A sequential method for a class of pseudoconcave fractional problems.

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Abstract

The aim of the paper is to maximize a pseudoconcave function which is the sum of a linear and a linear fractional function subject to linear constraints. Theoretical properties of the problem are first established and then a sequential method based on a simplex-like procedure is suggested.

Key words: Fractional programming, pseudoconcavity, post-optimality analysis.
2000 Mathematics Subject Classification 90C32, 26B25

1 Introduction

Optimization of linear and nonlinear fractional functions has been widely studied since the pioneering works of Charnes and Cooper [5] and Martos [13]. A particular attention is devoted to the sum of ratios fractional programs as it is confirmed in the extensive survey (with twelve hundred entries) appeared in [14]; another updated survey can be found in [7]. Some sequential methods are suggested for maximizing the sum of two linear fractional functions subject to linear constraints ([4, 6, 8, 9, 10, 11, 12]). This kind of generalized fractional problem can be reduced, by means of the Charnes-Cooper transformation, to maximize the sum of a linear and a linear fractional function. In finding a global optimal solution for this last class of problems, some difficulties arise since the function may have local maxima which are not global. Obviously, the problem becomes easier when the objective function verifies the local-global property; it is known ([2]) that this important property holds for the class of pseudoconcave functions which contains, properly, the class of concave functions. In this paper, firstly we will characterize in a simple way the pseudoconcavity of the sum of a linear and a linear fractional function and, successively, we will establish some theoretical results regarding the existence of optimal solutions. More exactly, we will give necessary and sufficient conditions for the supremum of the problem to be not finite or finite but not attained as maximum. The given results allow us to suggest a simplex-like procedure for solving our problem.

2 On the pseudoconcavity of a generalized fractional function.

In this section we characterize the pseudoconcavity of the function

$$f(x) = h^T x + \frac{c^T x + c_0}{d^T x + d_0}, \; x \in H = \{x \in \Re^n : d^T x + d_0 > 0\} \tag{2.1}$$

where $h, d, c \in \Re^n$, $c_0, d_0 \in \Re$; furthermore we assume $d \neq 0$ to avoid the linearity of the function.
We recall that a differentiable function f defined on an open convex set $S \subseteq \Re^n$ is pseudoconcave if the following logical implication holds:

$$x, y \in S, \; f(x) < f(y) \Rightarrow \; (y - x)^T \nabla f(x) > 0.$$

Although the class of pseudoconcave function is wider than the class of the concave function, it still maintains some fundamental optimization properties such as "a critical point is a global maximum point", and "a local maximum point is also global".
In general the function f is not pseudoconcave, since it may have local maximum points which are not global, as it is shown in the following example.

Example 2.1 *Consider the function* $f(x_1, x_2) = x_1 - x_2 + \frac{2x_1+7x_2+6}{x_1+x_2+1}$ *defined on the set* $S = \{(x_1, x_2) \in \Re^2 : x_1 \geq 0, \; 0 \leq x_2 \leq 4, \; x_1 - x_2 \leq 4\}$. *It is easy to verify that* $(0, 0)$ *is a local maximum while the global maximum is* $(8, 4)$.

Necessary and sufficient conditions for the pseudoconcavity of f can be deduced from the results obtained in [3], where the pseudoconvexity of the ratio between a quadratic function and an affine function is studied in the framework of advanced linear algebra. In order to have a self-contained paper, we will give a new simple proof which takes into account the special structure of the function and which is based on the following characterization of pseudoconcavity (see [1]):
f is pseudoconcave if and only if a critical point of a restriction of f on a line segment is a local maximum point for the restriction.
Throughout the paper we will use the following notations. Let $x_0 \in H$, $u \in \Re^n$ and let $x = x_0 + tu$ be the equation of a line segment contained in the half-space H. We have:

$$\varphi(t) = f(x_0 + tu) = h^T x_0 + th^T u + \frac{tc^T u + \bar{c}_0}{td^T u + \bar{d}_0} \tag{2.2}$$

$$\varphi'(t) = h^T u + \frac{c^T u \; \bar{d}_0 - d^T u \; \bar{c}_0}{(td^T u + \bar{d}_0)^2} \tag{2.3}$$

$$\varphi''(t) = \frac{-2d^T u \; (c^T u \; \bar{d}_0 - d^T u \; \bar{c}_0)}{(td^T u + \bar{d}_0)^3} \tag{2.4}$$

where $\bar{c}_0 = c^T x_0 + c_0$, $\bar{d}_0 = d^T x_0 + d_0$.
The following theorem holds.

Theorem 2.1 *The function f is pseudoconcave on the half-space H if and only if one of the following conditions holds:*
i) there exists $\alpha \leq 0$ such that $h = \alpha d$;
ii) there exists $k \in \Re$ such that $c = kd$ and $c_0 - kd_0 \leq 0$.

Proof Assume that f is pseudoconcave on H and consider, first, the case where the vector c is not proportional to d. We prove the existence of a real number $\alpha \leq 0$ such that $h = \alpha d$. Since $c \neq kd$, $\forall k \in \Re$, then there exist $\beta, \ \gamma \in \Re^n$ such that $\beta^T c = 0, \ \beta^T d > 0, \ \gamma^T d = 0, \ \gamma^T c > 0$.
Let $x_0 = \lambda\gamma + \mu\beta, \ \lambda, \mu \in \Re$. We have $\bar{c}_0 = \lambda\gamma^T c + c_0, \ \bar{d}_0 = \mu\beta^T d + d_0$ and thus, choosing $\mu > 0$ such that $\bar{d}_0 > 0$, x_0 becomes a feasible point.
Assume that h is not proportional to d or that $h = \alpha d$ with $\alpha > 0$. Then, there exists $u \in \Re^n$ such that $h^T u > 0, \ d^T u > 0$. Consider the feasible half-line $x = x_0 + tu, \ t \geq 0$ and the restriction (2.2). We can choose $\lambda > 0$ large enough such that $\varphi'(0) = \frac{h^T u \ \bar{d}_0^2 + c^T u \ \bar{d}_0 - d^T u \ \bar{c}_0}{\bar{d}_0^2} < 0$, so that $c^T u \ \bar{d}_0 - d^T u \ \bar{c}_0 < -h^T u \ \bar{d}_0^2 < 0$ and consequently $\varphi''(t) > 0, \ \forall t \geq 0$. Since $\lim_{t \to +\infty} \varphi'(t) = h^T u > 0$, there exists $t^* > 0$ such that $\varphi'(t^*) = 0, \ \varphi''(t^*) > 0$, so that t^* is a strict local minimum point for $\varphi(t)$ and this contradicts the pseudoconcavity of the function f. Consequently h is proportional to d with $\alpha \leq 0$ and thus i) follows.
Consider now the case $c = kd$. We prove that if condition i) is not verified, necessarily we have $c_0 - kd_0 \leq 0$. Let $u \in \Re^n$ be such that $h^T u > 0$ and $d^T u > 0$. Substituting $c = kd$ in (2.3), (2.4), we have:

$$\varphi'(t) = h^T u + \frac{-d^T u(c_0 - kd_0)}{(td^T u + \bar{d}_0)^2}, \ \varphi''(t) = \frac{2(d^T u)^2(c_0 - kd_0)}{(td^T u + \bar{d}_0)^3} \tag{2.5}$$

If we have $c_0 - kd_0 > 0$, then

$$\lim_{t \to (-\frac{\bar{d}_0}{d^T u})^+} \varphi'(t) = -\infty, \quad \lim_{t \to +\infty} \varphi'(t) = h^T u > 0,$$

so that there exists t^* such that $\varphi'(t^*) = 0, \ \varphi''(t^*) > 0$; consequently the restriction $\varphi(t)$ has a feasible critical point which is a strict local minimum and this is absurd. Then ii) holds.
Assume now that i) or ii) holds; we must prove that f is pseudoconcave on H. Assume that i) holds. If there exists a critical point t^* for a restriction $\varphi(t)$, then the real numbers $h^T u$ and $c^T u \ \bar{d}_0 - d^T u \bar{c}_0$ have opposite sign so that $\varphi''(t^*) < 0$; consequently t^* is a local maximum point for φ and this implies the pseudoconcavity of f.
If ii) holds, from (2.5) we have $\varphi''(t) \leq 0$ for every t such that $td^T u + \bar{d}_0 > 0$, so that every restriction of f is concave and, consequently, f is concave on H and, in particular, it is pseudoconcave. The proof is complete. □

Remark 2.1 *The proof given in Theorem 2.1 points out that in the case ii) the function is concave. This does not happen in the case i).*
For instance, it can be verified that the restriction of the pseudoconcave

function $f(x,y) = -x - y + \frac{2x+y+3}{x+y+1}$ *on the half-line* $y = 0,\ x > -1$ *is not concave.*

Remark 2.2 *If i) of Theorem 2.1 holds with* $\alpha = 0$*, then the function reduces to a linear fractional function which has been widely studied. Also in the case ii), if* $c_0 - kd_0 = 0$*, the function reduces to a linear function. For such reasons, in what follows we restrict our analysis to the cases* $\alpha \neq 0$ *and* $c_0 - kd_0 \neq 0$.

The following theorem points out that a pseudoconcave function has critical points (which are also global maximum points) only in a particular case.

Theorem 2.2 *Assume that the function* f *is pseudoconcave on the half-space* H*. Then,* f *has critical points if and only if* $h = \alpha d$*,* $c = kd$*,* $\alpha < 0$ *and* $c_0 - kd_0 < 0$*. In such a case the set of all critical points is whose equation is* $d^T x + d_0 = \delta$*, with* $\delta = \sqrt{\frac{c_0 - kd_0}{\alpha}}$.

Proof We have $\nabla f(x) = h + \frac{c(d^T x + d_0) - d(c^T x + c_0)}{(d^T x + d_0)^2}$, so that the gradient vanishes if and only if the vectors h, c, d are linearly dependent. Taking into account Theorem 2.1, the pseudoconcavity of f implies that the vectors h and c are proportional to vector d. Setting $h = \alpha d$, $c = kd$, we have $f(x) = \alpha d^T x + \frac{c_0 - kd_0}{d^T x + d_0} + k$ and thus $\nabla f(x) = \alpha d - \frac{d(c_0 - kd_0)}{(d^T x + d_0)^2}$. Consequently $\nabla f(x_0) = 0$ if and only if there exists a feasible point x_0 such that $\alpha = \frac{(c_0 - kd_0)}{(d^T x_0 + d_0)^2}$, that is if and only if α and $c_0 - kd_0$ have the same sign which is negative taking into account the pseudoconcavity of f.
The set of all critical points is $\{x \in H : \left(d^T x + d_0\right)^2 = \frac{c_0 - kd_0}{\alpha}\}$ and the thesis follows. □

3 A generalized fractional problem

In this section we consider the fractional problem

$$P : \max f(x) = h^T x + \frac{c^T x + c_0}{d^T x + d_0},\ x \in S = \{x \in \Re^n : Ax \leq b,\ x \geq 0\}$$

where $h, d, c \in \Re^n,\ d \neq 0,\ c_0, d_0 \in \Re,\ A$ is an $m \times n$ matrix, $b \in \Re^m$ and $d^T x + d_0 > 0,\ \ \forall x \in S$.
Firstly, we will establish a necessary condition for P to have a supremum not attained as a maximum. Successively, we will prove that when the problem has optimal solutions at least one belongs to an edge of S (in particular it may be a vertex). At last, assuming the pseudoconcavity of the objective function, we will establish necessary and sufficient conditions for the supremum to be infinite or finite but not attained as a maximum.
With this aim we recall that a vector $u \in \Re^n$ is called a recession direction of S if for every $y \in S$ the half-line $x = y + tu,\ t \geq 0$ is contained in S. A

recession direction u is said to be an extreme direction if it is not possible to express u as a convex combination of two distinct recession directions.

Theorem 3.1 *Let L be the supremum of problem P.*
i) L is attained as a maximum if and only if there exists a feasible point x_0 belonging to an edge of S such that $f(x_0) = L$.
ii) If L is not attained as a maximum then there exist a feasible point x_0 and an extreme direction u such that $L = \lim_{t \to +\infty} f(x_0 + tu)$.

Proof i) If the supremum L is attained as a maximum there exists a feasible point $\bar{x}$ such that $L = f(\bar{x})$. Consider the problem

$$\bar{P} : \max f(x), \; x \in \bar{S} = S \cap \{x \in \Re^n : d^T x + d_0 = d^T \bar{x} + d_0\}.$$

Obviously $\bar{x}$ is an optimal solution of $\bar{P}$ and since $\bar{P}$ is a linear problem the maximum is reached also at a vertex x_0 of $\bar{S}$ which belongs to an edge of S. The viceversa is obvious.
ii) Let $\{x_n\} \subset S$ be a sequence such that $f(x_n)$ converges to L and consider the following sequence of problems

$$P_n : \max_{x \in S_n} \left(f(x) = h^T x + \frac{c^T x + c_0}{d^T x + d_0} \right)$$

where $S_n = S \cap \{x \in \Re^n : d^T x + d_0 = d^T x_n + d_0\}$.
For every fixed x_n, the problem P_n is linear so that the following two exhaustive cases occur:
a) there exists n such that the supremum of P_n is $+\infty$;
b) for every n the supremum of P_n is attained as a maximum.
a) It is well known that the supremum of the linear problem P_n is $+\infty$ if and only if there exist a feasible point x_0 and an extreme direction u such that $\lim_{t \to +\infty} f(x_0 + tu) = +\infty$. In such a case the supremum of problem P is $+\infty$ and ii) holds (note that the half-line $x = x_0 + tu, t \geq 0$ is contained in S_n if and only if $d^T u = 0$).
b) In such a case the supremum is attained at a vertex y_n of S_n which belongs to an edge of S. Taking into account that $f(y_n) \geq f(x_n)$ we have $\lim_{n \to +\infty} f(y_n) = L$. Since S has a finite number of edges (in particular half-lines), there exists a subsequence $\{\hat{y}_n\}$ of $\{y_n\}$ contained in an edge of S. Since L is not attained as a maximum, the sequence $\{\hat{y}_n\}$ is necessary diverging in norm (if $\{\hat{y}_n\}$ converges to an element y_0, the continuity of f implies $f(y_0) = L$) and $f(\hat{y}_n) \neq L \; \forall n$. It follows that $\{\hat{y}_n\}$ is necessarily contained in a half-line $x = x_0 + tu, \; t \geq 0$, where x_0 is a vertex of S and u is an extreme direction. Let t_n be such that $\hat{y}_n = x_0 + t_n u$. We have $\lim_{n \to +\infty} f(\hat{y}_n) = \lim_{t_n \to +\infty} f(x_0 + t_n u) = L$ and ii) holds.
The proof is complete. □

Corollary 3.1 *Let L be the supremum of problem P. Then $L = +\infty$ if and only if there exists a feasible point x_0 and an extreme direction u such that $\lim_{t \to +\infty} f(x_0 + tu) = +\infty$.*

Let us note that Theorem 3.1 does not require any assumption of pseudoconcavity. When f is pseudoconcave, it is possible to establish necessary and sufficient conditions for the supremum to be not finite or to be finite but not attained as a maximum, as it is stated in the following theorems.

Theorem 3.2 *Consider problem P where* $f(x) = \alpha d^T x + \frac{c^T x + c_0}{d^T x + d_0}$, $\alpha < 0$. *Then the supremum of P is* $+\infty$ *if and only if there exists an extreme direction u such that* $d^T u = 0$, $c^T u > 0$. *In any other case the supremum is attained as a maximum.*

Proof Let u be an extreme direction and consider the restriction $f(x_0 + tu) = \alpha t d^T u + \alpha d^T x_0 + \frac{tc^T u + c^T x_0 + c_0}{td^T u + d^T x_0 + d_0}$. Note that $x = x_0 + tu$, $x_0 \in S$, $t \geq 0$ is feasible $\forall t \geq 0$ since $d^T u \geq 0$. We have $\lim_{t \to +\infty} f(x_0 + tu) = -\infty$ if and only if $d^T u > 0$ or $d^T u = 0$ and $c^T u < 0$; the limit is $+\infty$ if and only if $d^T u = 0$ and $c^T u > 0$ and it is finite if and only if $d^T u = 0$ and $c^T u = 0$; in this last case $f(x_0 + tu) = f(x_0)$, $\forall t \geq 0$. The thesis follows taking into account Theorem 3.1 and Corollary 3.1. □

Theorem 3.3 *Consider problem P where* $f(x) = h^T x + \frac{kd^T x + c_0}{d^T x + d_0}$ *and* $k \in \Re$ *is such that* $c_0 - kd_0 < 0$. *Then:*
i) the supremum of P is $+\infty$ *if and only if there exists an extreme direction u such that* $h^T u > 0$;
ii) the supremum of P is finite and not attained as a maximum if and only if there exists an extreme direction u such that $h^T u = 0$, $d^T u > 0$, *and there does not exist an extreme direction v such that* $h^T v > 0$.
In any other case, problem P has optimal solutions.

Proof Let u be an extreme direction and consider the restriction $f(x_0 + tu) = th^T u + h^T x_0 + \frac{c_0 - kd_0}{td^T u + d^T x_0 + d_0} + k$ on the feasible half-line $x = x_0 + tu$, $x_0 \in S$, $t \geq 0$. We have $\lim_{t \to +\infty} f(x_0 + tu) = -\infty$ if and only if $h^T u < 0$; the limit is $+\infty$ if and only if $h^T u > 0$ and it is finite but different from $f(x_0)$ if and only if $h^T u = 0$ and $d^T u > 0$. The thesis follows taking into account Theorem 3.1 and Corollary 3.1. □

Remark 3.1 *From Theorem 3.2 and Theorem 3.3 we have that the supremum of the pseudoconcave function* $f(x) = \alpha d^T x + \frac{c^T x + c_0}{d^T x + d_0}$, $\alpha < 0$, *is not finite or it is attained as a maximum, while the supremum of the pseudoconcave function* $f(x) = h^T x + \frac{kd^T x + c_0}{d^T x + d_0}$, $c_0 - kd_0 < 0$, *may be not finite, finite and not attained, or finite and attained as a maximum.*

4 Sequential methods

The pseudoconcavity of the function f implies that the rank of the set $\{h, c, d\}$ is at most 2, so that we have the following exhaustive subclasses of pseudoconcave functions:
I) $f(x) = \alpha d^T x + \frac{\gamma}{d^T x + d_0}$, $\alpha < 0$ and $\gamma \neq 0$ or $\alpha \neq 0$ and $\gamma < 0$.

II) $f(x) = \alpha d^T x + \frac{c^T x + c_0}{d^T x + d_0}$, $\alpha < 0$ and $c \neq kd \ \forall k \in \Re$.
III) $f(x) = h^T x + \frac{\gamma}{d^T x + d_0}$, $\gamma < 0$ and $h \neq \alpha d \ \forall \alpha \in \Re$.

The theoretical properties established in the previous sections allow us to suggest a sequential method for each subclass.

4.1 Case I

Problem P reduces to the problem

$$P_1 : \max \left(\alpha d^T x + \frac{\gamma}{d^T x + d_0} \right), \ x \in S, \ d^T x + d_0 > 0 \ \forall x \in S$$

where $\alpha < 0$ and $\gamma \neq 0$ or $\alpha \neq 0$ and $\gamma < 0$.
Setting $z = d^T x + d_0$, the function f is transformed into the function $\psi(z) = \alpha(z - d_0) + \frac{\gamma}{z}$, $z > 0$, whose derivative is $\psi'(z) = \frac{\alpha z^2 - \gamma}{z^2}$, $z > 0$.
If $\alpha < 0$ and $\gamma > 0$, the function $\psi(z)$ is decreasing so that solving P_1 is equivalent to solve the linear problem $\min(d^T x + d_0)$, $x \in S$.
If $\alpha > 0$ and $\gamma < 0$, the function $\psi(z)$ is increasing so that solving P_1 is equivalent to solve the linear problem $\max(d^T x + d_0)$, $x \in S$.
Taking into account the pseudoconcavity of f, it remains to consider the case $\alpha < 0$ and $\gamma < 0$. In such a case $\psi(z)$ is increasing in the interval $(0, \sqrt{\frac{\gamma}{\alpha}} = z^*]$ and decreasing in the half-line $[z^*, +\infty)$. If $d^T x + d_0 = z^*$ is a feasible level, then any point of $S \cap \{x : d^T x + d_0 = z^*\}$ is an optimal solution for problem P_1. If z^* does not correspond to a feasible level, let $\delta_{\min} = \min \ (d^T x + d_0)$, $x \in S$. If $\delta_{\min} > z^*$, then $\delta_{\min}$ is the maximum value of problem P_1; if $\delta_{\min} < z^*$, then solving problem P_1 is equivalent to solve the linear problem $\max \ (d^T x + d_0)$, $x \in S$.
Consequently the problem can be easily solved by means of linear programming.

4.2 Case II

Problem P reduces to the problem

$$P_2 : \max \left(\alpha d^T x + \frac{c^T x + c_0}{d^T x + d_0} \right), \ x \in S, \ d^T x + d_0 > 0 \ \forall x \in S$$

where $\alpha < 0$ and $c \neq kd \ \forall k \in \Re$.
Since the linear function $d^T x + d_0$ is lower bounded on S, the linear problem $P_d : \min \ (d^T x + d_0)$, $x \in S$ has optimal solutions. Let δ_0 be the minimum value of problem P_d and consider the linear program
$P_c : \max \ (c^T x + c_0)$, $x \in S \cap \{x : d^T x + d_0 = \delta_0\}$.
Taking into account Theorem 3.2, the supremum of problem P_2 is $+\infty$ if and only if the supremum of problem P_c is $+\infty$. If the supremum of P_c is finite, let x_0 be a vertex of S which is an optimal solution of P_c. Starting

from x_0, we suggest an algorithm for solving problem P_2.
Consider the linear parametric problem

$$P(\theta): \ \psi(\theta) = \max \ (c^T x + c_0), \ x \in S(\theta) = S \cap \{x : d^T x = d^T x_0 + \theta\}$$

and set $\Theta = \{\theta : \ S(\theta) \neq \emptyset\} = [0, \theta_{\max}]$, where $\theta_{\max}$ may be $+\infty$.
We have

$$\max_{x \in S} f(x) = \max_{\theta \in \Theta} \ \max_{x \in S(\theta)} f(x).$$

Setting $z(\theta) = \max\limits_{x \in S(\theta)} f(x)$, it results

$$\max_{x \in S} f(x) = \max_{\theta \in \Theta} z(\theta), \ z(\theta) = \alpha(\delta_0 - d_0 + \theta) + \frac{\psi(\theta)}{\delta_0 + \theta}.$$

If $z(\theta)$ increases (decreases), then the function $f(x)$ increases (decreases) so that a local maximum of $z(\theta)$ corresponds to a local maximum of $f(x)$, which is also global for the pseudoconcavity of the function.
The idea of the algorithm that we are going to describe is the following: corresponding to the vertex x_0, which is an optimal solution of $P(\theta_0)$, $\theta_0 = 0$, denote by B_0 the set of indices associated with the basic variables and set $x_0 = (x_{B_0}, 0)$. Applying sensitivity analysis we find $(x_{B_0}(\theta), 0) = (x_{B_0} + \theta u_{B_0}, 0)$ which is optimal for $P(\theta)$ for every θ belonging to the stability interval $[\theta_0, \theta_1] = \{\theta : \ x_{B_0}(\theta) \geq 0\}$. If $z'(0) \leq 0$, then $(x_{B_0}, 0)$ is the optimal solution of P_2. If there exists $\widetilde{\theta} \in [\theta_0, \theta_1]$ such that $z'(\widetilde{\theta}) = 0$, then $(x_{B_0}(\widetilde{\theta}), 0)$ is the optimal solution of P_2, otherwise for $\theta > \theta_1$ the feasibility is lost and it is restored applying the dual simplex algorithm. We find a new stability interval and we repeat the analysis. Proceeding in this way we develop a finite sequence of basis $B_k, k = 0, 1, ...$ and a finite numbers of stability intervals $[\theta_k, \theta_{k+1}], \ k = 0, 1, ...$
With the usual notations, corresponding to the basis B_k, we have:
$(x_{B_k}(\theta), 0) = (x_{B_k} + \theta u_{B_k}, 0), \ \psi(\theta) = c_{B_k}^T x_{B_k} + \theta c_{B_k}^T u_{B_k} + c_0, \ \theta \in [\theta_k, \theta_{k+1}]$ so that

$$z(\theta) = \alpha(\delta_0 - d_0 + \theta) + \frac{c_{B_k}^T x_{B_k} + \theta c_{B_k}^T u_{B_k} + c_0}{\delta_0 + \theta}, \ \theta \in [\theta_k, \theta_{k+1}] \quad (4.6)$$

$$z'(\theta) = \alpha + \frac{\xi_{B_k}}{(\delta_0 + \theta)^2}, \ \xi_{B_k} = \delta_0 c_{B_k}^T u_{B_k} - c_{B_k}^T x_{B_k} - c_0, \ \theta \in [\theta_k, \theta_{k+1}] \quad (4.7)$$

An algorithm for problem P_2
Step 0 Solve problem P_d and let δ_0 be its optimal value. Solve problem P_c. If P_c has no solutions, STOP: $\sup f(x) = +\infty$; otherwise let x_0 be an optimal solution of Problem P_c which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$. Set $k = 0$ and GO TO Step 1.
Step 1 Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $(x_{B_k}(\theta_k), 0) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$. Compute $\xi_{B_k} = \delta_0 c_{B_k}^T u_{B_k} - c_{B_k}^T x_{B_k} - c_0$. If $\xi_{B_k} \leq 0$, STOP: $(x_{B_k} + \theta_k u_{B_k}, 0)$ is the optimal

solution of P_2 otherwise GO TO Step 2.
Step 2 Compute $\widetilde{\theta} = -\delta_0 + \sqrt{-\frac{\xi_{B_k}}{\alpha}}$. If $\widetilde{\theta} \in [\theta_k, \theta_{k+1}]$, STOP: $\left(x_{B_k} + \widetilde{\theta} u_{B_k}, 0\right)$ is the optimal solution of P_2, otherwise let i such that $x_{B_{k_i}} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $k = k + 1$ and GO TO Step 1.

Example 4.1 *Consider the following problem*

$$\begin{cases} \max(-x_1 - x_2 + \frac{80x_1+60x_2-1}{x_1+x_2+1}) \\ x_1 - 4x_2 \leq 2 \\ x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

Step 0 $(0,0)$ is the unique solution of both problems $\min\limits_{(x_1,x_2)\in S}(x_1 + x_2 + 1)$ and P_c; we have $\delta_0 = 1$ and we go to Step 1.
Step 1 We have $x_{B_0}(\theta) = (x_3(\theta), x_4(\theta), x_1(\theta))^T = (2-\theta, 2, \theta)^T$, so that the stability interval is $[0,2]$ and $\xi_{B_0} = 81$. Since $\xi_{B_0} > 0$, we go to Step 2.
Step 2 We have $\widetilde{\theta} = -\delta_0 + \sqrt{-\frac{\xi_{B_0}}{\alpha}} = -1 + 9 = 8$. Since $\widetilde{\theta} > 2$, we perform a pivot operation according with the dual simplex algorithm. We obtain $x_{B_1}(\theta) = (x_2(\theta), x_4(\theta), x_1(\theta))^T = (-\frac{2}{5} + \frac{1}{5}\theta, \frac{12}{5} - \frac{1}{5}\theta, \frac{2}{5} + \frac{4}{5}\theta)^T$, we go to Step 1.
Step 1 The stability interval is $[2,12]$ and $\xi_{B_1} = 69$. Since $\xi_{B_1} > 0$, we go to Step 2.
Step 2. We have $\widetilde{\theta} = -\delta_0 + \sqrt{-\frac{\xi_{B_1}}{\alpha}} = -1 + \sqrt{69} \in [2,12]$, so that $(x_1, x_2) = (-\frac{2}{5} + \frac{4}{5}\sqrt{69}, -\frac{1}{5} + \frac{1}{5}\sqrt{69})$ is the optimal solution of the problem.

Remark 4.1 *Deleting the constraint $x_2 \leq 2$ in Example 4.1, we obtain a problem with an unbounded feasible region which have the same optimal solution of the original problem.*

4.3 Case III

Problem P reduces to the problem

$$P_3 : \max\ (h^T x + \frac{\gamma}{d^T x + d_0}),\ \gamma < 0,\ x \in S,\ d^T x + d_0 > 0\ \forall x \in S.$$

From a theoretical point of view, problem P_3 differs from Problem P_2 from the fact that now we may have a finite supremum not attained as a maximum. Referring to case II, we determine the optimal value δ_0 of problem P_d and successively we consider the linear program
$P_h : \max\ h^T x,\ x \in S \cap \{x : d^T x + d_0 = \delta_0\}$.
If the supremum of P_h is $+\infty$, then there exists an extreme direction u such that $d^T u = 0,\ h^T u > 0$, so that from Theorem 3.3, the supremum of P_3 is $+\infty$. If the supremum of P_h is finite, consider the linear parametric problem

$$P(\theta) :\ \phi(\theta) = \max\ h^T x,\ x \in S(\theta) = S \cap \{x : d^T x = d^T x_0 + \theta\}.$$

Setting $\Theta = \{\theta : \; S(\theta) \neq \emptyset\}$, we have

$$\max_{x \in S} f(x) = \max_{\theta \in \Theta} z(\theta), \; z(\theta) = \frac{\gamma}{\delta_0 + \theta} + \phi(\theta)$$

with $\phi(\theta) = h_{B_k}^T x_{B_k} + \theta h_{B_k}^T u_{B_k}, \; \theta \in [\theta_k, \theta_{k+1}]$, so that

$$z'(\theta) = h_{B_k}^T u_{B_k} - \frac{\gamma}{(\delta_0 + \theta)^2}, \quad \theta \in [\theta_k, \theta_{k+1}] \tag{4.8}$$

Referring to the stability interval $[\theta_k, \theta_{k+1}]$, we have:
if $h_{B_k}^T u_{B_k} > 0$, then the supremum of P_3 is $+\infty$;
if $h_{B_k}^T u_{B_k} = 0$ and $\theta_{k+1} = +\infty$, then the supremum of P_3 is $h_{B_k}^T x_{B_k}$ and it is not attained as a maximum;
if $h_{B_k}^T u_{B_k} < 0$, we have $z'(\widetilde{\theta}) = 0$ with $\widetilde{\theta} = -\delta_0 + \sqrt{\frac{\gamma}{h_{B_k}^T u_{B_k}}}$. If $\widetilde{\theta} < \theta_k$, then $(x_{B_k}(\theta_k), 0)$ is an optimal solution of P_3; if $\widetilde{\theta} \in [\theta_k, \theta_{k+1}]$, then $(x_{B_k}(\widetilde{\theta}), 0)$ is an optimal solution of P_3, otherwise we consider the vertex $(x_{B_k}(\theta_{k+1}), 0)$ and we apply the dual simplex algorithm in order to find a new stability interval; we repeat the analysis.

An algorithm for problem P_3
Step 0 Solve problem P_d and let δ_0 be its optimal value. Solve problem P_h. If P_h has no solutions, STOP: $\sup f(x) = +\infty$. Otherwise let x_0 be an optimal solution of Problem P_h which is also an optimal solution of Problem $P(\theta_0)$ with $\theta_0 = 0$. Set $k = 0$ and GO TO Step 1.
Step 1 Determine the stability interval $[\theta_k, \theta_{k+1}]$ associated with the optimal solution $(x_{B_k}(\theta_k), 0) = (x_{B_k} + \theta_k u_{B_k}, 0)$ of $P(\theta_k)$. Compute $h_{B_k}^T u_{B_k}$. If $h_{B_k}^T u_{B_k} > 0$ and $\theta_{k+1} = +\infty$, STOP: the supremum of P_3 is $+\infty$; if $h_{B_k}^T u_{B_k} > 0$ and θ_{k+1} is finite, go to Step 2; if $h_{B_k}^T u_{B_k} = 0$ and $\theta_{k+1} = +\infty$, STOP: the supremum of P_3 is $h_{B_k}^T x_{B_k}$ and it is not attained as a maximum; if $h_{B_k}^T u_{B_k} = 0$ and θ_{k+1} is finite, GO TO Step 2; if $h_{B_k}^T u_{B_k} < 0$, GO TO Step 3.
Step 2 Let i such that $x_{B_{k_i}} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $k = k + 1$ and GO TO Step 1.
Step 3 Compute $\widetilde{\theta} = -\delta_0 + \sqrt{\frac{\gamma}{h_{B_k}^T u_{B_k}}}$.
If $\widetilde{\theta} \in [\theta_k, \theta_{k+1}]$, STOP: $\left(x_{B_k} + \widetilde{\theta} u_{B_k}, 0\right)$ is the optimal solution of P_3; if $\widetilde{\theta} < \theta_k$, STOP: $(x_{B_k} + \theta_k u_{B_k}, 0)$ is the optimal solution of P_3; if $\widetilde{\theta} > \theta_{k+1}$, let i such that $x_{B_{k_i}} + \theta_{k+1} u_{B_{k_i}} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $k = k + 1$ and GO TO Step 1.

Example 4.2 *(The supremum is finite but not attained).*
Consider the following problem

$$\begin{cases} \max(3x_1 - 4x_2 + \frac{-5}{x_1 + 2x_2 + 1}) \\ 3x_1 - 4x_2 \leq 6 \\ -x_1 + x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

Step 0 $(0,0)$ is the unique solution of both problems $\min_{(x_1,x_2)\in S}(x_1+2x_2+1)$ and P_h and we have $\delta_0=1$. We go to Step 1.
Step 1 We have $x_{B_0}(\theta)=(x_3(\theta),x_4(\theta),x_1(\theta))^T=(6-3\theta,4+\theta,\theta)^T$, so that the stability interval is $[0,2]$. Since $h_{B_0}^T u_{B_0}=3>0$, we go to Step 2.
Step 2 We perform a pivot operation according with the dual simplex algorithm. We get $x_{B_1}(\theta)=(x_2(\theta),x_4(\theta),x_1(\theta))^T=(-\frac{3}{5}+\frac{3}{10}\theta,\frac{29}{5}+\frac{1}{10}\theta,\frac{6}{5}+\frac{2}{5}\theta)^T$, and we go to Step 1.
Step 1 The stability interval is $[2,+\infty]$. Since $h_{B_1}^T u_{B_1}=0$, the supremum of the problem is 6 and it is not attained as a maximum.

Computational results. We have implemented the algorithm related to Case II. Our implementation is based on Clp supported by OsiSolverInterface. The program has been tested using randomly generated problem instances. In order to guarantee the positivity of the function d^Tx+d_0 on the feasible set, we have randomly positively generated both d and d_0. Our preliminary experiment shows that this algorithm converges quickly and the number of iterations non-decreases as α approaches to 0.

5 Conclusion

In this paper we have considered the problem of maximizing the sum of a linear and a linear fractional function. Such a problem, equivalent to maxime the sum of two linear ratios, is not easy to solve since it may have several local maximum points. We have overcome this difficulty characterizing the pseudoconcavity of the objective function; the obtained results have allowed to propose a simple algorithm based on a suitable simplex-like procedure.
As we have already mentioned, some sequential methods have been suggested for the sum of two or more ratios; these interesting methods, which in general work on a compact feasible set and which do not converge in a finite number of steps, are necessarily more sofisticated (at least from a theoretical point of view). Computational comparisons and the extension of our approach to the general case are open problems.

Acknowledgement – The authors would like to thank Prof. Alberto Cambini and the two anonymous referees for their criticism and comments which helped to improve the presentation of our results.

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