Rigorous computation of invariant measures and fractal dimension for piecewise hyperbolic maps: 2D Lorenz like maps.

Stefano Galatolo [*], Isaia Nisoli [†]

February 25, 2014

Abstract

We consider a class of piecewise hyperbolic maps from the unit square to itself preserving a contracting foliation and inducing a piecewise expanding quotient map, with infinite derivative (like the first return maps of Lorenz like flows). We show how the physical measure of those systems can be rigorously approximated with an explicitly given bound on the error, with respect to the Wasserstein distance. We apply this to the rigorous computation of the dimension of the measure. We present a rigorous implementation of the algorithms using interval arithmetics, and the result of the computation on a nontrivial example of Lorenz like map and its attractor, obtaining a statement on its local dimension.

Contents

[*]Dipartimento di Matematica, Universita di Pisa, Via Buonarroti 1,Pisa. Email: galatolo@dm.unipi.it

[†]Instituto de Matemática - UFRJ Av. Athos da Silveira Ramos 149, Centro de Tecnologia - Bloco C Cidade Universitria - Ilha do Fundão. Caixa Postal 68530 21941-909 Rio de Janeiro - RJ - Brasil Email:nisoli@im.ufrj.br

1 Introduction

Overview Several important features of the statistical behavior of a dynamical system are "encoded" in the so called *Physical Invariant Measure*[1]. The knowledge of the invariant measure can give information on the statistical behavior for the long time evolution of the system. This strongly motivates the search for algorithms which are able to compute quantitative information on invariant measures of physical interest, and in particular algorithms giving an explicit bound on the error which is made in the approximation.

The problem of approximating the invariant measure of dynamical systems was broadly studied in the literature. Some algorithm is proved to converge to the real invariant measure in some classes of systems (up to errors in some given metrics), but results giving an explicit (rigorous) bound on the error are relatively few, and really working implementations, even fewer (e.g. [3, 10, 15, 20, 17]). Almost every (rigorous) implementation and almost all methods works in the case of one dimensional or expanding maps.

The case where contracting directions are present does not easily fit with known techniques, based on the choiche of a suitable functional analytic framework and on the related spectral properties of the transfer operator (or on Hilbert cones), because the involved functional spaces and the needed a priori estimations are not easy to be brought in the form which is necessary for an effective implementation.

The output of a computation with an explicit estimation for the error can be seen as a rigorously (computer aided) proved statement, and hence has a mathematical meaning. In our case, the rigorous approximation for invariant measures gives us the possibility to have rigorous quantitative estimations on

[1] Physical invariant measures are the ones which (in some sense that will be precised below) represent the statistical behavior of a large set of initial conditions.

some aspects of the statistical and geometrical behavior of the system we are interested in. In particular we will use it to have a statement on the dimension of its physical invariant measure.

About the general problem of computing invariant measures, it is worth to remark that some negative result are known. In [9] it is shown that *there are examples of computable*[2] *systems without any computable invariant measure.* This shows some subtlety in the general problem of computing the invariant measure up to a given error.

In this paper we focus on a class of Lorenz like maps, which are piecewise hyperbolic maps with unbounded derivatives preserving a *contracting foliation*, similar to the Poincaré map of the famous Lorenz system.

We consider maps F acting on $Q = I \times I$ (where $I = [-\frac{1}{2}, \frac{1}{2}]$) having the following properties:

1) $F : \Sigma \to \Sigma$ is of the form $F(x, y) = (T(x), G(x, y))$ (preserves the natural vertical foliation of the square) and:

2) $T : I \to I$ is onto and piecewise monotonic, with N, increasing, expanding branches with possibly infinite derivative: there are $c_i \in [0, 1]$ for $0 \le i \le N$ with $0 = c_0 < \cdots < c_N = 1$ such that $T|_{(c_i,c_{i+1})}$ is continuous and monotone for $0 \le i < N$. Furthermore, for $0 \le i < N$, $T|_{(c_i,c_{i+1})}$ is C^1 and $\inf_{x\in P} |T'(x)| > 1$.

3) F is uniformly contracting on each vertical leaf γ: there is a $\lambda < 1$ such that $|G(x, y_1) - G(x, y_2)| \le \lambda \cdot |y_1 - y_2|$;

4) $G : Q \to (0, 1)$ is C^1 on $P \times [0, 1]$, where $P = [0, 1] \setminus \cup_{0\le i<N} c_i$. Furthermore, $\sup |\partial G/\partial x| < \infty$ and $|(\partial G/\partial y)(x, y)| > 0$ for $(x, y) \in P \times [0, 1]$;

5) $\frac{1}{|T'|}$ has bounded variation.

About the regularity of T: we suppose that $\frac{1}{|T'|}$ has bounded variation to simplify the computation of the invariant measure of this induced map. We remark that in general, for Lorenz like systems this assumption should be replaced by generalized bounded variation (see [1, 13]). This kind of maps however still satisfy a Lasota-Yorke inequality, and the general strategy for the computation of the invariant measure should be similar to the one used here and explained in Section 11 for the bounded variation case.

We approach the computation of the invariant measure for the two dimensional map by some techniques which have been succesfully used to estimate decay of correlations in systems preserving a contracting foliation (see [1, 11]). In these systems, the physical invariant measure can be seen as the limit of iterates of a suitable absolutely continuous initial measure. Our strategy, in order to compute this measure with an explicit bound on the error, is to iterate a suitable initial measure a sufficient number of times and carefully estimate the

[2]Computable, here means that the dynamics can be approximated at any accuracy by an algorithm, see e.g. [9] for precise definition.

speed with which it approaches to the limit. This is not sufficient for the computation since there is a further technical problem: *the computer cannot perfectly simulate a real iteration.* Thus we need to understand how far simulated iterates are from real iterates.

Hence the algorithm and the estimation of the error involve two main steps:

a) we estimate how many iterates of a suitable starting measure[3] in the real system are necessary to approach the invariant measure at a given distance (see Theorem 2), and then

b) we estimate the distance between real iterates and the iterates of a suitable discretized model which can be implemented on a computer (see Proposition 5).

Altogether this allows to implement an algorithm which rigorously approximates the invariant measure by a suitable discretization of the system (in the paper we will consider the so called Ulam discretization method, which approximate the system by a Markov chain).

The results and the implementations which are presented are meant as a proof of concept, to solve the problem and run experiments in some nontrivial and interesting class of examples. We expect that a very similar strategy apply in many other cases of systems preserving a contracting foliation.

In the next sections we describe more precisely the problem and the technical tools we use to approach it: in section 2 we introduce some basic tools which are used in our construction.

In section 3 and 4 we show the general mathematical estimates which allows to implement the above two main steps a), b).

We then describe informally the algorithm which is meant to be implemented, and then in Section 6 we show how, by the approximated knowledge of the invariant measure and of the geometry of the system, it is possible to approximate its local dimension.

In section 7 we describe the implementation of the algorithm and some remarks which permitted us to optimize it.

The rigorous implementation of our algorithm is substantially made by interval arithmetics. It presents several technical issues; as an example we mention that since the map is two-dimensional the number of cells involved in the discretization increases, a priori, as the square of the size of the discretization. This seriously affect the speed of the computation and the possibility to reach a good level of precision. The presence of the contracting direction, and an attractor which is not two dimensional allows to find a suitable reduction of the discretization (restricting computations to a neighborhood of the attractor) which reduces the complexity of the problem (see Section 7.1).

In Section 8.3 we show the result of the computation of the invariant measure on an example of two dimensional Lorenz like map.

[3]The suitable measure to be iterated is constructed starting from a L^1 approximation of the absolutely continuous invariant measure of the induced map T.

The computation of the invariant measure also allows the rigorous approximation of the local dimension of the measure we are interested in. In Section 9 we show the result of the computation of the dimension of a non trivial example.

2 The general framework

In the next subsections we explain some preliminary notions and results used in the paper.

The transfer operator Let us consider the space $SM(X)$ of Borel measures with sign on X. A function T between metric spaces naturally induces a linear function $L_T : SM(X) \to SM(X)$ called the **transfer operator** (associated to T) which is defined as follows. If $\mu \in SM(X)$ then $L_T[\mu] \in SM(X)$ is the measure such that

$$L_T[\mu](A) = \mu(T^{-1}(A)).$$

Sometimes, when no confusion arises, we will denote L_T more simply by L.

Measures which are invariant for T are fixed points of L, hence the computation of invariant measures very often is done by computing some fixed points of this operator. The most applied and studied strategy is to find a finite dimensional approximation for L (restricted to a suitable function space) reducing the problem to the computation of the corresponding relevant eigenvectors of a finite matrix. In this case some quantitative stability result may ensure that the fixed point of the approximated operator is near to the real fixed point which was meant to be computed (see Section 11 for one example).

On the other hand, in many other interesting cases the invariant measure can be computed as the limit of the iterates of some suitable starting measure $\overline{\mu} = \lim_{n\to\infty} L^n(\mu_0)$. To estimate the error of the approximation is important to estimate the speed of convergence (in some topology). Another strategy is then to iterate the finite dimensional approximating operator a suitable number of times to "follow" the iterations of the original operator which will converge to the fixed point.

In this paper we consider a class of maps preserving a contracting foliation. For this kind of maps it is possible to compute the speed of convergence of suitable measures to the invariant one (see Section 3) and this is the main idea we apply to compute the invariant measure. A suitable starting measure has however to be computed. This is done by observing that this kind of maps induces a one dimensional map representing the dynamics between the leaves. We approximate the physical invariant measure for this map (up to small errors in L^1, this will be done by a suitable fixed point stability result, see Section 11) then we use this approximation to construct a suitable invariant measure to be iterated.

The Ulam method. We now describe a finite dimensional approximation of L which is useful to approximate invariant measures in the L^1 norm (see e.g. [4, 5, 6, 10, 17, 20]), and as we will see it also works with the Wasserstein distance in our case.

Let us suppose now that X is a manifold with boundary. Let us describe *Ulam's Discretization* method. In this method the space X is discretized by a partition I_δ (with k elements) and the system is approximated by a finite state Markov Chain with transition probabilities

$$P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i) \tag{1}$$

(where m is the normalized Lebesgue measure on X) and defining a corresponding finite-dimensional operator L_δ (L_δ depend on the whole chosen partition but simplifying we will indicate it with a parameter δ related to the size of the elements of the partition) we remark that in this way, to L_δ it corresponds a matrix $P_k = (P_{ij})$.

Alternatively L_δ can be seen in the following way: let F_δ be the $\sigma-$algebra associated to the partition I_δ, then:

$$L_\delta(f) = \mathbf{E}(L(\mathbf{E}(f|F_\delta))|F_\delta) \tag{2}$$

where $\mathbf{E}$ is the conditional expectation. Taking finer and finer partitions, in certain systems including for example piecewise expanding one-dimensional maps, the finite dimensional model converges to the real one and its natural invariant measure to the physical measure of the original system.

We use the Ulam discretization both when applying the fixed point stability result to compute the one dimensional invariant measure necessary to start the iteration, and when constructing an approximated operator to iterate the two dimensional starting measure.

The Wasserstein distance We are going to approximate the interesting invariant measure of our Lorenz like map up to small errors in the Wasserstein metric.

If X is a metric space, we denote by $SM(X)$ the set of Borel finite measures with sign on X. Let $g : X \to \mathbb{R}$; let

$$L(g) := \sup_{x,y} \frac{|g(x) - g(y)|}{|x - y|}$$

be the best Lipschitz constant of g and set $\|g\|_{\mathrm{Lip}} = \|g\|_\infty + L(g)$.

Let us consider the following slight modification of the classical notion of Wasserstein distance between probability measures: given two measures μ_1 and μ_2 on X, we define their distance as

$$W(\mu_1, \mu_2) = \sup_{g \ s.t. \ L(g) \leq 1, ||g||_\infty \leq 1} |\int_X g \ d\mu_1 - \int_X g \ d\mu_2|.$$

When μ_1 and μ_2 are probability measures, this is equavalent to the classical notion.

Let us denote by $||.||$ the norm relative to this notion of distance

$$||\mu|| = \sup_{\substack{\phi \in 1-\mathrm{Lip}(I) \\ ||\phi||_\infty \leq 1}} |\int \phi \, d\mu|.$$

Remark 1 *By definition it follows that if $\mu = \sum_1^m \mu_i$, $\nu = \sum_1^m \nu_i$*

$$W(\mu,\nu) \leq \sum_1^m W(\mu_i,\nu_i). \tag{3}$$

Moreover, (see [11]) if μ and ν are probability measures, and F is a λ-contraction ($\lambda < 1$), then

$$W(L_F(\mu), L_F(\nu)) \leq \lambda \cdot W(\mu,\nu).$$

3 Systems with contracting fibers, disintegration and effective estimation for the speed of convergence to equilibrium.

As explained before, we want to estimate how many iterations are needed for a suitable starting measure supported on a neighborhood of the attractor, to approach the invariant measure. This kind of estimation is similar to a decay of correlation one, and we use an approach similar to the one used in [1] to prove exponential decay of correlation for a class of systems with contracting fibers. Here a more explicit and sharper estimate is needed.

Let us introduce some notations: we will consider the sup distance on the square $Q = [-\frac{1}{2}, \frac{1}{2}]^2$, so that the diameter, $\mathrm{Diam}(Q) = 1$. This choice is not essential, but will avoid the presence of some multiplicative constants in the following, making notations cleaner.

The square Q will be foliated by stable, vertical leaves. We will denote the leaf with x coordinate by γ_x or, with a small abuse of notation when no confusion is possible, we will denote both the leaf and its coordinate with γ.

Given a measure μ and a function f, let $f\mu$ be the measure μ_1 such that $d\mu_1 = f d\mu$. Let μ be a measure on Q. In the following, such measures on Q will be often disintegrated in the following way: for each Borel set A

$$\mu(A) = \int_{\gamma \in I} \mu_\gamma(A \cap \gamma) d\mu_x \tag{4}$$

with μ_γ being probability measures on the leaves γ and μ_x is the marginal on the x axis which will be an absolutely continuous measure.

Let us consider a Lorenz like two dimensional map F and estimate explicitly the speed of convergence of iterates of two initial measures with absolutely continuous marginal.

Theorem 2 *Let $F : Q \to Q$ as above, let $\mu, \nu \in PM(\Sigma)$ be two measures with absolutely continuous marginals μ_x, ν_x. Then*

$$W(L_F^n(\mu), L_F^n(\nu)) \leq \lambda^n + ||\mu_x - \nu_x||_{L^1}.$$

Where we recall that λ is the contraction rate on the vertical leaves.

In the proof we use the following, proposition (see [1], Proposition 3) which allows to estimate the Wasserstein distance of two measures by its disintegration on stable leaves.

Proposition 3 *Let μ^1, μ^2 be measures on Q as above, such that for each Borel set A*

$$\mu^1(A) = \int_{\gamma \in I} \mu_\gamma^1(A \cap \gamma) d\mu_x^1 \text{ and } \mu^2(A) = \int_{\gamma \in I} \mu_\gamma^2(A \cap \gamma) d\mu_x^2,$$

where μ_x^i is absolutely continuous with respect to the Lebesgue measure. In addition, let us assume that

1. $\int_I W(\mu_\gamma^1, \mu_\gamma^2) d\mu_x^1 \leq \epsilon$
2. $V(\mu_x^1, \mu_x^2) \leq \delta$ *(where $V(\mu_x^1, \mu_x^2) = \sup_{|g|_\infty \leq 1} |\int g d\mu_x^1 - \int g d\mu_x^2|$ is the* ***total variation distance****).*

Then $|\int g d\mu^1 - \int g d\mu^2| \leq ||g||_{Lip} \cdot (\epsilon + \delta)$.

Remark 4 *Referring to Item 1 we garantee the left hand side to be well defined by assuming (without changing μ^2) that μ_γ^2 is defined in some way, for example $\mu_\gamma^2 = m$ (the one dimensional Lebesgue measure on the leaf) for each leaf where the density of μ_x^2 is null.*

Proof of Theorem 2. Let us consider $\{I_i\}_{i=1,\ldots,m}$ the intervals where the branches of T^n are defined. Let us consider $\varphi_i = 1_{I_i \times I}$ and let

$$\mu_i = \varphi_i \mu \quad \nu_i = \varphi_i \nu,$$

then $\mu = \sum \mu_i$, $\nu = \sum \nu_i$; thus by triangle inequality

$$W(L_F^n(\mu), L_F^n(\nu)) \leq \sum_{i=1,..,m} W(L_F^n(\mu_i), L_F^n(\nu_i)).$$

Let us denote by $T_i := T^n|_{I_i}$, remark that this is injective and recall that T^n is a L^1 contraction. Then by Proposition 3

$$\begin{aligned}
W(L_F^n(\mu_i), L_F^n(\nu_i)) &\leq \int_I W((L^n\mu_i)_\gamma, (L^n\nu_i)_\gamma)\; dL_{T^n}((\nu_i)_x) + ||L_T^n((\mu_i)_x) - L_T^n((\nu_i)_x)||_{L^1} \\
&\leq \int_I W(L_F^n((\mu_i)_{T_i^{-1}(\gamma)}), L_F^n((\nu_i)_{T_i^{-1}(\gamma)}))\; dL_{T^n}((\nu_i)_x) + ||(\mu_i)_x - (\nu_i)_x||_{L^1} \\
&\leq \lambda^n \int_I W((\mu_i)_{T_i^{-1}(\gamma)}, (\nu_i)_{T_i^{-1}(\gamma)})\; dL_{T^n}((\nu_i)_x) + ||(\mu_i)_x - (\nu_i)_x||_{L^1} \\
&= \lambda^n \int_{I_i} W((\mu_i)_\gamma, (\nu_i)_\gamma)\; d((\nu_i)_x) + ||(\mu_i)_x - (\nu_i)_x||_{L^1}.
\end{aligned}$$

Summarizing:

$$
\begin{aligned}
W(L_F^n(\mu), L_F^n(\nu)) \quad &\leq \quad \lambda^n \sum_i \int_{I_i} W((\mu_i)_\gamma, (\nu_i)_\gamma)\ d((\nu_i)_x) + ||(\mu_i)_x - (\nu_i)_x||_{L^1} \\
&= \quad \lambda^n \int_I W(\mu_\gamma, \nu_\gamma)\ d(\nu_x) + ||\mu_x - \nu_x|||_{L^1}.
\end{aligned}
$$

■

The two dimensional map F induces a one dimensional one T which is piecewise expanding. In this kind of maps the application of the Ulam method with cells of size δ, gives a way to approximate the absolutely continuous invariant measure f of T by a step function f_δ, which is the steady state of the associated Markov chain (see Section 11 for the details). We will use f_δ to construct a suitable starting measure for the iteration process. Denoting by $\overline{\mu}$ the physical invariant measure of F, we will consider the above iteration process by iterating $\overline{\mu}$ and a starting measure μ_0 supported on a suitable open neighborhood U of the attractor and having f_δ as marginal on the expanding direction.

4 Approximated 2D iterations

The general idea is to approach the invariant measure by iterating a suitable measure. We remark that we cannot simulate real iterates on a finite computer, and we can only work with approximated iterates. We will then estimate the distance between these and real ones.

Let us consider the grid partition $\mathcal{Q} = \{Q_{i,j}\}$ of Q, dividing Q in rectangles $Q_{i,j}$ of size $\delta \times \delta^{'}$ (where $\delta, \delta^{'}$ are the inverses of two integers). Let I_i be the relative subdivision of $[0,1]$ in δ' long segments. We also denote $I_{-1} = I_{\frac{1}{\delta'}+1} = \emptyset$. Let π_1 and π_2 be the two, natural projections of Q, respectively along the vertical and horizontal direction.

Let $Q_i = \cup_j Q_{i,j}$ be the horizontal, row strips and let $Q_j^t = \cup_i Q_{i,j}$ be the vertical ones. Moreover let $\varphi_i = 1_{Q_i}$ be the indicator function of Q_i; these are $\frac{1}{\delta'}$ many functions as shown in figure 1.

Let us denote by $\varphi_i m$, the measure having density φ_i (with respect to the Lebesgue one).

We will also need to perform some construction on measures. Let us introduce some notations which will be helpful.

Given a measure μ, let us now consider two projection operators averaging on vertical or horizontal segments, $P_{\_}$ and $P_{|} : PM(Q) \to PM(Q)$ defined by

$$P_{|}\mu = \sum_i \pi_1^*(\varphi_i \mu) \times \pi_2^*(\varphi_i m)$$

and $P_{\_}\mu$ the measure obtained similarly, averaging on horizontal segments in $Q_{i,j}$:

$$P_{\_}\mu = \sum_{i,j} \pi_1^*(1_{Q_{i,j}} m) \times \pi_2^*(1_{Q_{i,j}} \mu)$$

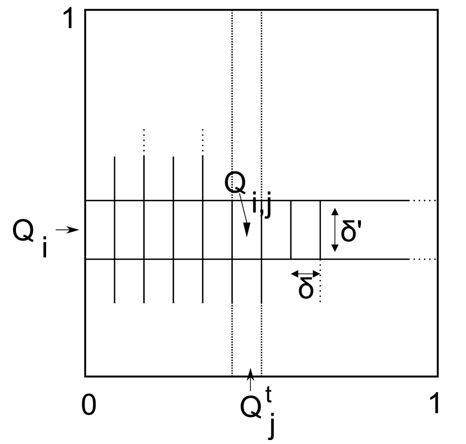


Figure 1: The partitioning scheme and the notation

so that $P_{\_}P_{|}\mu = \sum_{i,j} \mu(Q_{i,j}) 1_{Q_{i,j}} m = \mathbf{E}(\mu|\{Q_{i,J}\})$ and $\pi_1(P_{\_}P_{|}\mu) = \mathbf{E}(\pi_1(\mu)|\{I_i\})$.

Let us define

$$L_\delta = P_{\_}P_{|}LP_{\_}P_{|}.$$

This is a finite rank operator and is the Ulam discretization of L with respect to the rectangle partition.

We remark that L is not a contraction on the W distance, to realize it, consider a pair of Dirac-δ-measures on the expanding direction. This is a problem, in principle, when simulating real iterations of the system by approximate ones. The problem can be overcome disintegrating the measure along the stable leaves and *exploiting the fact that the measures we are interested in, are absolutely continuous on the expanding direction and the system, in some sense, will be stable for this kind of measures*. This can be already noticed in Theorem 2 where it can be seen that the (total variation) distance between the marginals does not increase by iterating the transfer operator.

Now let us define the measure μ_0 which is meant to be iterated and estimate $W(L^n\mu_0, L^n_\delta\mu_0)$. Let us consider the physical invariant measure of the system $\overline{\mu}$ and iterate a starting measure μ_0 supported on a suitable open neighborhood U of the attractor[4]. We suppose that U is such that $U \cap I_\gamma$ is a finite union of open intervals, where $I_\gamma = \{(x,y) \in [0,1] \times [0,1], x = \gamma\}$ is a vertical leaf at coordinate γ. Given U, we construct μ_0 in a way that it has the computed approximation f_δ of the one dimensional invariant mesure, as marginal on the expanding direction. We also construct the measure μ_0 in a way that there is on each stable leaf, a multiple of the Lebesgue measure m_{U_γ} on the union of intervals $U_\gamma = U \cap I_\gamma$.

More precisely

[4]This neighborhood will be constructed in the implementation by intersecting $F^n(Q)$ with a suitable grid and taking all the rectangles with non empty intersection.

$$\mu_0 = \ f_\delta \times \frac{m_{U_\gamma}}{m_{U_\gamma}(I_\gamma)}. \tag{5}$$

Proposition 5 *Let us consider a Lorenz like map F as described in the introduction, its transfer operator L and the finite dimensional Ulam approximation L_δ with grid size (δ, δ') a described above. Let L_T be the one dimensional transfer operator associated to the action of L on the $x-$marginals. Let μ_0 described above, and let $\epsilon \geq ||f - f_\delta||_{L^1}$, let moreover suppose that the whole space can be divided into two sets $X_1 = X - \mathcal{B} \times I, X_2 = \mathcal{B} \times I$ for some finite union of intervals $\mathcal{B} \subset I$ such that $\mu_0(X_2) \leq l$ (we have a bound for the measure of the bad part of the space) and $Lip(F|_{X_1}) \leq \overline{L}$. Then it holds for each n*

$$W(L^n\mu_0, L_\delta^n\mu_0) \leq \frac{2\delta'}{1-\lambda} + \delta \tag{6}$$

$$+\sum_{i=1}^{n-1} \min\{[\overline{L}^i(\delta + 2\delta') + (\overline{L}^{i-1} + \overline{L}^{i-2} + ... + 1)(2\delta^{'} + 2l + 3\epsilon)],$$

$$[\lambda^{n-i} + \frac{2\delta^{'}}{1-\lambda} + ||f_\delta - L_T f_\delta||_{L^1}]\}.$$

Remark 6 *We remark that that since f_δ is known, $||f_\delta||_{BV}$ can be recursively estimated, moreover, $||f_\delta - L_T f_\delta||_{L^1}$ can also be estimated quite sharply with some computation. Indeed, let $L_{T,\xi}$ be a Ulam discretization of L_T on a grid of size $\xi < \delta$. Remark that*

$$||f_\delta - L_T f_\delta||_{L^1} \leq ||f_\delta - L_{T,\xi} f_\delta||_{L^1} + ||L_{T,\xi} f_\delta - L_T f_\delta||_{L^1}$$

here $||f_\delta - L_{T,\xi} f_\delta||_{L^1}$ can be estimated explicitly by computation. On the other hand, by Lemma 23 [5]

$$||L_{T,\xi} f_\delta - L_T f_\delta||_{L^1} \leq \xi(2\lambda_1 + 1)||f_\delta||_{BV} + \xi B'||f_\delta||_{L^1}.$$

Here B' is the second coefficient of the Lasota Yorke inequality satisfied by the one dimensional map T (see Section 11) and $2\lambda_1$ is the first coefficient.

Before the proof we state a Lemma we will use in the following.

Remark 7 *If G_i is family of λ-contractions, μ, ν probability measures on the interval, $\mathcal{I} = \{I_i\}$ a partition whose diameter is δ, and $\Gamma_i(\mu) = \mathbf{E}(G_i(\mathbf{E}(\mu|\mathcal{I}))|\mathcal{I})$, then*

$$\begin{aligned} W(\Gamma_1 \circ ... \circ \Gamma_n(\mu), \Gamma_1 \circ ... \circ \Gamma_n(\nu)) &\leq 2\delta + 2\lambda\delta + 2\lambda^2\delta + ...2\lambda^{n-1}\delta \\ &\quad + \lambda^n(W(\mu,\nu) + 2\delta) \\ &\leq \lambda^n(W(\mu,\nu)) + \frac{2\delta}{1-\lambda}. \end{aligned}$$

[5] Here, since we are in dimension one, we have the freedom to chose ξ very small to minimize this part of the error without increasing too much the computation time.

Lemma 8 *Let $F:\Sigma\to\Sigma$ as above, $\mu,\nu\in PM(\Sigma)$ with absolutely continuous marginals μ_x,ν_x. Let us define $L_{|\delta}=P_{|}LP_{|}$, then*

$$||L^n_{|\delta}\mu - L^n_{|\delta}\nu|| \le \lambda^n + \frac{2\delta^{'}}{1-\lambda} + V(\mu_x,\nu_x).$$

Proof. The proof is similar to the one of Theorem 2. Let us consider $\{I_i\}_{i=1,..,m}$ the intervals where the branches of T^n are defined. Let us consider $\overline{\varphi}_i = 1_{I_i\times I}$ and $\mu_i=\overline{\varphi}_i\mu$, $\nu_i=\overline{\varphi}_i\nu$, then $\mu=\sum\mu_i$, $\nu=\sum\nu_i$, and then

$$W(L^n_{|\delta}(\mu),L^n_{|\delta}(\nu)) \le \sum W(L^n_{|\delta}(\mu_i),L^n_{|\delta}(\nu_i)). \tag{7}$$

Let us denote $T_i=T^n|_{I_i}$, as before. Recall that $L_{|\delta}$ and L_F have the same behavior on x marginals:$(L_{|\delta}(\mu))_x=(L_F(\mu))_x=L_T(\mu_x)$.

$$W(L^n_{|\delta}(\mu_i),L^n_{|\delta}(\nu_i)) \le \int_I W((L^n_{|\delta}\mu_i)_\gamma,(L^n_{|\delta}\nu_i)_\gamma)dL_{T^n}((\nu_i)_x) + ||L^n_T((\mu_i)_x)-L^n_T((\nu_i)_x)||_{L^1}$$

$$\le \int_I W(L^n_{|\delta}((\mu_i)_{T_i^{-1}(\gamma)}),L^n_{|\delta}((\nu_i)_{T_i^{-1}(\gamma)}))dL_{T^n}((\nu_i)_x) + ||(\mu_i)_x-(\nu_i)_x||_{L^1}$$

by the above Remark this is bounded by

$$\le \frac{2\mu_x(I_i)\delta'}{1-\lambda} + \lambda^n\int_I W((\mu_i)_{T_i^{-1}(\gamma)},(\nu_i)_{T_i^{-1}(\gamma)})dL_{T^n}((\nu_i)_x) + ||(\mu_i)_x-(\nu_i)_x||_{L^1}$$

$$\le \frac{2\mu_x(I_i)\delta'}{1-\lambda} + \lambda^n\int_{I_i} W_1((\mu_i)_\gamma,(\nu_i)_\gamma)d((\nu_i)_x) + ||(\mu_i)_x-(\nu_i)_x||_{L^1}$$

where the last step is by change of variable. Hence by Equation 7

$$W(L^n_{|\delta}(\mu),L^n_{|\delta}(\nu)) \le \frac{2\delta'}{1-\lambda} + \lambda^n\int_I W(\mu_\gamma,\nu_\gamma)d(\nu_x) + ||(\mu)_x-(\nu)_x||_{L^1}. \tag{8}$$

■

Proof of Proposition 5. We recall that in the following, we will consider probability measures having absolutely continuous marginals. Remark that

$$\begin{aligned} W(L^n\mu_0,L^n_\delta\mu_0) &\le W(L^n\mu_0,L^n_{|\delta}\mu_0)+W(L^n_{|\delta}\mu_0,L^n_\delta\mu_0)\\ &= ||L^n\mu_0-L^n_{|\delta}\mu_0|| + ||L^n_{|\delta}\mu_0-L^n_\delta\mu_0||.\end{aligned}$$

The two summands will be estimated separately in the following items:

1. Remark that $||L^n\mu_0-L^n_{|\delta}\mu_0|| = ||\sum_1^n L^{n-k}(L-L_{|\delta})L^{k-1}_{|\delta}\mu_0||$.

   Let us estimate $||L^{n-k}(L-L_{|\delta})L^{k-1}_{|\delta}\mu_0||$. Denoting $L^{k-1}_{|\delta}\mu_0=g_k$, we have $||((L-L_{|\delta})g_k)_\gamma|| \le 2\delta'$on each leaf γ (because $||(P_{|}g_k-g_k)_\gamma|| \le \delta'$ on each leaf, and L applied to two disintegrated measures having the same marginal does not increase distance of the respective measures induced on the leaves). Since the projections $\pi_1(Lg_k)=\pi_1(L_{|\delta}g_k)$ are the same, by Proposition 3, $||L^{n-k}(L-L_{|\delta})g_k|| \le \lambda^{n-k}2\delta'$ where λ is the rate of contraction of fibers. By this $||L^n\mu_0-L^n_{|\delta}\mu_0|| \le \frac{2\delta'}{1-\lambda}$.

2. In the same way as before $||L^n_{|\delta}\mu_0 - L^n_\delta \mu_0|| = ||\sum_1^n L^{n-k}_{|\delta}(L_{|\delta} - L_\delta)L^{k-1}_\delta \mu_0||$.

   Let us estimate $||L^{n-k}_{|\delta}(L_{|\delta} - L_\delta)L^{k-1}_\delta \mu_0||$. Denoting $L^{k-1}_\delta \mu_0 = f_k$, we have

$$\begin{aligned} ||L^{n-k}_{|\delta}(L_{|\delta} - L_\delta) f_k|| &\leq ||L^{n-k}_{|\delta}(P_| L P_| - P_| L P_- P_|) f_k|| + \\ & \quad ||L^{n-k}_{|\delta}(P_| L P_- P_| - P_- P_| L P_- P_|) f_k|| \end{aligned}$$

Let us estimate $||L^{n-k}_{|\delta}(P_- P_| L P_- P_| - P_| L P_- P_|) f_k||$. First let us consider $k = n$. In this case by transporting horizontally the measure to average inside each rectangle (recall that f_k is a probability measure)

$$||(P_- P_| L P_- P_| - P_| L P_- P_|) f_k|| \leq \delta.$$

Now let us face the case where $|k - n| \neq 0$. We will give two estimations for $||L^{n-k}_{|\delta}(P_- P_| L P_- P_| - P_| L P_- P_|) f_k||$; one will be suited when $n - k$ is small, and the other when it is large. Then we can take the minimum of the two estimations. The first estimation is based on splitting the space into two subsets, in the first subset the map is not too much expansive, the second set has small measure. This allow to estimate the maximal expansion rate of $L^{n-k}_{|\delta}$ with respect to the Wasserstein distance. The second estimation is based on disintegration, similar to theorem 2.

Now let us face the case where $|k - n| \neq 0$ is small.

We find an estimation for $||L^i_{|\delta}(\mu - \nu)||$ for a pair of probability measures μ and ν which is suitable when i is small. Let us divide the space X into two sets X_1, X_2 such that: $\mu(X_2), \nu(X_2) \leq l$, $Lip(F|_{X_1}) \leq \overline{L}$.

$$\begin{aligned} ||L_{|\delta}(\mu - \nu)|| &\leq 2\delta' + \sup_{\substack{Lip(g) \leq 1 \\ ||g||_\infty \leq 1}} |\int_X g d(LP_| \mu - LP_| \nu)| \\ &\leq 2\delta' + \sup_{\substack{Lip(g) \leq 1 \\ ||g||_\infty \leq 1}} |\int_{X_1} g \circ F \ d(P_| \mu - P_| \nu) + \int_{X_2} g \circ F \ d(P_| \mu - P_| \nu)| \\ &\leq 2\delta' + \overline{L}(||\mu - \nu|| + 2\delta') + 2l. \end{aligned}$$

We now iterate, we need that the above general assumptions are preserved. Recalling that $X_1 = X - \mathcal{B} \times I, X_2 = \mathcal{B} \times I$, we have that, since the map preserves the contracting foliation and since its one dimensional induced transfer operator L_T is a L^1 contraction then $||L^i_T f_\delta - f_\delta||_{L^1} \leq ||L^i_T f_\delta - f_\delta - L^i_T f + f||_{L^1} \leq 2\epsilon$, hence $L^i_{|\delta}\mu_0(X_2) \leq l + 2\epsilon, L^i_{|\delta}\overline{\mu}(X_2) \leq l + 2\epsilon$ and

$$\begin{aligned} ||L^{n-k}_{|\delta}(P_- P_| L P_- P_| - P_| L P_- P_|) f_k|| &\leq \overline{L}^{n-k}(\delta + 2\delta') + \overline{L}^{n-k-1}(2\delta' + 2l + 3\epsilon) \\ & \quad + \overline{L}^{n-k-2}(2\delta' + 2l + 3\epsilon) + ... + (2\delta' + 2l + 3\epsilon). \end{aligned}$$

Now let us face the case which seems to be suited when $|k - n| \neq 0$ is large; let us consider

$$V((P_- P_| L P_- P_| f_k)_x - (P_| L P_- P_| f_k)_x).$$

Recalling that L_T is the one dimensional transfer operator associated to T and $L_{T,\delta}$ is its Ulam discretization with a grid of size δ, since $f_\delta = (f_k)_x$ is invariant for the one dimensional approximated transfer operator then

$$L_{T,\delta}(\mu) = \pi_1(L_\delta(\mu \times m))$$

associated to L_δ then considering the disintegration and the marginals on the x axis

$$||(P_- P_| L P_- P_| f_k)_x - (P_| L P_- P_| f_k)_x||_{L^1} = ||f_\delta - L_T f_\delta||_{L^1}.$$

Thus by Lemma 8,

$$||L_{|\delta}^{n-k}(P_- P_| L P_- P_| - P_| L P_- P_|) f_k|| \leq \lambda^{n-k} + \frac{2\delta^{'}}{1-\lambda} + ||f_\delta - L_T f_\delta||_{L^1}.$$

Now, let us estimate $||L_{|\delta}^{n-k}(P_| L P_| - P_| L P_- P_|) f_k||$. We remark that $P_| f_k = P_- P_| f_k$. Indeed $f_k = (P_- P_| L P_- P_|) L_\delta^{k-2} \mu_0$ thus it has already averaged on the horizontal direction, this is not changed by applying $P_|$, and then applying again P_- has no effect. Hence $||L_{|\delta}^{n-k}(P_| L P_| - P_| L P_- P_|) f_k|| = 0$.

Summarizing, considering that we can take the minimum of the two different estimations and putting all small terms in a sum, we have Equation (6). ∎

5 The algorithm

The considerations made above justify an algorithm for the computation with explicit bound on the error for the physical invariant measure of Lorenz like systems we describe informally below.

Algorithm 9 *1. Input δ, δ'. Compute a L^1 approximation for the marginal one dimensional invariant measure f_δ of the one dimensional induced map T (see Section 11 for the details)*

2. *Input n. Use Theorem 2 to estimate $W(L_F^n(\mu_0), L_F^n(\overline{\mu}))$.*

3. *Use Proposition 5 to estimate the distance $W(L_\delta^n(\mu_0), L_F^n(\mu_0))$*

4. *Compute an approximation $\tilde{\mu}$ for $L_\delta^n(\mu_0)$ up to an error η.*

5. *Output $\tilde{\mu}$ and $W(L_F^n(\mu_0), L_F^n(\mu))+ W(L_\delta^n(\mu_0), L_F^n(\mu_0)) + \eta$.*

Proposition 10 *What is proved above implies that $\tilde{\mu}$ is such that*

$$W(\tilde{\mu}, \overline{\mu}) \leq W(L_F^n(\mu_0), L_F^n(\overline{\mu})) + W(L_\delta^n(\mu_0), L_F^n(\mu_0)) + \eta.$$

Of course this is an a posteriori estimation for the error. Hence it might be that the error of approximation is not satisfying. In this case one can restart the algorithm with a larger n and smaller δ, δ'.

Remark 11 *We remark that for each ε, there are integers m, n and grid sizes δ, δ', ξ such that the above algorithm applied to F^m computes a measure $\tilde{\mu}$ such that $W(\tilde{\mu}, \overline{\mu}) \leq \varepsilon$.*

Indeed choose m such that $\lambda^m \leq \frac{\varepsilon}{10}$ and $n = 2$ iterations. Choose δ such that $||f - f_\delta||_{L_1} \leq \frac{\varepsilon}{10}$ (see e.g. [10], Section 5.1 for the proof that such an approximation is possible up to any small error) then by Theorem 2, $||L^2\mu_0 - \overline{\mu}|| \leq \frac{\varepsilon}{5}$.

Let us suppose that δ and ξ are so small that $\delta + \frac{4\delta'}{1-\lambda} + ||f_\delta - L_{T,\xi} f_\delta||_{L^1} + \xi(2\lambda_1 + 1)||f_\delta||_{BV} + \xi B'||f_\delta||_{L^1} \leq \frac{\varepsilon}{10}$. This is possible because $||f_\delta - L_{T,\xi} f_\delta||_{L^1} \leq ||f_\delta - f_\xi||_{L^1} + ||f_\xi - L_{T,\xi} f_\delta||_{L^1} \leq 2||f_\delta - f_\xi||_{L^1}$ and $||f_\delta - f_\xi||_{L^1} \leq ||f_\delta - f||_{L^1} + ||f - f_\xi||_{L^1}$.

Then by Proposition 5, $||L^2\mu_0 - L^2_{(\delta,\delta')}\mu_0|| \leq \frac{\varepsilon}{5}$ and we have that $W(\tilde{\mu}, \overline{\mu})$ can be made as small as wanted.

It is clear that the choice of the parameters which is given above might be not optimal, and setting a suitable m or n we might achieve a better approximation. The purpose of this remark is just to show that our method can in principle approximate the physical measure up to any small error.

6 Dimension of Lorenz like attractors

We show how to use the computation of the invariant measure to compute the fractal dimension of a Lorenz like attractor.

We recall and use a result of Steinberger [22] which gives a relation between the entropy of the system and its geometrical features.

Let us consider a map $F : Q \to Q$, $F(x, y) = (T(x), G(x, y))$ satisfying the items 1)...4) in the Introduction, and

- $F((c_i, c_{i+1}) \times [0, 1]) \cap F((c_j, c_{j+1}) \times [0, 1]) = \emptyset$ for distinct i, j with $0 \leq i, j < N$.

Let us consider the projection $\pi : Q \to I$, set $V = \{(c_i, c_{i+1}), 1 \leq i \leq N\}$, consider $V_k = \bigvee_{i=0}^k T^{-i}V$. For $x \in E$ let $J_k(x)$ be the unique element of V_k which contains x. We say that V *is a generator* if the length of the intervals $J_k(x)$ tends to zero for $n \to \infty$ for any given x. For a topologically mixing piecewise expanding maps V is a generator. Set

$$\psi(x, y) = \log |T'(x)| \quad \text{and} \quad \varphi(x, y) = -\log |(\partial G/\partial y)(x, y)|.$$

The result we shall use to estimate the dimension is the following

Theorem 12 *[22, Theorem 1] Let F be a two-dimensional map as above and μ_F an ergodic, F-invariant probability measure on Q with the entropy $h_\mu(F) > 0$. Suppose V is a generator, $\int \varphi \, d\mu_F < \infty$ and $0 < \int \psi \, d\mu_F < \infty$. If the maps $y \mapsto \varphi(x, y)$ are uniformly equicontinuous for $x \in I \setminus \{0\}$ and $1/|T'|$ has finite universal p- Bounded Variation, then*

$$d_\mu(x, y) = h_\mu(F) \left(\frac{1}{\int \psi \, d\mu} + \frac{1}{\int \varphi \, d\mu} \right)$$

for μ-almost all $(x,y)\in Q$.

Remark 13 *We remark that since the right hand of the equation does not depend on (x,y), this implies that the system is exact dimensional.*

We also remark that $\int \psi \, d\mu$ can be computed by the knowledge of the measure of the 1 dimensional map under small errors in the L^1 norm and having a bound for its density (see Section 7).

The following should be more or less well known to the experts, however since we do not find a reference we present a rapid sketch of proof.

Lemma 14 *If (F,μ) as above is a computable dynamical system* [6] *then*

$$h_\mu(F) = h_{\mu_x}(T).$$

Proof. (sketch) We will use the equivalence between entropy and orbit complexity in computable systems ([8]). Since $h_\mu(F) \geq h_{\mu_x}(T)$ is trivial, we only have to prove the opposite inequality. What we are going to do is to show that from an approximate orbit for T and a finite quantity of information, one can recover (recursively) an approximated orbit for F.

We claim that, for most initial conditions x, starting from an r approximation $p_1,\ldots,p_n\in\mathbb{Q}$ for the T orbit of $\pi_1(x)$ (by r approximation we mean that $T^i(\pi_1(x))\in B(p_i,r)$, we recall that we take the sup norm on $\mathbb{R}^2$) we can recover a K approximation $x_1,...,x_n$ for the orbit of x by F (hence $F^i(x)\in B(x_i,K)$) for some K not depending on n. Let us denote the rectangle with edges $r,r'$and center x by $B(r,r',x)$. Let us consider

$$C=\sup_{x_1,x_2\in B(p_i,r)}\frac{|G(x_1,y)-G(x_2,y)|}{|x_1-x_2|}.$$

By Item 1, this is bounded.

Let us describe how to find the sequence x_i by p_i inductively. Suppose we have found x_i, such that $\pi_1 x_i = p_i$. Let us suppose r is so small that $\lambda r' + Cr \leq r'$. Let $K=\max(r,\lambda r'+Cr)$; by the contraction in the vertical direction

$$F(B(r',r,x_i))\cap\pi_1^{-1}(B(p_{i+1},r))\subseteq B(\lambda r'+Cr,r,x_{i+1})\subset B(x_{i+1},K).$$

for some x_{i+1} such that $\pi_1 x_{i+1}=p_{i+1}$. And if F is computable, such an x_{i+1} can be computed by the knowledge of x_i, p_i, F, r, r'.

Remark that if r is as above then $F(B(r',r,x_i))\cap\pi_1^{-1}(B(p_{i+1},r))\subseteq B(r,r',x_{i+1})$ and we can continue the process. Hence by the computability of the map, knowing x at a precision r (to start the process) and $p_1,\ldots,p_n$ we can recover suitable $x_1,\ldots,x_n$ at a precision K (by some algorithm, up to any accuracy).

[6] For the precise definition, see [8]. In practice, since the invariant measure is computable starting from the definition of F, this is satisfied by Remark 11 for example when F is given explicitly like in Equations 9 and 10.

This shows that from an encoding of $p_1, \ldots, p_n$ and a fixed quantity of information, one can recover a description of the orbit of x at a precision K. By this the orbit complexity of typical orbits in (I^2, F) is less or equal than the one in (I, T) and if everything is computable, these are equal to the respective entropies (see [8]). Thus $h_\mu(F) \leq h_{\mu_x}(T)$. ■

Remark 15 *By the above lemma and $h_\mu(F) = \int \psi \; d\mu$ then it follows that:*

$$d_\mu(x, y) = 1 + \frac{\int \psi \; d\mu}{\int \varphi \; d\mu}.$$

7 Implementation of the algorithms

Here we briefly discuss some numerical issue arising in the implementation of the algorithms and the choices we made to optimize it.

7.1 Reducing the number of elements of the partition

Our goal is to compute a Ulam like approximation of the 2-dimensional map. Since, as noticed in the introduction, the complexity of the problem and the number of cells involved in the discretization, grows quadratically and hence too fast if we consider the whole square. The idea is to restrict the dynamics to a suitable invariant set containing the attractor.

We remark that, since the image of the first iteration of the map $F(Q)$ is again invariant for the dynamics, we can restrict to the dynamics on some suitable set containing it (i.e. to the elements of the partition that intersect the image of the map) and compute the Ulam approximation of this restricted map. As a matter of fact, we could take an higher iteration of the map and narrow even more the size of the chosen starting region.

Therefore we have to find rigorously a subset of the indexes, such that the union of the elements of the partition with indexes in this subset contains the attractor. To do so we use the containment property of interval arithmetics; if $\tilde{F}$ is the interval extension of F and R is a rectangle in the continuity domain, then $\tilde{F}(R)$ is a rectangle such that $F(R) \subset \tilde{F}(R)$.

We want to compute rigorously a subset that contains the image of the map; at the same time we would like this set to be "small". We divide each of the continuity domains along the x in k homogeneous pieces and, at the same time we partition homogeneously in the y direction, constructing a partition $\mathcal{P}$, whose elements we denote by R, which is coarser than the Ulam partition.

To compute the indexes of the elements of the Ulam partition that intersect $F(Q)$, for each rectangle $R \in \mathcal{P}$ we take $\tilde{F}(R)$ and mark the indexes whose intersection with $\tilde{F}(R)$ is non empty. Doing so we obtain a subset of the indexes which is guaranteed to contain the image of the map and, therefore, the attractor.

This reduces dramatically the size of the problem; in our example with a size of 16384×1024 this permits us to reduce the number of coefficients involved in the computation from $2^{24} = 16384 \times 1024$ to 351198.

7.2 Computing the Ulam matrix

To compute with a given precision the coefficients of the Ulam approximation we have to compute $P_{ij} := m(F^{-1}R_j \cap R_i)/m(R_i)$; what we do is to find a piecewise linear approximation for the preimage of R_j. We remark that the preimages of the vertical sides of R_j are vertical lines, due to the fact that the map preserves the vertical foliation, while the preimage of the horizontal sides are lines which are graphs of functions $\phi(x)$.

We approximate $F^{-1}R_j$ by a polygon $\tilde{P}_j$ and compute the area of the intersection between the polygon and R_i with a precribed bound on the error.

We present a drawing to illustrate our ideas: figure 2. In the figure the intersection of the rectangle and the preimage, represented by the darkest region, is the region whose area we want to compute.

Denote by $R_{j,l}$ and $R_{j,u}$ the quotes of the lower ad upper sides of R_j; inside a continuity domain we can apply the implicit function theorem and we know that there exists ϕ_l and ϕ_u such that

$$G(x, \phi_l(x)) = R_{j,l} \quad G(x, \phi_u(x)) = R_{j,u}.$$

Computing P_{ij} is nothing else that computing the area of the intersection between R_i and the area between the graphs of ϕ_u and ϕ_l, i.e., computing rigorously the difference between the integrals of the two functions

$$\chi_u = \min\{\phi_u, R_{i,u}\}, \quad \chi_l = \max\{\phi_l, R_{i,l}\},$$

over the interval $R_i \cap T^{-1}(\pi_x(R_j))$, where T is the one dimensional map and π_x is the projection on the x coordinate.

We explain some of the ideas involved in the computation of the integral of χ_u; the procedure and the errors for χ_l follow from the same reasonings.

The main idea consists in approximating ϕ_u by a polygonal and estimate the error made in computing the area below its graph and its intersection point with the quote $R_{i,u}$.

In figure 2, ϕ_l is the preimage of the lower side of R_j, ϕ_u is the preimage of the upper side and $\tilde{\phi}_u$ is the approximated preimage of the upper side (with four vertices).

From straightforward estimates it is possible to see that the error made using the polygonal approximation $\tilde{\phi}_u$ when we compute the integral below the graph of ϕ_u depends on the second derivative of ϕ_u, while the error made on computing the intersection between ϕ_u and $R_{i,u}$ depends on the distortion of ϕ_u.

Please note that ϕ_u'' and the distortion of ϕ_u go to infinity near the discontinuity lines; at the same time, in our example the contraction along the y-direction is strongest near the discontinuity lines. Therefore, if the discretization is fine enough, *for rectangles near the discontinuity lines*, the image of R_i is strictly contained between two quotes; this implies that in this specific case

$$\chi_u \equiv R_{i,u} \quad \chi_l \equiv R_{i,l},$$

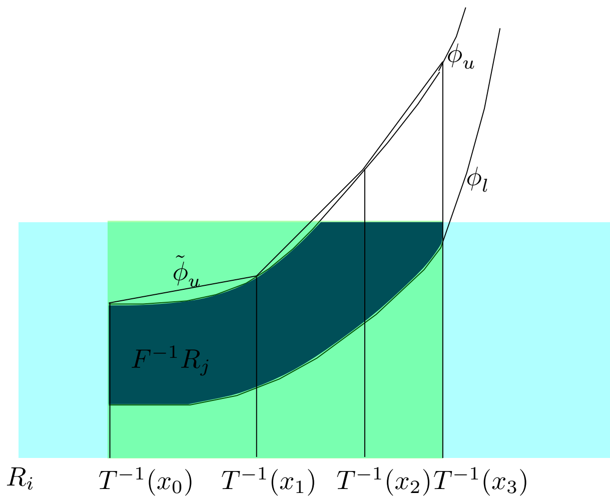


Figure 2: The rectangle R_i, the preimage $F^{-1}R_j$ with its upper boundary ϕ_u and its lower boundary ϕ_l and the linearization of the upper boundary $\tilde{\phi}_u$.

therefore

$$P_{ij} = \frac{m(F^{-1}R_j \cap R_i)}{m(R_i)} = \frac{\int_{T^{-1}(\pi_1(R_j))} \chi_u - \chi_l dm}{m(R_i)} = \frac{\int_{T^{-1}(\pi_1(R_j))} 1/\delta' dm}{m(R_i)}$$
$$= \frac{\int_{T^{-1}(\pi_1(R_j))} 1 dm}{m(\pi_1^{-1} R_i)} = \frac{m(T^{-1}(\pi_1(R_j)) \cap \pi_1(R_i))}{m(\pi_1(R_j))},$$

i.e. in these particular cases, the coefficients depend only on what happens along the x-direction.

8 Numerical experiments

In this section we show the results of a rigorous computation on a Lorenz-like map F . The C++ codes and the numerical data are available at:
http://im.ufrj.br/ nisoli/CompInvMeasLor2D

8.1 Our example

In our experiments we analize the fourth iterate (i.e. $F^4 : Q \to Q$) of the following two dimensional Lorenz map

$$F(x, y) = (T(x), G(x, y))$$

with

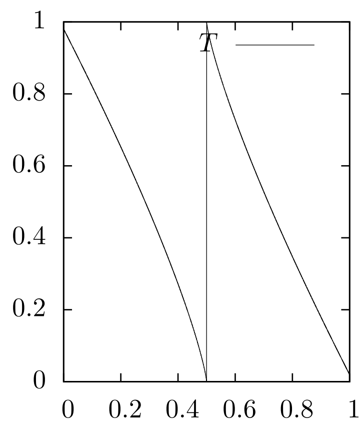
$$T(x) = \begin{cases} \theta|x-1/2|^{\alpha} & 0 \le x < 1/2 \\ 1-\theta|x-1/2|^{\alpha} & \frac{1}{2} < x \le 1 \end{cases} \quad (9)$$

with constants $\alpha = 51/64$, $\theta = 109/64$, and

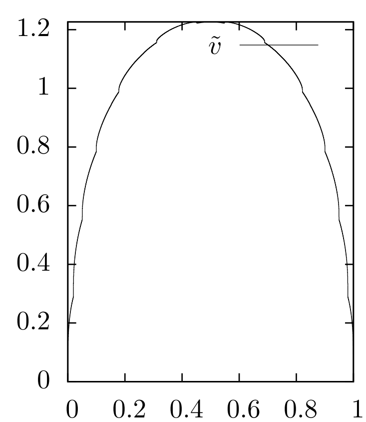
$$G(x,y) = \begin{cases} (y-1/2)|x-1/2|^{\beta} + 1/4 & 0 \le x < 1/2 \\ (y-1/2)|x-1/2|^{\beta} + 3/4 & \frac{1}{2} < x \le 1 \end{cases} \quad (10)$$

with $\beta = 396/256$.

The graph of T is plotted in figure 3a. In subsection 8.2 we give the results of the rigorous computation of the density plotted in figure 3b.



(a) Map (9)



(b) The invariant measure for map (9)

Figure 3: The one dimensional map: (9).

8.2 The Lorenz 1-dimensional map

The first step in our algorithm is the approximation of the invariant measure of the one dimensional induced map. The algoritm we use is the one described in [10] with the estimations described in Section 11.

We consider the fourth iterate T^4 of the Lorenz 1-dimensional map T given by Equation 9 with $\alpha = 51/64$ and $\theta = 109/64$ and estimate the coefficients of its Lasota Yorke inequality (see Section 11). We have that

$$\frac{2}{\min(d_i - d_{i+1})} \le 37.8247.$$

Moreover, $\sup|1/T'| \le 0.16$, we fix $l = 30$ obtaining that

$$\frac{1}{2}\int_{I_l} |\frac{T''}{(T')^2}| \le 0.46$$

and that $\lambda_1 \leq 0.763$.

We have therefore that the second coefficient of the Lasota Yorke inequality is $B \leq 285.053$. From the experiments, on a partition of 1048576 elements, with a matrix such that each component was computed with an error of 2^{-43} we have that the number of iterations needed to contract the zero average space is $N = 8$.

Therefore the rigorous error on the computation of the one dimensional measure is:[7]

$$||f - \tilde{v}||_{L^1} \leq 0.005.$$

8.3 Estimating the measure for the Lorenz 2-dimensional map

In our numerical experiments we used a partition of the domain of size $\delta = 1/16384 = 2^{-14}$ elements in the x direction and of $\delta' = 2^{-10}$ in the y-direction and reduced the number of elements we consider as explained in 7.1.

We computed the Ulam discretization of the fourth iterate of the Lorenz 2-dimensional map; using the method explained in subsection 7.1 our program needed to compute approximatively 351198 cofficients of the matrix.

As explained in Subsection 8.2 the one dimensional map satisfies a Lasota Yorke inequality with coefficients $\lambda_1 \leq 0.763$, $B \leq 285.53$ and we have a computed approximated density on a partition of size $\xi = 1/1048576 = 2^{-20}$ such that $||f - f_\xi||_{L^1} \leq 0.005$.

To estimate $||f_\delta||_{BV}$, as required in Proposition 5 we use the upper bound (Lemma 20)

$$||f_\delta||_{BV} \leq \mathrm{Var}(f_\delta) + 2||f_\delta||_{L^1},$$

which gave, constructing f_δ by averaging f_ξ on the coarser partition that

$$||f_\delta||_{BV} < 4.37, \quad ||f_\xi||_{BV} < 4.46, \quad ||f_\xi - f_\delta||_{L^1} \leq \frac{4.46}{16384} \leq 0.0003.$$

To apply Proposition 5 and Remark 6 we compute that in our example $\lambda \leq 0.014$, and we chose to take intervals of size $2/1048576$ near the discontinuity points to estimate l and $\bar{L}$, obtaining respectively that $l < 3.2 \cdot 10^{-5}$ and $\bar{L} < 1277$.

Since

$$||L_T f_\delta - f_\delta||_{L^1} \leq ||L_T f_\delta - L_{T,\xi} f_\delta||_{L^1} + ||L_{T,\xi} f_\delta - f_\delta||_{L^1},$$

[7] The additional parameters which are involved in this computation (see [10] for the meaning) are $B' \leq 67.83$, $N_\varepsilon = 7$, and the number of iterates necessary to contract the unit simplex to a diameter of 0.0001 was 10 (i.e., the numerical accuracy with which we know the eigenvector for the matrix). Therefore the rigorous error is estimated as:

$$||f - \tilde{v}||_{L^1} \leq \frac{2 \cdot 8 \cdot 285.053}{1048576} + 2 \cdot 7 \cdot 1048576 \cdot 2^{-43} + 0.0001 \leq 0.005.$$

then

$$||L_T f_\delta - f_\delta||_{L^1} \leq 0.00034.$$

Let μ_0 be as defined in (5). We apply Theorem 2 to estimate the distance $W(\overline{\mu}, L_F^3\mu_0)$ after 3 iterations, since $\lambda^3 \leq 3\cdot 10^{-6}$.

Let us consider the error estimate; let $\overline{\mu}$, with marginal f, be the physical invariant measure for F:

$$W(\overline{\mu}, L_F^3\mu_0) = W(L_F^3\overline{\mu}, L_F^3\mu_0) \leq ||f - f_\delta||_{L^1} + \lambda^3 \leq ||f - f_\xi||_{L^1} + ||f_\xi - f_\delta||_{L^1} + \lambda^3;$$

therefore:

$$W(\overline{\mu}, L_F^3\mu_0) \leq 0.0054.$$

Now, we need to take into account the fact that we are iterating an approximated operator; referring to Proposition 5 and looking at the data of our problem, we can see that, when taking the minimum, is always the second member that is chosen. Then, the explicit formula is:

$$W(L_F^n\mu_0, L_\delta^n\mu_0) \leq n\cdot\frac{2\delta'}{1-\lambda} + \delta + \sum_{i=1}^{n-1}\left(\lambda^{n-i} + ||L_T f_\delta - f_\delta||_{L^1}\right).$$

Therefore

$$\begin{aligned} W(L_F^3\mu_0, L_\delta^3\mu_0) &\leq 3\cdot\frac{2\delta'}{1-\lambda} + \delta + 2\cdot||L_T f_\delta - f_\delta||_{L^1} + \sum_{i=1}^{2}\lambda^{3-i} \\ &\leq 3\cdot 0.002 + 0.00007 + 0.00068 + 0.015 \leq 0.022 \end{aligned}$$

and

$$W(\overline{\mu}, L_\delta^3\mu_0) \leq 0.028.$$

In figure 4 we present an image of the computed density, on the partition 16384×1024.

9 Estimating the dimension

Here we use the results explained in Section 6 to rigorously approximate the dimension of the above computed invariant measure.

Inspecting (10), it is possible to see that in our case we have that $\partial_y G(x,y)$ is constant along the fibers. More explictly, by (9),(10) we have that:

$$\log(|\partial_x T(x)|) = \log(\theta) + \log(\alpha) + (\alpha-1)\log|x-1/2|,$$

$$\log(|\partial_y G(x,y)|) = \beta\log|x-1/2|$$

Therefore, if μ is the invariant measure for the Lorenz 2-dimensional map, to estimate the dimension, we have to estimate

$$\int_0^1 \log|x-1/2|d\mu_x,$$

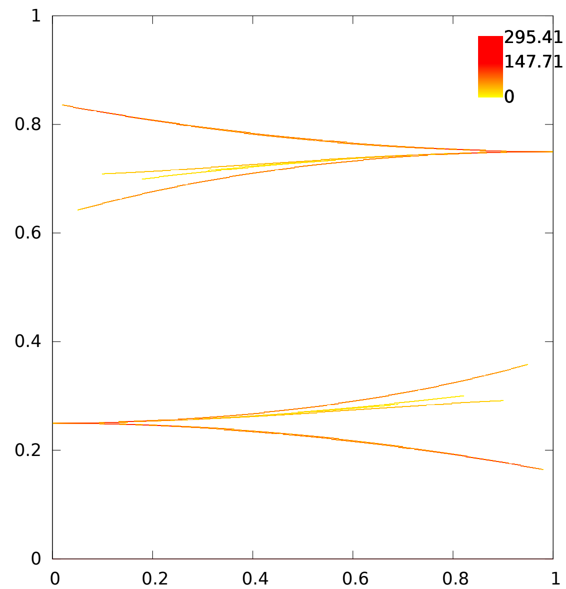


Figure 4: Density on a grid 16384×1024

where $d\mu_x$ has density f.

On one side, the function $\log|x-1/2|$ is unbounded, on the other side, we only know an approximation of the density, that we denote by f_δ. Let us estimate from above and from below of the integral.

To give the estimate from above, we take a small interval $(1/2-\epsilon_1, 1/2+\epsilon_1)$ and we define a new function

$$\tilde{\psi}_1(x) = \begin{cases} \log|x-1/2| & x \in [0,1] \setminus (1/2-\epsilon_1, 1/2+\epsilon_1) \\ \log|\epsilon_1| & x \in (1/2-\epsilon_1, 1/2+\epsilon_1) \end{cases} \tag{11}$$

Therefore we have:

$$\int_0^1 \log|x-1/2| df \leq \int_0^1 \tilde{\psi}_1 df_\delta + |\log(\epsilon_1)| \cdot ||f-f_\delta||_{L^1}.$$

Now, we want to estimate the integral from below; the idea is again to split the integral in two parts. By the Lasota-Yorke inequality we know that the BV norm of f is limited from above by the second coefficient of the Lasota Yorke inequality B; therefore we have that $||f||_\infty \leq B$.

Again, we take a small interval $(1/2-\epsilon_2, 1/2+\epsilon_2)$. We have that

$$\int_{1/2-\epsilon_2}^{1/2} \log|x-1/2| df \geq B \int_{1/2-\epsilon_2}^{1/2} \log|x-1/2| dx,$$

where dx is the Lebesgue measure on the interval $[0,1]$. Therefore

$$\int_{1/2-\epsilon_2}^{1/2} \log|x-1/2|df \geq B\epsilon_2(\log(\epsilon_2)-1).$$

Let

$$\tilde{\psi}_2(x) = \begin{cases} \log|x-1/2| & x \in [0,1] \setminus (1/2-\epsilon_2, 1/2+\epsilon_2) \\ 0 & x \in (1/2-\epsilon_2, 1/2+\epsilon_2) \end{cases} \tag{12}$$

Then we have that:

$$\int_0^1 \log|x-1/2|df \geq \int_0^1 \psi_2 df_\delta - |\log(\epsilon_2)| \cdot ||f-f_\delta||_{L^1} - 2B\epsilon_2|\log(\epsilon_2)-1|.$$

Using the computed invariant measure we have the following proposition.

Theorem 16 *The dimension of the physical invariant measure for the map described in Section 8.1 lies in the interval* $[1.24063, 1.24129]$.

Remark 17 *The high number of significative digits depends on the fact that, due to the properties of the chosen map, in this estimate we are using only the one dimensional approximation of the measure f_δ, which we know with high precision.*

10 A non-rigorous dimension estimate

As a control, we implemented a non-rigorous computation of the correlation dimension of the attractor, following the classical approach described in [12]. Let θ be the heavyside function, i.e., $\theta(x) = 0$ if $x \leq 0$ and $\theta(x) = 1$ if $x \geq 0$. Let x be a point on the attractor and $x_i := F^i(x)$, for $i = 1, \ldots n$; define

$$C(\varepsilon) = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \theta(\varepsilon - d(x_i, x_j)),$$

where $d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$ is the max distance; in the following, we denote by $B_\varepsilon(x)$ the ball with respect to the distance $d(x,y)$. This permits us to define a non rigorous estimator for the local dimension of μ, the so called correlation dimension:

$$\tilde{d}_\mu := \lim_{\varepsilon \to 0} \frac{\log C(\varepsilon)}{-\log(\varepsilon)}.$$

We implemented an algorithm that uses this idea and applied it to a non rigorous experiment where we fixed a family of tresholds $\varepsilon_k = 2^{-9-k}$ for $k = 0, \ldots, 18$ with an orbit (a pseudo orbit) of length $n = 2097152$, and interpolated the results (in a log-log scale) with least square methods. The linear coefficient of the interpolating line should be an approximation of $\tilde{d}_\mu$.

The linear coefficient we obtain from our computations is 1.236 which is near our rigorous estimate of $[1.24063, 1.24129]$.

11 Appendix: computing the invariant measure of piecewise expanding maps with infinite derivative

Approximating fixed points and the invariant measures. In this section we see how to estimate the invariant measure of a one dimensional piecewise expanding map to construct the starting measure for our iterative method.

The method we used is the one explained in [10]. In that paper piecewise expanding maps with finite derivative were considered, while here the map has infinite derivative. We briefly explain the method and show the estimation which allows to use it for the infinite derivative case.

In [10] the computation of invariant measures was faced by a fixed point stability result. The transfer operator is approximated by a suitable discretization (as the Ulam one described before) and the distance between the fixed point of the original operator and the dicretized one is estimated by the stability statement.

To use it we need some a priori estimation and some computation which is done by the computer.

Let us introduce the fixed point stability statement we are going to use.

Let us consider a restriction of the transfer operator to an invariant normed subspace (often a Banach space of regular measures) $\mathcal{B} \subseteq SPM(X)$ and let us still denote it by $L{:}\mathcal{B} \to \mathcal{B}$. Suppose it is possible to approximate L in a suitable way by another operator L_δ for which we can calculate fixed points and other properties (as an example, the Ulam discretization with a grid of size δ).

It is possible to exploit as much as possible the information contained in L_δ to approximate fixed points of L. Let us hence suppose that f, $f_\delta \in \mathcal{B}$ are fixed points, respectively of L and L_δ.

Theorem 18 (see [10]) Let $V = \{\mu \in \mathcal{B} \ s.t. \ \mu(X) = 0\}$. Suppose:

1. $||L_\delta f - Lf||_{\mathcal{B}} \le \epsilon$
2. $\exists N$ such that $\forall v \in V$, $||L_\delta^N v||_{\mathcal{B}} \le \frac{1}{2}||v||_{\mathcal{B}}$
3. $L_\delta^i|_V$ is continuous, let $C_i = \sup_{g \in V} \frac{||L_\delta^i g||_{\mathcal{B}}}{||g||_{\mathcal{B}}}$

Then

$$||f_\delta - f||_{\mathcal{B}} \le 2\epsilon \sum_{i \in [0, N-1]} C_i. \tag{13}$$

To apply the theorem we need to estimate the quantities related to the assumptions a), b), c).

Item a) can be obtained by the some approximation inequality showing that L_δ well approximates L and an estimation for the norm of f which can be recovered by the Lasota Yorke inequality.

In the following subsection we will prove a Lasota Yorke inequality for the kind of maps we are interested in (explicitly estimating its coefficients) involving the L^1 and bounded variation norm. This allows to estimate $||f||_{BV}$.

We then estimate (see [10] Lemma 10)

$$||L_\delta f - Lf||_{L^1} \leq 2\delta||f||_{BV}.$$

By this we complete the estimations needed for the first item.

About b), the required N is obtained by the rate of contraction of L_δ on the space of zero average measures and will be computed while running the algorithm by the computer (see [10] for the details).

Item c) also depend on the definition of L_δ; in the case of L^1 approximation with the Ulam method they can be bounded by 1.

For more details on the implementation of the algorithm, see [10] .

Lasota Yorke inequality with infinite derivative In the following, we see the estimations which are needed to bound the coefficients of the Lasota Yorke inequality when the map has infinite derivative.

Let us consider a class of maps which are locally expanding but they can be discontinuous at some point.

Definition 19 *We call a nonsingular function $T : ([0,1], m) \to ([0,1], m)$ piecewise expanding if*

- *There is a finite set of points $d_1 = 0, d_2, ..., d_n = 1$ such that $T|_{(d_i, d_{i+1})}$ is C^2.*
- $\inf_{x\in[0,1]} |D_xT| = \lambda^{-1} > 2$ *on the set where it is defined.*

Let us define a notion of bounded variation for measures: let

$$||\mu||_{BV} = \sup_{\phi\in C^1, |\phi|_\infty=1} |\mu(\phi')|$$

this is related to the usual notion of bounded variation for densities[8]: if $||\mu||_{BV} < \infty$ then μ is absolutely with bounded variation density (see [18]).

If f is a L^1 density, by a small abuse of notation, let us identify it with the associated measure. The following relates the above defined norm with the usual notion of variation

Lemma 20 *Let f be a bounded variation density, then*

$$||f||_{BV} \leq Var(f) + 2||f||_{L^1}$$

[8]Recall that the variation of a function g is defined as

$$\mathrm{Var}(g) = \sup_{(x_i)\in \text{Finite subdivisions of } [0,1]} \sum_{i\leq n} |g(x_i) - g(x_{i+1})|.$$

Proof. Let $\phi \in C^1, |\phi|_\infty = 1$; let $\tilde{\phi} = (\phi(1) - \phi(0)) \cdot x$ and $\phi_0 = \phi - \tilde{\phi}$. Then

$$|\int_0^1 \phi' f dm| = |\int \phi_0' f dm + \int (\phi(1) - \phi(0)) f dm| \le |\int \phi_0' f dm| + 2|\phi|_\infty \int |f| dm,$$

and, as ϕ varies, by integration by parts we have that

$$||f||_{BV} \le \mathrm{Var}(f) + 2||f||_{L^1}.$$

■

The following inequality can be established (see [10]) showing that for piecewise expanding maps the associated transfer operator is regularizing if one consider the a suitable norm.

Theorem 21 *If T is piecewise expanding as above and μ is a measure on $[0,1]$*

$$||L\mu||_{BV} \le \frac{2}{\inf T'} ||\mu||_{BV} + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + 2\mu(|\frac{T''}{(T')^2}|).$$

To use the above result in a computation, the problem is that $\mu(|\frac{T''}{(T')^2}|)$ cannot be estimated without having some information on μ. Hence some refinement is necessary. Remark that if μ has density f then $||\mu||_{BV} \ge 2||f||_\infty$.

To estimate $\mu(|\frac{T''}{(T')^2}|)$ we consider $I_l = \{x \ s.t. \ |\frac{T''}{(T')^2}| \ge l\}$. Let f be the density of μ

$$\begin{aligned}
\mu(|\frac{T''}{(T')^2}|) &= \int_{I - I_l} |\frac{T''}{(T')^2}| f dx + \int_{I_l} |\frac{T''}{(T')^2}| f dx \le \\
&\le l \int_{I - I_l} f dx + ||f||_\infty \int_{I_l} |\frac{T''}{(T')^2}| dx. \\
&\le l \int_{I - I_l} f dx + \frac{1}{2} ||\mu||_{BV} \int_{I_l} |\frac{T''}{(T')^2}| dx
\end{aligned}$$

If l is chosen such that $\frac{1}{2}\int_{I_l} |\frac{T''}{(T')^2}| + \frac{2}{\inf T'} = \lambda_1 < 1$ then we have the Lasota Yorke inequality which can be used for our purposes:

$$||L\mu||_{BV} \le \lambda_1 ||\mu||_{BV} + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + l\mu(1). \tag{14}$$

Remark 22 *We remark that once an inequality of the form*

$$||Lg||_{\mathcal{B}'} \le 2\lambda_1 ||g||_{\mathcal{B}'} + B' ||g||_{\mathcal{B}}.$$

is established (with $2\lambda_1 < 1$) then, iterating, we have

$$||L^n g||_{\mathcal{B}'} \le 2^n \lambda_1^n ||Lg||_{\mathcal{B}'} + \frac{1}{1 - 2\lambda} B' ||g||_{\mathcal{B}}$$

and the coefficient

$$B = \frac{1}{1 - 2\lambda_1} B'$$

can be used to bound the norm of the invariant measure.

11.1 An approximation inequality

Here we prove an inequality which is used in Remark 6.

Lemma 23 *For piecewise expanding maps, if L_δ is a Ulam discretization of size δ, for every measure f having bounded variation we have that*

$$||(L - L_\delta)f||_{L^1} \leq \delta(2\lambda_1 + 1)||f||_{BV} + \delta B'||f||_{L^1}$$

Where B' is the second coefficient of the Lasota Yorke Inequality related to the map.

Proof. It holds

$$||(L - L_\delta)f||_{L^1} \leq ||\mathbf{E}(L(\mathbf{E}(f|\mathcal{F}_\delta)|\mathcal{F}_\delta)) - \mathbf{E}(Lf|\mathcal{F}_\delta))||_{L^1} + ||\mathbf{E}(Lf|\mathcal{F}_\delta) - Lf||_{L^1},$$

But

$$\mathbf{E}(L(\mathbf{E}(f|\mathcal{F}_\delta)|\mathcal{F}_\delta)) - \mathbf{E}(Lf|\mathcal{F}_\delta)) = \mathbf{E}[L(\mathbf{E}(f|\mathcal{F}_\delta) - f)|\mathcal{F}_\delta].$$

Since both L and the conditional expectation are L^1 contractions

$$\mathbf{E}(L(\mathbf{E}(f|\mathcal{F}_\delta)|\mathcal{F}_\delta)) - \mathbf{E}(Lf|\mathcal{F}_\delta)) \leq ||\mathbf{E}(f|\mathcal{F}_\delta) - f||_{L^1}.$$

For a bounded variation measure f it is easy to see that $||\mathbf{E}(f|\mathcal{F}_\delta) - f||_{L^1} \leq \delta \cdot ||f||_{BV}$.

By this

$$\mathbf{E}(L(\mathbf{E}(f|\mathcal{F}_\delta)|\mathcal{F}_\delta)) - \mathbf{E}(Lf|\mathcal{F}_\delta)) \leq \delta||f||_{BV}.$$

On the other hand

$$||\mathbf{E}(Lf|\mathcal{F}_\delta) - Lf||_{L^1} \leq \delta||Lf||_{BV} \leq \delta(2\lambda_1||f||_{BV} + B'||f||_{L^1})$$

which gives

$$||(L - L_\delta)f||_{L^1} \leq \delta(2\lambda_1 + 1)||f||_{BV} + \delta B'||f||_{L^1} \tag{15}$$

■

References

[1] V.Araujio, S Galatolo, MJ Pacifico *Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors.* arXiv:1204.0703 (to appear on Math. Z.)

[2] W. Bahsoun, *Rigorous Numerical Approximation of Escape Rates* Nonlinearity, vol. 19, 2529-2542 (2006)

[3] W. Bahsoun, C. Bose *Invariant densities and escape rates: rigorous and computable estimation in the L^∞ norm.* Nonlinear Analysis, 2011, vol. 74, 4481-4495.

[4] M. Dellnitz, O. Junge *Set Oriented Numerical Methods for Dynamical Systems* Handbook of dynamical systems vol 2 - Elsevier, (2002).

[5] J. Ding and A. Zhou, *The projection method for computing multidimensional absolutely continuous invariant measures*, J. Stat. Phys. (1994), 77: 899-908.

[6] G. Froyland *Extracting dynamical behaviour via Markov models* in Alistair Mees, editor, Nonlinear Dynamics and Statistics: Proceedings, Newton Institute, Cambridge (1998): 283-324, Birkhauser, 2001.

[7] S. Galatolo, M. Hoyrup, C. Rojas *Dynamical systems, simulation, abstract computation* Cha. Sol. Fra. Volume 45, Issue 1, , Pages 1â€“14 (2012)

[8] S.Galatolo, M. Hoyrup and C. Rojas, *A constructive Borel-Cantelli lemma. Constructing orbits with required statistical properties*, Theor. Comput. Sci. (2009), 410: 2207-2222.

[9] S. Galatolo, M. Hoyrup, and C. Rojas. *Dynamics and abstract computability: computing invariant measures*, Disc. Cont. Dyn. Sys. (2011), 29(1): 193-212

[10] S. Galatolo, I Nisoli *An elementary approach to rigorous approximation of invariant measures* arXiv:1109.2342 (to appear on SIAM J. Appl. Dyn.l Sys.)

[11] S. Galatolo and M.J. Pacifico. *Lorenz like flows: exponential decay of correlations for the Poincare map, logarithm law, quantitative recurrence*. Ergodic Theory and Dynamical Systems, 30: 703–1737, Jan. 2010.

[12] H. Kantz, T. Schreiber *Nonlinear Time Series Analysis* Cambridge University Press, Cambridge (2004)

[13] G. Keller.*Generalized bounded variation and applications to piecewise monotonic transformations* Z. Wahrsch. Verw. Gebiete, 69(3):461–478, (1985).

[14] G Keller, C Liverani *Stability of the spectrum for transfer operators* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 no. 1, 141-152 (1999).

[15] O. Ippei *Computer-Assisted Verification Method for Invariant Densities and Rates of Decay of Correlations.* SIAM J. Applied Dynamical Systems 10(2): 788-816 (2011)

[16] A. Lasota, J.Yorke *On the existence of invariant measures for piecewise monotonic transformations* , Trans. Amer. Math. Soc. (1973), 186: 481-488.

[17] C. Liverani, *Rigorous numerical investigations of the statistical properties of piecewise expanding maps–A feasibility study*, Nonlinearity (2001), 14: 463-490.

[18] C. Liverani *Invariant measures and their properties. A functional analytic point of view*, Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics. Centro di Ricerca Matematica "Ennio De Giorgi": Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[19] S. Gouezel, C. Liverani *Banach spaces adapted to Anosov systems* Erg. Th. Dyn. Sys. (2006), 26: 189-217.

[20] M. Keane, R. Murray and L. S. Young, *Computing invariant measures for expanding circle maps*, Nonlinearity (1998), 11: 27-46.

[21] M. Pollicott and O. Jenkinson, *Computing invariant densities and metric entropy*, Comm. Math. Phys. (2000), 211: 687-703.

[22] T. Steinberger. Local dimension of ergodic measures for two-dimensional lorenz transformations. *Ergodic Theory and Dynamical Systems*, 20(3):911–923, 2000.

[23] Lai-Sang Young *What are SRB measures, and which dynamical systems have them?* J. Stat. Phys. (2002), 108: 733-754.