons, see [GW88, Jab15a, Jab15b].

*Proof of Theorem 1.2.* Let  $F: (N_1, d_1) \rightarrow (N_2, d_2)$  be an isometry between two nilpotent connected metric Lie groups. By Proposition 2.4 for  $i \in \{1, 2\}$  there exist left-invariant metric tensors  $g_i$  on  $N_i$  such that  $F: (N_1, g_1) \rightarrow (N_2, g_2)$  is a Riemannian isometry. By Remark 3.3, the map  $F$  is affine. In particular, we have [1.2.i].

Because of Lemma 3.2 we also deduce that the isometry group of a nilpotent connected metric Lie group  $N$  has the semi-direct product decomposition [1.2.ii]. Regarding [1.2.iii], given such a group  $N$  we use again Proposition 2.4 and have that  $N \subseteq \text{Isom}(N) \subseteq \text{Isom}(N, g)$ , for some left-invariant metric tensor  $g$  on  $N$ . By Remark 3.6, the group  $N^L$  is a maximal connected nilpotent subgroup inside  $\text{Isom}(N, g)$ , thus also inside  $\text{Isom}(N)$ . Since from [1.2.ii] we have  $N^L \triangleleft \text{Isom}(N)$ , so  $\text{Lie}(N^L)$  is an ideal of  $\text{Lie}(\text{Isom}(N))$ . Thus, by the maximality of  $N$ , we deduce the nilradical condition (3.5).  $\square$

#### 4. EXAMPLES FOR THE SHARPNESS OF THE ASSUMPTIONS

In this section we provide several examples to illustrate the sharpness of the assumptions in Theorem 1.2. Namely, we show that if one of the groups is not assumed connected and nilpotent then there may be isometries that are not affine.

Regarding the connectedness assumption, there are examples of Abelian metric Lie groups with finitely many components for which some isometries are not affine. One of



the simplest examples is the subgroup of  $\mathbb{C}$  consisting of the four points  $\{1, i, -1, -i\}$  equipped with the discrete distance. Here every permutation is an isometry. However, any automorphism needs to fix  $-1$ , since it is the only point of order 2.

Regarding the nilpotent assumption, there are both compact and non-compact examples. We remark that in any group equipped with a bi-invariant distance the involution is an isometry. Consequently, every compact group admits a distance for which the involution is an isometry. Such a map is a group isomorphism only if the group is Abelian. Nonetheless, we point out the following fact which is a consequence of the work of Baum–Browder and Ochiai–Takahashi, see [BB65, OT76] and also [Sch68, HK85].

**Corollary 4.1.** *Let  $G_1, G_2$  be connected compact simple metric Lie groups. If  $F: G_1 \rightarrow G_2$  is an isometry, then  $G_1$  and  $G_2$  are isomorphic as Lie groups. If, moreover,  $G_1, G_2$  are the same metric Lie group and  $F$  is homotopic to the identity map via isometries, then  $F$  is affine.*

We point out that there exist examples of pairs of metric Lie groups that are isomorphic as Lie groups and are isometric, but are not isomorphic as metric Lie groups: an example is the rototranslation group (see below) with different Euclidean distances.

Other interesting results for isometries between compact groups can be found in [Oze77] and [Gor80].

The conclusion of Corollary 4.1 may not hold for arbitrary connected metric Lie groups. In fact, we recall the following example, due to Milnor [Mil76, Corollary 4.8], of a group that is solvable and isometric to the Euclidean 3-space. Let  $G$  be the universal cover of the group of orientation-preserving isometries of the Euclidean plane, which is also called the rototranslation group. Such a group admits coordinates making it diffeomorphic to  $\mathbb{R}^3$  with the product

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In these coordinates, the Euclidean metric is left-invariant. On the one hand, one can check that the isometries that are also automorphisms of  $G$  form a 1-dimensional space. On the other hand, the isometries fixing the identity element and homotopic to the identity map form a group isomorphic to  $\mathrm{SO}(3)$ . Hence, we conclude that not all such isometries are affine. Moreover, this group gives an example of a non-nilpotent metric Lie group isometric (but not isomorphic) to a nilpotent connected metric Lie group, namely the Euclidean 3-space.

Notice that also the Riemannian metric with orthonormal frame  $\partial_x, \partial_y, 2\partial_z$  gives a left-invariant structure on  $G$ , which is isometric to the previous one, but there is

no isometric automorphism between the two structures. Hence, these spaces are not isomorphic as metric Lie groups.

A further study of metric Lie groups isometric to nilpotent metric Lie groups can be found in [CKL<sup>+</sup>]. In the simply connected case, such groups are exactly the solvable groups of type R.

We finally recall another example. The unit disc in the plane admits a group structure that makes the hyperbolic distance left-invariant. In this metric Lie group not all isometries are affine.

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