DIVISORIAL INERTIA AND CENTRAL ELEMENTS IN BRAID GROUPS

F.CALLEGARO, G.GAIFFI, AND P. LOCHAK

Abstract. Given a complex reflection group W, we will show how the generators of the centers of the parabolic subgroups of the pure braid group $P(W)$ can be represented by loops around irreducible divisors of the corresponding minimal De Concini-Procesi model $\overline{X}_W$. We will also show that a more subtle construction gives representations of the generators of the centers of the parabolic subgroups of the braid group $B(W)$ as loops in the (not smooth) quotient variety $\overline{X}_W/W$.

1. Introduction

This paper explores the connection between the minimal De Concini-Procesi wonderful model $\overline{X}_W$ associated with a complex reflection group W and the pure braid group $P(W)$: in particular we will show that the generators of the centers of the parabolic subgroups of $P(W)$ can be described by loops around the boundary components of $\overline{X}_W$.

Moreover, we will focus on the quotient $\overline{Y}_W$ of $\overline{X}_W$ with respect to the action of W. This is not a smooth variety, but we can still show that certain loops around the boundary components of $\overline{Y}_W$ represent the generators of the centers of the parabolic subgroups of $B(W)$, the braid group associated to W.

When a Garside structure is available on $B(W)$, Garside elements of parabolic subgroups of $B(W)$ will come into play, since the above mentioned loops represent their smallest central powers.

Let us describe our results more in detail. Given a subspace arrangement $\mathcal{A}$ in a vector space V over a field $\mathbb{K}$, with $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, De Concini and Procesi described in [DCP95a], [DCP95b] how to construct *wonderful models* for the complement of the arrangement. This construction was at first motivated by an approach to Drinfeld special solutions for Khniznik-Zamolodchikov equation (see [Dri91]). Moreover, in [DCP95a] it was shown, using the cohomology description of these models, that the mixed Hodge structure and the rational homotopy type of the complement of a complex subspace arrangement depend only on its intersection lattice.

Then real and complex De Concini-Procesi models turned out to play a relevant role in several fields of mathematical research: toric and tropical geometry (see [FY04], [FS05]), moduli spaces of curves and configuration spaces (see for instance [EHKR10]), box splines and index theory (see the exposition in [DCP10]), discrete geometry and combinatorics (see [Fei05], [Gai15], [Gai16]), representation theory of reflection groups (see for instance [Hen04], [HR08], [Rai09], [CG15]).

Among the models associated to an arrangement $\mathcal{A}$ there is a minimal one: we will recall the details of its construction in Section 2, in the case of the hyperplane arrangement associated to an irreducible complex reflection group W. Here we point out the fact that this model $\overline{X}_W$ adds to the complement of the arrangement a boundary which is a normal crossing divisor: its irreducible components are indexed by the parabolic subgroups of W. Now a loop in $\overline{X}_W$ around the irreducible component $\mathcal{D}_A$ (the inertia) associated to a parabolic subgroup W_A can be identified with a loop in the complement of the arrangement, that is to say, up to the choice of a fixed base point, with an element of $P(W)$. In Sections 4 and 5 we will prove (see Theorem 5.1 and Corollary 5.1) that this element is in fact the generator of the center of the parabolic subgroup of $P(W_A)$: in particular, when the parabolic subgroup is W itself, the inertia represents the generator of $Z(W)$.

In Section 6 we deal with the quotient $\overline{Y}_W = \overline{X}_W/W$. This is not a smooth variety, yet for every parabolic subgroup W_A of W we can consider in $\overline{Y}_W$ the image $\mathcal{L}_A$ of the divisor $\mathcal{D}_A$ via the quotient map. The subvariety $\mathcal{L}_A$ is not smooth too, but we will indicate a dense set of smooth points such that the inertia 'close' to these points represents the generator of the center of the parabolic subgroup $B(W_A)$ (see Theorem 6.1). We will call these smooth points the *Springer generic points* of $\mathcal{L}_A$, since the Springer regular elements defined in [Spr74] come into play.

We remark that to prove the above mentioned results we use the description of the center of $P(W)$ and $B(W)$ (and of their parabolic subgroups) provided by theorems from [BMR98], [Bes15], [DMM11] that are recalled in Section 3.

Finally we would like to mention that this paper may be considered a first step, which in our opinion turns out to be of independent topological interest, of a project whose aim is to transpose and adapt Grothendieck-Teichmüller theory to the setting of complex braid groups and their classifying spaces, thus obtaining what could be dubbed a 'Grothendieck-Artin theory' and an Artin (or Artin-Brieskorn) 'lego'. The ingredients of this project and a sketch of its first steps can be read in the last sections of [CGL15].

2. Minimal models

In [DCP95a, DCP95b] De Concini and Procesi introduced and described what they called *wonderful models* associated with subspace arrangements. We briefly recall their construction in the special case of *hyperplane* arrangements. In the next sections we will further specialize to arrangements associated with complex reflection groups.

2.1. **Irreducible subspaces.** Let V be a complex finite dimensional vector space which we identify with its dual by means of a given Hermitian nondegenerate pairing. An *hyperplane arrangement* in V is finite collection $\mathcal{A}$ of affine hyperplanes in V. The arrangement $\mathcal{A}$ is a *central arrangement* if $\cap\mathcal{A} \neq \emptyset$. In this case we assume that $O \in \cap\mathcal{A}$. Let $L(\mathcal{A})$ be the poset of all possible non-empty intersections of elements of $\mathcal{A}$, ordered by reverse inclusion. We call $\mathcal{A}$ *essential* if the maximal elements of $L(\mathcal{A})$ are points. In particular if $\mathcal{A}$ is central we have that $\mathcal{A}$ is essential if and only if $\cap\mathcal{A} = \{O\}$. Let $\mathcal{A}$ be a central hyperplane arrangement in V. For every subspace $B \subset V$, we write $B^\perp$ for its orthogonal and denote $\mathcal{A}^\perp$ the arrangement of *lines* in V, dual to $\mathcal{A}$:

$$\mathcal{A}^\perp = \{A^\perp \mid A \in \mathcal{A}\};$$

finally let $\mathcal{C}_\mathcal{A}$ (or $\mathcal{C}(\mathcal{A})$) be the closure of $\mathcal{A}^\perp$ in V under the sum.

Definition 2.1. *Given a subspace $U \in \mathcal{C}_{\mathcal{A}}$, a* decomposition *of U in $\mathcal{C}_{\mathcal{A}}$ is a collection $\{U_1, \dots, U_k\}$ $(k > 1)$ of non zero subspaces in $\mathcal{C}_{\mathcal{A}}$ such that*

(1) *$U = U_1 \oplus \cdots \oplus U_k$;*
(2) *for every subspace $A \in \mathcal{C}_{\mathcal{A}}$ such that $A \subset U$, we have $A \cap U_1, \dots, A \cap U_k \in \mathcal{C}_{\mathcal{A}}$ and $A = (A \cap U_1) \oplus \cdots \oplus (A \cap U_k)$.*

Definition 2.2 (Irreducible subspace and notation in the case of reflection groups)**.** *A nonzero subspace $F \in \mathcal{C}_{\mathcal{A}}$ which does not admit a decomposition is called* irreducible *and the set of irreducible subspaces is denoted $\mathcal{F}_{\mathcal{A}}$ (or $\mathcal{F}(\mathcal{A})$, or just $\mathcal{F}$). In the case when $\mathcal{A} = \mathcal{A}_W$ is the hyperplane arrangement associated with a complex reflection group W we write $\mathcal{F}_W$ (resp. $\mathcal{F}(W)$) instead of $\mathcal{F}_{\mathcal{A}_W}$ (resp. $\mathcal{F}(\mathcal{A}_W)$).*

Remark 2.1. *Consider a root system Φ in a complexified vector space V and its associated root arrangement, i.e. $\mathcal{A}$ is the (complex) hyperplane arrangement defined by the hyperplanes orthogonal to the roots in Φ. Then the building set of irreducibles is the set of the subspaces spanned by the irreducible root subsystems of Φ (see* [Yuz97]*).*

Definition 2.3. *A subset $\mathcal{S} \subset \mathcal{F}_{\mathcal{A}}$ is called ($\mathcal{F}_{\mathcal{A}}$-)nested, if given any subset $\{U_1, \dots, U_h\} \subseteq \mathcal{S}$ (with $h > 1$) of pairwise non comparable elements, we have $U_1 + \cdots + U_h \notin \mathcal{F}_{\mathcal{A}}$.*

Example 2.1. *Let us consider the case of the symmetric group $W = S_n$ and let $\mathcal{A}_{S_n}$ be its corresponding* essential *arrangement in $V = \mathbb{C}^n / < (1, 1, ..., 1) >$, i.e. we consider the hyperplanes defined by the equations $x_i - x_j = 0$ in V.*

Then $\mathcal{F}(S_n)$ consists of all the subspaces in V spanned by the irreducible *root subsystems, that is by the subspaces whose orthogonals are described by equations of the form $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ with $k \geq 2$. Therefore there is a one-to-one correspondence between the elements of $\mathcal{F}(S_n)$ and the subsets of $\{1, \cdots, n\}$ with at least 2 elements: to an $A \in \mathcal{F}(S_n)$ whose orthogonal is described by the equations $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ there corresponds the set $\{i_1, i_2, \dots, i_k\}$. As a consequence, a $\mathcal{F}(S_n)$-nested set $\mathcal{S}$ corresponds to a set of subsets of $\{1, \cdots, n\}$ with the property that its elements have cardinality ≥ 2 and if I and J belong to $\mathcal{S}$ then either $I \cap J = \emptyset$ or one of the two sets is included into the other.*

Example 2.2. *Let us consider the real reflection group W_{D_n} associated with the root system of type D_n $(n \geq 4)$. The reflecting hyperplanes have equations $x_i - x_j = 0$ and $x_i + x_j = 0$ in $V = \mathbb{C}^n$.*

The subspaces in $\mathcal{F}(W_{D_n})$ are all the subspaces of V spanned by the irreducible *root subsystems. They can be partitioned into two families. The subspaces in the first family are the* strong subspaces *$\overline{H}_{i_1, i_2, \dots, i_k}$ whose orthogonals are described by equations of the form $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0$ with $k \geq 3$ (if $k = 2$ this subspace is not irreducible). We can represent them by associating to $\overline{H}_{i_1, i_2, \dots, i_t}$ the subset $\{0, i_1, i_2, ..., i_k\}$ of $\{0, 1, ..., n\}$.*

The second family is made by the weak subspaces *$H_{i_1, i_2, \dots, i_k}(\epsilon_2, ..., \epsilon_k)$ whose orthogonals have equations of the form $x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k}$ where $\epsilon_i = \pm 1$ and $k \geq 2$.*

Let us suppose that $i_1 < i_2 < \cdots < i_k$; then we can represent these weak subspaces by associating to $H_{i_1, i_2, \dots, i_t}(\epsilon_2, ..., \epsilon_k)$ the weighted *subset $\{i_1, \epsilon_2 i_2, ..., \epsilon_k i_k\}$ of $\{1, ..., n\}$.*

According to the representation of the irreducibles by subsets of $\{0,1,....,n\}$ *described here, a* $\mathcal{F}(W_{D_n})$*-nested set is represented by a set* $\{A_1,...,A_m\}$ *of (possibly weighted) subsets of* $\{0,...,n\}$ *with the following properties:*

- *the subsets that contain 0 are not weighted; they are linearly ordered by inclusion;*
- *the subsets that do not contain 0 are weighted;*
- *if in a nested set there is no pair of type* $\{i,-j\},\{i,j\}$*, then for any pair of subsets* A_i, A_j*, we have that, forgetting their weights, they are one included into the other or disjoint; if* A_i, A_j *both represent weak subspaces one included into the other (say* $A_i \subset A_j$*), then their weights must be compatible. This means that, up to the multiplication of all the weights of* A_i *by* ± 1*, the weights associated to the same numbers must be equal.*
- *in a nested set there may be one (and only one) pair* $\{i,-j\},\{i,j\}$ *and in this case any other element* B *of the nested set satisfies (forgetting its weights)* $B\cap\{i,j\}=\emptyset$ *or* $\{0,i,j\}\subsetneq B$*.*

Example 2.3. *Let us consider the real reflection group* W_{G_2} *associated with the root system* G_2*. Since this is a two dimensional root system, the irreducible subspaces are the subspaces spanned by the roots and the whole space* $V=\mathbb{C}^2$*.*

Therefore the $\mathcal{F}(G_{G_2})$*-nested are the sets of cardinality 1 made by one irreducible subspace and the sets of cardinality 2 made by* V *and by the subspace spanned by one root.*

2.2. **Definition of the models and main properties.** Let $\mathcal{A}$ now be a (central) *hyperplane* arrangement in the complex space V. We denote its complement by $X_{\mathcal{A}}$ (or $X(\mathcal{A})$) and again write simply X_W (or $X(W)$) in the case of a complex reflection group W. Then we can consider the embedding

$$i : X(\mathcal{A}) \to V \times \prod_{D\in\mathcal{F}(\mathcal{A})} \mathbb{P}(V/D^{\perp})$$

where the first coordinate is the inclusion and the map from $X(\mathcal{A})$ to $\mathbb{P}(V/D^{\perp})$ is the restriction of the canonical projection $V\setminus D^{\perp}\to\mathbb{P}(V/D^{\perp})$.

Definition 2.4. *The minimal wonderful model* $\overline{X}(\mathcal{A})$ *is obtained by taking the closure of the image of the map* i*.*

Remark 2.2. *Actually in* [DCP95a] *not one but many wonderful models are associated with a given arrangement (see* [GS14] *for a classification in the case of root hyperplane arrangements); here we will focus on the minimal one. Note that among these models there is always a maximal one, obtained by substituting* $\mathcal{C}(A)$ *for* $\mathcal{F}(\mathcal{A})$ *in the definition above.*

De Concini and Procesi proved in [DCP95a] that the complement $\mathcal{D}$ of $X(\mathcal{A})$ in $\overline{X}(\mathcal{A})$ is a divisor with strict normal crossings whose irreducible components are naturally in bijective correspondence with the elements of $\mathcal{F}(\mathcal{A})$. They are denoted by $\mathcal{D}_F$ (or $\mathcal{D}(F)$) for $F\in\mathcal{F}(\mathcal{A})$.

Next if π is the projection of $\overline{X}(\mathcal{A})$ onto the first component V, one observes that the restriction of π to $X(\mathcal{A})$ is an isomorphism and $\mathcal{D}(F)$ can be characterized as the unique irreducible component of the divisor at infinity $\mathcal{D}$ such that $\pi(\mathcal{D}_F)=F^{\perp}$. A complete characterization of the boundary divisor $\mathcal{D}$ is then afforded by the observation that, if we consider a collection $\mathcal{T}$ of subspaces in $\mathcal{F}_{\mathcal{A}}$, then

$$\mathcal{D}_{\mathcal{T}} = \bigcap_{A \in \mathcal{T}} \mathcal{D}_A$$

is non empty if and only if $\mathcal{T}$ is $\mathcal{F}_{\mathcal{A}}$-nested; moreover in that case $\mathcal{D}_{\mathcal{T}}$ is smooth and irreducible, and it is part of the stratification associated with $\mathcal{D}$.

Nowadays this construction of the De Concini-Procesi wonderful models can be viewed as a member of a family of constructions which, starting from a 'good' stratified variety, produce models by blowing up a suitable subset of the strata. Among these constructions we recall the models described by Fulton-MacPherson in [FM94], and by MacPherson and Procesi in [MP98] (see also [Hu03], [Gai03], [Uly02], [Li09]). An interesting survey including tropical compactifications can be found in Denham's paper [Den14].

From this point of view, one considers V as a variety stratified by the subspaces in $\mathcal{F}_{\mathcal{A}}^{\perp}$ and the model $\overline{X}(\mathcal{A})$ is obtained by blowing up the strata in order of *increasing* dimensions, proceeding as follows: First choose an ordering $A_0, A_1, ..., A_k$ of the subspaces in $\mathcal{F}_{\mathcal{A}}^{\perp}$ which respects dimensions, i.e. such that $dim(A_i) \leq dim(A_j)$ if $i \leq j$. Assume that $V \in \mathcal{F}_{\mathcal{A}}^{\perp}$ (which will always be the case in the sequel) and so that $A_0 = \{0\}$. Then one starts by blowing up the stratified variety V at the origin 0, obtaining a variety X_0; at the next step one blows up X_0 along the proper transform of A_1 in order to obtain X_1, and so on... The end result, after a finite number of steps (=the cardinality of $\mathcal{F}_{\mathcal{A}}$) is the wonderful model $\overline{X}(\mathcal{A})$.

3. The center of the complex braid groups

From now on we restrict attention to arrangements arising from complex reflection groups. As mentioned in the introduction, in order to describe the inertia elements we will need a few pieces of information on the centers of the attending braid groups, which we recall in this short group theoretic section. Let W be an irreducible finite complex reflection group and let $B = B(W)$ and $P = P(W)$ denote as usual the associated full and pure braid groups. We write $Z(G)$ for the center of a group G.

In [BMR98] central elements $\beta \in Z(B)$ and $\pi \in Z(P)$ were introduced; they are of infinite order, with $\beta^{|Z(W)|} = \pi$. We recall the following results from [BMR98, Bes15, DMM11]:

Theorem 3.1. *The center $Z(B(W))$ is infinite cyclic, generated by β.*

Theorem 3.2. *The center $Z(P(W))$ infinite cyclic, generated by π.*

Theorem 3.3. *There is a short exact sequence:*

$$1 \to Z(P(W)) \to Z(B(W)) \to Z(W) \to 1. \tag{1}$$

We also recall from [BMR98] that the center $Z(W)$ has order

$$|Z(W)| = \gcd(d_1, d_2, \ldots, d_r)$$

where $d_1, \ldots, d_r$ are the degrees of W.

Theorems 3.1, 3.2 and 3.3 were conjectured in [BMR98] and proved there for infinite series $G(de, e, n)$ and rank 2 cases. Moreover in [BMR98] Theorem 3.1 is proven for all Shephard groups. Some of the remaining cases of Theorem 3.1 are proved in [Bes15] and an argument due to Bessis and reported in [DMM11] completes the proof of Theorem 3.1. Theorem 3.3 is proven in [DMM11] and together with Theorem 3.1 it implies Theorem 3.2.

Remark 3.1. *The center of the irreducible Coxeter groups of type A_n, D_{2n-1} and E_6 is trivial, while it is isomorphic to $\mathbb{Z}/2$ for all the other irreducible Coxeter groups. The cardinality of the center of the other irreducible finite* complex *reflection groups can be determined using the table of the degrees given in* [BMR98].

We recall that if n is the rank of a complex reflection group W, the latter is called *well generated* if it can be generated by n reflections. In general, for a well generated complex braid group B, there are many monoids such that B can be presented as a group of fractions of the monoid. In several cases these monoids admit a Garside structure, that in general is not unique.

For example we have the following result for Artin-Tits groups:

Theorem 3.4 ([DDG+15, IX, Prop. 1.29])**.** *Assume that (W, Σ) is a Coxeter system of spherical type and B (resp. B^+) is the associated Artin-Tits group (resp. monoid). Let Δ be the lifting of the longest element of W. Then (B^+, Δ) is a Garside monoid, and B is a Garside group. The element Δ is the right-lcm of Σ, which is the atom set of B^+, and $\mathrm{Div}(\Delta)$ is the smallest Garside family of B^+ containing* 1.

Morover, with respect to this Garside structure, we have:

Proposition 3.1. [DDG+15, IX, Corollary 1.39] *The center of an irreducible Artin-Tits group of spherical type is infinite cyclic, generated by the smallest central power of the Garside element Δ.*

All the finite Coxeter groups are well generated; actually the only irreducible complex reflection groups which are *not* well generated are $G(de, e, n)$ for $d \neq 1$ and $e \neq 1$, some groups of rank 2 (namely G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19} and G_{22}), and G_{31}. Now if W is well generated, $B(W)$ can be equipped with another Garside structure, unrelated to the previous one: for any Coxeter element $c \in W$ we can define a *dual braid monoid* Garside structure on $B(W)$, see [Bes15, Theorem 8.2] and β appears as the *smallest central power of the Garside element* with respect to that structure (see [Bes15, Theorem 12.3]).

Example 3.1. *In the classical braid group on n strands Br_n, let σ_i, $i = 1, \ldots, n-1$ be the standard set of generators. The Garside element according to the natural Garside structure reads*

$$\Delta = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1),$$

representing a global half twist. Let $c \in \mathfrak{S}_n$ be the Coxeter element $(1, 2, \ldots, n)$. The Garside element according to the dual *braid monoid Garside structure is*

$$\Delta^* = a_{1,2}\cdots a_{n-1,n},$$

where $a_{i,j} = \sigma_i \cdots \sigma_{j-2}\sigma_{j-1}{\sigma_{j-2}}^{-1}\cdots\sigma_i^{-1}$. So we can rewrite Δ^* as

$$\Delta^* = \sigma_1 \cdots \sigma_{n-1}.$$

Setting the n points at the vertices of a regular n-gon, Δ^* thus represents a $1/n$-th twist (see for example [DDG+15, I, Sec. 1.3 and IX, Prop. 2.7]*). This confirms that $\Delta^2 = (\Delta^*)^n$ is a generator of the center of Br_n. Since Δ^2 belongs to the pure braid group PBr_n, it is also a generator of its center.*

Example 3.2. *The Artin group of type D_n is the braid group $B(W)$ associated to the reflection group $W = W_{D_n}$, also denoted by $G(2,2,n)$ in the Shephard-Todd classification. The group $B(W_{D_n})$ is generated by the elements $s_1, s_1', s_2, \dots, s_{n-1}$ with $s_1, \dots, s_{n-1}$ and $s_1', s_2, \dots, s_{n-1}$ satisfying the relations of Br_n, together with $s_1 s_1' = s_1' s_1$.*

The Garside element is

$$\Delta = (s_1 s_1' s_2 \cdots s_{n-1})^{n-1}$$

and the center of the full braid group is generated by Δ if n is even, by Δ^2 if n is odd (see Table 5 in [BMR98, Appendix I]*). The center of the pure Artin-Tits group $P(W_{D_n})$ is generated by Δ^2. According to* [Bes03, Section 5.1] *the Garside element corresponding to a dual Garside monoid structure for D_n is*

$$\Delta^* = (s_1 s_1' s_3 s_5 \cdots)(s_2 s_4 \cdots)$$

and the image of Δ^* in W is the Coxeter element c.

Example 3.3. *The Artin group of type G_2 is the braid group associated to the reflection group W_{G_2}. The group $B(W_{G_2})$ is generated by two elements s, t satisfying the relation $(st)^3 = (ts)^3$. The Garside element is $\Delta = (st)^3$ (see Table 5 in* [BMR98, Appendix I]*) which is also the generator of the center. The center of the pure Artin-Tits group $P(W_{G_2})$ is generated by Δ^2, while the Garside element corresponding to a dual Garside monoid structure (see* [Bes03, Section 5.1]*) is $\Delta^* = st$, whose image in the Coxeter group is the Coxeter element c.*

We remark that in general there is no relation between the natural Garside structure (that is known only for finite type Artin-Tits groups) and the dual Garside structure (that are defined for all well generated complex braid groups). Moreover, several dual Garside structures can be defined, one for every Coxeter element. Nevertheless, the Garside elements in those structures are related, since their smallest central powers generate the center, that is infinite cyclic.

4. Inertia and the center

Let again W be an irreducible complex reflection group, $\mathcal{A} = \mathcal{A}_W$ the corresponding hyperplane arrangement which we assume to be essential (see the definition in Section 2.1), in the complex vector space V of dimension n. We write as usual $X = X_W = V \setminus \mathcal{A}_W$, $\overline{X} = \overline{X}_W$ for the associated minimal wonderful model, obtained by a finite sequence of blowups of V viewed as a stratified variety, along strata with non decreasing dimensions. We let $P = P(W) = \pi_1(X_W)$ denote the pure braid group associated with W.

In particular, let X_0 be the first step in this process, namely the blowup of the space V at its origin 0; let $\mathcal{D}_0 \subset X_0$ be the corresponding exceptional divisor. We write $\tilde{A}$ for the proper transform in X_0 of a subspace $A \subset V$ and identify X with the complement in X_0 of the proper transforms of the hyperplanes in $\mathcal{A}$ and of $\mathcal{D}_0$:

$$X \simeq X_0 \setminus \left(\mathcal{D}_0 \cup \bigcup_{H \in \mathcal{A}} \tilde{H} \right).$$

Using this we denote by $z \in P$ the topological *inertia* around the divisor $\mathcal{D}_0$. It can be viewed as the homotopy class of a counterclockwise loop in $X_0 \setminus \left(\bigcup_{H \in \mathcal{A}} \tilde{H} \right)$ around $\mathcal{D}_0$, identified with a loop in X (cf. e.g. [BMR98, Appendix I]). Here we use the natural complex orientation of the normal bundle of a hypersurface in a complex variety. Note that z is *a priori* defined, as it should be, as a *conjugacy class* in the pure braid group P; we did not specify a basepoint for the fundamental group $\pi_1(X) \simeq P$. However z will be shown to be central in P and so *a posteriori* it turns out to be well-defined as an element of P. Note as well that z also represents the inertia around the divisor $\mathcal{D}_V$ in $\overline{X}$: it is the homotopy class of a loop in the big open part of X_0, which identifies with that of $\overline{X}$; indeed they both identify with X.

We recall from the definition of blow up (see for example [GH78, Chapter 1.4]) that X_0 is defined as the closure of the image of the map

$$V \setminus \{O\} \to V \times \mathbb{P}(V),$$

therefore there is a well-defined projection $\pi : X_0 \to \mathbb{P}(V)$. It defines a line bundle

$$\begin{tikzcd} \mathbb{C} \arrow[r] & X_0 \arrow[d, "\pi"] \\ & \mathbb{P}(V) \end{tikzcd}$$

whose 0-section is precisely the divisor $\mathcal{D}_0$. This line bundle is the normal bundle of $\mathcal{D}_0$ in X_0 and a representative of z is given by a loop around the origin in the fiber at a generic point.

If we fix a hyperplane $H \in \mathcal{A}$, we can restrict the line bundle to the complement of $\mathbb{P}(H)$ in $\mathbb{P}(V)$, which is affine. The fibre over any point $[v] \in \mathbb{P}(V) \setminus \mathbb{P}(H)$ is a line $l \times \{[v]\} = \{(\lambda v, [v]) \mid \lambda \in \mathbb{C}\} \subset V \times \mathbb{P}(V)$. We can fix a translation $H + \delta$ of H in V that doesn't contain the origin. Then $H + \delta$ intersects the line l in a point $p_{[v]} = l \cap (H + \delta)$. This defines a nonzero section of π that trivializes our restricted line bundle and we thus get a *trivial* bundle:

$$\begin{tikzcd} \mathbb{C} \arrow[r] & X_0 \setminus \tilde{H} \arrow[d, "\pi"] \\ & \mathbb{P}(V) \setminus \mathbb{P}(H). \end{tikzcd}$$

Now recalling that $X \subset V$, consider the restriction of π to the preimage of $\mathbb{P}(X)$, namely:

$$\pi : X_0 \setminus \big(\bigcup_{H \in \mathcal{A}} \tilde{H} \big) \to \mathbb{P}(X).$$

From the argument above, since X is contained in the complement of an hyperplane (simply choose any of the hyperplanes in the arrangement $\mathcal{A}$), we have that the restriction is actually a trivial line bundle.

Moreover, since $\mathcal{D}_0$ is the 0-section, we can restrict to $X \to \mathbb{P}(X)$ and we obtain the trivial fiber bundle

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] & X \arrow[d, "\pi"] \\ & \mathbb{P}(X). \end{tikzcd}$$

whose fiber is the punctured affine line ($\simeq \mathbb{C}^*$).

So X factors as $X \simeq \mathbb{P}(X) \times \mathbb{C}^*$ and the inertia element z is represented by a nontrivial loop in the second factor. Given the base point $x_0 \in X \subset V$, we can choose (cf. [BMR98, Section 2.A]) as a representative of z the map

$$t \mapsto ([x_0], e^{2\pi\imath t}).$$

This way we have essentially proved

Theorem 4.1. *The inertia element $z = z_W$ generates the center of $P(W)$. If the group $B(W)$ is an Artin-Tits group equipped with the classical Garside structure or a well generated complex braid group equipped with the dual Garside structure z represents, up to orientation, the smallest power of the Garside element that belongs to $P(W)$ and is central.*

Proof. This immediately follows from the factorization $X \simeq \mathbb{P}(X) \times \mathbb{C}^*$, the description of $Z(P(W))$ given in Theorem 3.2 and the results on well generated groups recalled in Section 3. □

The same argument of Theorem 4.1 will be used inductively (i.e. applied to subspaces and subarrangements) in the next section to show the relation between the inertia around the other divisors and the center of the corresponding parabolic subgroups.

5. Inertia and the divisor at infinity

With the same setting as above, let $A \in \mathcal{F}_W$ be an irreducible subspace and define the parabolic subgroup

$$W_A = \{w \in W \mid w \text{ fixes } A^\perp \text{ pointwise}\}.$$

Noting that W_A is itself an irreducible complex reflection group with an essential action on A we let as usual $\mathcal{A}_{W_A}$ be the associated hyperplane arrangement in $A \subset V$ and by $X_{W_A} = X(W_A)$ its complement in A. Note that the ambient space A is not made explicit in the notation but this is harmless in the sequel. If we define, as is usual in the theory of hyperplane arrangements:

$$(\mathcal{A}_W)_{A^\perp} = \{H \in \mathcal{A}_W \mid A^\perp \subset H\},$$

then the hyperplanes in $\mathcal{A}_{W_A}$ can be seen as the intersections with A of the hyperplanes of $(\mathcal{A}_W)_{A^\perp}$. Further, we will denote, according to the standard notation used for hyperplane arrangements, by $(\mathcal{A}_W)^{A^\perp}$ the hyperplane arrangement in $A^\perp \subset V$ defined by the intersections with $A^\perp$ of those hyperplanes of the original arrangement $\mathcal{A}_W$ which do not contain $A^\perp$.

With this setup we can now generalize the construction of the previous section and define an inertia generator (or rather a conjugacy class) attached to any irreducible subspace A of the arrangement $\mathcal{A}_W$:

Definition 5.1. *The inertia class $z_A \in P(W)$ associated with the divisor $\mathcal{D}_A \subset \mathcal{D}$, is the homotopy class of a counterclockwise loop around $\mathcal{D}_A$ in the big open part of $\overline{X}_W$ (which can be identified with X_W).*

Now consider the restriction to $X_W = X(W)$ of the natural projection

$$\pi_{A^\perp} : V \to A^\perp. \tag{2}$$

Intersecting $X(W)$ with an open tubular neighbourhood of $A^\perp \subset V$ we find that the fiber over a point of the complement of $(\mathcal{A}_W)^{A^\perp}$ in $A^\perp$ is isomorphic to $X(W_A)$:

if the tubular neighborhood is small enough, the hyperplanes of $\mathcal{A}$ that do not fix $A^\perp$ are "far away" from this fiber.

This yields a natural inclusion

$$i_A : X(W_A) \hookrightarrow X = X(W). \tag{3}$$

Let $z_{W_A} \in Z(P(W_A)) \subset P(W_A)$ denote the inertia element as constructed in the previous section, using W_A (resp. A) instead of W (resp. V). More precisely, if we denote by $\mathcal{D}_{A,W_A}$ the divisor in $\overline{X}(W_A)$ associated to the maximal element of the building set, that is A, then z_{W_A} is the inertia around $\mathcal{D}_{A,W_A}$.

Remark 5.1. *Here and in the sequel we will indicate with $P(W_A)$ two isomorphic groups: the pure braid group associated with the complex reflection group W_A and the parabolic subgroup of $P(W)$ associated with the subspace A. It will be clear from the context which is the group we are dealing with.*

Theorem 5.1. *The map*

$$(i_A)_* : P(W_A) \to P(W)$$

induced by the inclusion $i_A : X(W_A) \hookrightarrow X = X(W)$ is injective*. It maps the inertia element $z_{W_A} \in P(W_A)$ to $z_A \in P = P(W)$.*

Proof. We start by showing that $(i_A)_*$ is injective. This fact is already known from [BMR98, Section 2.D], but we reprove it using our notation. To this end consider the sub-arrangement $(\mathcal{A}_W)_{A^\perp}$ in V defined above (notice that in general $(\mathcal{A}_W)_{A^\perp}$ is *not* essential). The respective complements in the ambient space V determine the inclusion $X(W) \subset X((\mathcal{A}_W)_{A^\perp})$ and the composition

$$X(W_A) \overset{i_A}{\hookrightarrow} X(W) \hookrightarrow X((\mathcal{A}_W)_{A^\perp})$$

is easily seen to be a homotopy equivalence, hence induces an isomorphism on fundamental groups. This shows that $(i_A)_*$ is injective.

Moreover (see also [BMR98, Section 2.D]) the standard generators of $P(W_A)$, which are given by loops in $X(W_A)$ around hyperplanes, map via i_A to loops in X around hyperplanes and these determine standard generators of $P(W)$.

Let now x be a point of the complement of $(\mathcal{A}_W)^{A^\perp}$ in $A^\perp$. Let us consider the affine subspace $A_x = \pi_{A^\perp}^{-1}(x) \subset V$ and the intersection $U_x = A_x \cap B_x$ with an open ball B_x centered at x, small enough to avoid the hyperplanes not containing x. Denote by $\overline{U}_x$ the proper transform of U_x in $\overline{X}$. Then $\overline{U}_x$ is isomorphic to $\overline{X}(W_A)$, since of all the blowups that contribute to the construction of $\overline{X}(W)$ only the ones that involve subspaces that contain $A^\perp$ have an effect on U_x. Furthermore, the big open parts of the varieties $\overline{U}_x$ and $\overline{X}(W_A)$ are identified by the projection $\pi_A : V \to A$. In the isomorphism mentioned above the intersection $\overline{U}_x \cap \mathcal{D}_A$ corresponds to the divisor $\mathcal{D}_{A,W_A}$ in $\overline{X}(W_A)$.

We then notice that a loop in $\overline{U}_x$ around $\overline{U}_x \cap \mathcal{D}_A$ is also a loop in $\overline{X}$ around $\mathcal{D}_A$, which is tantamount to saying that i_A maps a representative of the inertia $z_{W_A} \in P(W_A)$, that lies in the big open part of $\overline{X}(W_A)$, to a loop that is homotopic to a representative of $z_A \in P(W)$. □

Remark 5.2. *Take another wonderful model associated with the arrangement $\mathcal{A}_W$, for example the maximal one (see Remark 2.2), and consider the inertia class z_A around the divisor $\mathcal{D}_A$, which is a component of the divisor at infinity of our given model and is associated to a reducible (i.e. not irreducible) subspace A. Then if A*

decomposes as a direct sum of irreducible subspaces, $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, its associated inertia class can be written as a product of commuting factors:

$$z_A = z_{A_1} z_{A_2} \cdots z_{A_k}.$$

The following statement is an immediate corollary of Theorems 4.1 and 5.1:

Corollary 5.1. *The inertia class $z_A \in P(W_A) \subset P$ attached to the irreducible divisor $\mathcal{D}_A \subset \mathcal{D} \subset \overline{X}$ generates the center of the parabolic subgroup $P(W_A) \subset P$. If moreover $B(W_A)$ is an Artin-Tits group equipped with the classical Garside structure or a well generated complex braid group equipped with the dual braid monoid Garside structure, up to a change of orientation, z_A is (the image via $\pi_1(i_A)$ of) the smallest power of the Garside element of $B(W_A)$ that belongs to $P(W_A)$ and is central.*

6. Inertia in the quotient model

Let us now consider the action of W on the spaces X_W and $\overline{X}_W$. We denote by Y_W the quotient space X_W/W, which is a smooth variety (or scheme), and by $\overline{Y}_W = \overline{X}_W/W$ the quotient of the model, which in general is *not* smooth; there may be singular points on the divisor at infinity.

So let us focus on this divisor at infinity, i.e. on the quotient of the boundary components of $\overline{X}_W$. First we notice that W acts naturally on the building set $\mathcal{F}_W$; for every $A \in \mathcal{F}_W$ and $w \in W$ we write $w(\mathcal{D}_A) = \mathcal{D}_{wA}$, $\mathcal{O}(\mathcal{D}_A) = \bigcup_{w \in W} \mathcal{D}_{wA}$. We denote by $\mathcal{L}_A$ the quotient

$$\mathcal{L}_A = \mathcal{O}(\mathcal{D}_A)/W.$$

Clearly if A and B are in the same orbit of the action of W on $\mathcal{F}_W$, then $\mathcal{L}_A = \mathcal{L}_B$. It turns out that even in the example of W=A_3 the quotient divisor $\mathcal{L}_A$ is not smooth, but we will indicate a dense set of smooth points (see Proposition 6.2 below). The cardinality of the fibers of the projection map

$$\mathcal{O}(\mathcal{D}_A) \to \mathcal{L}_A$$

is determined by the cardinality of the stabilizers of the points in $\mathcal{D}_A$, so that we are interested in getting some information about these stabilizers.

Let us start by recalling that in [FK03] Feichtner and Kozlov provide a description of these groups. In order to describe the points in $\overline{X}_W$ they use the following encoding (see also [FM94]) which records the information coming from the projections of $\overline{X}_W$ onto the factors of the product $V \times \prod_{A \in \mathcal{F}_W} \mathbb{P}(V/A^{\perp})$.

Every point $\omega \in \overline{X}_W$ is represented by a list:

$$\omega = (x, A_1, l_1, A_2, l_2, ..., A_k, l_k)$$

where:

- x is the point in V given by the image of ω in the projection $\pi : \overline{X}_W \to V$;
- A_1 is the smallest subspace in $\mathcal{F}_W$ that contains x, and it appears in the list only if $A_1 \neq V$. If $A_1 = V$ then the encoding stops here: $\omega = (x)$;
- l_1 is the line in A_1 given by the image of ω in the projection $\pi : \overline{X}_W \to \mathbb{P}\left(V/(A_1)^{\perp}\right)$ (identifying A_1 with $V/(A_1)^{\perp}$);
- A_2 is the smallest subspace in $\mathcal{F}_W$ that contains A_1 and l_1, and it appears in the list only if $A_2 \neq V$, otherwise the list stops : $\omega = (x, A_1, l_1)$;
- l_2 is the line in A_2 defined by the image of ω in the projection $\pi : \overline{X}_W \to \mathbb{P}(V/(A_2)^{\perp})$

and so on...

Proposition 6.1 (see Proposition 4.2 in [FK03])**.** *The stabilizer stab ω of the point $\omega = (x, A_1, l_1, A_2, l_2, ..., A_k, l_k)$ is equal to*

$$\text{stab } x \cap \text{stab } l_1 \cap \text{stab } l_2 \cap \cdots \cap \text{stab } l_k$$

where by stab l_i *we mean the subgroup of W that sends l_i to itself (not necessarily fixing l_i pointwise), i.e. the stabilizer of the point $[l_i] \in \mathbb{P}(V/(A_i)^{\perp})$.*

It is immediate to check that if ω is a generic point of the divisor $\mathcal{D}_A$, more precisely if it doesn't lie in an intersection of $\mathcal{D}_A$ with other irreducible components of the boundary, it can be representated by a *triple* $(x, A, l = l_1)$, with stabilizer

$$\text{stab } x \cap \text{stab } l = W_A \cap \text{stab } l.$$

In other words the stabilizer of a generic point $\omega = (x, A, l) \in \mathcal{D}_A$ coincides with the stabilizer in W_A of $[l] \in \mathbb{P}(V/(A)^{\perp})$. Since we identify $\mathbb{P}(V/(A)^{\perp})$ with $\mathbb{P}(A)$, the problem of describing stab ω for a generic ω is reduced to the study of the stabilizers of the points of the projective space $\mathbb{P}(A)$ under the action of the parabolic subgroup W_A, for every $A \in \mathcal{F}_W$. The following notion of *regular element* of a complex reflection group (from Springer's paper [Spr74]) now comes into play.

Definition 6.1. *Given an irreducible finite reflection group G in a vector space V, an element $g \in G$ is called* regular *if it has an eigenvector that does not lie in any of the reflecting hyperplanes of G. If g is not a multiple of the identity, we call such an eigenvector a* Springer regular vector *of V.*

A classification of regular elements of irreducible real finite reflection groups has been provided by Springer in [Spr74]. Now we say that a generic point $\omega = (x, A, l)$ in $\mathcal{D}_A$ is a *Springer generic point* if the line l is *not* spanned by a Springer regular vector in A for the action of W_A. Finally a point $y \in \mathcal{L}_A$ is *Springer generic* if it is the image of a Springer generic point of $\mathcal{D}_A$.

Now we note that given an irreducible subspace A, and $\omega \in \mathcal{D}_A$ Springer generic, its stabilizer $W_A \cap \text{stab } l$ is trivial if the center of W_A is trivial, and otherwise it is a cyclic group generated by a multiple of the identity in $GL(V)$. Among the *real* irreducible finite reflection groups, as recalled in Remark 3.1, only A_n, D_{2n-1} and E_6 have trivial center, while in the other cases the center is $\mathbb{Z}/2$.

Proposition 6.2. *For $A \in \mathcal{F}_W$, the component $\mathcal{L}_A \subset \overline{Y}_W$ of the divisor at infinity is smooth at the Springer generic points.*

Proof. Let $\omega = (x, A, l)$ be a Springer generic point. The statement is trivial if the stabilizer of ω is trivial. If stab$\,\omega$ is not trivial we can assume that $Z(W_A) \simeq \mathbb{Z}/m$; the assumption that ω is Springer generic implies that there exists an element $\rho \in W_A$, with $\rho_{|A} = \epsilon_m I_{|A}, \rho_{|A^{\perp}} = Id_{|A^{\perp}}$, where ϵ_m is an m-th primitive root of unity and stab$\,\omega = \langle \rho \rangle = Z(W_A)$. So ρ fixes pointwise the divisor $\mathcal{D}_A$ in $\overline{X}_W$. Moreover, a point $(x, A, \langle v \rangle), t)$ ($t \in \mathbb{C}$ small enough) in the normal bundle of $\mathcal{D}_A$ maps onto a tubular neighborhood of $\mathcal{D}_A$ via

$$((x, A, \langle v \rangle), t) \mapsto x + tv$$

with $\rho(x + tv) = x + \epsilon_m tv$. Since ρ acts via a multiple of the identity on the normal bundle of $\mathcal{D}_A$ in a neighborhood of ω, the quotient $\mathcal{L}_A$ is smooth near ω. $\square$

Theorem 6.1. *Let W be an irreducible complex reflection group acting on the complex vector space V. Let $\zeta_W \in \pi_1(Y_W) = B(W)$ be the inertia generator around a Springer generic point of $\mathcal{L}_V$. The loop ζ_W is a generator of $Z(B(W))$.*

If the group $B(W)$ is an Artin-Tits group equipped with the classical Garside structure or a well generated complex braid group equipped with the dual Garside structure, then ζ_W is, up to orientation, the smallest central power of the Garside element.

Proof. Up to homotopy we can assume that ζ_W is a loop around a point $y \in \mathcal{L}_V$ which is the projection of a Springer generic point $\omega \in \overline{X}_W$.

Recall that in Theorem 4.1 we showed that the inertia z_W around the divisor $\mathcal{D}_V$ generates the center of $P(W)$. If $\operatorname{stab}\omega$ is trivial, a neighborhood of y in $\overline{Y}_W$ is homeomorphic to a neighborhood of ω in $\overline{X}_W$ and ζ_W is the image of z_W via the homomorphism of topological fundamental groups induced by the quotient map $X_W \to Y_W$. As we are assuming that $\operatorname{stab}\omega$ is trivial, the center $Z(W)$ is trivial. The short exact sequence (1) of Theorem 3.3 yields an isomorphism $Z(P(W)) = Z(B(W))$ and the result follows.

Assume now that $\operatorname{stab}\omega$ is not trivial. Since ω is Springer generic, $\operatorname{stab}\ \omega = \langle\rho\rangle$, with $\rho = \epsilon_m Id \in W$ (see above). We need to show that a) ζ_W belongs to the center $Z(B(W))$, and that b) it generates that center.

To prove a) we recall from Section 4 that there is a trivial bundle $\pi : X_W \to \mathbb{P}(X_W)$ with fiber $\mathbb{C}^*$ determining a decomposition $X_W \simeq \mathbb{P}(X_W) \times \mathbb{C}^*$. Hence a loop around ω in $\overline{Y}_W$ can be represented by a path in X_W

$$t \mapsto ([v], e^{\frac{2\pi\imath t}{m}}) \text{ for } t \in [0,1].$$

In particular we can represent ζ_W as a path in $X_W \subset V$

$$t \mapsto e^{\frac{2\pi\imath t}{m}} v \text{ for } t \in [0,1],$$

where v is not a Springer regular vector. We know from the classical result of [Che55] and [ST54] that V/W is an affine space with coordinates given by homogeneous polynomials on V, say $p_1(x), \dots, p_n(x)$, of degrees $d_1, \dots, d_n$.

Hence ζ_W is represented by a loop γ in $Y_W \subset V/W$ given by

$$t \mapsto \gamma(t) = (e^{\frac{2d_1\pi\imath t}{m}} p_1(v), \dots, e^{\frac{2d_n\pi\imath t}{m}} p_n(v)) \text{ for } t \in [0,1].$$

Let $\gamma' : [0,1] \to Y_W$ be another closed path with the same base point in Y_W. We claim that γ and γ' commute. In fact the following map $H : [0,1] \times [0,1] \to Y_W$

$$H(t,t') = (e^{\frac{2d_1\pi\imath t}{m}} \gamma'_1(t'), \dots, e^{\frac{2d_n\pi\imath t}{m}} \gamma'_n(t')) \text{ for } (t,t') \in [0,1] \times [0,1],$$

provides a homotopy between $\gamma \circ \gamma'$ and $\gamma' \circ \gamma$. This proves that indeed ζ_W is central.

In order to show b), i. e. that ζ_W generates the center of $B(W)$, consider again the short exact sequence (1) of Theorem 3.3. Since $Z(W) = \mathbb{Z}/m$ we get

$$1 \to Z(P(W)) \to Z(B(W)) \to \mathbb{Z}/m \to 1.$$

By construction we know that z_W is a generator of $Z(P(W))$ and it maps to ζ_W^m. Moreover ζ_W maps to the generator of $\mathbb{Z}/m$. Since $Z(B(W))$ is infinite cyclic, it follows that ζ_W generates $Z(B(W))$. $\square$

We now observe that the inclusion $i_A : X(W_A) \hookrightarrow X = X(W)$ given in Equation (3) (see Section 5) is W_A-equivariant, with the W_A-action compatible with the inclusion $W_A \subset W$. Hence it induces an inclusion $i_A : Y(W_A) \hookrightarrow Y = Y(W)$ and a corresponding map between the fundamental groups. As in Section 5 we generalize the definition of inertia for an irreducible subspace A.

Definition 6.2. *The inertia class $\zeta_A \in B(W)$ associated with the divisor $\mathcal{L}_A$ is the homotopy class of an oriented loop around a Springer generic point of $\mathcal{L}_A$ in the big open part of $\overline{Y}_W$ (which can be identified with Y_W).*

Now let us denote by $\mathcal{L}_{A,W_A}$ the quotient divisor of $\overline{Y}(W_A)$ associated to the maximal element of the building set, namely A; then we denote by ζ_{W_A} the inertia around $\mathcal{L}_{A,W_A}$. Below we use $B(W_A)$ to denote both the braid group associated with the complex reflection group W_A and the (isomorphic) parabolic subgroup of $B(W)$ associated with the subspace A. It will be clear from the context which group we are dealing with (see also Remark 5.1).

Theorem 6.2. *The map*

$$(i_A)_* = \pi_1(i_A) : B(W_A) \to B = B(W)$$

induced by the inclusion $i_A : Y(W_A) \hookrightarrow Y = Y(W)$ is injective*. It maps the inertia element $\zeta_{W_A} \in B(W_A)$ to $\zeta_A \in B$. Moreover the loop ζ_A is a generator of $Z(B(W_A))$, the center of the parabolic subgroup $B(W_A)$.*

If the group $B(W_A)$ is an Artin-Tits group equipped with the classical Garside structure or a well generated complex braid group equipped with the dual Garside structure then ζ_A is, up to orientation, the smallest central power of the Garside element of $B(W_A)$.

Proof. Injectivity follows from the 5-Lemma (essentially, this rephrases the proof in [BMR98, Section 2.D]): we already know that the natural maps $P(W_A) \to P(W)$ and $W_A \to W$ are injective and they fit into the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & P(W_A) \arrow[r] \arrow[d, "(i_A)_*"] & B(W_A) \arrow[r] \arrow[d, "(i_A)_*"] & W_A \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P(W) \arrow[r] & B(W) \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

where the first and last vertical maps are injective.

We can choose as a representative of $\zeta_A \in \pi_1(Y_W) = B(W)$ a loop around the projection $y \in \mathcal{L}_A$ of a Springer generic point $\omega \in \mathcal{D}_A$. Since the stabilizer $\operatorname{stab}\omega$ is the center of W_A, the argument used in the proof of Theorem 5.1 shows that $\zeta_{W_A} \in B(W_A)$ maps to $\zeta_A \in B = B(W)$. The second part of the statement then follows from Theorem 6.1.

□

References

[Bes03] David Bessis. The dual braid monoid. *Ann. Sci. École Norm. Sup. (4)*, 36(5):647–683, 2003.

[Bes15] David Bessis. Finite complex reflection arrangements are $K(\pi, 1)$. *Ann. of Math. (2)*, 181(3):809–904, 2015.

[BMR98] Michel Broué, Gunter Malle, and Raphaël Rouquier. Complex reflection groups, braid groups, Hecke algebras. *J. Reine Angew. Math.*, 500:127–190, 1998.

[CG15] Filippo Callegaro and Giovanni Gaiffi. On models of the braid arrangement and their hidden symmetries. *Int. Math. Res. Not.*, :Art. ID rnv009, 33 pages, 2015.

[CGL15] Filippo Callegaro, Giovanni Gaiffi, and Pierre Lochak. Garside elements, inertia and Galois action on braid groups. `arXiv:1507.07208`, 2015.

[Che55] Claude Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

[DCP95a] Corrado De Concini and Claudio Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3):459–494, 1995.

[DCP95b] Corrado De Concini and Claudio Procesi. Hyperplane arrangements and holonomy equations. *Selecta Math. (N.S.)*, 1(3):495–535, 1995.

[DCP10] Claudio De Concini and Corrado Procesi. *Topics in Hyperplane Arrangements, Polytopes and Box-Splines.* Springer, Universitext, 2010.

[DDG+15] Patrick Dehornoy, François Digne, Eddy Godelle, Daan Kramer, and Jean Michel. *Foundations of Garside theory.* Zürich: European Mathematical Society (EMS), 2015.

[Den14] Graham Denham. Toric and tropical compactifications of hyperplane complements. *Ann. Fac. Sci. Toulouse Math. (6)*, 23(2):297–333, 2014.

[DMM11] François Digne, Ivan Marin, and Jean Michel. The center of pure complex braid groups. *J. Algebra*, 347:206–213, 2011.

[Dri91] Vladimir Geršonovič Drinfel′d. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Leningrad Math. J.*, 2(4):829–860, 1991.

[EHKR10] Pavel Etingof, André Henriques, Joel Kamnitzer, and Eric M. Rains. The cohomology ring of the real locus of the moduli space of stable curves of genus 0 with marked points. *Ann. of Math. (2)*, 171(2):731–777, 2010.

[Fei05] Eva Maria Feichtner. De Concini-Procesi wonderful arrangement models: a discrete geometer's point of view. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 333–360. Cambridge Univ. Press, Cambridge, 2005.

[FK03] Eva Maria Feichtner and Dmitry N. Kozlov. Abelianizing the real permutation action via blowups. *Int. Math. Res. Not.*, 2003(32):1755–1784, 2003.

[FM94] William Fulton and Robert MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[FS05] Eva Maria Feichtner and Bern Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62:437–468, 2005.

[FY04] Eva Maria Feichtner and Sergey Yuzvinski. Chow rings of toric varieties defined by atomic lattices. *Invent. math.*, 155(3):515–536, 2004.

[Gai03] Giovanni Gaiffi. Models for real subspace arrangements and stratified manifolds. *Int. Math. Res. Not.*, 2003(12):627–656, 2003.

[Gai15] Giovanni Gaiffi. Permutonestohedra. *J. Algebraic Combin.*, 41(1):125–155, 2015.

[Gai16] Giovanni Gaiffi. Exponential formulas for models of complex reflection groups. `arXiv:1507.02090`, to appear in *European J. Combin.*, 2016.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GS14] Giovanni Gaiffi and Matteo Serventi. Families of building sets and regular wonderful models. *European J. Combin.*, 36:17–38, 2014.

[Hen04] Anthony Henderson. Representations of wreath products on cohomology of De Concini-Procesi compactifications. *Int. Math. Res. Not.*, 2004(20):983–1021, 2004.

[HR08] Antony Henderson and Eric M. Rains. The cohomology of real De Concini-Procesi models of Coxeter type. *Int. Math. Res. Not.*, 2008:Art. ID rnn001, 29 pages, 2008.

[Hu03] Yi Hu. A compactification of open varieties. *Trans. Amer. Math. Soc.*, 355(12):4737–4753, 2003.

[Li09] Li Li. Wonderful compactification of an arrangement of subvarieties. *Michigan Math. J.*, 58(2):535–563, 2009.

[MP98] Robert MacPherson and Claudio Procesi. Making conical compactifications wonderful. *Selecta Math. (N.S.)*, 4(1):125–139, 1998.

[Rai09] Eric M. Rains. The action of S_n on the cohomology of $\overline{M}_{0,n}(\mathbb{R})$. *Selecta Math. (N.S.)*, 15(1):171–188, 2009.

[Spr74] Tonny Albert Springer. Regular elements of finite reflection groups. *Invent. Math.*, 25:159–198, 1974.

[ST54] Geoffrey Colin Shephard and John Arthur Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[Uly02] Alexander P. Ulyanov. Polydiagonal compactification of configuration spaces. *J. Algebraic Geom.*, 11(1):129–159, 2002.

[Yuz97] Sergey Yuzvinsky. Cohomology bases for the De Concini-Procesi models of hyperplane arrangements and sums over trees. *Invent. Math.*, 127(2):319–335, 1997.

F.C.: Dipartimento di matematica, Largo Bruno Pontecorvo, 5 56127 Pisa, Italia
E-mail address: callegaro@dm.unipi.it

G.G.: Dipartimento di matematica, Largo Bruno Pontecorvo, 5 56127 Pisa, Italia
E-mail address: gaiffi@dm.unipi.it

P.L.: CNRS and Institut Mathématique de Jussieu, Université P. et M. Curie, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: pierre.lochak@imj-prg.fr