ISOMETRIC EMBEDDINGS OF SNOWFLAKES INTO FINITE-DIMENSIONAL BANACH SPACES

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Abstract. We consider a general notion of snowflake of a metric space by composing the distance with a nontrivial concave function. We prove that a snowflake of a metric space X isometrically embeds into some finite-dimensional normed space if and only if X is finite. In the case of power functions we give a uniform bound on the cardinality of X depending only on the power exponent and the dimension of the vector space.

1. Introduction

The study of isometric embeddings of metric spaces into infinite-dimensional Banach spaces has a long tradition. Classical results are due to Fréchet, Urysohn, Kuratowski, Banach, [Fré10, Ury27, Kur35, Ban55]. For an introduction to the subject we refer to Heinonen's survey [Hei03]. The case of embeddings into finite-dimensional Banach spaces is more difficult, even when one considers bi-Lipschitz embeddings in place of isometric embeddings. It is a wide open problem to give intrinsic characterizations of those metric spaces that admit bi-Lipschitz embeddings into some Euclidean space. See for example [Sem99, Luo96, LP01, Seo11, LN14].

The situation is quite different for quasisymmetric maps (see [Hei01, Chapter 10–12] for an introduction to the theory of quasisymmetric embeddings). A metric space quasisymmetrically embeds into some Euclidean spaces if and only if it is doubling (see [Hei01, Theorem 12.1]). More specifically, Assouad proved the following result (see [Ass83], and also [NN12, DS13]): if (X, d) is a doubling metric space and $\alpha \in (0, 1)$, then the metric space (X, d^α) admits a bi-Lipschitz embedding into some Euclidean space. If d_E is the Euclidean distance and $\alpha = \log 2/\log 3$, then the metric space $([0, 1], d_E^\alpha)$ is bi-Lipschitz equivalent to the von Koch snowflake curve; so (X, d^α) is said to be the α-*snowflake* of (X, d).

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The Assouad Embedding Theorem is sharp in that there are examples (none of which are trivial) of doubling spaces that do not admit bi-Lipschitz embeddings into any Euclidean space, even though each of their α-snowflakes do. See [Sem96a, Sem96b, Laa02, CK10]. We also stress that it has been known that snowflakes of doubling spaces in general do not isometrically embed in any Euclidean space. Indeed, the space $([0,1], d_E^{1/2})$ does not, see [Hei03, Remark 3.16(b)].

The main aim of this paper is to show that if some α-snowflake of a metric space isometrically embeds into a finite-dimensional Banach space, then the metric space in question is finite. Our main result is the following.

Theorem 1.1. *For any $n \in \mathbb{N}$ and $\alpha \in (0,1)$ there is an $N \in \mathbb{N}$ such that if a metric space (X,d) has cardinality at least N, then (X,d^α) does not admit an isometric embedding into any n-dimensional normed space.*

The techniques that we use in Section 3.2 for the proof of Theorem 1.1 can also be used to study more general notions of snowflakes. For this purpose, we introduce general snowflaking functions. We say that a function $h : \mathbb{R}_\geq \to \mathbb{R}_\geq$ is a *snowflaking function* if the following hold:

(S1) $h(0) = 0$.
(S2) h is concave.
(S3) $\frac{h(t)}{t} \to \infty$, as $t \to 0$.
(S4) $\frac{h(t)}{t} \to 0$, as $t \to \infty$.

Let h be a snowflaking function. Then function h is weakly increasing and, if d is a metric on a set X, then $h \circ d$ is also a metric on X. Given a snowflaking function h and a metric space (X,d) we say that the metric space $(X, h \circ d)$ is the *h-snowflake* of (X,d). If $h(t) = t^\alpha$ for some $\alpha \in (0,1)$, then $(X, h \circ d)$ is the α-snowflake of (X,d). Other terms for generalized snowflake and α-snowflake are *metric transform* and *power transform*, respectively, see [DL10]. In Section 3.1 we prove the following.

Theorem 1.2. *Let h be a snowflaking function and (X,d) a metric space. If the h-snowflake of (X,d) admits an isometric embedding into some finite-dimensional Banach space, then X is finite.*

Remark 1.3. Note that for general snowflaking functions there may not be any bound on the number of points one can embed, see Remark 3.18. If one removes either of the requirements (S3) or (S4), then we say that $(X, h \circ d)$ is a degenerate snowflake (at zero or at infinity, respectively). Indeed, in such cases the conclusion of Theorem 1.2 does not hold in general, see Proposition 3.14.

We conclude the introduction with a few other simple observations about embeddings into Euclidean spaces. Every α-snowflake of $\mathbb{R}^n$, $\alpha \in (0,1)$, isometrically embeds into the Hilbert space ℓ^2 of square summable sequences, see [Hei03, Remark 3.16(d)]. For any $n \in \mathbb{N}$, there is a metric space of cardinality n such that for any $\alpha \in (0,1)$ its

α-snowflake can be isometrically embedded into $\mathbb{R}^{n-1}$ (just take the standard basis vectors of $\mathbb{R}^n$). There is a 4-point metric space having an α-snowflake that cannot be isometrically embedded into $\mathbb{R}^4$, and so cannot be isometrically embedded into any Euclidean space, (just take the vertices of the (3,1) complete bipartite graph). Every finite metric space has an α-snowflake admitting an isometric embedding into some Euclidean space, see Corollary 2.2.

2. Corollaries

In this section we record two corollaries. We start with a simple corollary of Theorem 1.2:

Corollary 2.1. *Suppose that (X, d) is infinite. For any snowflaking function h and $n \in \mathbb{N}$ there is a $\delta > 0$ such that $(X, h \circ d)$ does not admit a $(1 + \delta)$-bi-Lipschitz embedding into any normed space of dimension n.*

Proof. We first observe that as a consequence of John Ellipsoid Theorem we have a compactness property of normed n-dimensional spaces: a sequence of normed spaces $(V_k)_{k\in\mathbb{N}}$ can be written in suitable coordinates as $V_k = (\mathbb{R}^n, \|\cdot\|_k)$ so that $\|\cdot\|_k$ are uniformly comparable to the Euclidean norm and thus by Ascoli-Arzelà Theorem the functions $\|\cdot\|_k$ subconverge to a norm $\|\cdot\|$. Therefore, if $(X, d), h, n$ form a counterexample to the corollary for any $\delta_k = 1/k$, then the $(1 + \delta_k)$-bi-Lipschitz embeddings $(X, h \circ d) \to V_k$ (sub)converge after suitable translations, again by Ascoli-Arzelà Theorem, to an isometric embedding $(X, h \circ d) \to V$, contradicting Theorem 1.2. □

There is no bi-Lipschitz version of Theorem 1.1. In fact, in the above corollary the constant δ cannot be chosen independently of X, not even for α-snowflakes. Consider the metric on $\mathbb{N}$ given by $d(n, m) = \sum_{i=n}^{m} L^i$, for L large enough in terms of δ and α. Then $(\mathbb{N}, d^\alpha)$ can be $(1 + \delta)$-bi-Lipschitzly embedded into $\mathbb{R}$.

From Theorem 1.1 we actually get a characterisation of finite spaces:

Corollary 2.2. *Given a metric space (X, d), there is an α-snowflake of (X, d) that isometrically embeds into some finite-dimensional normed space if and only if X is finite.*

Indeed, on the one hand Theorem 1.1 shows that there is a bound (depending on α and n) on the cardinality of a metric space whose α-snowflake admits an isometric embedding into $\mathbb{R}^n$. On the other hand, any finite metric space can be isometrically embedded in some Euclidean space after some α-snowflaking, see for example [DM90]. More results on embeddings of snowflakes of finite metric spaces into Euclidean spaces can be found in the survey article [Mae13].

3. Proofs of the main results

In the following two subsections we prove Theorem 1.1 and Theorem 1.2. In the final subsection we prove that if h is a degenerate snowflaking function, then there exists an infinite metric space whose h-snowflake isometrically embeds into the 2-dimensional Euclidean space.

Throughout this section V is an n-dimensional normed space with norm $\|\cdot\|$. For the remainder of this section we fix an inner product $\langle\cdot,\cdot\rangle$ on V for which John ellipsoid property holds. Namely, we have

$$B \subset B_V \subset \sqrt{n}B, \tag{3.1}$$

where B_V is the $\|\cdot\|$-unit ball and B is the the unit ball in the l_2 metric associated to the inner product $\langle\cdot,\cdot\rangle$. For $u, v \in V$ we denote the length given by the inner product by $uv := \sqrt{\langle u-v, u-v\rangle}$.

Given two vectors $u, v \in V$ let $\angle(u, v)$ be their angle with respect to the above inner product $\langle\cdot,\cdot\rangle$:

$$\angle(u, v) := \arccos \frac{\langle u, v\rangle}{\langle u, u\rangle\langle v, v\rangle}.$$

Given three points $x, y, z \in V$ we set $\angle_y(x, z) := \angle(x-y, z-y)$.

The proofs of Theorem 1.1 and Theorem 1.2 rely on the following Ramsey-theoretic result. Explicit bounds on the number of points that one can have in $\mathbb{R}^n$ without forming an angle larger than a given bound can be found in [EF83]. A proof of Lemma 3.2 can also be found in [KS11].

Lemma 3.2. *For any $n \in \mathbb{N}$ and $0 < \beta < \pi$ there is an $N \in \mathbb{N}$ such that if $S \subseteq \mathbb{R}^n$ has cardinality at least N, then there are distinct $x, y, z \in S$ such that $\beta \le \angle_z(x, y) \le \pi$.*

3.1. **Proof of Theorem 1.1.**

The proof of Theorem 1.1 as well as the proof of Theorem 1.2 combines two observations: snowflaking forbids the formation of large angles whereas the fact that we have many points forces such angles to exist. In the proof of Theorem 1.1 the special form of the snowflaking function allows us to directly prove a bound on the cardinality of the snowflaked space that can be embedded in the normed space.

Proof of Theorem 1.1. We start with an improved version of the triangle inequality. Let $\alpha \in (0, 1)$ and $x, y, z \in X$ with $d(x, z) \le d(z, y)$. We claim that

$$d^\alpha(x, y) \le d^\alpha(z, y) + \alpha d^\alpha(x, z). \tag{3.3}$$

Indeed, notice that by concavity the graph of the function $t \mapsto t^\alpha$ lies below its tangents. Hence by triangle inequality, we have

$$\begin{aligned} d^\alpha(x, y) &\le (d(x, z) + d(z, y))^\alpha \le (d(z, y))^\alpha + \alpha(d(z, y))^{\alpha-1} d(x, z) \\ &\le d^\alpha(z, y) + \alpha(d(x, z))^{\alpha-1} d(x, z) = d^\alpha(z, y) + \alpha d^\alpha(x, z). \end{aligned}$$

Fix n and α and let $\theta \in (0, \frac{\pi}{2})$ be such that $\sin\theta \leq \frac{1-\alpha}{4n}$. We let $N \in \mathbb{N}$ be the constant in Lemma 3.2 with $\beta = \pi - \theta$. We suppose towards a contradiction that (X, d) has cardinality N and that there is an isometric embedding ι of (X, d^α) into an n-dimensional normed space $(V, \|\cdot\|)$. By Lemma 3.2 there exist three isometrically embedded points $x, y, z \in \iota(X)$ such that $\pi - \theta < \angle_z(x, y) \leq \pi$.

We may assume that $\|x - z\| \leq \|y - z\|$. Then by (3.3) we have

$$\|x - y\| \leq \|z - y\| + \alpha\|x - z\|. \tag{3.4}$$

Let $z' \in \mathbb{R}^n$ be the orthogonal projection of z on the line passing through x and y, i.e. $z' := x + \frac{\langle z-x, y-x\rangle}{\|y-x\|^2}(y - x)$. Notice that z' is in between x and y, since $\angle_z(x, y) > \frac{\pi}{2}$. Hence, we have

$$\|x - y\| = \|x - z'\| + \|z' - y\| \geq \|x - z\| + \|z - y\| - 2\|z - z'\|. \tag{3.5}$$

On the one hand, from (3.4) and (3.5) we have

$$(1 - \alpha)\|x - z\| \leq 2\|z - z'\|. \tag{3.6}$$

On the other hand, by the choice of θ and the John Ellipsoid Theorem we get

$$\begin{aligned}\|z - z'\| &\leq \sqrt{n} zz' = \sqrt{n}\sin\angle_x(y, z)\cdot xz \leq \sqrt{n}\sin\theta\cdot xz\\ &\leq \sqrt{n}\frac{1-\alpha}{4n}\sqrt{n}\|x - z\| < \frac{1-\alpha}{2}\|x - z\|\end{aligned}$$

contradicting (3.6). □

3.2. **Proof of Theorem 1.2.** In the proof of Theorem 1.2 we shall use similar arguments as in the proof of Theorem 1.1. However, in the proof of Theorem 1.2 the choice of a sequence of points giving the contradiction depends not only on the snowflaking function h, but also on the first element of the sequence. Therefore no upper bound (depending on h and n) on the number of points that can be snowflake embedded can in general be obtained in Theorem 1.2.

Proof of Theorem 1.2. Suppose to the contrary that X is infinite and that there exists an isometric embedding $\iota\colon (X, h \circ d) \to V$ where V is an n-dimensional normed space. We divide our proof into two cases. An infinite bounded subset of $\mathbb{R}^n$ is not discrete, so one of the following holds:

(i) $\iota(X)$ is unbounded;
(ii) $\iota(X)$ is not discrete.

If (i) holds we will arrive at a contradiction with the condition (S4) of a snowflaking function. If (ii) holds, a contradiction follows with (S3).

Case (i): Suppose $\iota(X)$ is unbounded
Observe that (S4) implies the existence of a function $T\colon \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ such that for any $t > 0$ and $S \geq T(t)$ we have

$$\frac{t}{h(t)}\frac{h(S)}{S} \leq \frac{1}{2}. \tag{3.7}$$

Combining (3.7) with (S1) and (S2) we get, for any $t > 0$ and $S \geq T(t)$,

$$h(S+t) \leq h(S) + t\frac{h(S)}{S} = h(S) + \frac{t}{h(t)}\frac{h(S)}{S}h(t) \leq h(S) + \frac{1}{2}h(t). \tag{3.8}$$

Now fix $x_0, x_1 \in X$, $x_0 \neq x_1$. Since (X,d) is unbounded, there exists a point $x_2 \in X$ with $\angle_{\iota(x_2)}(\iota(x_0), \iota(x_1)) \leq \pi/2$ and $d(x_2, x_i) > T(d(x_0, x_1))$ for $i = 0, 1$. We continue inductively. Suppose $(x_i)_{i=0}^{N-1} \subset X$ have been chosen. Now we select $x_N \in X$ satisfying

$$\angle_{\iota(x_N)}(\iota(x_i), \iota(x_j)) \leq \pi/2 \text{ and } d(x_i, x_N) > T(d(x_i, x_j)), \quad \text{for all } i, j < N. \tag{3.9}$$

Let $\delta \in (0, \frac{\pi}{2})$ be such that $\sin\delta \leq \frac{1}{8n}$.

By Lemma 3.2 there exist x, y, z in $(\iota(x_\ell))_{\ell=0}^{\infty}$ such that $\angle_z(x,y) > \pi - \delta > \pi/2$. By the condition (3.9), we have that the point among x, y, z chosen last cannot be z; thus by symmetry, we may assume it is y. Hence, there exist $i, j, k \in \mathbb{N}$ with $k > \max\{i, j\}$ such that $x = \iota(x_j)$, $y = \iota(x_k)$, $z = \iota(x_i)$.

From (3.9), we have that $d(x_i, x_k) \geq T(d(x_j, x_i))$. Hence, since h is weakly increasing, from (3.8) we have

$$\|x - y\| = h(d(x_j, x_k)) \leq h(d(x_i, x_k) + d(x_j, x_i)) \leq \|z - y\| + \frac{1}{2}\|x - z\|. \tag{3.10}$$

We now continue almost verbatim as in the proof Theorem 1.1. Let z' be the orthogonal projection of z to the line passing through x and y. Notice that z' is in between x and y, since $\angle_z(x,y) > \frac{\pi}{2}$. Hence, we have

$$\|x - y\| = \|x - z'\| + \|z' - y\| \geq \|x - z\| + \|z - y\| - 2\|z - z'\|. \tag{3.11}$$

On the one hand, from (3.10) and (3.11) we have

$$\frac{1}{2}\|x - z\| \leq 2\|z - z'\|. \tag{3.12}$$

On the other hand, by the choice of δ and the John Ellipsoid Theorem we get

$$\begin{aligned} \|z - z'\| &\leq \sqrt{n}zz' = \sqrt{n}\sin\angle_x(y,z)\cdot xz \leq \sqrt{n}\sin\delta\cdot xz \\ &\leq \sqrt{n}\frac{1}{8n}\sqrt{n}\|x - z\| < \frac{1}{4}\|x - z\| \end{aligned}$$

contradicting (3.12).

Case (ii): Suppose $\iota(X)$ is not discrete
This time we observe that (S3) implies the existence of a function $\tilde{T}\colon \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ such

that $\tilde{T}(r) \le r$ for all r and for any $S > 0$ and $0 < t \le \tilde{T}(S)$ we have (3.7), and hence (3.8), using (S1) and (S2).

Let y be an accumulation point of X. First we select $x_0 \in X \setminus \{y\}$. Next we take a radius $r_0 > 0$ so that for all $y_1, y_2 \in B(y, r_0)$ we have both $\angle_{\iota(x_0)}(\iota(y_1), \iota(y_2)) \le \pi/2$ and $d(y_1, y_2) < \tilde{T}(d(x_0, y_i))$ for $i = 0, 1$. Now we select a point $x_1 \in B(y, r_0) \setminus \{y\}$. We continue inductively. Suppose $(x_i)_{i=0}^{N-1} \subset X$ have been chosen. Now we take a radius $r_{N-1} < r_{N-2}$ such that for all $y_1, y_2 \in B(y, r_{N-1})$ we have

(3.13)
$$\angle_{\iota(x_i)}(\iota(y_1), \iota(y_2)) \le \pi/2 \text{ and } d(y_1, y_2) < \tilde{T}(d(x_i, y_j)), \quad \text{for all } i < N, j = 1, 2.$$

Then we select a point $x_N \in B(y, r_{N-1}) \setminus \{y\}$.

With the points $\{x_i\}$ we now arrive at a contradiction with the same argument as in the case (i). Let δ be as in case (i). Again by Lemma 3.2 there exist x, y, z in $\{\iota(x_\ell)\}_{\ell \in \mathbb{N}}$ such that $\angle_z(x, y) > \pi - \delta$, but this time by the condition (3.13) z cannot be the first-chosen point. Thus, there exist $i, j, k \in \mathbb{N}$ with $k < \min\{i, j\}$ such that $x = \iota(x_j)$, $y = \iota(x_k)$, $z = \iota(x_i)$. From (3.13) and the definition of $\tilde{T}$ we get (3.10). Now we continue verbatim the proof in the case (i). □

3.3. **Necessity of (S3) and (S4).** We end this paper by showing that the conditions (S3) and (S4) of generalized snowflakes are indeed needed for Theorem 1.2 to hold.

Proposition 3.14. *Suppose $h\colon \mathbb{R}_\ge \to \mathbb{R}_\ge$ satisfies (S1) and (S2) but fails to satisfy (S3) or (S4). Then there is an infinite metric space (X, d) such that $(X, h \circ d)$ admits an isometric embeddeding into the 2-dimensional Euclidean space.*

Proof. We only treat the case where (S4) fails. The case where (S3) fails is proved in a similar way.

We will construct a sequence of points $(x_i)_{i=0}^\infty$ in $\mathbb{R}^2$ such that $(X, d) := (\{x_i\}_{i=1}^\infty, h^{-1} \circ d_E)$ is a metric space. For this purpose we now fix a sequence $(\alpha_i)_{i=1}^\infty$ of positive angles such that $\sum_{i=1}^\infty \alpha_i < \frac{\pi}{2}$. Depending on the sequence $(\alpha_i)_{i=1}^\infty$ and the function h we will construct an increasing sequence $(t_i)_{i=1}^\infty$ of positive real numbers that determines the Euclidean distance between x_{i-1} and x_i. For notational convenience we let $c(t) = \frac{h(t)}{t}$. Notice that by assumption $c(t) \searrow c > 0$ as $t \to \infty$.

For every $i \in \mathbb{N}$, consider the value

$$\eta_i(t) := 2c(t)(c(t) - c) - c(t)^2 \cos(\pi - \alpha_i) - c^2.$$

Since $\lim_{t\to\infty} \eta_i(t) = -c^2(\cos(\pi - \alpha_i) + 1) < 0$, we can choose $t_1 > 0$ so that $\eta_1(t_1) < 0$ and then for every $i > 1$ continue iteratively by selecting $t_i > t_{i-1}$ so that $\eta_i(t_i) < 0$. Consequently, for all $s, t \ge t_i$, since $c(s), c(t), c(s+t) \in [c, c(t_i)]$, we have the estimate

(3.15)
$$\begin{aligned} &c(s)(c(s+t) - c) + c(t)(c(s+t) - c) - (c(s)c(t)\cos(\pi - \alpha_i) + c(s+t)^2) \\ &\qquad \le 2c(t_i)(c(t_i) - c) - (c(t_i)^2 \cos(\pi - \alpha_i) + c^2) = \eta_i(t_i) < 0. \end{aligned}$$

Now, using the sequences $(\alpha_i)_{i=1}^\infty$ and $(t_i)_{i=1}^\infty$ we define the sequence $(x_i)_{i\in\mathbb{N}}$ as follows. We set $x_0 := (0,0)$, $x_1 := (h(t_1),0)$, and inductively for $n \geq 2$ declare

$$x_n := x_{n-1} + \left(h(t_n)\cos(\sum_{j=1}^{n-1}\alpha_j), h(t_n)\sin(\sum_{j=1}^{n-1}\alpha_j)\right).$$

In order to see that (X,d) is a metric space we need to check that the triangle inequality holds. For this purpose let $0 \leq i < j < k$ be three integers. Let d_E be the Euclidean metric. Since $\sum_{i=1}^\infty \alpha_i < \frac{\pi}{2}$ we have that

$$d_E(x_i,x_k) \geq \max\{d_E(x_i,x_j), d_E(x_j,x_k)\}.$$

Therefore the only nontrivial inequality that we have to verify is

$$h^{-1}(d_E(x_i,x_k)) \leq h^{-1}(d_E(x_j,x_k)) + h^{-1}(d_E(x_j,x_k)).$$

When we denote $s := h^{-1}(d_E(x_i,x_j))$ and $t := h^{-1}(d_E(x_j,x_k))$ the above inequality is equivalent to

$$d_E(x_i,x_k) \leq h(s+t). \tag{3.16}$$

For all $r_1, r_2 > 0$ by (S1) and (S2) we have $h(r_1) \leq h(r_1+r_2) - cr_2$, which implies

$$c(r_1) - c(r_1+r_2) \leq \frac{r_2}{r_1}(c(r_1+r_2) - c). \tag{3.17}$$

Because of $\sum_{i=1}^\infty \alpha_i < \frac{\pi}{2}$ we have $s,t \geq t_j$. Therefore, by applying the law of cosines, (3.17) with $(r_1,r_2) = (t,s)$ and with $(r_1,r_2) = (s,t)$, and finally (3.15) we obtain

$$\begin{aligned}
d_E(x_i,x_k)^2 - h(s+t)^2 &= d_E(x_i,x_j)^2 + d_E(x_j,x_k)^2 \\
&\quad - 2d_E(x_i,x_j)d_E(x_j,x_k)\cos(\angle_{x_j}(x_i,x_k)) - h(s+t)^2 \\
&\leq h(s)^2 + h(t)^2 - 2h(s)h(t)\cos(\pi-\alpha_j) - h(s+t)^2 \\
&= s^2c(s)^2 + t^2c(t)^2 - 2stc(s)c(t)\cos(\pi-\alpha_j) - (s+t)^2c(s+t)^2 \\
&= s^2(c(s)^2 - c(s+t)^2) + t^2(c(t)^2 - c(s+t)^2) \\
&\quad - 2st(c(s)c(t)\cos(\pi-\alpha_j) + c(s+t)^2) \\
&= s^2(c(s)+c(s+t))(c(s)-c(s+t)) \\
&\quad + t^2(c(t)-c(s+t))(c(t)-c(s+t)) \\
&\quad - 2st(c(s)c(t)\cos(\pi-\alpha_j) + c(s+t)^2) \\
\text{(Using (3.17))} \qquad &\leq 2st\big(c(s)(c(s+t)-c) + c(t)(c(s+t)-c) \\
&\quad - (c(s)c(t)\cos(\pi-\alpha_j) + c(s+t)^2)\big) \\
\text{(Using (3.15))} \qquad &\leq 0
\end{aligned}$$

and thus (3.16) holds. □

Remark 3.18. The proof of Proposition 3.14 can be modified to show that there is a snowflake function h (satisfying all the conditions (S1)–(S4)) such that for every $n \in \mathbb{N}$ there exists a metric space (X_n, d_n) with cardinality n so that $(X_n, h \circ d_n)$ embeds isometrically into the 2-dimensional Euclidean space.

Indeed, we can start by defining $h(t) = \sqrt{t}$ on $[0,1]$ so that (S1) and (S3) are satisfied. By the proof of Proposition 3.14 we know that we can then choose $T_3 > 1$ such that with the definition $h(t) = h(1) + \frac{1}{2}(t-1)$ on $[1, T_3]$ we can find a 3 point metric space (X_3, d_3) so that all the distances are between 1 and T_3 and $(X_3, h \circ d_3)$ embeds isometrically into the 2-dimensional Euclidean space. Continuing inductively we can choose for every $n > 3$ a real number $T_n > T_{n-1} + 1$ such that with the definition $h(t) = h(T_{n-1}) + \frac{1}{n}(t - T_{n-1})$ on $[T_{n-1}, T_n]$ we can find an n point metric space (X_n, d_n) such that all the distances are between T_{n-1} and T_n and $(X_n, h \circ d_n)$ embeds isometrically into the 2-dimensional Euclidean space. The conditions (S2) and (S4) are now also satisfied.

References

[Ass83] Patrice Assouad, *Plongements lipschitziens dans* $\mathbf{R}^n$, Bull. Soc. Math. France **111** (1983), no. 4, 429–448.

[Ban55] Stefan Banach, *Théorie des opérations linéaires*, Chelsea Publishing Co., New York, 1955.

[CK10] Jeff Cheeger and Bruce Kleiner, *Differentiating maps into* L^1*, and the geometry of BV functions*, Ann. of Math. (2) **171** (2010), no. 2, 1347–1385.

[DL10] Michel Marie Deza and Monique Laurent, *Geometry of cuts and metrics*, Algorithms and Combinatorics, vol. 15, Springer, Heidelberg, 2010.

[DM90] Michel Marie Deza and Hiroshi Maehara, *Metric transforms and Euclidean embeddings*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 661–671.

[DS13] Guy David and Marie Snipes, *A non-probabilistic proof of the Assouad embedding theorem with bounds on the dimension*, Anal. Geom. Metr. Spaces **1** (2013), 36–41.

[EF83] Paul Erdős and Zoltán Füredi, *The greatest angle among n points in the d-dimensional Euclidean space*, Combinatorial mathematics (Marseille-Luminy, 1981), North-Holland Math. Stud., vol. 75, North-Holland, Amsterdam, 1983, pp. 275–283.

[Fré10] Maurice Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann. **68** (1910), no. 2, 145–168.

[Hei01] Juha Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[Hei03] ———, *Geometric embeddings of metric spaces*, Report. University of Jyväskylä Department of Mathematics and Statistics, vol. 90, University of Jyväskylä, Jyväskylä, 2003.

[KS11] Antti Käenmäki and Ville Suomala, *Nonsymmetric conical upper density and k-porosity*, Trans. Amer. Math. Soc. **363** (2011), no. 3, 1183–1195.

[Kur35] Casimir Kuratowski, *Quelques problèmes concernant les espaces métriques nonséparables*, Fund. Math. **25** (1935), 534–545.

[Laa02] Tomi J. Laakso, *Plane with* A_∞*-weighted metric not bi-Lipschitz embeddable to* $\mathbb{R}^N$, Bull. London Math. Soc. **34** (2002), no. 6, 667–676.

[LN14] Vincent Lafforgue and Assaf Naor, *A doubling subset of* L_p *for* $p > 2$ *that is inherently infinite dimensional*, Geom. Dedicata **172** (2014), 387–398.

[LP01] Urs Lang and Conrad Plaut, *Bilipschitz embeddings of metric spaces into space forms*, Geom. Dedicata **87** (2001), no. 1-3, 285–307.

[Luo96] Kerkko Luosto, *Ultrametric spaces bi-Lipschitz embeddable in* $\mathbf{R}^n$, Fund. Math. **150** (1996), no. 1, 25–42.

[Mae13] Hiroshi Maehara, *Euclidean embeddings of finite metric spaces*, Discrete Math. **313** (2013), no. 23, 2848–2856.

[NN12] Assaf Naor and Ofer Neiman, *Assouad's theorem with dimension independent of the snowflaking*, Rev. Mat. Iberoam. **28** (2012), no. 4, 1123–1142.

[Sem96a] Stephen Semmes, *Good metric spaces without good parameterizations*, Rev. Mat. Iberoamericana **12** (1996), no. 1, 187–275.

[Sem96b] ———, *On the nonexistence of bi-Lipschitz parameterizations and geometric problems about* A_∞*-weights*, Rev. Mat. Iberoamericana **12** (1996), no. 2, 337–410.

[Sem99] ———, *Bilipschitz embeddings of metric spaces into Euclidean spaces*, Publ. Mat. **43** (1999), no. 2, 571–653.

[Seo11] Jeehyeon Seo, *A characterization of bi-Lipschitz embeddable metric spaces in terms of local bi-Lipschitz embeddability*, Math. Res. Lett. **18** (2011), no. 6, 1179–1202.

[Ury27] Paul Urysohn, *Sur un espace métrique universel*, Bull. Sc. Math. 2e série **51** (1927), 43–64.

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