Macroscopic and microscopic behavior of narrow elastic ribbons

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Abstract

A one-dimensional model for a narrow ribbon is derived from the plate theory of Kirchhoff by means of a power expansion in the width variable. The energy found coincides with the corrected Sadowsky's energy. Furthermore, we derive the Euler-Lagrange equations and use them to study an equilibrium configuration of a twisted ribbon. Within this example we also describe how to construct the fine scale oscillations that develop in the deformed configuration.

Dedicated to the memory of Walter Noll.

1 Introduction

Geometrically a ribbon is a body with a rectangular parallelepiped configuration having thickness ε much smaller than the width h which, in turn, is much smaller than the length ℓ. Customarily, the theory of finite deformations of an extremely thin elastic ribbon is studied by associating to the two-dimensional rectangular region, of width h and length ℓ, the Kirchhoff bending energy, [22].

No stretching energy is introduced by assuming the material to be inextensible. This is the starting point of the seminal work of Sadowsky [28, 20]. For h extremely small Sadowsky, in [28], proposed to model an isotropic ribbon through a one-dimensional energy that depends on the curvature and the torsion of the deformed image of the interval $(0, \ell)$. Nowadays, this energy is known as Sadowsky's energy. A formal justification of the Sadowsky's energy was given by Wunderlich [35, 32] also by means of a parametrization of the deformed configuration of the ribbon as a ruled surface. His ingenious argument is more geometrical than mechanical. Interesting generalizations and clarifications on the argument may be found in [1, 3, 4, 5, 6, 10, 11, 21, 29, 30], moreover a comprehensive collection of papers on ribbons is [13].

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The deduction of a one-dimensional model for ribbons was recently obtained by means of Γ-convergence in [14] in the isotropic case and in [15] for a general material symmetry. The obtained asymptotic energy density turns out to display two regimes only one of which agrees with the functional proposed by Sadowsky. Another key difference to Sadowsky's theory is that the asymptotic functional does not depend only on the deformation of the midline, but rather on a curve endowed with a triad of directors satisfying several constraints.

The starting point in [14, 15] was the Kirchhoff bending energy as done by Sadowsky; in [16] analogous results were obtained starting from von Kàrmàn models. The same asymptotic models found in [14, 15] were also recently obtained from the nonlinear three-dimensional theory of elasticity in [17], *i.e.,* by letting ε and h simultaneously to zero. Other models for ribbons deduced from the three-dimensional theory of nonlinear elasticity may be found in [18, 19].

The present paper is concerned with the derivation of a ribbon model starting from the Kirchhoff bending energy. We follow the typical mechanical approach used to derive plate and rod theories, namely, in our contest, on a power series expansion in the width variable. The power series of the energy is essentially truncated at order h^2. The one-dimensional energy obtained depends on the symmetric tensor $\mathbf{K}^\circ$ which represents the second fundamental form of the deformed surface restricted to the midline of the ribbon. This line is naturally endowed with three directors; three of the four components of the tensor $\mathbf{K}^\circ$ may be written in terms of these directors. Moreover the inextensibility of the ribbon leads to the constraint $\det \mathbf{K}^\circ = 0$, among others. Due to this constraint the energy is not convex and, as a consequence, the energy minimization problem generally has no solution. Indeed, what may happen is that a minimizing sequence of isometric deformation may have a limit that does not satisfy the constraint $\det \mathbf{K}^\circ = 0$; this, for instance, happens if the minimizing sequence develops fine scale oscillations. This difficulty is resolved by using a so-called relaxed energy in place of the original bending energy. The relaxed energy automatically accounts for the fine scale oscillations and, as a consequence, is defined also for symmetric tensor having determinant different from zero. Solutions for which the determinant is different from zero should then be interpreted as the limit of a sequence of finely oscillating deformations. Of course, microscopic oscillation effects are expected to be lost for moderately thin ribbons, where the membrane energy is important, and the ribbon cannot be considered inextensible. This situation is of no less importance since the competition and interplay between bending and stretching energy can produce exotic behaviors, such as for instance temperature-induced sharp shape transitions between helicoidal to twisted configurations [33, 31, 34].

The theory developed has some analogies with the theory of nets modeled as a continuum of fibers by Tchebychev and Rivlin [26, 27]. In Tchebychev nets two families of inextensible cords cross each other without sliding. Pipkin in [25] observed that the minimization problem may not have a kinematically admissible minimizer. He therefore introduced the relaxed problem in which the inextensibility constraint is weakened: the cords can grow shorter but not longer. He named his model 'inextensible nets with slack'.

In the case of ribbons the $\det \mathbf{K}^\circ = 0$ constraint is completely absorbed in the relaxed bending energy density. The model that we obtain coincides with those found in [14, 15]. Our derivation, contrary to that of Wunderlich [35] makes no use of ruled surfaces.

The paper is organized as follows. In Section 2 we briefly review the Kirchhoff theory for plates and in the following section we derive the model for a narrow ribbon. In Section 4 we explicitly write the bending energy density of the ribbon. Several results used in this Section are proved in Appendices A.1 and A.2. The Euler-Lagrange equations are then derived in Section 5. In the last Section of the paper we study an equilibrium configuration of a twisted ribbon. This example is particularly interesting since the constraint $\det \mathbf{K}^\circ = 0$ does not hold. We therefore show how to construct a sequence of isometric deformations whose limit is the equilibrium configuration. In this construction we use, like Wunderlich, ruled surfaces.

Standard notation is used throughout. Vectors and tensors are denoted in boldface letters; Latin indices take values in $\{1, 2, 3\}$, while Greek in $\{1, 2\}$. We adopt Einstein's summation convention, and we denote with a dot, $\cdot$, the standard inner product and with the wedge, $\wedge$, the standard cross product. The transpose of the second order tensor $\mathbf{A}$ is denoted by $\mathbf{A}^\top$, while the cofactor by $\mathbf{A}^* = (\det \mathbf{A})\mathbf{A}^{-\top}$ for an invertible matrix $\mathbf{A}$. We recall that if $\mathbf{A}, \mathbf{B}$ are two tensors that can be represented by 2×2 matrices, then

$$\det(\mathbf{A} - \mathbf{B}) = \det \mathbf{A} - \mathbf{A}^* \cdot \mathbf{B} + \det \mathbf{B}. \tag{1}$$

Finally, the partial derivative with respect to the variable x_i is simply denoted by ∂_i.

2 Plate model

We consider deformations of an inextensible elastic plate that initially occupies a rectangular domain $\omega = (0, \ell) \times (-h/2, h/2)$, of length ℓ and width h, in the x_1, x_2 plane. Let $\mathbf{e}_\alpha$, $\alpha = 1, 2$, be an orthonormal system of base vectors oriented as the initial domain with $\mathbf{e}_1$ parallel to the x_1 axis. Deformations of the plate are specified by a mapping $\boldsymbol{\chi}$ that takes the point x of ω to the point $\boldsymbol{\chi}(x)$ of the three-dimensional space. Let $\mathbf{e}_3$ be a unit vector orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$ so that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal system of the three-dimesional space. The vectors $\mathbf{e}_\alpha$, $\alpha = 1, 2$, will be considered as two or three components vectors as convenient; the context should make it clear. The deformation gradient, $\nabla\boldsymbol{\chi}$, is denoted by F and is represented by a 3×2 matrix. Since the plate is inextensible we have that

$$\mathsf{F}\mathbf{e}_\alpha \cdot \mathsf{F}\mathbf{e}_\beta = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta, \tag{2}$$

for $\alpha, \beta = 1, 2$. Equivalently, we may write

$$\partial_\alpha\boldsymbol{\chi} \cdot \partial_\beta\boldsymbol{\chi} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta. \tag{3}$$

Let $\mathbf{n}$ be the unit normal to the deformed plate defined by

$$\mathbf{n} = \mathsf{F}\mathbf{e}_1 \wedge \mathsf{F}\mathbf{e}_2, \tag{4}$$

and let

$$\mathbf{K} = -\mathsf{F}^{\top}\nabla\mathbf{n} \tag{5}$$

be the second fundamental form of $\boldsymbol{\chi}$. The symmetric tensor $\mathbf{K}$ is represented by a 2×2 matrix whose components, with respect to the $\mathbf{e}_\alpha$ basis, are $K_{\alpha\beta} = -\partial_\alpha\boldsymbol{\chi}\cdot\partial_\beta\mathbf{n} = \partial^2_{\alpha\beta}\boldsymbol{\chi}\cdot\mathbf{n}$. Since it is impossible to have deformations that simultaneously change both principal curvatures without stretching, by the inextensibility of the plate we must have that

$$\det\mathbf{K} = 0; \tag{6}$$

as follows from Gauss's Theorema Egregium.

As energy density of the plate we simply take a quadratic form in $\mathbf{K}$:

$$Q(x_1, \mathbf{K}) = \frac{1}{2}\mathbb{D}(x_1)\mathbf{K}\cdot\mathbf{K}$$

where the bending stiffness tensor $\mathbb{D}$ maps symmetric second order tensors to symmetric second order tensors. For simplicity, we assume that $\mathbb{D}$ depends only on the x_1 coordinate. We further assume that $\mathbb{D}$ is strictly positive definite and has the major symmetry, i.e., there exists a constant d such that $\mathbb{D}(x_1)\mathbf{A}\cdot\mathbf{A} \geq d|\mathbf{A}|^2$ and $\mathbb{D}(x_1)\mathbf{A}\cdot\mathbf{B} = \mathbb{D}(x_1)\mathbf{B}\cdot\mathbf{A}$ for every symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ and for every x_1. Hence the total bending energy in a given deformation $\boldsymbol{\chi}$ is

$$\mathcal{E}^h(\boldsymbol{\chi}) = \frac{1}{2}\int_\omega \mathbb{D}\mathbf{K}\cdot\mathbf{K}\,da = \frac{1}{2}\int_0^\ell\int_{-h/2}^{+h/2}\mathbb{D}\mathbf{K}\cdot\mathbf{K}\,dx_2dx_1, \tag{7}$$

where $\mathbf{K}$ is the curvature associated to $\boldsymbol{\chi}$.

3 A model for a narrow ribbon

Hereafter, we assume that the width h of the plate is much smaller than its length ℓ. Under this assumption it appears natural to formally expand the deformation $\boldsymbol{\chi}$ in series with respect to x_2, by writing

$$\boldsymbol{\chi}(x_1, x_2) = \boldsymbol{\chi}^\circ(x_1) + \sum_{i=1}^{\infty}\boldsymbol{\chi}^{(i)}(x_1)x_2^i,$$

deduce the formal expansion

$$\mathbf{K}(x_1, x_2) = \mathbf{K}^\circ(x_1) + \sum_{i=1}^{\infty}\mathbf{K}^{(i)}(x_1)x_2^i,$$

and then substitute this last expansion into (7). If we content ourselves with $O(h^2)$ approximation of the energy, and if we reckon that $\mathbb{D}$ does not depend on

x_2 (symmetry with respect to x_2 would indeed suffice), the resulting expression of the energy is

$$\mathcal{E}^h(\boldsymbol{\chi}) = \frac{h}{2}\int_0^\ell \mathbb{D}\mathbf{K}^\circ \cdot \mathbf{K}^\circ \, dx_1 + o(h^3). \tag{8}$$

We now render this expression more explicit. To do so we find it convenient to adopt the following notation: given any field $f(x_1, x_2)$, we set $f^\circ(x_1) := f(x_1, 0)$ to denote the trace of f on the midline of the ribbon. By making use of this notation, we introduce the *directors*:

$$\mathbf{d}_1 = (\mathsf{F}\mathbf{e}_1)^\circ, \qquad \mathbf{d}_2 = (\mathsf{F}\mathbf{e}_2)^\circ, \qquad \mathbf{d}_3 = \mathbf{n}^\circ, \tag{9}$$

and we observe that these vectors constitute an orthonormal basis:

$$\mathbf{d}_i \cdot \mathbf{d}_j = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{10}$$

as can be checked by means of (2) and (4).

Keeping in mind that the operation of taking the trace on the midline commutes with that of performing differentiation with respect to x_1, we see that the first director is tangent to the midline of the ribbon

$$\mathbf{d}_1 = (\partial_1 \boldsymbol{\chi})^\circ = \partial_1 \boldsymbol{\chi}^\circ. \tag{11}$$

The second director

$$\mathbf{d}_2 = (\partial_2 \boldsymbol{\chi})^\circ$$

is tangent to the ribbon on the midline and "represents" the orientation of the cross section of the ribbon. The third director is perpendicular to the ribbon on the midline.

Taking $\alpha = \beta = 1$ in (3), differentiating with respect to x_2, and evaluating the result at $x_2 = 0$, we find

$$0 = (\partial_2\partial_1\boldsymbol{\chi})^\circ \cdot (\partial_1\boldsymbol{\chi})^\circ = \partial_1(\mathsf{F}\mathbf{e}_2)^\circ \cdot (\mathsf{F}\mathbf{e}_1)^\circ,$$

that is

$$\mathbf{d}_2' \cdot \mathbf{d}_1 = 0. \tag{12}$$

This is a constraint that must be satisfied by the directors. From the definition of the second fundamental form we find that

$$K^\circ_{1\beta} = (\partial^2_{1\beta}\boldsymbol{\chi})^\circ \cdot \mathbf{n}^\circ = \partial_1(\partial_\beta\boldsymbol{\chi})^\circ \cdot \mathbf{n}^\circ = \mathbf{d}_\beta' \cdot \mathbf{d}_3. \tag{13}$$

The components $K^\circ_{11} = \mathbf{d}_1' \cdot \mathbf{d}_3$ and $K^\circ_{12} = \mathbf{d}_2' \cdot \mathbf{d}_3$ are the curvatures of the ribbon on the midline. It is convenient to introduce the *curvature*

$$\mathbf{k} := \frac{1}{2}\sum_{i=1}^{3} \mathbf{d}_i \wedge \mathbf{d}_i',$$

which, as is easily seen, satisfies the identity

$$\mathbf{d}_i' = \mathbf{k} \wedge \mathbf{d}_i,$$

which in the components $k_i := \mathbf{k} \cdot \mathbf{d}_i$ rewrites as

$$k_1 = \mathbf{d}_2' \cdot \mathbf{d}_3 = K^\circ_{12}, \quad k_2 = \mathbf{d}_3' \cdot \mathbf{d}_1 = -K^\circ_{11}, \quad k_3 = \mathbf{d}_1' \cdot \mathbf{d}_2. \tag{14}$$

Then, k_1 measures torsion, k_2 measures flexure with respect to $\mathbf{d}_2$ and k_3 measure flexure with respect to $\mathbf{d}_3$. Equation (12) implies that the ribbon does not bend around $\mathbf{d}_3$:

$$k_3 = 0. \tag{15}$$

We shall simply refer to k_2 as the *curvature* of the midline curve $\boldsymbol{\chi}^\circ$ and to k_1 as the *twist* of the cross-section of the ribbon. The component K°_{22} can not be expressed in terms of only the directors. Hereafter, we set $\gamma := K^\circ_{22}$ so to write

$$K^\circ_{\alpha\beta} = \begin{pmatrix} \mathbf{d}_1' \cdot \mathbf{d}_3 & \mathbf{d}_2' \cdot \mathbf{d}_3 \\ \mathbf{d}_2' \cdot \mathbf{d}_3 & \gamma \end{pmatrix} = \begin{pmatrix} -k_2 & k_1 \\ k_1 & \gamma \end{pmatrix}. \tag{16}$$

By Gauss's Theorema Egregium, surfaces that satisfy (3) have Gaussian curvature everywhere equal to zero. Evaluating (6) at $x_2 = 0$ we deduce that

$$\det \mathbf{K}^\circ = 0. \tag{17}$$

This is a severe constraint on the deformations, indeed note that if $k_2 = 0$ then also $k_1 = 0$: in words, twist is not allowed if the midline is kept straight.

With (8) in mind, we set

$$\mathcal{E}^\circ(\mathbf{r}, \mathbf{d}_i, \gamma) := \frac{1}{2} \int_0^\ell \mathbb{D}\mathbf{K}^\circ \cdot \mathbf{K}^\circ \, dx_1 = \int_0^\ell Q(x_1, \mathbf{K}^\circ) \, dx_1, \tag{18}$$

where $\mathbf{K}^\circ$ is given in terms of $\mathbf{d}_i$ and γ by (16), and

$$\mathbf{r} := \boldsymbol{\chi}^\circ$$

is related to $\mathbf{d}_1$ by (11), that is

$$\mathbf{d}_1 = \mathbf{r}'. \tag{19}$$

Because of the constraint (17), minimizing the energy (18) is the same as minimizing the energy

$$\mathcal{E}_\circ(\mathbf{r}, \mathbf{d}_i, \gamma) := \int_0^\ell W(x_1, \mathbf{K}^\circ) \, dx_1, \tag{20}$$

where, the density W is defined by

$$W(x_1, \mathbf{L}) = \begin{cases} Q(x_1, \mathbf{L}) & \text{if } \det \mathbf{L} = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{21}$$

We view (20) as the *microscopic energy*. The energy density W, when finite, is a strictly positive quadratic function over a non convex set. Indeed, as it is easy to check, the set of 2-by-2 matrices with zero determinant is not convex.

Thus, W is not convex and hence the microscopic energy (20) may not have a minimizer. What may go wrong is that a minimizing sequence $(\mathbf{r}_n, (\mathbf{d}_i)_n, \gamma_n)$, which determines (according to (16)) a $\mathbf{K}_n^\circ$ with null determinant, may approach a limit $\mathbf{K}^\circ$ with determinant not everywhere equal to zero. This can be achieved for instance when the n–th element of the sequence is obtained by alternating, on a scale of order $1/n$ or smaller, two deformations with finite energy, *i.e.*, $\det \mathbf{K}_n^\circ = 0$ everywhere, but with an "average" (weak limit of the sequence) $\mathbf{K}^\circ$ with determinant different from zero. Hereafter, we shall say that $\mathbf{K}^\circ$ is *generated* by $\mathbf{K}_n^\circ$. Thus, we may extend the definition of energy so to make it finite also for deformations with $\det\mathbf{K}^\circ \neq 0$: the energy that we may associate to a $\mathbf{K}^\circ$ generated by $\mathbf{K}_n^\circ$ is the limit of the energies associated to $\mathbf{K}_n^\circ$. The *macroscopic energy* or *relaxed energy* is the least energy that is generated by a sequence of deformations with finite energy; this corresponds to, see for instance [9, 12],

$$\mathcal{E}_\circ^{**}(\mathbf{r}, \mathbf{d}_i, \gamma) := \int_0^\ell W^{**}(x_1, \mathbf{K}^\circ)\, dx_1,$$

where W^{**} is the *convex envelope* of W.

This energy may be further reduced, since the field γ is neither related to $\mathbf{r}$ nor to $\mathbf{d}_i$, we minimize once and for all over γ to obtain

$$\min_\gamma \mathcal{E}_\circ^{**}(\mathbf{r}, \mathbf{d}_i, \gamma) = \int_0^\ell \overline{W}(x_1, k_1, k_2)\, dx_1,$$

where

$$\overline{W}(x_1, k_1, k_2) := \min_\gamma W^{**}\Big(x_1, \begin{pmatrix} -k_2 & k_1 \\ k_1 & \gamma \end{pmatrix}\Big). \tag{22}$$

We interpret

$$\overline{\mathcal{E}}(\mathbf{r}, \mathbf{d}_i) := \int_0^\ell \overline{W}(x_1, k_1, k_2)\, dx_1. \tag{23}$$

as the *macroscopic energy* of the ribbon. A key observation here is that the macroscopic energy is always less or equal to the microscopic energy; indeed,

$$\begin{aligned} \mathcal{E}_\circ(\mathbf{r}, \mathbf{d}_i, \gamma) &= \int_0^\ell W(x_1, \mathbf{K}^\circ)\, dx_1 \geq \int_0^\ell W^{**}(x_1, \mathbf{K}^\circ)\, dx_1 \\ &\geq \int_0^\ell \overline{W}(x_1, k_1, k_2)\, dx_1 = \overline{\mathcal{E}}(\mathbf{r}, \mathbf{d}_i). \end{aligned}$$

The energy of the ribbon (23) coincides with that obtained by means of $\Gamma-$convergence in [15].

4 Representation of the energy density of the ribbon

In order to have an explicit representation of the energy density $\overline{W}$ of the ribbon it is necessary to first evaluate the *convex envelope* W^{**} of the energy W defined in (21). It turns out that, given any symmetric 2-by-2 matrix $\mathbf{L}$,

$$W^{**}(\mathbf{L}) = Q(\mathbf{L}) + \alpha^+(\det \mathbf{L})^+ + \alpha^-(\det \mathbf{L})^-, \tag{24}$$

where

$$\alpha^- = \min_{\det \mathbf{L}=1} Q(\mathbf{L}), \qquad \alpha^+ = \min_{\det \mathbf{L}=-1} Q(\mathbf{L}), \tag{25}$$

and where

$$(\det \mathbf{L})^+ = \begin{cases} \det \mathbf{L} & \text{if } \det \mathbf{L} > 0, \\ 0 & \text{else,} \end{cases} \qquad (\det \mathbf{L})^- = \begin{cases} -\det \mathbf{L} & \text{if } \det \mathbf{L} < 0, \\ 0 & \text{else.} \end{cases}$$

Here and in what follows, for brevity we do not write the dependence of W on the variable x_1. This representation has been obtained in [15] by means of convex duality. In Appendix A.1, besides giving a new proof, we prove that the convex envelope can be obtained by a single "lamination". That is: for every symmetric matrix $\mathbf{L}$ there exist two symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ and a $t \in [0, 1]$ such that

$$W^{**}(\mathbf{L}) = (1-t)Q(\mathbf{A}) + tQ(\mathbf{B}), \qquad \text{with} \qquad \mathbf{L} = (1-t)\mathbf{A} + t\mathbf{B}. \tag{26}$$

The quantities $\mathbf{A}, \mathbf{B}$, and t can be evaluated by using the symmetric matrices $\mathbf{D}^-$ and $\mathbf{D}^+$ that minimize problems (25). More precisely, $\mathbf{D}^-$ and $\mathbf{D}^+$ are two symmetric matrices with $\det \mathbf{D}^- = -1$ and $\det \mathbf{D}^+ = +1$ such that

$$\alpha^- = Q(\mathbf{D}^+), \qquad \alpha^+ = Q(\mathbf{D}^-). \tag{27}$$

If $\det \mathbf{L} > 0$ then the real number t and the matrix $\mathbf{B}$ are given by

$$t = \frac{1}{2 - 2\lambda(\sqrt{\lambda^2+1} - \lambda)}, \qquad \mathbf{B} = \mathbf{L} - \eta \mathbf{D}^-$$

where λ and η are defined by

$$\lambda := \frac{\mathbf{L}^* \cdot \mathbf{D}^-}{2\sqrt{\det \mathbf{L}}}, \qquad \eta := \sqrt{\det \mathbf{L}}(\sqrt{\lambda^2+1} - \lambda).$$

While, if $\det \mathbf{L} < 0$ we have that

$$t = \frac{1}{2 + 2\lambda(\lambda - \sqrt{\lambda^2+1})}, \qquad \mathbf{B} = \mathbf{L} - \eta \mathbf{D}^+, \tag{28}$$

with

$$\lambda := \frac{\mathbf{L}^* \cdot \mathbf{D}^+}{2\sqrt{-\det \mathbf{L}}}, \qquad \eta := \sqrt{-\det \mathbf{L}}(\lambda - \sqrt{\lambda^2+1}). \tag{29}$$

In both cases, $\det \mathbf{L} > 0$ and $\det \mathbf{L} < 0$, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{\mathbf{L} - t\mathbf{B}}{1-t}. \tag{30}$$

The elastic constants $\alpha^{\pm}$ are computed in Appendix A.2, here we briefly outline the results. We denote by

$$\mathbb{D}_{\alpha\beta\gamma\delta} := \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \cdot \mathbb{D}\, \mathbf{e}_\gamma \otimes \mathbf{e}_\delta.$$

For an orthotropic ribbon with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$, we have

$$\mathbb{D}_{1122} = \mathbb{D}_{1222} = 0,$$

and it is found that

$$\alpha^- = \sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} + \mathbb{D}_{1122}, \quad \alpha^+ = \min\{\sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} - \mathbb{D}_{1122}, 2\mathbb{D}_{1212}\}.$$

In particular, for the isotropic strain energy

$$Q(\mathbf{L}) = d_\mu |\mathbf{L}|^2 + \frac{d_\lambda}{2}(L_{11} + L_{22})^2,$$

where d_μ and d_λ are two elastic constants, we find

$$\alpha^- = 2(d_\mu + d_\lambda), \qquad \alpha^+ = 2d_\mu.$$

In this case, the convex envelope of W, (24), turns out to be, see Appendix A.2 for details,

$$W^{**}(\mathbf{L}) = \left(d_\mu + \frac{d_\lambda}{2}\right)\left(|\mathbf{L}|^2 + 2|\det \mathbf{L}|\right). \tag{31}$$

One can also check, by a direct computation, that $\alpha^- = Q(\mathbf{D}^+)$, and $\alpha^+ = Q(\mathbf{D}^-)$ for

$$\mathbf{D}^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{D}^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{32}$$

In the isotropic case, a simple computation leads to

$$\min_{L_{22}\in\mathbb{R}} W^{**}(\mathbf{L}) = \left(d_\mu + \frac{d_\lambda}{2}\right) \begin{cases} \dfrac{(L_{11}^2 + L_{12}^2)^2}{L_{11}^2} & \text{if } |L_{11}| > |L_{12}|, \\ 4L_{12}^2 & \text{else if } |L_{11}| \leq |L_{12}|, \end{cases}$$

and hence, the energy density of the ribbon (22) is

$$\overline{W}(k_1, k_2) = \left(d_\mu + \frac{d_\lambda}{2}\right) \begin{cases} \dfrac{(k_2^2 + k_1^2)^2}{k_2^2} & \text{if } |k_2| > |k_1|, \\ 4k_1^2 & \text{else if } |k_2| \leq |k_1|. \end{cases} \tag{33}$$

This is the corrected Sadowsky energy density found in [14]. Indeed, in the regime in which the absolute value of the curvature k_2 is larger than the absolute value of the twist, the energy density coincides with that proposed by Sadowsky [28].

5 Euler-Lagrange equations

In order to derive the Euler-Lagrange equations we follow the approach of [24], based on incorporating the constraints into the energy functional through Lagrange multipliers. To this effect, we introduce the Lagrangian associatead to the energy (23)

$$\mathcal{L}(\mathbf{r},\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{n},m_3)=\int_0^\ell \overline{W}(k_1,k_2)+m_3k_3\,dx_1+\int_0^\ell \mathbf{n}\cdot(\mathbf{r}'-\mathbf{d}_1)\,dx_1.$$

For notational simplicity, we have dropped the dependence of $\overline{W}$ on the variable x_1. Here m_3 and $\mathbf{n}$ are the Lagrange multipliers associated to the constraints $k_3=0$ and $\mathbf{r}'=\mathbf{d}_1$, respectively, see (15) and (19). We consider an ε-parametrized family of admissible perturbations

$$\begin{aligned}
&\mathbf{r}_\varepsilon=\mathbf{r}+\varepsilon\dot{\mathbf{r}},\\
&\mathbf{d}_{i\varepsilon}=\exp(\varepsilon\dot{\mathbf{\Theta}})\mathbf{d}_i,\qquad i=1,2,3,\\
&\mathbf{n}_\varepsilon=\mathbf{n}+\varepsilon\dot{\mathbf{n}},\\
&m_{3,\varepsilon}=\mathsf{m}+\varepsilon\dot{m}_3,
\end{aligned}\tag{34}$$

of a putative stationary point $(\mathbf{r},\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{n},m_3)$. Here $\dot{\mathbf{n}}$, $\dot{m}_3$, $\dot{\mathbf{r}}$, and $\dot{\mathbf{\Theta}}$ are smooth functions taking values, respectively, in $\mathbb{R}^3$, $\mathbb{R}$, $\mathbb{R}^3$, and $\mathbb{R}^{3\times 3}$, with $\dot{\mathbf{\Theta}}$ a skew-symmetric matrix. We leave it for later the determination of boundary conditions and, accordingly, we restrict attention to perturbations that vanish at the endpoints:

$$\dot{\mathbf{r}}(0)=\dot{\mathbf{r}}(L)=\mathbf{0},\qquad \dot{\mathbf{\Theta}}(0)=\dot{\mathbf{\Theta}}(L)=\mathbf{0}.$$

We note on passing that, since

$$\mathbf{Q}_\varepsilon:=\exp(\varepsilon\dot{\mathbf{\Theta}})\tag{35}$$

is a proper rotation such that

$$\mathbf{Q}_0=\mathbf{I},\tag{36}$$

the ordered list of vectors $\{\mathbf{d}_{i\varepsilon}\}_{i=1,2,3}$ constitutes a positively–oriented orthonormal triad which depends smoothly on ε and coincides with the triad $\{\mathbf{d}_i\}_{i=1,2,3}$ for $\varepsilon=0$. We also recall, for later use, that the exponential matrix commutes with its derivative, namely, $\frac{d}{d\varepsilon}\mathbf{Q}_\varepsilon=\mathbf{Q}_\varepsilon\dot{\mathbf{\Theta}}=\dot{\mathbf{\Theta}}\mathbf{Q}_\varepsilon$. In particular, we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{Q}_\varepsilon=\dot{\mathbf{\Theta}}.\tag{37}$$

As a consequence, if $\dot{\boldsymbol{\theta}}$ is the axial vector of $\dot{\mathbf{\Theta}}$, then

$$\dot{\mathbf{d}}_i:=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{d}_{i\varepsilon}=\dot{\mathbf{\Theta}}\mathbf{d}_i=\dot{\boldsymbol{\theta}}\wedge\mathbf{d}_i\tag{38}$$

for every $i = 1, 2, 3$.

The next step is to evaluate the functional $\mathcal{L}$ at the typical perturbed configuration. The result is a differentiable function of ε, whose derivative at $\varepsilon = 0$ we require to vanish. We set

$$m_\alpha := \partial_{k_\alpha} \overline{W}(k_1, k_2), \qquad \alpha = 1, 2,$$

and we define the *referential moment vector* and the *referential curvature vector* as

$$\mathsf{m} := (m_1, m_2, m_3), \qquad \mathsf{k} := (k_1, k_2, k_3).$$

We denote by $k_{i\varepsilon}$ the quantities obtained by means of (14) but with $\mathbf{d}_i$ replaced by $\mathbf{d}_{i\varepsilon}$ and we set $\mathsf{k}_\varepsilon := (k_{1\varepsilon}, k_{2\varepsilon}, k_{3\varepsilon})$. On letting $\dot{\mathsf{k}} := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathsf{k}_\varepsilon$, we may write, on intergrating by parts,

$$\begin{aligned}\dot{\mathcal{L}} :=& \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\mathbf{r}_\varepsilon, \mathbf{d}_{1\varepsilon}, \mathbf{d}_{2\varepsilon}, \mathbf{d}_{3\varepsilon}, \mathbf{n}_\varepsilon, m_{3\varepsilon}) \\ =& \int_0^\ell \mathsf{m} \cdot \dot{\mathsf{k}} + \dot{m}_3 k_3 \, dx_1 + \int_0^\ell \dot{\mathbf{n}} \cdot (\mathbf{r}' - \mathbf{d}_1) - \mathbf{n}' \cdot \dot{\mathbf{r}} - \mathbf{n} \cdot \dot{\boldsymbol{\theta}} \wedge \mathbf{d}_1 \, dx_1.\end{aligned}$$

One can write the relation between the curvature vector and the directors in the perturbed configuration as:

$$k_{i\varepsilon} = \frac{1}{2} \epsilon_{ijk} \mathbf{d}'_{j\varepsilon} \cdot \mathbf{d}_{k\varepsilon},$$

an identity whose differentiation with respect to ε yields

$$\begin{aligned}\dot{k}_i &= \frac{1}{2}\epsilon_{ijk}\dot{\mathbf{d}}'_j \cdot \mathbf{d}_k + \frac{1}{2}\epsilon_{ijk}\mathbf{d}'_j \cdot \dot{\mathbf{d}}_k \\ &= \frac{1}{2}\epsilon_{ijk}(\dot{\boldsymbol{\theta}} \wedge \mathbf{d}_j)' \cdot \mathbf{d}_k + \frac{1}{2}\epsilon_{ijk}\mathbf{d}'_j \cdot \dot{\boldsymbol{\theta}} \wedge \mathbf{d}_k \\ &= \frac{1}{2}\epsilon_{ijk}\dot{\boldsymbol{\theta}}' \wedge \mathbf{d}_j \cdot \mathbf{d}_k + \underbrace{\frac{1}{2}\epsilon_{ijk}\dot{\boldsymbol{\theta}} \wedge \mathbf{d}'_j \cdot \mathbf{d}_k + \frac{1}{2}\epsilon_{ijk}\mathbf{d}'_j \cdot \dot{\boldsymbol{\theta}} \wedge \mathbf{d}_k}_{=0} \\ &= \frac{1}{2}\epsilon_{ijk}\mathbf{d}_j \wedge \mathbf{d}_k \cdot \dot{\boldsymbol{\theta}}' \\ &= \mathbf{d}_i \cdot \dot{\boldsymbol{\theta}}',\end{aligned}$$

where we have used the identity $\mathbf{d}_i = \frac{1}{2}\epsilon_{ijk}\mathbf{d}_j \wedge \mathbf{d}_k$. Now, on letting

$$\mathbf{m} := m_i \mathbf{d}_i, \tag{39}$$

we can write

$$\mathsf{m} \cdot \dot{\mathsf{k}} = \sum_{i=1}^3 m_i \dot{k}_i = \sum_{i=1}^3 m_i \mathbf{d}_i \cdot \dot{\boldsymbol{\theta}}' = \mathbf{m} \cdot \dot{\boldsymbol{\theta}}',$$

and we arrive at the following expression:

$$
\begin{aligned}
\dot{\mathcal{L}} &= \int_0^\ell \mathbf{m}\cdot\dot{\boldsymbol{\theta}}' + \dot{m}_3 k_3 - \mathbf{n}\cdot\dot{\boldsymbol{\theta}}\wedge\mathbf{d}_1\,dx_1 + \int_0^\ell \dot{\mathbf{n}}\cdot(\mathbf{r}'-\mathbf{d}_1) - \mathbf{n}'\cdot\dot{\mathbf{r}}\,dx_1 \\
&= -\int_0^\ell (\mathbf{m}'+\mathbf{d}_1\wedge\mathbf{n})\cdot\dot{\boldsymbol{\theta}} + \dot{m}_3 k_3\,dx_1 + \int_0^\ell \dot{\mathbf{n}}\cdot(\mathbf{r}'-\mathbf{d}_1) - \mathbf{n}'\cdot\dot{\mathbf{r}}\,dx_1
\end{aligned}
$$

On requiring that $\dot{\mathcal{L}}$ vanish for every perturbation we obtain the *constraint equations*:

$$\mathbf{r}' = \mathbf{d}_1, \qquad k_3 = 0,$$

and the *equilibrium equations*:

$$\mathbf{n}' = \mathbf{0}, \qquad \mathbf{m}' + \mathbf{d}_1\wedge\mathbf{n} = \mathbf{0}. \tag{40}$$

The equilibrium equations can be rendered in referential form by introducing the components

$$n_i := \mathbf{n}\cdot\mathbf{d}_i,$$

so that

$$\mathbf{n} = n_i\mathbf{d}_i. \tag{41}$$

On substituting (41) into the first equilibrium equation and (39) into the second equilibrium equation we obtain, in the order,

$$
\begin{aligned}
&n_i'\mathbf{d}_i + n_i\mathbf{k}\wedge\mathbf{d}_i = \mathbf{0}, \\
&m_i'\mathbf{d}_i + m_i\mathbf{d}_i' + \mathbf{d}_1\wedge\mathbf{n} = \mathbf{0}.
\end{aligned} \tag{42}
$$

Now we introduce the rotation matrix

$$\mathbf{D} = \mathbf{d}_i\otimes\mathbf{e}_i, \tag{43}$$

and we observe that

$$\mathbf{D}^T\mathbf{d}_i = \mathbf{e}_i. \tag{44}$$

We also notice that, given two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$, the following identity holds true:

$$\mathbf{D}^T(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{D}^T\mathbf{a})\wedge(\mathbf{D}^T\mathbf{b})$$

Thus, in particular, we have

$$\mathbf{D}^T(\mathbf{k}\wedge\mathbf{d}_i) = \boldsymbol{\mathsf{k}}\wedge\mathbf{e}_i,$$

and

$$\mathbf{D}^T(\mathbf{d}_1\wedge\mathbf{n}) = \mathbf{e}_1\wedge\boldsymbol{\mathsf{n}}.$$

Thus on premultiplying both equations in (42) by $\mathbf{D}^T$, we arrive at

$$
\begin{aligned}
&n_i'\mathbf{e}_i + n_i\boldsymbol{\mathsf{k}}\wedge\mathbf{e}_i = \mathbf{0}, \\
&m_i'\mathbf{e}_i + m_i\boldsymbol{\mathsf{k}}\wedge\mathbf{e}_i + \mathbf{e}_1\wedge\boldsymbol{\mathsf{n}} = \mathbf{0},
\end{aligned}
$$

which are equivalent to

$$
\begin{aligned}
&\boldsymbol{\mathsf{n}}' + \boldsymbol{\mathsf{k}}\wedge\boldsymbol{\mathsf{n}} = \mathbf{0}, \\
&\boldsymbol{\mathsf{m}}' + \boldsymbol{\mathsf{k}}\wedge\boldsymbol{\mathsf{m}} + \mathbf{e}_1\wedge\boldsymbol{\mathsf{n}} = \mathbf{0}.
\end{aligned}
$$

6 Example: a twisted ribbon

A class of deformations that are of particular importance for the mechanics of a ribbon are the twisting deformations.

We consider an experiment where an isotropic ribbon, while being subject to a *tensile force* N, is twisted under controlled *terminal rotation* Θ.

We model this experiment as follows. We identify the middle line of the ribbon with a straight line passing through the origin and parallel to the unit vector $\mathbf{e}_1$. We clamp the end $x_1 = 0$ to a fixed base support by asking

$$\mathbf{r}(0) = \mathbf{0}, \qquad \text{and} \qquad \mathbf{d}_i(0) = \mathbf{e}_i, \qquad \text{for } i = 1, 2, 3.$$

The condition that the end $x_1 = \ell$ be clamped and subject to a tensile force N is rendered by the following set of equations:

$$\begin{gathered} \mathbf{n}(\ell) \cdot \mathbf{e}_1 = N, \quad \mathbf{r}(\ell) \cdot \mathbf{e}_2 = 0, \quad \mathbf{r}(\ell) \cdot \mathbf{e}_3 = 0, \\ \mathbf{d}_1(\ell) = \mathbf{e}_1, \\ \mathbf{d}_2(\ell) = \cos\Theta\, \mathbf{e}_2 + \sin\Theta\, \mathbf{e}_3, \quad \mathbf{d}_3(L) = -\sin\Theta\, \mathbf{e}_2 + \cos\Theta\, \mathbf{e}_3. \end{gathered}$$

The boundary–value problem. For the reader's convenience we recapitulate the relevant field equations.

- The referential equilibrium equations:

$$\begin{aligned} &\mathbf{n}' + \mathbf{k} \wedge \mathbf{n} = \mathbf{0}, \\ &\mathbf{m}' + \mathbf{k} \wedge \mathbf{m} + \mathbf{e}_1 \wedge \mathbf{n} = \mathbf{0}. \end{aligned} \tag{45a}$$

- The compatibility equation:

$$\mathbf{d}_i' = \mathbf{k} \wedge \mathbf{d}_i, \qquad i = 1, 2, 3. \tag{45b}$$

- The constitutive equations:

$$m_\alpha = \frac{\partial \overline{W}(k_1, k_2)}{\partial k_\alpha}, \qquad \alpha = 1, 2. \tag{45c}$$

- The constraint equations:

$$\begin{aligned} &\mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}, \\ &k_3 = 0, \\ &\mathbf{r}' = \mathbf{d}_1. \end{aligned} \tag{45d}$$

The reference state. A solution of the twist problem is the following

$$\begin{aligned} \mathbf{r}(x_1) &= x_1 \mathbf{e}_1, \\ \mathbf{d}_1(x_1) &= \mathbf{e}_1, \\ \mathbf{d}_2(x_1) &= +\cos(\Theta x_1/\ell)\mathbf{e}_2 + \sin(\Theta x_1/\ell)\mathbf{e}_3, \\ \mathbf{d}_3(x_1) &= -\sin(\Theta x_1/\ell)\mathbf{e}_2 + \cos(\Theta x_1/\ell)\mathbf{e}_3. \end{aligned} \tag{46}$$

The corresponding referential curvatures are

$$\begin{aligned} k_1(x_1) &= \Theta/\ell, \\ k_2(x_1) &= 0, \\ k_3(x_1) &= 0. \end{aligned} \tag{47}$$

The corresponding referential tension and moment are, respectively,

$$\begin{aligned} \mathbf{n}(x_1) &= (N, 0, 0), \\ m_1(x_1) &= \partial_{k_1}\overline{W}(k_1, k_2) =: GI_t\Theta, \\ m_2(x_1) &= \partial_{k_2}\overline{W}(k_1, k_2) = 0, \\ m_3(x_1) &= 0. \end{aligned} \tag{48}$$

The bending moments m_1 and m_2 are determined by the corresponding curvatures through the constitutive equations (33). Instead, the bending moment m_3 is computed by making use of the equilibrium equation $m_2' + k_3 m_1 - k_1 m_3 - n_3 = 0$. In the second equation of (48) we have set

$$GI_t := 4\,(2d_\mu + d_\lambda)\,\frac{1}{\ell}.$$

Computation of $\overline{W}$ in terms of Q. The solution found is the *macroscopic* solution, *i.e.,* the solution of the macroscopic problem. We now reconstruct the *microscopic* solution. To do so we need to express the energy in terms of the quadratic form Q.

From (16) and (47) we find

$$\mathbf{K}^\circ = \begin{pmatrix} 0 & k_1 \\ k_1 & \gamma \end{pmatrix},$$

where $k_1 = \Theta/\ell$ is the amount of twist, see (47), and γ has to be determined from (22). Since

$$W^{**}(\mathbf{L}) = \frac{GI_t\ell}{8}\left(|\mathbf{L}|^2 + 2|\det \mathbf{L}|\right),$$

see (31), we have that $W^{**}(\mathbf{K}^\circ) = GI_t\ell/8(4k_1^2 + \gamma^2)$ and thence the mimimum in (22) is achieved for

$$\gamma = 0.$$

We therefore have

$$\mathbf{K}^\circ = \begin{pmatrix} 0 & k_1 \\ k_1 & 0 \end{pmatrix}, \qquad \overline{W}(k_1, 0) = W^{**}(\mathbf{K}^\circ). \tag{49}$$

Thus, we are in the case of negative Gaussian curvature ($\det \mathbf{K}^\circ < 0$). We look for $\mathbf{A}$, $\mathbf{B}$, and t such that $\det \mathbf{A} = \det \mathbf{B} = 0$ and $\mathbf{K}^\circ = (1-t)\mathbf{A} + t\mathbf{B}$.

Since

$$(\mathbf{K}^\circ)^* = \begin{pmatrix} 0 & -k_1 \\ -k_1 & 0 \end{pmatrix},$$

on recalling (29) and (32) we have

$$\lambda = \frac{(\mathbf{K}^\circ)^* \cdot \mathbf{D}^+}{2\sqrt{-\det \mathbf{L}}} = 0. \tag{50}$$

By (28) and (29) we find

$$t = \frac{1}{2 + 2\lambda(\lambda - \sqrt{\lambda^2+1})} = \frac{1}{2}, \tag{51}$$

and

$$\eta = \sqrt{-\det \mathbf{K}^\circ}(\lambda - \sqrt{\lambda^2+1}) = -k_1, \tag{52}$$

so that by (28) and (30) we deduce that

$$\mathbf{B} = \mathbf{K}^\circ - \eta \mathbf{D}^- = \begin{pmatrix} k_1 & k_1 \\ k_1 & k_1 \end{pmatrix}, \qquad \mathbf{A} = \frac{\mathbf{K}^\circ - t\mathbf{B}}{1-t} = \begin{pmatrix} -k_1 & k_1 \\ k_1 & -k_1 \end{pmatrix}.$$

According to (26) and (49), we have

$$\mathbf{K}^\circ = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}, \quad \text{and} \quad \overline{W}(k_1, 0) = W^{**}(\mathbf{K}^\circ) = \frac{1}{2}Q(\mathbf{A}) + \frac{1}{2}Q(\mathbf{B}). \tag{53}$$

Microscopic oscillations. The macroscopic curvature $\mathbf{K}^\circ$ has determinant different from zero and therefore it is microscopically inaccessible. Equation (53) shows that it can be generated by mixing, on a fine scale and in equal proportions, the "curvatures" $\mathbf{A}$ and $\mathbf{B}$, which have both null determinant. The oscillations occurring in a twisted ribbon are also studied in [7, 8, 23].

In a ribbon of finite width h, given a matrix with a null determinant, like $\mathbf{A}$ and $\mathbf{B}$, it is always possible to find a deformation whose second fundamental form is the given matrix, essentially by solving (5). Instead, given two matrices $\mathbf{A}$ and $\mathbf{B}$ with null determinant it is possible to find a (smooth) deformation whose fundamental form takes only the values $\mathbf{A}$ and $\mathbf{B}$ only if $\mathbf{A}$ is a multiple of $\mathbf{B}$, see Appendix A.3 for a proof of this statement. Since in the case under consideration the matrices $\mathbf{A}$ and $\mathbf{B}$ are not proportional, it is necessary to introduce a boundary layer between the regions where the curvature is equal to $\mathbf{A}$ and $\mathbf{B}$. Since

$$\mathbf{A} = -k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

the boundary layer can be introduced as follows. Let $f_n : \mathbb{R} \to [-1, 1]$ be a periodic function whose period is similar to that represented in Fig. 1 and let

$$\mathbf{K}^{(n)}(x_1) = k_1 f_n(nx_1) \begin{pmatrix} f_n(nx_1) \\ 1 \end{pmatrix} \otimes \begin{pmatrix} f_n(nx_1) \\ 1 \end{pmatrix}.$$

Then, $\mathbf{K}^{(n)}$ is a periodic function, whose period size is $1/n$, having null determinant everywhere and with average (and weak limit) equal to $\mathbf{K}^\circ$. Moreover,

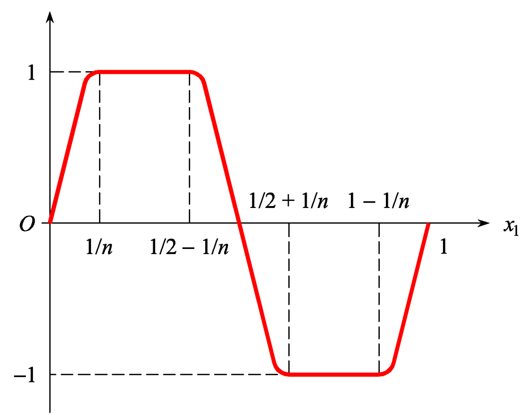

Figure 1: Period of the function f_n.

the function $\mathbf{K}^{(n)}$ is equal to $\mathbf{A}$ and $\mathbf{B}$ except on n boundary layers of size $2/n^2$. Since the energy on these layers is bounded they will contribute, to the total energy, by a constant times $n\ 2/n^2$. Thus, the contribution of the layers will approach zero as the oscillations increase, *i.e.*, as n goes to $+\infty$. From the second equation of (53) we can therefore deduce that

$$\lim_{n\to\infty}\int_0^\ell Q(\mathbf{K}^{(n)})\,dx_1=\int_0^\ell W^{**}(\mathbf{K}^\circ)\,dx_1=\int_0^\ell \overline{W}(k_1,0)\,dx_1.$$

Reconstruction of the microscopic deformation from the microscopic curvature. With (14) in mind, we set

$$k_1^{(n)}:=K_{12}^{(n)}=k_1 f_n^2(n\cdot),\quad k_2^{(n)}:=-K_{11}^{(n)}=-k_1 f_n^2(n\cdot),$$

and we construct a set of directors $\mathbf{d}_i^{(n)}$ such that

$$(\mathbf{d}_2^{(n)})'\cdot\mathbf{d}_3^{(n)}=k_1^{(n)},\quad (\mathbf{d}_3^{(n)})'\cdot\mathbf{d}_1^{(n)}=k_2^{(n)},\quad\text{and}\quad (\mathbf{d}_1^{(n)})'\cdot\mathbf{d}_2^{(n)}=0.$$

This task is accomplished by introducing the skew–symmetric tensor

$$\widetilde{\mathbf{K}}^{(n)}=\begin{pmatrix}0&0&+k_2^{(n)}\\0&0&-k_1^{(n)}\\-k_2^{(n)}&+k_1^{(n)}&0\end{pmatrix},\tag{54}$$

and by finding a proper rotation $\mathbf{D}^{(n)}(x_1)$ that solves the Cauchy problem:

$$\begin{cases}(\mathbf{D}^{(n)})'=\mathbf{D}^{(n)}\widetilde{\mathbf{K}}^{(n)}\quad\text{in}\quad(0,\ell),\\ \mathbf{D}^{(n)}(0)=\mathbf{I}.\end{cases}\tag{55}$$

The set of directors is then found by setting

$$\mathbf{d}_i^{(n)} := \mathbf{D}^{(n)} \mathbf{e}_i,$$

compare with (43). We are now in a position, for small h, to construct a isometric deformation $\boldsymbol{\chi}^{(n)} : \omega \to \mathbb{R}^3$ by means of a ruled surface *à la* Wunderlich [35], see also [14]. We first determine the microscopic deformation of the midline

$$\mathbf{r}^{(n)}(x_1) := \int_0^{x_1} \mathbf{d}_1^{(n)}(s)\, ds,$$

compare the third equation of (45d), and then we set

$$\begin{aligned}
\Phi^{(n)}(\xi_1, \xi_2) &:= \xi_1 \mathbf{e}_1 + \xi_2(\cos\vartheta^{(n)} \mathbf{e}_1 + \sin\vartheta^{(n)} \mathbf{e}_2), \\
v^{(n)}(\xi_1, \xi_2) &:= \mathbf{r}^{(n)}(\xi_1) + \xi_2(\cos\vartheta^{(n)} \mathbf{d}_1^{(n)} + \sin\vartheta^{(n)} \mathbf{d}_2^{(n)}),
\end{aligned}$$

where $(\cos\vartheta^{(n)}, \sin\vartheta^{(n)})$ is the eigenvector associated to the null eigenvalue of $\mathbf{K}^{(n)}$. Then, the microscopic deformation is

$$\boldsymbol{\chi}^{(n)} := v^{(n)} \circ (\Phi^{(n)})^{-1}.$$

Tedious calculations show that the second fundamental form of $\boldsymbol{\chi}^{(n)}$ is exactly $\mathbf{K}^{(n)}$ and also that $\boldsymbol{\chi}^{(n)}$ approaches $\boldsymbol{\chi}$ as n goes to infinity. We refrain from doing these calculations and we refer to [14] for further details.

A Appendices

A.1 Convexification

The aim of this appendix is to determine the convex envelope of

$$W(\mathbf{L}) = \begin{cases} Q(\mathbf{L}) & \text{if } \det \mathbf{L} = 0, \\ +\infty & \text{else,} \end{cases}$$

where Q is a positive definite quadratic form defined on 2-by-2 symmetric matrices.

Consider the function

$$Q_{\det}(\mathbf{L}; \alpha) = Q(\mathbf{L}) + \alpha \det \mathbf{L}$$

with α a real number. For fixed α, $Q_{\det}(\cdot; \alpha)$ is a quadratic function - recall that $\mathbf{L}$ may be represented by a 2×2 matrix - and hence it is convex if and only if $Q_{\det}(\mathbf{L}; \alpha) \geq 0$ for every $\mathbf{L}$. This inequality is satisfied for all $\mathbf{L}$ with determinant equal to zero, since Q is positive definite. If $\det \mathbf{L} > 0$, the condition $Q_{\det}(\mathbf{L}; \alpha) \geq 0$ for every $\mathbf{L}$ holds if and only if

$$Q(\mathbf{L}/\sqrt{\det \mathbf{L}}) + \alpha \geq 0 \quad \text{for every } \mathbf{L},$$

that is equivalent to

$$\alpha \geq -Q(\mathbf{L}) \quad \text{for every } \mathbf{L} \text{ with } \det \mathbf{L} = 1,$$

or

$$\alpha \geq -\min_{\det \mathbf{L}=1} Q(\mathbf{L}) =: -\alpha^-. \tag{56}$$

Similarly, if $\det \mathbf{L} < 0$, the condition $Q_{\det}(\mathbf{L}; \alpha) \geq 0$ for every $\mathbf{L}$ holds if and only if

$$\alpha \leq \min_{\det \mathbf{L}=-1} Q(\mathbf{L}) =: \alpha^+. \tag{57}$$

Note that $\alpha^-, \alpha^+ > 0$, since Q is positive definite. Hence, $Q_{\det}(\cdot; \alpha)$ is convex if and only if $-\alpha^- \leq \alpha \leq \alpha^+$, and

$$W_c(\mathbf{L}) = \sup_{-\alpha^- \leq \alpha \leq \alpha^+} Q_{\det}(\mathbf{L}; \alpha) \tag{58}$$

is also convex, for the point-wise supremum of convex functions is convex. It immediately follows that

$$W_c(\mathbf{L}) = Q(\mathbf{L}) + \alpha^+ (\det \mathbf{L})^+ + \alpha^- (\det \mathbf{L})^-, \tag{59}$$

where $(\det \mathbf{L})^+$ is the positive part of $\det \mathbf{L}$, and $(\det \mathbf{L})^-$ is the negative part, i.e., $(\det \mathbf{L})^- = -\det \mathbf{L}$ if $\det \mathbf{L} < 0$ and zero otherwise.

Clearly, $W(\mathbf{L}) \geq Q_{\det}(\mathbf{L}; \alpha)$ for any $\mathbf{L}$ and for any α, hence $W(\mathbf{L}) \geq W_c(\mathbf{L})$. Since W_c is convex it also follows that

$$W^{**}(\mathbf{L}) \geq W_c(\mathbf{L}).$$

In the rest of the section we will show that the inequality above is indeed an equality, and that the lower bound can be achieved by a single lamination. More precisely, we show that given $\mathbf{L}$ we can find two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, and $0 < t < 1$ such that

$$(1-t)\mathbf{A} + t\mathbf{B} = \mathbf{L}, \qquad \text{and} \qquad (1-t)W(\mathbf{A}) + tW(\mathbf{B}) = W_c(\mathbf{L}).$$

By the definition of W, the statement above is equivalent to: given $\mathbf{L}$ there exist two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ with $\det \mathbf{A} = \det \mathbf{B} = 0$, and $0 < t < 1$ such that

$$(1-t)\mathbf{A} + t\mathbf{B} = \mathbf{L}, \qquad \text{and} \qquad (1-t)Q(\mathbf{A}) + tQ(\mathbf{B}) = W_c(\mathbf{L}).$$

In this statement we may eliminate the tensor $\mathbf{A}$ by noticing that: given $\mathbf{L}, \mathbf{B}$ and t, as above, there exists a symmetric tensor $\mathbf{A}$ with $\det \mathbf{A} = 0$ and satisfying the equation

$$(1-t)\mathbf{A} = \mathbf{L} - t\mathbf{B}$$

if and only if $\det(\mathbf{L} - t\mathbf{B}) = 0$. Since

$$\begin{aligned}(1-t)Q(\mathbf{A}) + tQ(\mathbf{B}) &= \frac{1}{1-t}Q((1-t)\mathbf{A}) + tQ(\mathbf{B}) \\ &= \frac{1}{1-t}Q(\mathbf{L} - t\mathbf{B}) + tQ(\mathbf{B}) \\ &= Q(\mathbf{L}) + \frac{t}{1-t}Q(\mathbf{L} - \mathbf{B}),\end{aligned}$$

the statement to be proven is: given $\mathbf{L}$ there exist a symmetric tensor $\mathbf{B}$ with $\det \mathbf{B} = 0$, and $0 < t < 1$ such that

$$\det(\mathbf{L} - t\mathbf{B}) = 0, \qquad \text{and} \qquad Q(\mathbf{L}) + \frac{t}{1-t}Q(\mathbf{L} - \mathbf{B}) = W_c(\mathbf{L}). \tag{60}$$

Recalling (56) and (57), let $\mathbf{D}^-, \mathbf{D}^+$ be the two symmetric matrices with $\det \mathbf{D}^- = -1$ and $\det \mathbf{D}^+ = 1$ such that

$$\alpha^- = Q(\mathbf{D}^+), \qquad \alpha^+ = Q(\mathbf{D}^-). \tag{61}$$

Note that the tensors $\mathbf{D}^-$ and $\mathbf{D}^+$ only depend on the bending stiffness tensor $\mathbb{D}$. We now show that if $\det \mathbf{L} > 0$ there exist a symmetric tensor $\mathbf{B}$ with null determinant and $0 < t < 1$ such that

$$\det(\mathbf{L} - t\mathbf{B}) = 0, \quad \det(\mathbf{L} - \mathbf{B}) = -\frac{1-t}{t}\det \mathbf{L}, \tag{62}$$

and

$$\mathbf{D}^- = \frac{\mathbf{L} - \mathbf{B}}{\sqrt{-\det(\mathbf{L} - \mathbf{B})}}. \tag{63}$$

Assuming, for the moment, the validity of this statement we deduce (60) in the case $\det \mathbf{L} > 0$:

$$\begin{aligned}Q(\mathbf{L}) + \frac{t}{1-t}Q(\mathbf{L} - \mathbf{B}) &= Q(\mathbf{L}) + \frac{t}{1-t}(-\det(\mathbf{L} - \mathbf{B}))Q(\mathbf{D}^-) \\ &= Q(\mathbf{L}) + Q(\mathbf{D}^-)\det \mathbf{L} \\ &= Q(\mathbf{L}) + \alpha^+ \det \mathbf{L} = W_c(\mathbf{L}),\end{aligned}$$

where we have used, in order, (63), (62), (61), and (59). Hence, assuming (62) and (63) we have shown that if $\det \mathbf{L} > 0$ then $W^{**}(\mathbf{L}) = W_c(\mathbf{L})$. A similar proof can be made for $\det \mathbf{L} < 0$, while the result is trivial for $\det \mathbf{L} = 0$.

We therefore only need to show the validity of (62) and (63). In doing it, we shall provide formulae that deliver $\mathbf{B}$ and t.

For $\det \mathbf{L} > 0$, set $\mathbf{B} = \mathbf{L} - \eta\mathbf{D}^-$ where the real number η is chosen by imposing that $\det \mathbf{B} = 0$. With (1), this amount to solve the second order equation

$$\det \mathbf{L} - \eta \mathbf{L}^* \cdot \mathbf{D}^- - \eta^2 = 0, \tag{64}$$

where we used the fact that $\det \mathbf{D}^- = -1$. Of the two solutions we choose

$$\eta = \frac{-\mathbf{L}^* \cdot \mathbf{D}^- + \sqrt{(\mathbf{L}^* \cdot \mathbf{D}^-)^2 + 4 \det \mathbf{L}}}{2}.$$

To keep the notation compact, we set

$$\lambda = \frac{\mathbf{L}^* \cdot \mathbf{D}^-}{2\sqrt{\det \mathbf{L}}}$$

so that

$$\eta = \sqrt{\det \mathbf{L}}(\sqrt{\lambda^2 + 1} - \lambda).$$

Now that $\mathbf{B}$ has been fixed, we find t by solving $(62)_1$. We first write

$$\mathbf{L} - t\mathbf{B} = \mathbf{L} - t(\mathbf{L} - \eta\mathbf{D}^-) = (1-t)\mathbf{L} + t\eta\mathbf{D}^-,$$

and by using (1), we write $(62)_1$ as

$$(1-t)^2 \det \mathbf{L} + (1-t)t\eta\mathbf{L}^* \cdot \mathbf{D}^- - t^2\eta^2 = 0,$$

which simplifies, thanks to (64), to

$$\det \mathbf{L} + t(\eta\mathbf{L}^* \cdot \mathbf{D}^- - 2 \det \mathbf{L}) = 0.$$

Thus

$$t = \frac{-\det \mathbf{L}}{\eta\mathbf{L}^* \cdot \mathbf{D}^- - 2 \det \mathbf{L}} = \frac{1}{2 - 2\lambda(\sqrt{\lambda^2 + 1} - \lambda)}.$$

We may easily verify that $0 < t < 1$. Indeed, $t > 0$ if and only if $2\lambda(\sqrt{\lambda^2+1} - \lambda) < 2$, which clearly holds if $\lambda \leq 0$, while for $\lambda > 0$ is equivalent to $\sqrt{\lambda^2+1} < \frac{1}{\lambda} + \lambda$ which is trivially verified. Similarly we may check that $t < 1$.

From the definition of $\mathbf{B}$, we have that $\eta\mathbf{D}^- = \mathbf{L} - \mathbf{B}$. Taking the determinant, we find $-\eta^2 = \det(\mathbf{L} - \mathbf{B})$ from which (63) follows. Finally, by applying (1) twice, we obtain that

$$\begin{aligned} \det(\mathbf{L} - \mathbf{B}) = \det \mathbf{L} - \mathbf{L}^* \cdot \mathbf{B} &= \det \mathbf{L} - \frac{\det \mathbf{L}}{t} + \frac{\det \mathbf{L} - \mathbf{L}^* \cdot (t\mathbf{B})}{t} \\ &= \frac{t-1}{t} \det \mathbf{L} + \frac{\det(\mathbf{L} - t\mathbf{B})}{t}, \end{aligned}$$

and recalling $(62)_1$ we deduce $(62)_2$.

We close this appendix by summarizing the results found:

$$W^{**}(\mathbf{L}) = Q(\mathbf{L}) + \alpha^+(\det \mathbf{L})^+ + \alpha^-(\det \mathbf{L})^-,$$

where

$$\alpha^- = \min_{\det \mathbf{L} = 1} Q(\mathbf{L}) = Q(\mathbf{D}^+), \qquad \alpha^+ = \min_{\det \mathbf{L} = -1} Q(\mathbf{L}) = Q(\mathbf{D}^-).$$

If $\det \mathbf{L} > 0$, set

$$\lambda = \frac{\mathbf{L}^* \cdot \mathbf{D}^-}{2\sqrt{\det \mathbf{L}}}, \qquad t = \frac{1}{2 - 2\lambda(\sqrt{\lambda^2+1} - \lambda)},$$
$$\eta = \sqrt{\det \mathbf{L}}(\sqrt{\lambda^2+1} - \lambda), \quad \mathbf{B} = \mathbf{L} - \eta \mathbf{D}^-;$$

while, if $\det \mathbf{L} < 0$, set

$$\lambda = \frac{\mathbf{L}^* \cdot \mathbf{D}^+}{2\sqrt{-\det \mathbf{L}}}, \qquad t = \frac{1}{2 + 2\lambda(\lambda - \sqrt{\lambda^2+1})},$$
$$\eta = \sqrt{-\det \mathbf{L}}(\lambda - \sqrt{\lambda^2+1}), \quad \mathbf{B} = \mathbf{L} - \eta \mathbf{D}^+.$$

Then, with

$$\mathbf{A} = \frac{\mathbf{L} - t\mathbf{B}}{1-t},$$

we have that

$$(1-t)\mathbf{A} + t\mathbf{B} = \mathbf{L}, \qquad \text{and} \qquad (1-t)Q(\mathbf{A}) + tQ(\mathbf{B}) = W^{**}(\mathbf{L}).$$

A.2 Evaluation of $\alpha^\pm$ for various symmetries

We first turn our attention to the computation of the constants α^+ and α^- in the definition of the convexification W^{**} in (24). According to (25) we must minimize the quadratic form $Q(\mathbf{L})$ over the manifolds $\{\mathbf{L} : \det \mathbf{L} = \pm 1\}$. We accomplish this task through the method of Lagrange multipliers by seeking stationary points of the *augmented functional* $(\mathbf{L}, \beta) \mapsto Q(\mathbf{L}) + \beta(\det \mathbf{L} \pm 1)$. We note that the determinant of a 2-by-2 matrix is a quadratic form. Thus, there exists a fourth–order tensor $\mathbb{E}$ such that

$$\det \mathbf{L} = L_{11}L_{22} - L_{12}^2 = \frac{1}{2}\mathbb{E}\mathbf{L} \cdot \mathbf{L}. \tag{65}$$

Thus, the augmented functional is

$$(\mathbf{L}, \beta) \mapsto \frac{1}{2}(\mathbb{D} + \beta\mathbb{E})\mathbf{L} \cdot \mathbf{L} \mp \beta. \tag{66}$$

We now argue that the minima $\alpha^\pm$ have the following property:

$$\mathbb{D} \mp \alpha^\mp \mathbb{E} \text{ is singular,} \tag{67}$$

that is $\alpha^\mp$ are *generalized eigenvalues of* $\mathbb{D}$ *with respect to* $\mathbb{E}$. To verify the last statement, let us recall the definition of $\mathbf{D}^\pm$, see (27), that is

$$\mathbf{D}^\pm = \operatorname{argmin}_{\det \mathbf{L} = \pm 1} Q(\mathbf{L}). \tag{68}$$

Then there exist $\beta^\mp$ such that the pair $(\mathbf{D}^\pm, \beta^\mp)$ is a stationary point of the augmented functional defined in (66). The stationarity conditions for such pair are

$$(\mathbb{D} + \beta^{\mp}\mathbb{E})\mathbf{D}^{\pm} = \mathbf{0}, \tag{69a}$$

$$\frac{1}{2}\mathbb{E}\mathbf{D}^{\pm} \cdot \mathbf{D}^{\pm} \mp 1 = 0. \tag{69b}$$

It follows from the second stationarity condition that $\mathbf{D}^{\pm} \neq \mathbf{0}$. This implies that the tensor $\mathbb{D} + \beta^{\mp}\mathbb{E}$ is singular. Moreover, the minimum $\alpha^{\mp}$ coincides with the Lagrange multiplier $\beta^{\mp}$, up to sign:

$$\alpha^{\mp} = Q(\mathbf{D}^{\pm}) = \frac{1}{2}\mathbb{D}\mathbf{D}^{\pm} \cdot \mathbf{D}^{\pm} \overset{(69a)}{=} -\frac{\beta^{\mp}}{2}\mathbb{E}\mathbf{D}^{\pm} \cdot \mathbf{D}^{\pm} \overset{(69b)}{=} \mp\beta^{\mp}. \tag{70}$$

To carry our calculation further on, we observe that

$$\mathbb{D}\mathbf{L} \cdot \mathbf{L} = \begin{pmatrix} \mathbb{D}_{1111} & \mathbb{D}_{1122} & \mathbb{D}_{1112} \\ \mathbb{D}_{1122} & \mathbb{D}_{2222} & \mathbb{D}_{1222} \\ \mathbb{D}_{1122} & \mathbb{D}_{1222} & \mathbb{D}_{1212} \end{pmatrix} \begin{pmatrix} L_{11} \\ L_{22} \\ 2L_{12} \end{pmatrix} \cdot \begin{pmatrix} L_{11} \\ L_{22} \\ 2L_{12} \end{pmatrix}. \tag{71}$$

and that

$$\mathbb{E}\mathbf{L} \cdot \mathbf{L} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} L_{11} \\ L_{22} \\ 2L_{12} \end{pmatrix} \cdot \begin{pmatrix} L_{11} \\ L_{22} \\ 2L_{12} \end{pmatrix}. \tag{72}$$

Thus, on denoting by $\mathbb{d}$ and $\mathbb{e}$ the matrices appearing in the definitions of the quadratic forms (71) and (72), we can write (67) as

$$\det(\mathbb{d} \mp \alpha^{\mp}\mathbb{e}) = 0. \tag{73}$$

Orthotriopic response. In the special case, when the material is orthotropic with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$, we have

$$\mathbb{D}_{1122} = \mathbb{D}_{1222} = 0.$$

In this case, the characteristic polynomial

$$p(\beta) := \det(\mathbb{d} + \beta\mathbb{e}) = \left(\mathbb{D}_{1212} - \frac{\beta}{2}\right)(\mathbb{D}_{1111}\mathbb{D}_{2222} - (\mathbb{D}_{1122} + \beta)^2)$$

has the following set of roots:

$$\{-\sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} - \mathbb{D}_{1122}, \sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} - \mathbb{D}_{1122}, 2\mathbb{D}_{1212}\}.$$

The positive definiteness of $\mathbb{D}$ implies that the first root is negative, and that the remaining roots are positive. The negative root must coincide with $\beta^- = -\alpha^-$, that is

$$\alpha^- = \sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} + \mathbb{D}_{1122}.$$

On the other hand, the minimality of α^+ yields

$$\alpha^+ = \min\{\sqrt{\mathbb{D}_{1111}\mathbb{D}_{2222}} - \mathbb{D}_{1122}, 2\mathbb{D}_{1212}\}.$$

We note that the foregoing calculation can be used also when the plate is orthotropic with respect to a basis $(\mathbf{e}'_1, \mathbf{e}'_2)$, not necessarily coincident with $(\mathbf{e}_1, \mathbf{e}_2)$. In this case, the relevant strain energy is $W'(\mathbf{L}) = W(\mathbf{R}\mathbf{L}\mathbf{R}^T)$, where $\mathbf{R} = \mathbf{e}'_\alpha \otimes \mathbf{e}_\alpha$ is a rotation matrix. Then it can be easily shown that the convexification of Q' is given by

$$(W')^{**}(\mathbf{L}) = W^{**}(\mathbf{R}\mathbf{L}\mathbf{R}^T).$$

Isotropic response. For the isotropic strain energy $Q(\mathbf{L}) = d_\mu|\mathbf{L}|^2 + \frac{d_\lambda}{2}(L_{11} + L_{22})^2$ the relevant components of $\mathbb{D}$ are $\mathbb{D}_{1111} = \mathbb{D}_{2222} = 2d_\mu + d_\lambda$, $\mathbb{D}_{1122} = d_\lambda$, $\mathbb{D}_{1212} = d_\mu$. Then

$$\alpha^- = 2(d_\mu + d_\lambda), \qquad \alpha^+ = 2d_\mu,$$

and so

$$\begin{aligned} W^{**}(\mathbf{L}) &= d_\mu|\mathbf{L}|^2 + \frac{d_\lambda}{2}(L_{11} + L_{22})^2 + 2d_\mu(\det \mathbf{L})^+ + 2(d_\mu + d_\lambda)(\det \mathbf{L})^- \\ &= d_\mu|\mathbf{L}|^2 + \frac{d_\lambda}{2}|\mathbf{L}|^2 + d_\lambda \det \mathbf{L} + 2d_\mu(\det \mathbf{L})^+ + 2(d_\mu + d_\lambda)(\det \mathbf{L})^- \\ &= \left(d_\mu + \frac{d_\lambda}{2}\right)|\mathbf{L}|^2 + 2d_\mu|\det \mathbf{L}| + d_\lambda(\det \mathbf{L} + 2(\det \mathbf{L})^-) \\ &= \left(d_\mu + \frac{d_\lambda}{2}\right)\left(|\mathbf{L}|^2 + 2|\det \mathbf{L}|\right). \end{aligned}$$

A.3 Compatibility among curvatures

The aim of this Appendix is to prove the following statement.

Let $\omega, \omega_\mathbf{A}$ and $\omega_\mathbf{B}$ be three two-dimensional open sets, whit $\omega = \omega_\mathbf{A} \cup \omega_\mathbf{B} \cup \Gamma$, where $\Gamma = \overline{\omega}_\mathbf{A} \cap \overline{\omega}_\mathbf{B}$ is a smooth curve and $\overline{\omega}_\mathbf{A}$ denotes the closure of $\omega_\mathbf{A}$. Let $\mathbf{A}$ and $\mathbf{B}$ be two 2-by-2 symmetric matrices with null determinant. If there exists a deformation $\mathbf{y}$ continuously differentiable on ω and twice differentiable on $\overline{\omega}_\mathbf{A}$ and on $\overline{\omega}_\mathbf{B}$ such that

$$\mathbf{K}_\mathbf{y} = \mathbf{A} \text{ on } \overline{\omega}_\mathbf{A}, \quad \text{and} \quad \mathbf{K}_\mathbf{y} = \mathbf{B} \text{ on } \overline{\omega}_\mathbf{B},$$

where $\mathbf{K}_\mathbf{y}$ denotes the second fundamental form of $\mathbf{y}$, then the curve Γ is straight and there exists a constant σ for which $\mathbf{A} = \sigma\mathbf{B}$.

We assume that a deformation $\mathbf{y}$ as described in the statement above exists and we study the consequences. Since $\mathbf{y}$ is continuously differentiable we have that $\nabla\mathbf{y}$ and the normal $\mathbf{n} = \partial_1\mathbf{y} \wedge \partial_2\mathbf{y}$ are continuous on ω. Let $\mathbf{t}$ be the unit tangent to the curve Γ. Since $(\nabla\mathbf{n})\mathbf{t} = \partial_\mathbf{t}\mathbf{n}$ and since $\mathbf{K} = -(\nabla\mathbf{y})^\top\nabla\mathbf{n}$ we have that

$$\mathbf{A}\mathbf{t} = \mathbf{B}\mathbf{t}, \quad \text{on } \Gamma.$$

Thus $\det(\mathbf{A} - \mathbf{B}) = 0$, and the identity $\det(\mathbf{A} - \mathbf{B}) = \det \mathbf{A} - \mathbf{A}^* \cdot \mathbf{B} + \det \mathbf{B}$ implies that

$$\mathbf{A}^* \cdot \mathbf{B} = 0.$$

Let $(\lambda, \mathbf{a}_1)$ and $(0, \mathbf{a}_2)$ be the pairs of eigenvalues and eigenvectors of $\mathbf{A}$, so that

$$\mathbf{A} = \lambda \mathbf{a}_1 \otimes \mathbf{a}_1.$$

Then $\mathbf{A}^* = \lambda \mathbf{a}_2 \otimes \mathbf{a}_2$ and $\mathbf{A}^* \cdot \mathbf{B} = 0$ implies that $\mathbf{B}\mathbf{a}_2 \cdot \mathbf{a}_2 = 0$. We may therefore write

$$\mathbf{B} = \beta \mathbf{a}_1 \otimes \mathbf{a}_1 + \tilde{\beta}(\mathbf{a}_1 \otimes \mathbf{a}_2 + \mathbf{a}_2 \otimes \mathbf{a}_1),$$

and, since $\det \mathbf{B} = 0$, conclude that

$$\mathbf{B} = \beta \mathbf{a}_1 \otimes \mathbf{a}_1.$$

Setting $\sigma = \lambda/\beta$ we have that $\mathbf{A} = \sigma \mathbf{B}$. From the identity $\mathbf{A}\mathbf{t} = \mathbf{B}\mathbf{t}$ we deduce that $\mathbf{a}_1 \cdot \mathbf{t} = 0$, which implies that $\mathbf{t}$ is constant and, in turn, that the curve Γ is a straight segment.

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