MATRIX APPROACH TO HYPERCOMPLEX APPELL POLYNOMIALS

LIDIA ACETO, H. R. MALONEK, AND G. TOMAZ

Dedicated to Professor Francesco A. Costabile on his 70th birthday

Abstract. Recently the authors presented a matrix representation approach to real Appell polynomials essentially determined by a nilpotent matrix with natural number entries. It allows to consider a set of real Appell polynomials as solution of a suitable first order initial value problem. The paper aims to confirm that the unifying character of this approach can also be applied to the construction of homogeneous Appell polynomials that are solutions of a generalized Cauchy-Riemann system in Euclidean spaces of arbitrary dimension. The result contributes to the development of techniques for polynomial approximation and interpolation in non-commutative Hypercomplex Function Theories with Clifford algebras.

1. Introduction

In [1] the authors presented a matrix representation approach to all types of Appell polynomial sequences $\{p_k(x)\}_{k\geq 0}$ of one real variable which relies essentially on the matrix H defined by

$$(H)_{ij} = \begin{cases} i, & i = j+1 \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,\ldots,m. \tag{1}$$

In fact, considering that a sequence of real polynomials of degree k $\{p_k(x)\}_{k\geq 0}$ is called *Appell polynomial sequence* if

$$\frac{d}{dx}p_k(x) = k\,p_{k-1}(x), \quad k = 1,2,\ldots, \tag{2}$$

cf. [3], by introducing the vector $\mathbf{p}(x) = [p_0(x) \;\; p_1(x) \;\; \cdots \;\; p_m(x)]^T$, from relation (2) we obtain the first order differential equation

$$\frac{d}{dx}\mathbf{p}(x) = H\,\mathbf{p}(x), \tag{3}$$

whose general solution is

$$\mathbf{p}(x) = e^{Hx}\,\mathbf{p}(0). \tag{4}$$

It is evident that the role of H is twofold: it acts both as *derivation matrix* and also as some type of *creation matrix*; in fact, the different kinds of Appell polynomials are uniquely determined from (4) by choosing the entries of the initial value vector $\mathbf{p}(0)$. In some sense, this matrix reveals the arithmetical tape which glues the different types of Appell polynomials together.

The prototype of an Appell sequence are the monomials $p_k(x) = x^k$. The Bernoulli polynomials as the most prominent representatives of Appell polynomials are in Numerical Analysis almost so important as orthogonal polynomials are [14, 15, 32]. It is well known that among the classical orthogonal polynomials only Hermite polynomials are also Appell polynomials, cf. [36]. But even Laguerre and Chebyshev polynomials (so far not studied in the context of Appell polynomials) can be treated as Appell polynomials using a straightforward substitution suggested in [13], cf. [1].

Although an old subject, during the last two decades the interest in Appell polynomials and their applications has significantly increased. As very few examples, recent applications of Appell polynomials exist in fields like statistics, cf. [35], linear elasticity [5] or approximation of 3D-mappings in [30]. Based on a probabilistic approach, there is a nice contribution concerning Appell polynomials in the univariate case as well as in the multivariate case in [2]. As examples of other types of Appell polynomials we refer to iterated Appell polynomials (e.g. [25]) and concerning a determinantal approach we refer to [26]. We also notice that some approaches to q-Appell polynomials from a determinantal point of view and from Umbral Calculus have been introduced recently in [18] and [19], respectively. Results in the framework of noncommutative Clifford algebras and related to the Gel'fand-Tsetlin branching approach gave evidence to Appell polynomial sequences with shift, cf. [33], as sequences of orthogonal polynomials in several variables, cf. [7, 27]. Operational approaches based on Appell polynomials for generalizing Jacobi, Laguerre, Gould-Hopper, and Chebyshev polynomials are used in the recent papers [8, 9, 11]. Employing methods of representation theory, they are also tools for applications in quantum physics [40].

Some authors were concerned with finding new characterizations of Appell polynomials themselves through new approaches. We mention, for instance, the approach developed in [39], which makes use of the generalized Pascal functional matrices and the characterization proposed in [16] which is based on a determinantal definition. As previously quoted, the authors introduced in [1] a matrix approach in the real case which has already been applied in the context of "image

synthesis" [34], and in connection with "evolution equations" for the construction of a new algorithm of deflation [38].

In addition, Appell polynomial sequences were subject to innumerable generalizations, mostly depending from the type of applications where they could be advantageously used, including multivariate commutative or noncommutative settings.

This paper intends to confirm the advantageously use of the matrix approach to real Appell polynomial sequences developed in [1] for the case of monogenic polynomials in arbitrary dimensions and in a noncommutative hypercomplex setting, i.e. for Clifford-algebra valued polynomials in the kernel of a generalized Cauchy-Riemann operator, cf. [17]. In [24] it has been shown that monogenic functions are hypercomplex differentiable. Therefore the classical definition in (2) by the derivative property can be analogously used for defining monogenic Appell sequences with respect to the hypercomplex derivative, cf. [6, 20, 21, 31].

The paper is organized as follows. Basic concepts and notations used in Hypercomplex Function Theory are given in Section 2. In Section 3 the extension to the hypercomplex case of the unifying matrix approach developed in [1] is introduced. Since the hypercomplex generalized Cauchy-Riemann operator is a linear combination of two different types of first order differential operators (a scalar one and a vector operator in the sense of the underlying Clifford algebra) also two different nilpotent matrices H and $\tilde{H}$ are involved. Their relationship is essential for guaranteeing the desired Appell property. The relation between H and $\tilde{H}$ reveals in a new and condensed matrix form the transition from the real to the hypercomplex case. Applying the results of Section 3 and the concept of a transfer matrix, in Section 4 some of the main hypercomplex monogenic counterparts of real Appell sequences are listed.

2. Basic concepts and notation

For an independent reading, in this section we repeat briefly some notions and results useful in the sequel, mainly following [10, 17, 23, 29].

Definition 2.1. *Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ provided with a non-commutative product according to the multiplication rules $e_i e_j + e_j e_i = -2\delta_{ij}$, $i, j = 1, 2, \ldots, n$, where δ_{ij} is the Kronecker symbol. The associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$ is the set of numbers $\alpha \in \mathcal{C}\ell_{0,n}$ of the form $\sum_A \alpha_A e_A$, with the basis $\{e_A : A \subseteq \{1, \ldots, n\}\}$ formed by $e_A = e_{h_1} e_{h_2} \ldots e_{h_r}$, $1 \leq h_1 < \cdots < h_r \leq n, e_\emptyset = e_0 = 1$ and where*

the components α_A are real numbers. The conjugate of α is defined by $\bar{\alpha} = \sum_A \alpha_A \bar{e}_A$, with $\bar{e}_A = \bar{e}_{h_r}\bar{e}_{h_{r-1}}\dots\bar{e}_{h_1}$; $\bar{e}_k = -e_k, k = 1,\dots,n, \bar{e}_0 = e_0 = 1$.

In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying the element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ with an element of the real vector space $\mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \dots, e_n\} \subset \mathcal{C}\ell_{0,n}$. For our purpose we only consider such elements $x \in \mathcal{A}_n$ of the form $x = x_0 + \sum_{k=1}^{n} e_k x_k = x_0 + \underline{x}$, called *paravectors* (naturally, x_0 and $\underline{x}$ are called the *scalar part* and the *vector part* of x, respectively). Similarly to the complex case the conjugate $\bar{x}$ and the norm $|x|$ of x are given by $\bar{x} = x_0 - \underline{x}$ and $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = (x_0^2 + x_1^2 + \dots + x_n^2)^{1/2}$, respectively.

In Hypercomplex Function Theory $\mathcal{C}\ell_{0,n}-$valued functions are studied. They are functions $f : \Omega \subseteq \mathbb{R}^{n+1} \cong \mathcal{A}_n \to \mathcal{C}\ell_{0,n}$ defined in an open subset by $f(z) = \sum_A f_A(z)e_A$, where $f_A(z)$ are real valued functions. We will focus on a special class of these functions analogous to complex holomorphic functions and connected with them via the following concept.

Definition 2.2. *A function f is called left (right) monogenic in Ω if it is a solution of the differential equation $\overline{\partial} f = 0$ ($f\overline{\partial} = 0$) where*

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}}), \quad \text{with } \partial_0 := \frac{\partial}{\partial x_0}, \quad \text{and } \partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k},$$

generalizes the Cauchy-Riemann operator ($n = 1, e_1 \equiv i, z = x + iy$)

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The operator $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is called the conjugate generalized Cauchy-Riemann operator or the hypercomplex differential operator.

Hereafter we only deal with left monogenic functions and we shall refer to them simply as monogenic functions (right monogenic functions are treated analogously). Notice that in the case of a paravector-valued function f of the variable $x \in \mathcal{A}_n$, the hypercomplex partial differential equation $\overline{\partial} f = 0$ is, except for the complex case ($n = 1$), equivalent to an over-determined Cauchy-Riemann system of $\frac{n(n+1)+2}{2}$ first order differential equations for the $(n + 1)$ component functions of f.

We remark that hypercomplex differentiability as generalization of complex differentiability has to be understood in the following way: a function f defined in an open domain $\Omega \subseteq \mathbb{R}^{n+1}$ is hypercomplex differentiable supposed there exists in each point of Ω a uniquely defined areolar derivative f'. Then f is real differentiable and $f' := \partial f$. On

the other hand, f is hypercomplex differentiable in Ω if and only if f is monogenic, cf. [24]. Consequently, if a hypercomplex function is monogenic then the existence of the hypercomplex derivative is guaranteed and can be obtained as result of the application of the conjugate generalized Cauchy-Riemann operator.

3. Sequences of monogenic hypercomplex Appell polynomials

Considering that for each $x \in \mathcal{A}_n$,

$$\overline{\partial} x^n = \frac{1}{2}(1-n), \qquad n \in \mathbb{N},$$

only in the complex case ($n = 1$) the function x^n belongs to the set of monogenic functions. Obviously, this fact causes problems for the consideration of monogenic polynomials in the ordinary way and, moreover, for the whole understanding of a suitable analog to power series in monogenic function theory, cf. [28]. Indeed, in some sense the problem of embedding integer powers of x in a theory of monogenic functions was in the 90-ties of the last century for more than a decade a driving force for modifying or extending the class of monogenic functions. The result was, for example, the consideration of a *Modified Clifford Analysis* (H. Leutwiler et al.) or the introduction of *holomorphic Cliffordian functions* (G. Laville et al.). The third way to overcome the problem 'inside' of the class of ordinary monogenic functions, namely by generalizing Appell's concept of power-like polynomials will be explained now (for further details, see [10]).

Motivated by relation (2) and the previously mentioned concept of hypercomplex derivative, Appell sequences of homogeneous monogenic polynomials in the framework of Clifford Analysis have been introduced in the following way, [21, 31]:

Definition 3.1. *A sequence of multivariate homogeneous polynomials $\{\phi_k(x)\}_{k\geq 0}$ of degree k in the variable $x \in \mathcal{A}_n$ is called an hypercomplex Appell sequence with respect to the hypercomplex differential operator ∂ if the following conditions are satisfied:*

(i): $\overline{\partial}\phi_k(x) = 0$, *i.e.,* $\phi_k(x)$ *is monogenic for each* $k \geq 0$;
(ii): $\partial\phi_k(x) = k\phi_{k-1}(x), \quad k = 1, 2, \ldots$.

Remark 3.2. *Hereafter, if $x = x_0 + \underline{x} \in \mathcal{A}_n$, we shall refer to $\phi_k(x)$ by using equivalently the following notations: $\phi_k(x_0 + \underline{x})$ or $\phi_k(x_0, \underline{x})$.*

Before generalizing the matrix approach to hypercomplex Appell polynomials, it is important to observe that, setting $p_0(x) \equiv c_0 \neq 0$,

from (2) the following explicit representation for the truncated sequence $\{p_k(x)\}_{k=0}^m$ of Appell polynomials in one real variable occurs:

$$\begin{aligned} p_0(x) &= c_0 \\ p_1(x) &= c_1 + c_0\, x \\ p_2(x) &= c_2 + 2\, c_1\, x + c_0\, x^2 \\ &\vdots \\ p_m(x) &= c_m + \binom{m}{1} c_{m-1}\, x + \binom{m}{2} c_{m-2}\, x^2 + \cdots + \binom{m}{m} c_0\, x^m, \end{aligned}$$

or, equivalently, in a more compact form

$$p_k(x) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) c_j\, x^{k-j}, \quad k = 0, 1, \ldots, m, \qquad c_0 \neq 0. \tag{5}$$

In particular, in this notation

$$\mathbf{p}(0) = [c_0\ c_1\ \ldots\ c_m]^T. \tag{6}$$

The formal replacement in (5) and (6) of the real variable x by x_0, the scalar part of the paravector $x = x_0 + \underline{x} \in \mathcal{A}_n$, as well as that of c_j by $\tilde{c}_j\, \underline{x}^j$ leads to multivariate homogeneous polynomials

$$\phi_k(x) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \tilde{c}_j\, \underline{x}^j\, x_0^{k-j}, \quad k = 0, 1, \ldots, m, \tag{7}$$

with the vector $\boldsymbol{\phi}(0, \underline{x}) = [\tilde{c}_0\, \underline{x}^0\ \ \tilde{c}_1\, \underline{x}^1\ \ \ldots\ \ \tilde{c}_m\, \underline{x}^m]^T$ of initial values on the hyperplane $x_0 = 0$. Consequently, recalling that the entries of the lower triangular *generalized Pascal matrix* $P(x_0) \equiv e^{Hx_0}$ are given by

$$(P(x_0))_{ij} = \left\{ \begin{array}{ll} \binom{i}{j} x_0^{i-j}, & i \geq j \\ 0, & \text{otherwise}, \end{array} \right. \qquad i, j = 0, 1, \ldots, m, \tag{8}$$

and denoting by $\boldsymbol{\phi}(x) = [\phi_0(x)\ \ \phi_1(x)\ \ \cdots\ \ \phi_m(x)]^T$, the corresponding matrix form of (7) is

$$\boldsymbol{\phi}(x) = e^{Hx_0} \boldsymbol{\phi}(0, \underline{x}),$$

which represents the counterpart in the hypercomplex framework of (4). Introducing the diagonal matrix $D_{\tilde{c}} = \mathrm{diag}[\tilde{c}_0\ \ \tilde{c}_1\ \cdots\ \tilde{c}_m]$, and the vector

$$\boldsymbol{\xi}(\underline{x}) = [1\ \ \underline{x}\ \ \underline{x}^2\ \ \cdots\ \ \underline{x}^m]^T \tag{9}$$

the previous relation becomes

$$\boldsymbol{\phi}(x) = e^{Hx_0} D_{\tilde{c}}\, \boldsymbol{\xi}(\underline{x}). \tag{10}$$

Now, the entries of such a vector are hypercomplex Appell polynomials with respect to ∂ if the two properties in Definition 3.1 are verified. It is evident that this should lead to impose some constraints on the diagonal coefficients of $D_{\tilde{c}}$. In order to deduce them, we prove the following result.

Proposition 3.1. *Let the matrix $\tilde{H}$ be defined by*

(11)
$$(\tilde{H})_{ij} = \begin{cases} -(n+i-1), & i = j+1 \wedge j \text{ even} \\ -i, & i = j+1 \wedge j \text{ odd} \\ 0, & \text{otherwise,} \end{cases} \qquad i,j = 0,1,\ldots,m.$$

Then the following vector differential equation is true:

$$\partial_{\underline{x}}\, \boldsymbol{\xi}(\underline{x}) = \tilde{H}\boldsymbol{\xi}(\underline{x}). \tag{12}$$

Proof. To obtain in matrix form the action of $\partial_{\underline{x}}$ on the vector $\boldsymbol{\xi}(\underline{x})$ it is sufficient to use the following relation from [17, p. 219]

$$\partial_{\underline{x}}(\underline{x}^k) = \begin{cases} -k\,\underline{x}^{k-1}, & k \text{ even} \\ -(n+k-1)\,\underline{x}^{k-1}, & k \text{ odd.} \end{cases}$$

Straightforward calculations lead to (12). □

Notice that (12) means that $\tilde{H}$ is the derivation matrix corresponding to $\partial_{\underline{x}}$.

Remark 3.3. *In the complex case ($n = 1$) we obtain simply $\tilde{H} = -H$. Denoting by $w = x_0 + \underline{x}$, with $\underline{x} = ix_1$, it is easily checked that for $\boldsymbol{\xi}(w) = [1 \;\; w \;\; w^2 \;\; \cdots \;\; w^m]^T$ one has $\overline{\partial}\boldsymbol{\xi}(w) = \mathbf{0}^T$, the null vector, and $\partial\boldsymbol{\xi}(w) = H\boldsymbol{\xi}(w)$, i.e., the entries of $\boldsymbol{\xi}(w)$ are complex Appell polynomials $w^s = (x_0 + ix_1)^s$, $s = 0, \ldots m$.*

By applying to both sides in (10) the hypercomplex differential operator we obtain

$$\begin{aligned} \bar{\partial}\boldsymbol{\phi}(x) &= \frac{1}{2}(\partial_0 + \partial_{\underline{x}})(e^{Hx_0} D_{\tilde{c}}\; \boldsymbol{\xi}(\underline{x})) = \frac{1}{2}\left[(\partial_0 e^{Hx_0}) D_{\tilde{c}}\; \boldsymbol{\xi}(\underline{x}) + e^{Hx_0} D_{\tilde{c}}(\partial_{\underline{x}}\boldsymbol{\xi}(\underline{x}))\right] \\ &= \frac{1}{2}\left[(He^{Hx_0}) D_{\tilde{c}}\; \boldsymbol{\xi}(\underline{x}) + e^{Hx_0} D_{\tilde{c}}\; (\tilde{H}\boldsymbol{\xi}(\underline{x}))\right] \\ &\overset{(13)}{=} \frac{1}{2} e^{Hx_0}\left[H D_{\tilde{c}} + D_{\tilde{c}}\; \tilde{H}\right] \boldsymbol{\xi}(\underline{x}). \end{aligned}$$

Due to the uniqueness theorem for monogenic functions (cf. [29] or [23, p.180]) $D_{\tilde{c}}$ must be non-singular and we can suppose that all $\tilde{c}_k \neq 0,\ k = 0, 1, \ldots$. Otherwise the restriction of a component $\phi_j(x_0, \underline{x})$ of $\boldsymbol{\phi}(x)$ with $\tilde{c}_j = 0$ to the hyperplane $x_0 = 0$ would have the value

$\phi_j(0,\underline{x}) = 0$ and, consequently, be identically zero. But this contradicts the property of belonging to an Appell sequence.

We can now prove the following result.

Theorem 3.1. *Suppose $\tilde{c}_0 \neq 0$ is a given real number. The polynomials in (10) are monogenic if their remaining coefficients satisfy the conditions*

$$\tilde{c}_{2k} = \tilde{c}_{2k-1} = \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n}{2}\right)_k}\,\tilde{c}_0, \qquad k = 1, 2, \ldots;\ n > 1, \tag{14}$$

where $(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r \geq 0$, denotes the Pochhammer symbol.

Proof. For $\boldsymbol{\phi}(x)$ being monogenic, from (13) the diagonal matrix $D_{\tilde{c}}$ should be of such a form that

$$H D_{\tilde{c}} + D_{\tilde{c}}\ \tilde{H} = O, \tag{15}$$

the null matrix, which means that H and $-\tilde{H}$ become similar matrices. By using the definitions of H and $D_{\tilde{c}}$, the product $HD_{\tilde{c}}$ is given by

$$(HD_{\tilde{c}})_{ij} = \begin{cases} (j+1)\tilde{c}_j, & i = j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, from (11) the entries of $D_{\tilde{c}}\ \tilde{H}$ are

$$(D_{\tilde{c}}\tilde{H})_{ij} = \begin{cases} (n+j)\ \tilde{c}_{j+1}, & i = j+1 \wedge j \text{ even} \\ (j+1)\ \tilde{c}_{j+1}, & i = j+1 \wedge j \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity (15) is evident for $i \neq j+1$. For the other cases ($i = j+1 \wedge j$ odd, and $i = j+1 \wedge j$ even), one has

$$(j+1)\tilde{c}_{j+1} = (j+1)\tilde{c}_j, \qquad (n+j)\tilde{c}_{j+1} = (j+1)\tilde{c}_j,$$

respectively, or, equivalently,

$$\tilde{c}_{2k} = \tilde{c}_{2k-1}, \qquad \tilde{c}_{2k-1} = \frac{2k-1}{n+2k-2}\tilde{c}_{2k-2}.$$

From these relations the assertion follows. □

Under the hypothesis of Theorem 3.1 the polynomials $\phi_k(x)$ are monogenic. It remains to check if property (ii) in Definition 3.1 is

satisfied. Applying (15) we get

$$\begin{aligned}\partial\boldsymbol{\phi}(x) &= \frac{1}{2}\left[(\partial_0 - \partial_{\underline{x}})(e^{Hx_0}D_{\tilde{c}}\ \boldsymbol{\xi}(\underline{x}))\right] \\ &= \frac{1}{2}\left[He^{Hx_0}D_{\tilde{c}}\ \boldsymbol{\xi}(\underline{x}) - e^{Hx_0}D_{\tilde{c}}\ \tilde{H}\boldsymbol{\xi}(\underline{x})\right] \\ &= \frac{1}{2}e^{Hx_0}\left[HD_{\tilde{c}} - D_{\tilde{c}}\ \tilde{H}\right]\boldsymbol{\xi}(\underline{x}) = He^{Hx_0}D_{\tilde{c}}\ \boldsymbol{\xi}(\underline{x}) = H\boldsymbol{\phi}(x),\end{aligned}$$

which, actually, is the corresponding matrix representation of the property (ii) in Definition 3.1.

Remark 3.4. *Following the recent article* [12] *on special properties of hypercomplex Appell polynomials like, for instance, three-term recurrence relations, further extensions including also all orthogonal Appell polynomial sequences obtained by the Gel'fand-Tsetlin procedure, cf.* [7, 27]*, can be constructed.*

In order to get such extensions, the suitable formal replacement in (5) and (6) should be x *by* x_0 *and* c_j *by* $\tilde{c}_j\,\underline{x}^j\,Q_s(\underline{x})$*, where* $Q_s(\underline{x})$ *is an arbitrary chosen monogenic polynomial of fixed degree* $s = 0, 1, 2, \ldots$*.*

One should notice that $Q_s(\underline{x})$*, as an arbitrary chosen polynomial monogenic with respect to the generalized Cauchy-Riemann operator, is automatically a generalized constant since it does not depend from* x_0 *and therefore belongs also to the kernel of the conjugated generalized Cauchy-Riemann operator. This fact implies that from the point of view of the hypercomplex derivative the polynomial* $Q_s(\underline{x})$ *behaves like an ordinary constant number, i.e., its hypercomplex derivative is constant zero. Moreover, one could come to the conclusion that the first polynomial of hypercomplex Appell sequences could be such an initial monogenic generalized constant* $Q_s(\underline{x})$ *and all polynomials of higher degree would have* $Q_s(\underline{x})$ *as common factor. This idea is realized in the paper* [33].

The matrix form of these multivariate homogeneous polynomials is

$$\boldsymbol{\phi}(x) = Q_s(\underline{x})\ e^{Hx_0}D_{\tilde{c}}\ \boldsymbol{\xi}(\underline{x})$$

and the constraints to be imposed on the diagonal entries of $D_{\tilde{c}}$ *in order to get hypercomplex Appell polynomials are*

$$\tilde{c}_{2k} = \tilde{c}_{2k-1} = \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n+2s}{2}\right)_k}\,\tilde{c}_0, \qquad k = 1, 2, \ldots;\ n > 1,\ s = 0, 1, 2, \ldots.$$

Such constraints are achieved like in Theorem 3.1, but taking into account that the action of $\partial_{\underline{x}}$ *on the vector* $\boldsymbol{\xi}(\underline{x})$ *is performed by the*

matrix $\tilde{H}^s\, Q_s(\underline{x})$*, where*

$$(\tilde{H}^s)_{ij} = \begin{cases} -(n+i+2s-1), & i = j+1 \wedge j \ \textit{even} \\ -i, & i = j+1 \wedge j \ \textit{odd} \\ 0, & \textit{otherwise,} \end{cases} \qquad i,j = 0,1,\ldots,m.$$

This matrix results from the fact that

$$\partial_{\underline{x}}(\underline{x}^k Q_s(\underline{x})) = \begin{cases} -k\,\underline{x}^{k-1}\, Q_s(\underline{x}), & k \ \textit{even} \\ -(n+k+2s-1)\,\underline{x}^{k-1}\, Q_s(\underline{x}), & k \ \textit{odd.} \end{cases}$$

4. Special families of monogenic Appell polynomials

We recall that the real Appell polynomials, $p_k(x)$, may also be characterized in terms of their exponential generating function of the form (see, for example, [4])

$$f(t)e^{tx} = \sum_{n=0}^{+\infty} p_n(x)\frac{t^n}{n!},$$

where

$$f(t) = \sum_{s=0}^{+\infty} c_s \frac{t^s}{s!}, \qquad c_0 \neq 0. \tag{16}$$

The creation matrix H and formula (16) are the essential tools to get the *transfer matrix* $M = f(H)$ (see [1, Theorem 3.2]) whose action is to change the vector of monomial powers $\boldsymbol{\xi}(x)$ into the associated Appell vector $\mathbf{p}(x)$, i.e.,

$$\mathbf{p}(x) = f(H)\boldsymbol{\xi}(x) = M\boldsymbol{\xi}(x).$$

The choice of the coefficients (14) with $\tilde{c}_0 = 1$ gives the sequence $\{\mathscr{P}_k^{(n)}(x)\}_{k\geq 0}$ introduced in [21] which allows to define a generalized exponential generating function in the hypercomplex setting by

$$\mathrm{Exp}_n(x) \equiv e^{x_0}F(\underline{x}) = \sum_{k=0}^{+\infty} \frac{\mathscr{P}_k^{(n)}(x)}{k!}, \qquad x \in \mathcal{A}_n, \tag{17}$$

where

$$F(\underline{x}) = \sum_{s=0}^{+\infty} \tilde{c}_s \frac{\underline{x}^s}{s!}.$$

Due to the homogeneity of $\mathscr{P}_k^{(n)}(x)$, we have

$$Exp_n(tx) = \sum_{k=0}^{+\infty} \mathscr{P}_k^{(n)}(x)\frac{t^k}{k!}, \tag{18}$$

showing that the generating function of the hypercomplex Appell sequence $\{\mathscr{P}_k^{(n)}(x)\}_{k\geq 0}$ is of the form $G(x,t) = \mathrm{Exp}_n(tx),\ x \in \mathcal{A}_n, t \in \mathbb{R}$.

Remark 4.1. *In the complex case,* $G(x,t) = e^{tx},\ x \in \mathbb{C},\ t \in \mathbb{R}$ *is the generating function of the basic Appell sequence* $\{x^k\}_{k\geq 0}$. *Thus,* $\mathscr{P}_k^{(n)}(x)$ *behave like monomial functions in the sense of the complex powers* $w^k = (x_0 + ix_1)^k,\ k \geq 0$.

Using the hypercomplex exponential function (17) we define generalized hypercomplex Appell polynomials, in general non-homogeneous, as follows:

Definition 4.2. *The sequence* $\{\psi_k(x)\}_{k\geq 0}$, *whose generating function is* $G(x,t) = f(t)Exp_n(tx),\ x \in \mathcal{A}_n$ *and* $f(t)$ *a formal power series as in (16), is called generalized hypercomplex Appell sequence.*

The function $f(t)$ coincides with the one appearing in the generating function of real Appell polynomials and the Cauchy product of (16) by (18) leads to

$$f(t)\ \mathrm{Exp}_n(tx) = \sum_{k=0}^{+\infty} \psi_k(x)\frac{t^k}{k!},$$

with

$$\psi_k(x) = \sum_{s=0}^{k} \binom{k}{s} c_s \mathscr{P}_{k-s}^{(n)}(x).$$

Therefore, arguments similar to the ones used in the proof of Theorem 3.2 in [1] lead to the following result.

Theorem 4.1. *Let* H *be the creation matrix defined by (1). If* $G(x,t) = f(t)Exp_n(tx)$ *is the generating function of the generalized hypercomplex Appell sequence, then the transfer matrix* M *of order* $m + 1$ *is the nonsingular matrix* $f(H)$.

However, in this case, $M = f(H)$ transforms the vector

$$\boldsymbol{\xi}(\mathscr{P}^{(n)}(x)) = [\mathscr{P}_0^{(n)}(x)\ \ \mathscr{P}_1^{(n)}(x) \cdots \mathscr{P}_m^{(n)}(x)]^T$$

into the generalized hypercomplex Appell vector $\boldsymbol{\psi}(x) = [\psi_0(x)\ \cdots\ \psi_m(x)]^T$, that is

$$\boldsymbol{\psi}(x) = f(H)\,\boldsymbol{\xi}(\mathscr{P}^{(n)}(x)).$$

We notice that, when $\underline{x} = 0$, $\boldsymbol{\xi}(\mathscr{P}^{(n)}(x_0)) \equiv \boldsymbol{\xi}(x_0)$ while the components of $\boldsymbol{\psi}(x_0)$ are the classical real Appell polynomials entries of the vector $\mathbf{p}(x_0)$.

Applying the results of [1] and [37], different kinds of generalized hypercomplex Appell polynomials can be derived by an appropriate choice

of $f(t)$. The transfer matrices for generalized hypercomplex Bernoulli, Frobenius-Euler, and monic Hermite polynomials are

$$\left(\sum_{k=0}^{m}\frac{H^k}{(k+1)!}\right)^{-1}, \qquad (1-\lambda)(e^H-\lambda I)^{-1}, \qquad \sum_{k=0}^{m}\frac{(-H^2)^k}{2^{2k}k!},$$

respectively.

The form of the transfer matrices stresses again the special role of the creation matrix H. For the value $\lambda=-1$ in the transfer matrix of generalized hypercomplex Frobenius-Euler polynomials, we get generalized hypercomplex Euler polynomials (see [22]).

Acknowledgements

This work was supported by Portuguese funds through the CIDMA-Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e Tecnologia"), within project PEst-OE/MAT/UI4106/2014.

References

[1] L. Aceto, H. R. Malonek, G. Tomaz, A unified matrix approach to the representation of Appell polynomials, Integral Transforms Spec. Funct. 26 (2015) 426-441.

[2] M. Anshelevich, Appell polynomials and their relatives, Int. Math. Res. Not. 65 (2004) 3469-3531.

[3] P. Appell, Sur une classe de polynômes, Ann. Sci. École Norm. Sup. 9 (2) (1880) 119-144.

[4] R. P. Boas, R. C. Buck, Polynomial expansions of analytic functions, Springer, Berlin, 1964.

[5] S. Bock, On monogenic series expansions with applications to linear elasticity, Adv. Appl. Clifford Algebras 24 (4) (2014) 931-943.

[6] S. Bock, K. Gürlebeck, On a generalized Appell system and monogenic power series, Math. Methods Appl. Sci. 33 (4) (2010) 394-411.

[7] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Gel'fand-Tsetlin procedure for the construction of orthogonal bases in Hermitean Clifford Analysis. In: T.E. Simos et al.(Eds.): Numerical Analysis and Applied Mathematics-ICNAAM 2010, AIP Conf. Proc. 1281 (2010) 1508-1511.

[8] I. Cação, D. Eelbode, Jacobi polynomials and generalized Clifford algebra-valued Appell sequences, Math. Methods Appl. Sci. 37 (2014) 1527-1537.

[9] I. Cação, M. I. Falcão, H. R. Malonek, Laguerre derivative and monogenic Laguerre polynomials: an operational approach, Math. Comput. Modelling 53 (2011) 1084-1094.

[10] I. Cação, M. I. Falcão, H. R. Malonek, Matrix representations of a basic polynomial sequence in arbitrary dimension, Comput. Methods Funct. Theory 12 (2) (2012) 371-391.

[11] I. Cação, H. R. Malonek, On an hypercomplex generalization of Gould-Hopper and related Chebyshev polynomials. In: B. Murgante et al. (Eds.): Computational Science and Its Applications-ICCSA 2011 (LNCS 6784, Part III). Springer-Verlag, Berlin (2011) 316-326.

[12] I. Cação, M. I. Falcão and H. R. Malonek, Three-term recurrence relations for systems of Clifford algebra-valued orthogonal polynomials, Adv. Appl. Clifford Algebras, Published online: 08 September 2015, 15pp., doi:10.1007/s00006-015-0596-z

[13] B. C. Carlson, Polynomials satisfying a binomial theorem, J. Math. Anal. Appl. 32 (1970) 543-558.

[14] F. Costabile, Expansions of real functions in Bernoulli polynomials and applications, Conf. Semin. Mat. Univ. Bari No. 273 (1999), 13 pp.

[15] F. Costabile, F. Dell'Accio, M. I. Gualtieri, A new approach to Bernoulli polynomials, Rend. Mat. Appl. (7) 26 (2006), no. 1, 1-12.

[16] F. Costabile, E. Longo, A determinantal approach to Appell polynomials, J. Comput. Appl. Math. 234 (2010) 1528-1542.

[17] R. Delanghe, F. Sommen, V. Souček, Clifford Algebra and Spinor-Valued Functions. A function theory for the Dirac operator, Mathematics and its Applications (Dordrecht) 53 Kluwer Academic Publishers, 1992.

[18] M. Eini Keleshteri, N. I. Mahmudov, A study on q-Appell polynomials from a determinantal point of view, Appl. Math. Comput. 260 (2015) 351-369.

[19] M. Eini Keleshteri, N. I. Mahmudov, A q-umbral approach to q-Appell polynomials, arXiv 1505-05067v1-math-NT, 19 May 2015.

[20] M. I. Falcão, J. F. Cruz, H. R. Malonek, Remarks on the generation of monogenic functions. In: K. Gürlebeck, C. Könke (Eds.), 17^{th} Inter. Conf. on the Appl. of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar (2006) 12-14.

[21] M. I. Falcão, H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. In: T. E. Simos, G. Psihoyios, C. Tsitouras (Eds.), AIP Conference Proceedings 936 (2007) 738-741.

[22] G. Frobenius, Über die Bernoulli'schen Zahlen und die Euler'schen Polynome. Sitzungsberichte der Preussischen Akademie der Wissenschaften (1910) 809–847.

[23] K. Gürlebeck, K. Habetha, W. Sprößig, Holomorphic Functions in the Plane and n-Dimensional Space, Translated from the 2006 German original. Birkhäuser Verlag, Basel, 2008.

[24] K. Gürlebeck, H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{m+1}$ and its applications, Complex Variables 39 (1999) 199-228.

[25] S. Khan, N. Raza, 2-Iterated Appell polynomials and related numbers, Appl. Math. Comput. 219 (17) (2013) 9469-9483.

[26] S. Khan, M. Riyasat, A determinantal approach to Sheffer-Appell polynomials via monomiality principle, J. Math. Anal. Appl. 421 (2015) 806-829.

[27] R. Lávička, Complete orthogonal Appell systems for spherical monogenics, Complex Anal. Oper. Theory, 6 (2012) 477-489.

[28] H. R. Malonek, Power series representation for monogenic functions in $\mathbb{R}^{n+1}$ based on a permutational product, Complex Variables, Theory Appl. 15 (1990) 181-191.

[29] H. R. Malonek, in: S.-L. Eriksson (Eds.), Selected topics in hypercomplex function theory. In: Clifford algebras and potential theory, University of Joensuu, Research Reports 7 (2004) 111-150.
[30] H. R. Malonek, M. I. Falcão, 3D-mappings by means of monogenic functions and their approximation, Math. Methods Appl. Sci. 33 (2010) 423-430.
[31] H. R. Malonek, M. I. Falcão, Special monogenic polynomials-properties and applications. In: T. E. Simos, G. Psihoyios, C. Tsitouras (Eds.), AIP Conference Proceedings 936 (2007) 764-767.
[32] G. V. Milovanovic, Families of Euler-MacLaurin formulae for composite Gauss-Legendre and Lobatto quadratures, Bull. Cl. Sci. Math. Nat. Sci. Math. No. 38 (2013) 63-81.
[33] D. Peña Peña, Shifted Appell Sequences in Clifford Analysis, Results. Math. 63 (2013) 1145-1157.
[34] P. E. Roman, T. Asahi, S. Casassus, Hermite-Gaussian functions for image synthesis, In: Asia-Pacific Signal and Information Processing Association, 2014 Annual Summit and Conference (APSIPA) (2014) 1-9.
[35] P. Salminen, Optimal stopping, Appell polynomials, and Wiener-Hopf factorization. Stochastics 83 (2011), no.4-6, 611-622.
[36] J. Shohat, The relation of the classical orthogonal polynomials to the polynomials of Appell, Amer. J. Math. 58 (1936) 453-464
[37] G.Tomaz, H. R. Malonek, Matrix Approach to Frobenius-Euler Polynomials. In: B. Murgante et al. (Eds.): Computational Science and Its Applications-ICCSA 2014 (LNCS 8579, Part I). Springer (2014) 75-86.
[38] R. M. Yamaleev, Pascal matrix representation of evolution of polynomials, Int. J. Appl. Comput. Math. 1 (4) (2015) 513-525.
[39] Y. Yang, H. Youn, Appell polynomials sequences: a linear algebra approach. JP J. Algebra Number Theory Appl. 13 (2009), no. 1, 65-98.
[40] S. Weinberg, The quantum theory of fields. Vol. I. Foundations. Corrected reprint of the 1995 original. Cambridge University Press, Cambridge, 1996.

Lidia Aceto, Università di Pisa, Dipartimento di Matematica, via F. Buonarroti 1/C - 56127 Pisa, Italy
E-mail address: lidia.aceto@unipi.it

CIDMA - Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: hrmalon@ua.pt

UDI/IPG - Research Unit for Inland Development, 6300-559 Guarda, Portugal
E-mail address: gtomaz@ipg.pt