Endogenous restricted participation in general financial equilibrium

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Abstract

We consider an incomplete market model with numeraire assets. Each household faces an individual constraint on its participation in the asset market. In related literature, the constraint is described by a function whose sole argument is the asset portfolio. On the contrary, in our analysis the constraint depends not only on the asset portfolio, but also on asset and good prices - hence the reference to *endogenous* (in contrast to *exogenous*) in the title.

Economies are described by endowments of commodities, utility functions, asset yield matrices, and restriction functions. We study two specifications of the constraint function. The first one is *homogeneous* of degree zero with respect to spot prices. The second one does not exhibit that property. We then consistently distinguish between *homogeneous* and *nonhomogeneous* economies. After having established existence of equilibria for both types of economies, we study indeterminacy for each of them and show the following results. For an open and dense subset of the set of homogeneous economies, equilibria are finite and regular, up to innocuous price normalizations. There exists an open and nonempty set of nonhomogeneous economies, whose associated equilibria exhibit real indeterminacy.

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1 Introduction

The importance of restrictions on financial markets cannot be overestimated. Even a superficial observation of everyday economic life or a hasty reading of finance papers supplies a list of examples of such phenomenon, a list which can only be incomplete.

Asymmetric information between lender and borrower is a key reason for the need of designing personalized contracts based on observable variables linked to the capability of honoring obligations. Access to credit can be limited by personal expected income and wealth both in the amount and in the structure of repayments. The presence of collateral in terms of goods or assets is a quite standard way of easing credit restrictions.

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Commodities or assets may be freely tradable within but not across countries. In international finance, it is commonly assumed that markets are segmented, i.e., some commodities or assets are argued to be non-tradable across countries.

Some individuals or institutions are allowed to be only on one side of the market.

Country specific credit system regulations and different degree of competitiveness of the banking system can heavily influence the access to credit for large number of individuals.

Government credit interventions are common in most countries. Student loans, lower interest rates for first home buyers, better credit conditions for investments in less developed areas or in productions considered socially valuable, like renewable energy resources, are simple examples of those interventions.

There are stocks which cannot be traded by some households or there exist limits on the fraction of the portfolio which could be invested in some markets, say, of fast developing countries.

Some assets have intrinsic restrictions: pension funds are not available for present consumption; human capital investments are not liquid; housing investments are also illiquid and subject to significant transaction costs. "Together, pensions, human capital, and housing constitute a substantial part of a typical household's assets."[1].

Impossibility of short sales, transaction costs, information costs and obligations to carry life insurance, or hold compensating balances or meet margin requirements are other examples of commonly encountered restrictions.

Empirical evidence of portfolio restrictions and the key role that pervading phenomenon plays in financial markets have generated a large amount of contributions on this topic in finance literature.

Many papers focus on how financial constraints on asset markets affects equilibrium consumption and prices, and in particular, how they are related to asset pricing puzzles (see among others Allen and Gale (1994), Basak and Croitoru (2000), Basak and Cuoco (1998)), market crashes (Hong and Stein (2003)) and the arbitrage opportunities for some investors (Zigrand (2006) and related literature there).

Our goal is to study restricted participation on financial markets using a General Equilibrium model with Incomplete financial market (GEI). Each household has to choose a consumption vector and an asset demand as in a standard model with incomplete or even potentially complete markets, with the additional constraint that asset demand is *restricted* to belong to a household specific portfolio set.

Some contributions on the topic are indeed available in the literature. Siconolfi (1988) and, in a multiperiod framework, Angeloni and Cornet (2006) show existence of equilibria assuming the portfolio set is a closed, convex subset of a Euclidean space containing zero for each households and a neighborhood of zero for some of them. Balasko, Cass and Siconolfi (1990) analyze the case in which the portfolio constraint set is a linear subspace. Polemarchakis and Siconolfi (1997) prove existence for a case of restricted participation when asset payoffs are denominated in multiple commodities. In a model with numeraire asset and outside money with restrictions for both type of assets, Carosi (2001) proves generic inefficiency of equilibria and effectiveness of monetary policy. Martins Da-Rocha and Triki (2005) present an original proof of existence. Won and Hahn (2007) discuss the presence of redundant assets. Hens, Herings and Predtetchinskii (2006) consider a GEI model with one good per spot and give conditions for the existence of arbitrage possibilities when households cannot exchange at all some assets. In the same framework, Herings and Schmedders (2006) study the case of transaction costs proportional to the units (or values) of traded assets and present homotopy arguments to study equilibria. Basak, Cass, Licari and Pavlova (2008) describe the indeterminacy generating capability of participation restrictions, presenting an economy with two goods and two households for which adding a constraint may generate additional inefficient (and sunspot) equilibria.

The model closer to ours is that one presented in Cass, Siconolfi and Villanacci (2001), where portfolio sets are described by restriction functions. Imposing some differentiability and regularity properties on those functions, they show generic regularity of equilibria, if assets are of the numeraire type. In their analysis, however, the constraint is described by a function whose sole argument is

[1]See Detemple and Sundaresan (1999) and Detemple and Serrat (2003).

the asset portfolio[2]. On the contrary, in our analysis the constraint depends not only on the asset portfolio, but also on asset and good prices – hence the reference to *endogenous* (in contrast to *exogenous*) in the title.

We study two specifications of this function. In the first, households constraint sets are *homogeneous* (of degree zero) with respect to spot prices, a fact which permits the same price normalizations as in the standard general equilibrium model with incomplete markets and numeraire assets. In the second, we consider restrictions which do not exhibit the homogeneity property: purely nominal changes, i.e., price changes, may affect the possibilities for participation on the asset market. We will refer to the restriction functions with one or the other of those two specifications as *homogeneous* and *nonhomogeneous* restriction functions, respectively. The chosen way of describing restrictions allows to encompass in a simple manner some of the examples of restricted participation described above - see Section 2.

We can now present the main results of the paper. We consider a pure exchange, general equilibrium model, with two periods and a finite number of states, goods, assets, and households. Assets bestow the right to receive a certain quantity of the numeraire commodity. Economies are described by endowments of commodities, utility functions, asset yield matrices, and restriction functions.

After having established existence of equilibria for both types of economies, we study indeterminacy for each of them. For an open and dense subset of the set of homogeneous economies, we show that equilibria are finite and regular, up to innocuous price normalizations. On the contrary, there exists an open and nonempty set of homogeneous economies exhibiting real indeterminacy.[3] We therefore show that the presence of restricted participation in the asset market may generate real indeterminacy even in the presence of numeraire assets. Indeterminacy is clearly an interesting feature of a model. Consistently with the rational expectation assumption, all agents have to anticipate the same equilibrium price among those associated with the given fundamentals. The presence of "many" equilibria could be a way to explain the well known excess volatility puzzle, and even market bubbles or crashes. Those phenomena could be interpreted as the result of expectation driven behaviors of households attempting to select one equilibrium over another one.

As explained in some detail in Section 4.2, an interesting and, at the best of our knowledge, original by-product of our analysis is a general and easy to use strategy to show smooth (not only continuous) dependence of equilibria from economies in the case of infinite dimensional economy spaces. That methodology, based on a result by Glöckner (2006), can be used to get that result in many general equilibrium models, not last the case of pure exchange economies.

The paper is organized as follows. In Section 2 we first describe the set-up of the model and state the existence result for arbitrary economy (Theorem 2). We then introduce the crucial homogeneity assumption and some examples of homogenous and nonhomogeneous economies. In Section 3, we present our determinacy and indeterminacy results. Theorem 3 shows there exists an open and dense subset of the space of homogenous economies whose associated equilibria are finite in number, and which depend smoothly on *all* elements defining economies, both finite and *infinite* dimensional ones. That theorem then generalizes the main result in Cass, Siconolfi and Villanacci (2001), when differentiably concave utility functions are considered. Theorem 4 shows that there exists an open and nonempty set of nonhomogeneous economies, whose associated equilibria exhibit real indeterminacy, i.e., the set of equilibrium allocations contains a smooth manifold of dimension one. We conclude the section providing a discussion about the conjecture of existence of an open and nonempty set of nonhomogeneous economies exhibiting determinacy of equilibria. The Appendix contains the proofs of the three theorems stated in the previous sections.

[2] A sentence in Cass, Siconolfi and Villanacci (2001) describes the main goal of the present paper. "While it seems likely that our present analysis can be sharpened to incorporate explicit dependence of the constraints on endogenous variables other than just portfolio holdings, this question remains to be more thoroughly investigated."

[3] A reasonable conjecture is that the set of nonhomogenous economies for which the associated equilibrium allocation set is finite contains an open and nonempty set. However, it is hard to prove it or to find a counterexample, as discussed at the end of Section 3.

2 Set-up of the model and existence result

Our model is the by now very standard two-period, pure exchange economy with uncertainty and both commodities and assets. Spot commodity markets open in the first and second periods, and there are $C \geq 2$ types of commodities traded at each spot, denoted by $c \in \mathcal{C} = \{1, 2, ..., C\}$. Asset markets open in just the first period, and there are $A \geq 1$ (inside) assets traded, denoted by $a \in \mathcal{A} = \{1, 2, ..., A\}$. We will also denote spots by $s \in \mathcal{S} = \{0, 1, ..., S\}$, $S \geq 1$, where $s = 0$ corresponds to the first period, today, and $s \geq 1$ the possible states of the world in the second period, tomorrow. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} = \{1, 2, ..., H\}$.

The time line for this model is as follows: today, households exchange commodities and assets, and consumption takes place. Then, tomorrow, uncertainty is resolved, households honor their financial obligations, and they again exchange and then consume commodities.

$x_h^c(s)$ is the consumption of commodity c in state s by household h, with parallel notation for the endowment of commodities, $e_h^c(s)$. Both consumption $x_h = (x_h^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ and endowment $e_h = (e_h^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ are elements of $\mathbb{R}^G_{++}$, where $G = (S+1)C$ is the total number of goods.

Household h's preferences are represented by a utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$. As in most of the literature on smooth economies we will adopt throughout

Assumption u. For all $h \in \mathcal{H}$,

u1. $u_h \in C^2(\mathbb{R}^G_{++})$;

u2. u_h is differentiably strictly increasing, that is, $Du_h(x_h) \gg 0$;

u3. u_h is differentiably strictly concave[4], i.e.,

$$\Delta x \neq 0 \text{ and } \text{ implies } \quad \Delta x^T D^2 u_h(x_h) \Delta x < 0;$$

u4. u_h has upper contour sets closed in the standard topology of $\mathbb{R}^G$, that is, for any $\underline{x} \in \mathbb{R}^G_{++}$, $\left\{x \in \mathbb{R}^G_{++} : u_h(x) \geq u_h(\underline{x})\right\}$ is closed in the topology of $\mathbb{R}^G$.

The set of utility functions satisfying Assumption u is denoted by $\overline{\mathcal{U}}$ and $\mathcal{U} = \overline{\mathcal{U}}^H$.

We will also use the following standard notation: $p^c(s)$ is the price of commodity c at spot s and $p = (p^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ is the corresponding commodity price vector; q^a is the price of asset a and $q = (q^a, a \in \mathcal{A})$ is the corresponding asset price vector; $y^a(s)$ is the yield in state s of asset a in *units of the numeraire commodity*, which, for specificity, we designate to be C, and

$$Y = \begin{bmatrix} y^1(1) & \dots & y^a(1) & \dots & y^A(1) \\ \vdots & & \vdots & & \vdots \\ y^1(s) & \dots & y^a(s) & \dots & y^A(s) \\ \vdots & & \vdots & & \vdots \\ y^1(S) & \dots & y^a(S) & \dots & y^A(S) \end{bmatrix}$$

is the corresponding yield matrix; $y(s) = (y^a(s), a \in \mathcal{A})$ is the vector of asset yields in state s; z_h^a is the quantity of asset a held by household h, $z_h = (z_h^a, a \in \mathcal{A})$ is the corresponding asset portfolio and $z = (z_h, h \in \mathcal{H}) \in \mathbb{R}^{AH}$.

Concerning the financial side of the economy, and consistently with our restricted participation framework, we assume that

- there exists a given set of assets which, in number and kind, may even be sufficient for complete markets,
- each household h has only partial access, in a personalized manner to the available set of assets.

[4]In fact, to show existence of equilibria, it is sufficient to assume that u_h is differentiably strictly quasi-concave, i.e., $\Delta x \neq 0$ and $Du_h(x_h)\Delta x = 0 \Rightarrow \Delta x^T D^2 u_h(x_h) \Delta x < 0$.

In other words, while there may be just a "few" or "many" assets, the market imperfection we consider is not incompleteness of numbers of assets, but rather restrictions on households' opportunities for transacting in assets.
It greatly simplifies our analysis (but, for the reason just mentioned, is not without loss of generality) to assume that

Assumption Y. rank $Y = A \leq S$.

Let $\mathcal{Y}$ be the set of $S \times A$ matrices satisfying the above assumption.

There are $J \geq 1$ potential participation constraints for each household. Let $\mathcal{J} = \{1, ..., J\}$ with generic element j. Then, the restriction function for household h is

$$\begin{aligned} r_h &: \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^J \\ (z_h, p, q) &\mapsto \left(r^j_h (z_h, p, q), j \in \mathcal{J} \right). \end{aligned}$$

For each nonempty subset $\mathcal{J}_h \subseteq \mathcal{J}$, denote its cardinality by J_h, and let

$$\begin{aligned} r^{\mathcal{J}_h}_h &: \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^{J_h} \\ (z_h, p, q) &\mapsto \left(r^j_h (z_h, p, q), j \in \mathcal{J}_h \right). \end{aligned}$$

We now introduce assumptions on restriction functions.

Assumption r.

r1. For all $h \in \mathcal{H}$, r_h is $C^2(\mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A; \mathbb{R}^J)$;

r2. For all $h \in \mathcal{H}$, $j \in \mathcal{J}$, $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, r^j_h is quasi-concave in z_h;

r3. For all $h \in \mathcal{H}$, $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $r_h (0, p, q) \geq 0$;

r4. For all $h \in \mathcal{H}, (z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A$, $\mathcal{J}_h \subseteq \mathcal{J}$ such that $\mathcal{J}_h \neq \varnothing$,

$$r^{\mathcal{J}_h}_h (z_h, p, q) = 0 \quad \Rightarrow \quad \text{rank } D_{z_h} r^{\mathcal{J}_h}_h (z_h, p, q) = J_h;$$

r5. For all $a \in \mathcal{A}$, there exists $h \in \mathcal{H}$ such that, for every $(z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A$,

$$D_{z^a_h} r_h(z_h, p, q) = 0.$$

A word about each of these assumptions is in order. Assumption r1 allows to employ differential techniques, Assumptions r2 assures that the portfolio set of each household is convex and Assumption r3 permits no participation on the asset market. Assumption r4 implies that, for each household, changes of the asset portfolio have a nontrivial effect on the possibility of accessing the financial market along the boundary of the portfolio set. Finally Assumption r5 requires that, for each asset, there exists a household whose participation restrictions do not depend on that asset demand. Note also that Assumptions r2 and r4 are needed for characterizing the solutions of household's maximization problems in terms of Kuhn-Tucker conditions. Moreover Assumptions r4 and r5 allow to get the desired property of the homotopy function used to prove existence of equilibria and they are also crucial in rank computations related to generic determinacy and indeterminacy of equilibria.

Let $\mathcal{R}$ be the set of restriction functions satisfying Assumptions r1-r5 above, with generic element $r = (r_h)^H_{h=1}$. An economy is $E = (e, u, Y, r) \in \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{R} = \mathcal{E}$.

For given $(p, q, E) \in \mathbb{R}^G_{++} \times \mathbb{R}^A \times \mathcal{E}$, household $h \in \mathcal{H}$ maximization problem is as follows.

Problem (Ph)

$$\max_{(x_h,z_h)} \; u_h(x_h) \quad s.t.$$

$$\begin{aligned}
&p(0)\,x_h(0) + qz_h \le p(0)\,e_h(0)\\
&p(s)\,x_h(s) - p^C(s)\,y(s)\,z_h \le p(s)\,e_h(s) \quad s \in \{1,...,S\}\\
&r_h(z_h,p,q) \ge 0
\end{aligned} \tag{1}$$

Denote the constraint set of the above problem by $C_h(p,q,E)$.

Observe that normalizations of spot by spot prices are not possible because of the dependence of the restriction functions on (p,q). In fact, nominal changes of prices may in general affect the constraint set of some household's maximization problem. Therefore the appropriate definition of equilibrium is as follows.

Definition 1 $\left((x_h,z_h)_{h\in\mathcal{H}},p,q\right) \in \left(\mathbb{R}^G_{++}\times\mathbb{R}^A\right)^H \times \mathbb{R}^G_{++}\times\mathbb{R}^A = \Theta$ *is an equilibrium for the economy $E \in \mathcal{E}$ if for each h, (x_h,z_h) solves Problem (Ph) at (p,q,E) and (x,z) solves market clearing conditions at e*

$$\begin{aligned}
&\sum_{h=1}^{H}(x_h - e_h) = 0,\\
&\sum_{h=1}^{H} z_h = 0.
\end{aligned} \tag{2}$$

The set of equilibria and the set of equilibrium allocations for economy E are denoted by $\Theta(E)$ and $X(E)$, respectively. Moreover, for every $E \in \mathcal{E}$, we define the set of the *normalized equilibria* as

$$\Theta_n(E) = \left\{\left((x_h,z_h)_{h\in\mathcal{H}},p,q\right) \in \Theta(E) : \forall s \in \mathcal{S}, p^C(s) = 1\right\},$$

and the associated set of equilibrium allocations as $X_n(E)$.

An indispensable preliminary result in every general equilibrium model is existence of equilibria.[5]

Theorem 2 *For every $E \in \mathcal{E}$, $\Theta_n(E) \neq \varnothing$.*

As discussed in the Introduction, the possibility of normalizing prices is linked to a crucial property of restriction functions. In fact, they may or may not exhibit a homogeneity property. Throughout the paper we will consider the following general form of homogeneity.

Assumption r6.[6] r is such that for all $h \in \mathcal{H}$, $(p,q,e,u,Y) \in \mathbb{R}^G_{++}\times\mathbb{R}^A\times\mathbb{R}^{GH}_{++}\times\mathcal{U}\times\mathcal{Y}$ and $\delta \in \mathbb{R}^{S+1}_{++}$,

$$C_h(p,q,e,u,Y,r) = C_h(\delta\Box p, \delta(0)\,q, e,u,Y,r), \tag{3}$$

where $\delta\Box p = \left(\delta(s)\,p(s) \in \mathbb{R}^C_{++},\ s \in \mathcal{S}\right)$.

In the following we denote by $\mathcal{R}_{ho}$ the subset of $\mathcal{R}$ whose elements satisfy Assumption r6, as well. We call them homogeneous restriction functions. $\mathcal{E}_{ho} = \mathbb{R}^{GH}_{++}\times\mathcal{U}\times\mathcal{Y}\times\mathcal{R}_{ho}$ is the set of homogeneous economies and $\mathcal{R}_{nh} = \mathcal{R}\backslash\mathcal{R}_{ho}$ and $\mathcal{E}_{nh} = \mathbb{R}^{GH}_{++}\times\mathcal{U}\times\mathcal{Y}\times\mathcal{R}_{nh}$ are the sets of nonhomogeneous restriction functions and economies, respectively. Before proceeding in our analysis, we provide some

[5] Proofs of all theorems are deferred to the Appendix.

[6] Assumption r6 is strictly more general than any of the following two, and in fact, Assumption r6' is strictly more general than r6".

r6'. r is such that for all $h \in \mathcal{H}$, $(p,q) \in \mathbb{R}^G_{++}\times\mathbb{R}^A$ and $\delta \in \mathbb{R}^{S+1}_{++}$,

$$\{z_h \in \mathbb{R}^A : r_h(z_h,p,q)\} = \{z_h \in \mathbb{R}^A : r_h(z_h,\delta\Box p,\delta(0)q)\}.$$

r6". r is such that for all $h \in \mathcal{H}$, $(z_h,p,q) \in \mathbb{R}^A\times\mathbb{R}^G_{++}\times\mathbb{R}^A$ and $\delta \in \mathbb{R}^{S+1}_{++}$,

$$r_h(z_h,\delta\Box p,\delta(0)\,q) = r_h(z_h,p,q).$$

examples of exogenous and endogenous restriction functions. We informally describe household h's restriction function through inequalities.[7]

Impossibility of short sales

$$z_h^a \geq 0 \tag{4}$$

and constraints on borrowing

$$z_h^a \geq -m \tag{5}$$

with $m > 0$, are clearly exogenous. Observe that some apparently exogenous restriction functions like (5) can be easily thought as endogenous. In fact m could be assumed to be fixed today and in a near future, but it would change in a quite different economic situation (described in terms of prices and fundamentals). All the examples presented below describe economies with restrictions explicitly depending on prices.

Limits on the fraction of the portfolio that could be invested in some "markets" can be written as

$$\sum_{a' \in \mathcal{A}'} q^{a'} z_h^{a'} \leq \alpha \sum_{a \notin \mathcal{A}'} q^a z_h^a + m \tag{6}$$

where $\mathcal{A}' \subseteq \mathcal{A},\ \alpha \in (0,1)$ and $m \geq 0$ is a sort of minimal amount of "anyway admissible investment". The restriction function does not satisfy (3) if and only if $m > 0$.

Constraints on borrowing in the first period and on the amount of future obligations can be described by the following inequalities,

$$\begin{aligned} -qz_h &\leq \alpha(0)\ p(0)\, e_h(0) \\ -p^C(s)\, y(s)\, z_h &\leq \alpha(s)\ p(s)\, e_h(s) \quad s \in \{1, .., S\} \end{aligned} \tag{7}$$

where $(\alpha(s))_{s \in \mathcal{S}} \in (0,1)^{S+1}$. Inequalities in (7) simply say that amount borrowed today and debts tomorrow cannot exceed a proportion of individual wealth in the corresponding spot. The restriction function resulting from (7) is clearly nonhomogeneous.

Other simple examples with straightforward interpretation are given by

$$-qz_h \leq f\Big(p(0)\, e_h(0)\Big)$$

$$-qz_h \leq g\Big(p(0)\, e_h(0), \ldots, p(S)\, e_h(S)\Big)$$

where $f : \mathbb{R}_{++} \to (0,1)$ and $g : \mathbb{R}_{++}^{S+1} \to (0,1)$ are smooth, and smooth versions of

$$-qz_h \leq \min\Big\{\alpha_s\ p(s)\, e_h(s) : s = 0, \ldots, S\Big\}$$

$$\max\left\{-qz_h, -p^C(0)\, y(0)\, z_h, \ldots, -p^C(S)\, y(S)\, z_h\right\} \leq \min\Big\{\alpha_s\ p(s)\, e_h(s) : s = 0, \ldots, S\Big\}.$$

Summarizing to get nonhomogeneity, it suffices that either a restriction function nontrivially depends on different spot prices or it is nonlinear in any of them. In general, nonhomogeneous constraints are more common if there are forms of imperfect price indexation of the "rules" defining the extent of access to markets.

3 Determinacy and indeterminacy of equilibria

In this section, we provide two results: generic determinacy of normalized equilibria for homogenous economies and a robust example of indeterminacy for nonhomogeneous economies. Then, we briefly discuss how determinacy of equilibria is hard to be shown in the nonhomogeneous case.

For every $\delta = (\delta(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{S+1}$, $r \in \mathcal{R}$ and $E = (e, u, Y, r) \in \mathcal{E}$, define $r^\delta \in \mathcal{R}$ as

[7] Since Assumptions r1-r6 deal with the all vector $(r_h)_{h \in \mathcal{H}}$, we assume that restriction functions of not explicitly mentioned households are chosen in order to satisfy the needed requirements.

$$r_h^\delta(z_h, p, q) = r_h(z_h, \delta \Box p, \delta(0) q) \quad \forall h \in \mathcal{H},$$

and $E^\delta = (e, u, Y, r^\delta) \in \mathcal{E}$. Defined $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^{S+1}$, of course, $E^{\mathbf{1}} = E$. Observe that, for every $E \in \mathcal{E}$, we have

$$\Theta(E) = \bigcup_{\delta \in \mathbb{R}^{S+1}_{++}} \left\{ \left((x_h, z_h)_{h \in \mathcal{H}}, \delta \Box p, \delta(0) q \right) \in \Theta : \left((x_h, z_h)_{h \in \mathcal{H}}, p, q \right) \in \Theta_n(E^\delta) \right\} \tag{8}$$

and

$$X(E) = \bigcup_{\delta \in \mathbb{R}^{S+1}_{++}} X_n(E^\delta). \tag{9}$$

For every $E \in \mathcal{E}_{ho}$, we have that

$$\Theta(E) = \bigcup_{\delta \in \mathbb{R}^{S+1}_{++}} \left\{ \left((x_h, z_h)_{h \in \mathcal{H}}, \delta \Box p, \delta(0) q \right) \in \Theta : \left((x_h, z_h)_{h \in \mathcal{H}}, p, q \right) \in \Theta_n(E) \right\} \tag{10}$$

and

$$X(E) = X_n(E). \tag{11}$$

Roughly speaking, equalities (9) and (11) say what follows. For arbitrary economy E, equilibrium allocations can be found looking at normalized equilibrium allocations for E^δ, for all possible value of δ. For homogeneous economies, equilibrium allocations are normalized equilibrium allocations. What the above observations suggest and which is investigated in the paper is if for each $E \in \mathcal{E}_{nh}$

$$\delta_1 \neq \delta_2 \quad \Rightarrow \quad X_n(E^{\delta_1}) \neq X_n(E^{\delta_2}),$$

and then $X(E)$ contains an infinite number of equilibrium allocations.

In fact, we are going to show that typically in the space of homogeneous economies, associated normalized equilibria are finite (and smoothly depend on economies) - see Theorem 3 below. On the other hand, there exist robust examples of nonhomogeneous economies for which equilibrium allocations exhibit indeterminacy - see Theorem 4 below. The intuition of both results can be given in terms of a simple count of significant equations and variables in the two cases.

In the homogeneous case, $S+1$ Walras' laws hold and $S+1$ price normalizations are possible. Therefore, the number of significant equations (market clearing conditions) and the number of significant variables $((p^c(s), \ s \in \mathcal{S}, c \neq C), q)$ are the same and equal to $G + A - (S+1)$.

In the general case price normalizations are not possible. Since Walras' laws still hold, there are $S+1$ extra price variables, which can potentially cause real indeterminacy. In fact, rescaling of spot prices changes households' constraint sets, via the restriction constraints. That change *may* affect households' demands and therefore equilibrium prices and allocations, as long as those constraints are effectively binding for at least some households. It is then natural to conjecture that nontrivial effects on households' demand do arise at least for some economies.

We are now ready to state our main results.

Consider the Hausdorff topological vector space

$$T = \mathbb{R}^{GH}_{++} \times \left[C^2(\mathbb{R}^G_{++}) \right]^H \times \mathcal{M}_{S \times A} \times \left[C^2(\mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A; \mathbb{R}^J) \right]^H, \tag{12}$$

endowed with the product topology of the natural topologies on the spaces $\mathbb{R}^{GH}_{++}$, $\left[C^2(\mathbb{R}^G_{++}) \right]^H$, $\mathcal{M}_{S \times A}$, i.e., the space of $S \times A$ matrices and $\left[C^2(\mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A; \mathbb{R}^J) \right]^H$. In the following, we endow $\mathcal{E}$, $\mathcal{E}_{ho} \subseteq T$ with the topologies induced by T.

Theorem 3 *There exists an open and dense set $\mathcal{D} \subseteq \mathcal{E}_{ho}$ such that, for every $E \in \mathcal{D}$,*

$$\Theta_n(E) = \{\theta_i\}_{i=1}^k, \tag{13}$$

where $k \in \mathbb{N}$ and depends on E. Moreover, there exist an open neighborhood $V(E) \subseteq \mathcal{E}_{ho}$ of E and, for every $i \in \{1, \dots, k\}$, an open neighborhood $O(\theta_i) \subseteq \Theta$ of θ_i and $g_i : V(E) \to O(\theta_i)$ such that:

1. g_i *is* C^1, $g_i(E) = \theta_i$, *and* $O(\theta_i) \cap O(\theta_j) = \varnothing$ *if* $i \neq j$,

2. $\{(E,\theta) \in V(E) \times O(\theta_i) : \theta \in \Theta_n(E)\} =$ graph g_i,

3. $\{(E,\theta) \in V(E) \times \Theta : \theta \in \Theta_n(E)\} = \bigcup_{i=1}^{k}$ graph g_i.

Theorem 4 *There exists an open and nonempty set* $\mathcal{O} \subseteq \mathcal{E}$ *such that, for every* $E \in \mathcal{O}$, $X(E)$ *contains a* C^1*-manifold of dimension* 1. *In particular* $\mathcal{O} \subseteq \mathcal{E}_{nh}$.

We conclude the section with a discussion about determinacy of equilibria for nonhomogeneous economies. A reasonable conjecture is that the set of those economies for which the associated equilibrium allocation set is finite contains an open and nonempty set of $\mathcal{E}$. An attempt to construct such a set could be the following one.

Consider the incomplete market model and, to simplify matters, an economy for which there exists a unique equilibrium. Add restriction functions "far away from that equilibrium". Then the initial equilibrium will not be affected by those "insignificant" restrictions. Using regularity would then allow to get the desired openness result.

The problem with the argument above is that while the added restriction do not disturb the equilibrium under consideration, they may create other equilibria. A simple Edgeworth-Bowley box diagram can illustrate that situation in the case of a standard two households, two goods exchange economy. Take an arbitrary equilibrium (p^*, x^*). Then add to household 1's maximization problem the "insignificant" constraint

$$x_1^1 \leq \alpha \tag{14}$$

with $\alpha > x_1^{1*}$. In this new model, the old equilibrium survives, but as illustrated in the picture below, the new different equilibrium (p^{**}, x^{**}) arises: $x_1^{1**} = \alpha \neq x_1^{1*}$. Observe also that the new equilibrium is not Pareto Optimal and therefore it is different from any equilibrium of the model without the added constraint (14).

The above observation can be rephrased in terms of a simple "count of equations and unknowns" approach. Take the system defining equilibria in an incomplete market economy. Adding a restriction function r_1^1 to household 1's constraints means adding a "new" variable μ_1^1, i.e., the multiplier associated with r_1^1, and a "Kuhn-Tucker equation" of the type $\min\left\{r_1^1, \mu_1^1\right\} = 0$. Observe that μ_1^1 appears in other "old" equations, and "old" variables p, q are arguments of r_1^1. Clearly, adding an equation and a variable to a given system may change the solution set in an arbitrary manner.

4 Appendix

In the present appendix, we introduce some preliminary definitions and then prove the three theorem stated in the previous sections.

First of all observe that in Definition 1 of equilibrium, $S+1$ Walras' laws do hold. It is then useful to define $x_h^{\setminus}(s) = (x_h^c(s), c \neq C)$, $x_h^{\setminus} = \left(x_h^{\setminus}(s), s \in \mathcal{S}\right)$ and similarly $e_h^{\setminus}(s) = (e_h^c(s), c \neq C)$, $e_h^{\setminus} = \left(e_h^{\setminus}(s), s \in \mathcal{S}\right)$. Then, the significant market clearing conditions at e are in fact

$$\begin{array}{l} \sum_{h=1}^{H} \left(x_h^{\setminus} - e_h^{\setminus}\right) = 0, \\ \sum_{h=1}^{H} z_h = 0. \end{array} \tag{15}$$

Moreover define

$$p^{\setminus}(s) = (p^c(s), c \neq C), \quad p^{\setminus} = \left(p^{\setminus}(s), s \in \mathcal{S}\right),$$

$$\overline{p}(s) = \left(p^{\setminus}(s), 1\right), \quad \overline{p} = (\overline{p}(s), s \in \mathcal{S}).$$

We are going to study normalized equilibria in terms of the system of equations of Kuhn-Tucker conditions associated with households' maximization problems, and market clearing conditions. Define then

$$\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{AH} \times \mathbb{R}_{++}^{(S+1)H} \times \mathbb{R}^{JH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{A}$$

with generic element

$$\xi = \left((x_h, z_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^{\backslash}, q \right) = \left(x, z, \lambda, \mu, p^{\backslash}, q \right).$$

Consider then $E \in \mathcal{E}$ and $\delta \in \mathbb{R}_{++}^{S+1}$. It is immediate to prove that if

$$\left((x_h, z_h)_{h \in \mathcal{H}}, \overline{p}, q \right) \in \Theta_n(E^{\delta})$$

then there exists $(\lambda_h, \mu_h)_{h \in \mathcal{H}} = (\lambda, \mu) \in \mathbb{R}_{++}^{(S+1)H} \times \mathbb{R}^{JH}$ such that

$$\xi = \left((x_h, z_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^{\backslash}, q \right)$$

solves the system $F_{\Delta}(\xi, \delta, E) = 0$ where

$$F_{\Delta} : \Xi \times \mathbb{R}_{++}^{S+1} \times \mathcal{E} \to \mathbb{R}^{\dim \Xi},$$

$$F_{\Delta} \left(x, z, \lambda, \mu, p^{\backslash}, q, \delta, E \right) =$$

$$\left[\begin{array}{ll} \underset{h \in \mathcal{H}, s \in \mathcal{S}}{(h.1.s)} & D_{x_h(s)} u_h(x_h) - \lambda_h(s) \overline{p}(s) \\ \underset{h \in \mathcal{H}}{(h.2.0)} & -\overline{p}(0)(x_h(0) - e_h(0)) - q z_h \\ \underset{h \in \mathcal{H}, s \in \mathcal{S} \backslash \{0\}}{(h.2.s)} & -\overline{p}(s)(x_h(s) - e_h(s)) + y(s) z_h \\ \underset{h \in \mathcal{H}, a \in \mathcal{A}}{(h.3.a)} & -\lambda_h(0) q^a + \sum_{s=1}^{S} \lambda_h(s) y^a(s) + \sum_{j=1}^{J} \mu_h^j D_{z_h^a} r_h^j (z_h, \delta \square \overline{p}, \delta(0) q) \\ \underset{h \in \mathcal{H}, j \in \mathcal{J}}{(h.4.j)} & \min \left\{ \mu_h^j, r_h^j (z_h, \delta \square \overline{p}, \delta(0) q) \right\} \\ (M.x) & \sum_{h=1}^{H} \left(x_h^{\backslash} - e_h^{\backslash} \right) \\ (M.z) & \sum_{h=1}^{H} z_h \end{array} \right] \tag{16}$$

Moreover, if

$$\xi = \left((x_h, z_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^{\backslash}, q \right)$$

solves the system $F_{\Delta}(\xi, \delta, E) = 0$, then

$$\left((x_h, z_h)_{h \in \mathcal{H}}, \overline{p}, q \right) \in \Theta_n(E^{\delta}).$$

Finally, define $F(\xi, E) = F_{\Delta}(\xi, \mathbf{1}, E)$.

4.1 Existence of equilibria

Theorem 2 is a consequence of the following fact.

Theorem 5 *For every $E \in \mathcal{E}$, there exists $\xi \in \Xi$ such that $F(\xi, E) = 0$.*

Theorem 5 is proved applying the following well known result[8].

[8] A proof of Theorem 6 can be found, for instance, in Villanacci et al. (2002), Chapter 7.

Theorem 6 *Let M and N be two C^2 boundaryless manifolds of the same dimension, $y \in N$ and $F, G : M \to N$ be continuous functions. Assume that G is C^1 in an open neighborhood U of $G^{-1}(y)$, y is a regular value for G restricted to U, $\#G^{-1}(y)$ is odd and there exists a continuous homotopy $H : M \times [0,1] \to N$ from F to G such that $H^{-1}(y)$ is compact. Then $F^{-1}(y) \neq \varnothing$.*

Proof of Theorem 5. [9]Observe preliminarily that it is possible to prove that for every $h \in \mathcal{H}$ there exists a continuous function $\widetilde{z}_h : \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^A$ such that, for every $(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $r_h(\widetilde{z}_h(p,q), p, q) \gg 0$. Let $E = (e, u, Y, r) \in \mathcal{E}$ be fixed. Then it is well known that there exists a Pareto optimal allocation x^* for u such that $\sum_{h=1}^{H} x^*_h = \sum_{h=1}^{H} e_h$. Define

$$F(\xi) = F(\xi, E), \quad \forall\, \xi \in \Xi,$$

and consider the system in the unknowns $\xi = (x, \lambda, z, \mu, p^\setminus, q) \in \Xi$ and $\tau \in [0,1]$, given by

$$\begin{cases} \underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)} u_h(x_h) - \lambda_h(s)\,\overline{p}(s) & = 0 \\ \underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\left(x_h(0) - ((1-\tau)\,e_h(0) + \tau x^*_h(0))\right) - q z_h & = 0 \\ \underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}(s)\left(x_h(s) - ((1-\tau)\,e_h(s) + \tau x^*_h(s))\right) + y(s)\, z_h & = 0 \\ \underset{h\in\mathcal{H},a\in\mathcal{A}}{(h.3.a)} & -\lambda_h(0)\, q^a + \sum_{s=1}^{S} \lambda_h(s)\, y^a(s) + \sum_{j=1}^{J} \mu^j_h (1-\tau)\, D_{z^a_h} r^j_h\left((1-\tau)\, z_h + \tau \widetilde{z}_h(\overline{p}, q), \overline{p}, q\right) & = 0 \\ \underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \min\left\{\mu^j_h, r^j_h\left((1-\tau)\, z_h + \tau\widetilde{z}_h(\overline{p}, q), \overline{p}, q\right)\right\} & = 0 \\ (M.x) & \sum_{h=1}^{H}\left(x^\setminus_h - \left((1-\tau)\, e^\setminus_h + \tau x^{*\setminus}_h\right)\right) & = 0 \\ (M.z) & \sum_{h=1}^{H} z_h & = 0 \end{cases} \tag{17}$$

Define now

$$H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$$

$$(\xi, \tau) \mapsto \text{ left hand side of system (17)},$$

and

$$G : \Xi \to \mathbb{R}^{\dim \Xi}, \;\; \xi \mapsto H(\xi, 1).$$

Observe that

$$H(\xi, 0) = F(\xi).$$

It is then possible to verify that all assumptions of Theorem 6 are satisfied. ■

Both in the endogenous and exogenous settings, Assumptions r1-r4, consistently defined, rule out economies where some households are excluded from the trade of some assets. Roughly speaking, $z^a_h = 0$, written as $z^a_h \geq 0$ and $-z^a_h \geq 0$, violates Assumption r4[10]. On the other hand, the strategy of proof we follow can easily accommodate those kinds of restrictions[11].

4.2 Generic regularity of normalized equilibria for homogeneous economies

The goal of this section is to prove that there exists an open and dense subset of homogeneous economies for which associated normalized equilibria are finite in number and depends smoothly on the economies themselves, a so-called generic regularity result. Two main difficulties arise to accomplish the desired result. The presence of nonnegative constraints implies the function describing the equilibria is not C^1. Using a now standard argument due to Cass, Siconolfi and

[9]See Carosi, Gori and Villanacci (2009) for a detailed proof of Theorem 5.

[10]More precisely, if Assumptions r1-r4 are satisfied, the admissible portfolio set has nonempty interior, as it follows from the property of $\widetilde{z}$ described in the proof of above Theorem 5.

[11]See Carosi, Gori and Villanacci (2009).

Villanacci (2001) that problem is solved. Moreover, we do need a *smooth* dependence of equilibrium variables on *all* elements defining economies, both finite and *infinite* dimensional ones. A general implicit function theorem allows to get the desired result and it is presented below.

Mas Colell (1986), among others, shows that there exists a *continuous* dependence of equilibria on economies. In the paper, we present a general and relatively simple method to address the analysis of smooth dependence. We use a version of the implicit function theorem presented by Glöckner (2006) and described below. Once the generic regularity result is obtained for the finite dimensional part of the economy space, Glöckner's Theorem provides the desired result for the whole space if it is shown that the equilibrium function is "smooth". As described in Lemma 9, verifying that smoothness condition is straightforward if the equilibrium function is the left hand side of the so called "extended" system of Kuhn-Tucker and market clearing conditions. The above described strategy of proof can be easily applied to a variety of general equilibrium models. The proof of Theorem 3 goes through the following steps: brief description of the needed version of the implicit function theorem; construction of two suitable open and dense sets $\mathcal{D}_1$ and $\mathcal{D}_2$; proof that $\mathcal{D}_1 \cap \mathcal{D}_2$ is the desired set.

Let T be a topological Hausdorff vector space, $V \subseteq T$ be an open set and $f : V \to \mathbb{R}^n$. We say $f \in C^0(V; \mathbb{R}^n)$ if f is continuous while $f \in C^1(V; \mathbb{R}^n)$ if it is continuous, there exists the limit

$$df(v, w) = \lim_{t \to 0} \frac{f(v + tw) - f(v)}{t}, \quad \forall v \in V, w \in T,$$

and the function $df : V \times T \to \mathbb{R}^n$ is continuous.
Given now *any* (not necessarily open) set $X \subseteq T$, and $f : X \to \mathbb{R}^n$, we say $f \in C^0(X, \mathbb{R}^n)$ if f is continuous with respect to the topology on X induced by T, while, as in the finite dimensional setting, $f \in C^1(X; \mathbb{R}^n)$ if for all $v_0 \in X$ there exists an open neighborhood of v_0 in T, say $V(v_0)$, and a function $\overline{f} : V(v_0) \to \mathbb{R}^n$ such that $\overline{f} \in C^1(V(v_0); \mathbb{R}^n)$ and $f(x) = \overline{f}(x)$, for all $v \in V(v_0) \cap X$. Those definitions allow to state the following implicit function theorem which is a simplified version of Theorem 2.3 in Glöckner (2006).

Theorem 7 *Let us consider $f : O \times V \to \mathbb{R}^n$, where O is an open subset of $\mathbb{R}^n$ and V is an open subset of a topological Hausdorff vector space T. Assume $f \in C^1(O \times V; \mathbb{R}^n)$ and let $(x_0, v_0) \in O \times V$ such that $f(x_0, v_0) = 0$ and*

$$df((x_0, v_0), (\cdot, 0)) : \mathbb{R}^n \to \mathbb{R}^n, \quad w \longmapsto \lim_{t \to 0} \frac{f((x_0, v_0) + t(w, 0)) - f(x_0, v_0)}{t}$$

is invertible. Then there exist $O(x_0) \subseteq O$ open neighborhood of x_0, $V(v_0) \subseteq V$ open neighborhood of v_0 and $g : V(v_0) \to O(x_0)$ such that

1. $g \in C^1(V(v_0); O(x_0))$,
2. $g(v_0) = x_0$,
3. $\{(x, v) \in O(x_0) \times V(v_0) : f(x, v) = 0\} = \{(x, v) \in O(x_0) \times V(v_0) : x = g(v)\}$.

Note that if $f \in C^1(O \times V; \mathbb{R}^n)$ then, for all $v \in V$, $f_v : O \to \mathbb{R}^n$ defined as $f_v(x) = f(x, v)$ belongs to $C^1(O, \mathbb{R}^n)$. Moreover, since O is an open set of a euclidean space, $df((x, v), (\cdot, 0))$ can be identified with the Jacobian matrix $D_x(f_v)(x)$. Then in Theorem 7 the condition

$$df((x_0, v_0), (\cdot, 0)) : \mathbb{R}^n \to \mathbb{R}^n \text{ is invertible,}$$

is equivalent to require the $n \times n$ matrix $D_x(f_{v_0})(x_0)$ to be nonsingular.

Lemma 8 *The set*

$$\mathcal{D}_1 = \left\{ E \in \mathcal{E}_{ho} : \forall \xi \in (F_E)^{-1}(0), \forall h \in \mathcal{H}, \forall j \in \mathcal{J} \text{ either } \mu_h^j > 0 \text{ or } r_h^j(z_h, \overline{p}, q) > 0 \right\}$$

is open and dense in $\mathcal{E}_{ho}$.

Proof. First of all observe that F is continuous on $\Xi \times \mathcal{E}_{ho}$ and

$$\text{proj} : F^{-1}(0) \to \mathcal{E}_{ho}, \quad (\xi, E) \longmapsto E \tag{18}$$

is proper. Then openness follows. Density is showed in the remaining part of the proof.

It is sufficient to show that for all $(u, Y, r) \in \mathcal{U} \times \mathcal{Y} \times \mathcal{R}_{ho}$ the set

$$\mathcal{D}_1(u, Y, r) = \left\{ e \in \mathbb{R}^{GH}_{++} : (e, u, Y, r) \in \mathcal{D}_1 \right\}$$

is dense in $\mathbb{R}^{GH}_{++}$. Fix (u, Y, r) and define

$$G : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim \Xi},$$

as

$$G(\xi, e) = F(\xi, e, u, Y, r).$$

Given now $(\xi, e) \in G^{-1}(0)$, for all $h \in \mathcal{H}$, we consider the following partition of $\mathcal{J}$:

$$\begin{aligned} \mathcal{J}^1_h(\xi, e) &= \{ j \in \mathcal{J} : r^j_h(z_h, \overline{p}, q) > 0, \mu^j_h = 0 \}, \\ \mathcal{J}^2_h(\xi, e) &= \{ j \in \mathcal{J} : r^j_h(z_h, \overline{p}, q) = 0, \mu^j_h > 0 \}, \\ \mathcal{J}^3_h(\xi, e) &= \{ j \in \mathcal{J} : r^j_h(z_h, \overline{p}, q) = 0, \mu^j_h = 0 \}. \end{aligned} \tag{19}$$

Obviously

$$\mathcal{D}_1(u, Y, r) = \left\{ e \in \mathbb{R}^{GH}_{++} : \forall \xi \in (G_e)^{-1}(0) \text{ and } \forall h \in \mathcal{H},\ \mathcal{J}^3_h(\xi, e) = \varnothing \right\}.$$

Let us call $\mathfrak{Q}_h$ the family of all possible *tri-partitions* $\mathbf{Q}_h = \left\{ \mathcal{Q}^1_h, \mathcal{Q}^2_h, \mathcal{Q}^3_h \right\}$ of the set $\mathcal{J}$ and $Q^i_h = \# \mathcal{Q}^i_h$ for $i \in \{1, 2, 3\}$. Define then $\mathfrak{Q} = \times_{h \in \mathcal{H}} \mathfrak{Q}_h$, with generic element $\mathbf{Q} = (\mathbf{Q}_h, h \in \mathcal{H})$, and

$$\mathfrak{Q}^* = \{ \mathbf{Q} \in \mathfrak{Q} : \exists h \in \mathcal{H} \text{ such that } \mathcal{Q}^3_h \neq \varnothing \}.$$

Fixed $\mathbf{Q} \in \mathfrak{Q}^*$, define

$$G^{\mathbf{Q}} : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim \Xi + k(\mathbf{Q})},$$

where $k(\mathbf{Q}) = \sum_{h \in \mathcal{H}} Q^3_h > 0$, as

$$G^{\mathbf{Q}}(x, \lambda, z, \mu, p^{\setminus}, q, e) =$$

$$\left[\begin{array}{ll} \underset{h \in \mathcal{H}}{(h.1)} & D_{x_h} u_h(x_h) - \lambda_h \Phi(\overline{p}) \\ \underset{h \in \mathcal{H}}{(h.2)} & -\Phi(\overline{p})(x_h - e_h) + \left[\begin{array}{c} -q \\ Y \end{array} \right] z_h \\ \underset{h \in \mathcal{H}}{(h.3)} & \lambda_h \left[\begin{array}{c} -q \\ Y \end{array} \right] + \mu_h D_{z_h} r_h(z_h, \overline{p}, q) \\ \underset{h \in \mathcal{H}, j \in \mathcal{Q}^1_h \cup \mathcal{Q}^3_h}{(h.4.j)} & \mu^j_h \\ \underset{h \in \mathcal{H}, j \in \mathcal{Q}^2_h \cup \mathcal{Q}^3_h}{(h.5.j)} & r^j_h(z_h, \overline{p}, q) \\ (M.x) & \sum_{h=1}^{H} (x^{\setminus}_h - e^{\setminus}_h) \\ (M.z) & \sum_{h=1}^{H} z_h \end{array} \right]$$

We are going to show that for every $\mathbf{Q} \in \mathfrak{Q}^*$, 0 is a regular value for $G^{\mathbf{Q}}$ - see below. If that is the case from the transversality theorem (see, for example, Hirsch (1976), Theorem 2.7, page 79) the set

$$\mathcal{E}^{bl\mathbf{Q}}(u, Y, r) = \left\{ e \in \mathbb{R}^{GH}_{++} : \exists \xi \in \left(G^{\mathbf{Q}}_e \right)^{-1}(0) \right\}$$

has zero measure. Defining then

$$\mathcal{E}^{bl}(u,Y,r)=\left\{e\in\mathbb{R}^{GH}_{++}:\exists\xi\in\ (G_e)^{-1}(0)\ \text{such that}\ \exists h\ \text{such}\ \mathcal{J}^3_h(\xi,e)\neq\varnothing\right\}$$

we have

$$\mathcal{E}^{bl}(u,Y,r)\subseteq\cup_{\mathbf{Q}\in\mathfrak{Q}^*}\mathcal{E}^{bl\mathbf{Q}}(u,Y,r).$$

The inclusion holds because given $e\in\mathcal{E}^{bl}(u,Y,r)$ and $\xi\in\ (G_e)^{-1}(0)$, it suffices to take $\mathbf{Q}'=\left(\mathcal{J}^i_h(\xi,e),i=1,2,3\text{ and }h\in\mathcal{H}\right)\in\mathfrak{Q}$ to get $G^{\mathbf{Q}'}(\xi,e)=0$.[12] Then

$$\mathcal{D}_1(u,Y,r)=\mathbb{R}^{GH}_{++}\backslash\mathcal{E}^{bl}(u,Y,r)$$

and since $\mathcal{E}^{bl}(u,Y,r)$ has measure zero $\mathcal{D}_1(u,Y,r)$ has full measure in $\mathbb{R}^{GH}_{++}$ and then, in particular, it is dense.

Let us finally prove that, for all $\mathbf{Q}\in\mathfrak{Q}^*$, 0 is a regular value for $G^{\mathbf{Q}}$. Observe that from Assumption r4, if there exists h such that $Q^2_h\cup Q^3_h>A$ then $\left(G^{\mathbf{Q}}\right)^{-1}(0)=\varnothing$. Therefore in what follows we assume that for all $h\in\mathcal{H}$ it is $Q^2_h\cup Q^3_h\leq A$. By reordering the equations of the system $G^{\mathbf{Q}}(\xi,e)=0$ to simplify our argument, we have

$$\left\{\begin{array}{lll}
\underset{h\in\mathcal{H}}{(h.1)} & D_{x_h}u_h(x_h)-\lambda_h\Phi(\overline{p}) & =0\\
\underset{h\in\mathcal{H}}{(h.2)} & -\Phi(\overline{p})(x_h-e_h)+\left[\begin{smallmatrix}-q\\Y\end{smallmatrix}\right]z_h & =0\\
\underset{h\in\mathcal{H}}{(h.3)} & \lambda_h\left[\begin{smallmatrix}-q\\Y\end{smallmatrix}\right]+\mu_hD_{z_h}r_h(z_h,\overline{p},q) & =0\\
\underset{h\in\mathcal{H},j\in\mathcal{Q}^1_h}{(h.4'.j)} & \mu^j_h & =0\\
\underset{h\in\mathcal{H},j\in\mathcal{Q}^2_h\cup\mathcal{Q}^3_h}{(h.5.j)} & r^j_h(z_h,\overline{p},q) & =0\\
(M.x) & \sum\limits_{h=1}^{H}\left(x^{\backslash}_h-e^{\backslash}_h\right) & =0\\
(M.z) & \sum\limits_{h=1}^{H}z_h & =0\\
\underset{h\in\mathcal{H},j\in\mathcal{Q}^3_h}{(h.4''.j)} & \mu^j_h & =0
\end{array}\right. \tag{20}$$

The computation of the Jacobian matrix of the function $G^{\mathbf{Q}}(\xi,e)$ is presented in the table below.

(a) The components of the functions are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the corresponding partial Jacobian is displayed.

(b) The $*$ next to a matrix indicates that it is a full row rank matrix.

(c) The desired full rank result is obtained as follows. In each super-row, use the starred matrix to clean up that super-row, being sure that in that super-column there are only zero matrices. An order in which the appropriate elementary (super) column operations have to be performed is that one indicated in the last column of the table.

The above *three step procedure* will be used in some other rank computations below.

Observe that $Dr_h^{\mathcal{Q}^2_h\cup\mathcal{Q}^3_h}$ has full row rank because of Assumption r4 and that the elementary

[12] Observe that the opposite inclusion does not hold because in the definition of $G^{\mathbf{Q}}$ negative values of r^j_h (for $j\in\mathcal{Q}^1_h$) and of μ^j_h (for $j\in\mathcal{Q}^2_h$) are allowed.

column operation for the super-row $(M.z)$ is performed using Assumption r5.

	x_h	λ_h	z_h	$\left(\mu_h^j\right)_{j\in\mathcal{Q}_h^1\cup\mathcal{Q}_h^2}$	$\left(\mu_h^j\right)_{j\in\mathcal{Q}_h^3}$	$e_h^{\backslash}$	$e_h^{\cdot}$	
$\underset{h\in\mathcal{H}}{(h.1)}$	D^2u_h*	$-\Phi\left(\overline{p}\right)^T$						3
$\underset{h\in\mathcal{H}}{(h.2)}$	$-\Phi\left(\overline{p}\right)$		$\left[\begin{smallmatrix}-q\\ Y\end{smallmatrix}\right]^T$			$\Phi^{\backslash}(\overline{p})$	$I*$	1
$\underset{h\in\mathcal{H}}{(h.3)}$		$\left[\begin{smallmatrix}-q\\ Y\end{smallmatrix}\right]^T*$	$\circledcirc$	$\left[Dr_h^{\mathcal{Q}_h^1\cup\mathcal{Q}_h^2}\right]^T$	$\left[Dr_h^{\mathcal{Q}_h^3}\right]^T$			4
$\underset{h\in\mathcal{H},j\in\mathcal{Q}_h^1}{(h.4'.j)}$				$I*$				6
$\underset{h\in\mathcal{H},j\in\mathcal{Q}_h^2\cup\mathcal{Q}_h^3}{(h.5.j)}$			$Dr_h^{\mathcal{Q}_h^2\cup\mathcal{Q}_h^3}*$					7
$(M.x)$	$\widehat{I}$					$-I*$		2
$(M.z)$			$I*$					5
$\underset{h\in\mathcal{H},j\in\mathcal{Q}_h^3}{(h.4''.j)}$					$I*$			8

(21)

where the symbol $\circledcirc$ indicates a nonzero matrix whose values are insignificant for our argument, I is an identity matrix of appropriate dimension, $e_h^{\cdot} = \left(e_h^C(s), s\in\mathcal{S}\right)$,

$$\Phi\left(\overline{p}\right) = \begin{bmatrix} p^1(0) & \ldots & p^{C-1}(0) & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & p^1(S) & \ldots & p^{C-1}(S) & 1 \end{bmatrix}_{(S+1)\times G}$$

$$\Phi^{\backslash}\left(\overline{p}\right) = \begin{bmatrix} p^1(0) & \ldots & p^{C-1}(0) & & & & \\ & & & \ddots & & & \\ & & & & p^1(S) & \ldots & p^{C-1}(S) \end{bmatrix}_{(S+1)\times(G-(S+1))}$$

and

$$\widehat{I} = \begin{bmatrix} I_{C-1} & 0 & & & \\ & & \ddots & & \\ & & & I_{C-1} & 0 \end{bmatrix}_{(G-(S+1)\times G)}$$

■

Let us introduce now the following objects by using a generality which will be useful later. Call $\mathfrak{P}_h$ the family of all possible bi-partitions $\mathbf{P}_h = \left\{\mathcal{P}_h^1, \mathcal{P}_h^2\right\}$ of the set $\mathcal{J}$. Define $\mathfrak{P} = \underset{h\in\mathcal{H}}{\times}\mathfrak{P}_h$, with generic element $\mathbf{P} = (\mathbf{P}_h, h\in\mathcal{H})$. Fixed $\mathbf{P}$, consider

$$\mathcal{F}_\Delta^{\mathbf{P}} : \Xi\times\mathbb{R}_{++}^{S+1}\times T\to\mathbb{R}^{\dim\Xi},$$

$$\mathcal{F}_\Delta^{\mathbf{P}}(x,\lambda,z,\mu,p^{\backslash},q,\delta,e,u,Y,r) =$$

$$\begin{bmatrix}
\underset{h\in\mathcal{H}}{(h.1)} & D_{x_h}u_h(x_h) - \lambda_h\Phi(\overline{p}) \\
\underset{h\in\mathcal{H}}{(h.2)} & -\Phi(\overline{p})(x_h - e_h) + \begin{bmatrix} -q \\ Y \end{bmatrix} z_h \\
\underset{h\in\mathcal{H}}{(h.3)} & \lambda_h \begin{bmatrix} -q \\ Y \end{bmatrix} + \mu_h D_{z_h} r_h(z_h, \delta\square\overline{p}, \delta(0)q) \\
\underset{h\in\mathcal{H},j\in\mathcal{P}_h^1}{(h.4.j)} & \mu_h^j \\
\underset{h\in\mathcal{H},j\in\mathcal{P}_h^2}{(h.5.j)} & r_h^j(z_h, \delta\square\overline{p}, \delta(0)q) \\
(M.x) & \sum_{h=1}^H (x_h^{\backslash} - e_h^{\backslash}) \\
(M.z) & \sum_{h=1}^H z_h
\end{bmatrix}.$$

where T is defined in (12).

Lemma 9 *For all* $\mathbf{P} \in \mathfrak{P}$, $\mathcal{F}^{\mathbf{P}}_{\Delta} \in C^1(\Xi \times \mathbb{R}^{S+1}_{++} \times T, \mathbb{R}^{\dim\Xi})$.

Proof. Of course $\mathcal{F}^{\mathbf{P}}_{\Delta}$ is continuous. We have to show that

$$d\mathcal{F}^{\mathbf{P}}_{\Delta} : (\Xi \times \mathbb{R}^{S+1}_{++} \times T) \times (\mathbb{R}^{\dim\Xi} \times \mathbb{R}^{S+1} \times T) \to \mathbb{R}^{\dim\Xi}$$

is well defined and is continuous. Consider then

$$(\xi, \delta, E) \in \Xi \times \mathbb{R}^{S+1}_{++} \times T, \quad (\xi^*, \delta^*, E^*) \in \mathbb{R}^{\dim\Xi} \times \mathbb{R}^{S+1} \times T,$$

It suffices to show that

$$\lim_{t \to 0} \frac{\mathcal{F}^{\mathbf{P}}_{\Delta}(\xi + t\xi^*, \delta + t\delta^*, E + tE^*) - \mathcal{F}^{\mathbf{P}}_{\Delta}(\xi, \delta, E)}{t}$$

exists and it is continuous and that can be easily done. ■

It then follows that for all $\mathbf{P} \in \mathfrak{P}$,

$$F^{\mathbf{P}} : \Xi \times \mathcal{E}_{ho} \to \mathbb{R}^{\dim\Xi}, \quad F^{\mathbf{P}}(\xi, E) = \mathcal{F}^{\mathbf{P}}_{\Delta}(\xi, \mathbf{1}, E),$$

is C^1 .

Lemma 10 *The set*

$$\mathcal{D}_2 = \left\{ E \in \mathcal{E}_{ho} : \forall \xi \in (F_E)^{-1}(0), \forall \mathbf{P} \in \mathfrak{P}, F^{\mathbf{P}}(\xi, E) = 0 \Rightarrow \det D_\xi F^{\mathbf{P}}(\xi, E) \neq 0 \right\}$$

is open and dense in $\mathcal{E}_{ho}$.

Proof. Openness follows from continuity. To show density, it is sufficient to show that for all $(u, Y, r) \in \mathcal{U} \times \mathcal{Y} \times \mathcal{R}_{ho}$ the set

$$\mathcal{D}_2(u, Y, r) = \left\{ e \in \mathbb{R}^{GH}_{++} : (e, u, Y, r) \in \mathcal{D}_2 \right\}$$

is dense in $\mathbb{R}^{GH}_{++}$. Fix (u, Y, r) and $\mathbf{P} \in \mathfrak{P}$ and define

$$G^{\mathbf{P}} : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim\Xi},$$

$$G^{\mathbf{P}}(\xi, e) = F^{\mathbf{P}}(\xi, e, u, Y, r) =$$

$$\left[\begin{array}{ll} \underset{h \in \mathcal{H}}{(h.1)} & D_{x_h} u_h(x_h) - \lambda_h \Phi(\overline{p}) \\ \underset{h \in \mathcal{H}}{(h.2)} & -\Phi(\overline{p})(x_h - e_h) + \left[\begin{array}{c} -q \\ Y \end{array} \right] z_h \\ \underset{h \in \mathcal{H}}{(h.3)} & \lambda_h \left[\begin{array}{c} -q \\ Y \end{array} \right] + \mu_h D_{z_h} r_h(z_h, \overline{p}, q) \\ \underset{h \in \mathcal{H}, j \in \mathcal{P}^1_h}{(h.4.j)} & \mu^j_h \\ \underset{h \in \mathcal{H}, j \in \mathcal{P}^2_h}{(h.5.j)} & r^j_h(z_h, \overline{p}, q) \\ (M.x) & \sum_{h=1}^{H} (x^{\setminus}_h - e^{\setminus}_h) \\ (M.z) & \sum_{h=1}^{H} z_h \end{array} \right] \tag{22}$$

Define also

$$\mathcal{E}^{reg}(u, Y, r) = \left\{ e \in \mathbb{R}^{GH}_{++} : \forall \mathbf{P} \in \mathfrak{P}, 0 \text{ is a regular value for } G^{\mathbf{P}}(\cdot, e) \right\}.$$

From an argument very similar to the one proposed in Lemma 8, we can prove that, for all $\mathbf{P} \in \mathfrak{P}$

$$\mathcal{E}^{reg,\mathbf{P}}(u,Y,r) = \left\{e \in \mathbb{R}^{GH}_{++} : 0 \text{ is a regular value for } G^{\mathbf{P}}(\cdot, e)\right\},$$

is a full measure set. Since $\mathcal{D}_2(u,Y,r) \supseteq \mathcal{E}^{reg}(u,Y,r) = \cap_{\mathbf{P}\in\mathfrak{P}}\mathcal{E}^{reg,\mathbf{P}}(u,Y,r)$, we get the desired result. ■

Proof of Theorem 3. Define

$$\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2.$$

Of course, $\mathcal{D}$ is open and dense in $\mathcal{E}_{ho}$ and $\mathcal{D}$ is the set of economies $E \in \mathcal{E}_{ho}$ such that for all $\xi \in \Xi$ such that $F(\xi, E) = 0$ the following conditions hold:

$$\forall h \in \mathcal{H}, \forall j \in \mathcal{J} \text{ either } r^j_h(z_h, \overline{p}, q) > 0 \text{ or } \mu^j_h > 0, \tag{23}$$

$$F(\cdot, E) \text{ is } C^1 \text{ in a neighborhood of } \xi, \tag{24}$$

$$\det D_\xi F(\xi, E) \neq 0. \tag{25}$$

We are then left with showing that for any $E \in \mathcal{D}^*$ all conditions in Theorem 3 are satisfied.

From the existence result (Theorem 2) and properness of the projection defined in (18), we get

$$\{\xi \in \Xi : F(\xi, E) = 0\} = \{\xi_i\}^k_{i=1}, \tag{26}$$

where $k \in \mathbb{N}$ and depends on E. Then, from Theorem 7, there exist an open neighborhood $V(E) \subseteq \mathcal{E}_{ho}$ of E and, for every $i \in \{1, \ldots, k\}$, an open neighborhood $O(\xi_i) \subseteq \Xi$ of ξ_i and $\varphi_i : V(E) \to O(\xi_i)$ such that:

1. $\varphi_i \in C^1(V(E); O(\xi_i))$, $\varphi_i(E) = \xi_i$, $O(\xi_i) \cap O(\xi_j) = \varnothing$ if $i \neq j$,
2. $\{(\xi, E) \in O(\xi_i) \times V(E) : F(\xi, E) = 0\} = \{(\xi, E) \in O(\xi_i) \times V(E) : \xi = \varphi_i(E)\}$, (27)
3. $\{(\xi, E) \in \Xi \times V(E) : F(\xi, E) = 0\} = \bigcup^k_{i=1} \{(\xi, E) \in \Xi \times V(E) : \xi = \varphi_i(E)\}$.

Of course, (26) and (27) imply the desired conditions. ■

4.3 Indeterminacy of equilibria for an open set of nonhomogeneous economies

In this section we prove Theorem 4. To do so, we preliminarily present some simple facts about incomplete and complete markets.

Define

$$\Sigma = \left(\mathbb{R}^G_{++} \times \mathbb{R}^{S+1}_{++} \times \mathbb{R}^A_{++}\right)^H \times \mathbb{R}^{G-(S+1)} \times \mathbb{R}^A$$

with generic element

$$\sigma = \left((x_h, \lambda_h, z_h)^H_{h=1}, p^\setminus, q\right),$$

and the function

$$F^{IM} : \Sigma \times \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^{\dim \Sigma}$$

$$F^{IM}(\sigma, e, u, Y) =$$

$$\begin{bmatrix} \underset{h\in\{1,\ldots,H\}}{(IM.h.1)} & D_{x_h} u_h(x_h) - \lambda_h \Phi(\overline{p}) \\ \underset{h\in\{1,\ldots,H\}}{(IM.h.2)} & -\Phi(\overline{p})(x_h - e_h) + \left[\begin{smallmatrix} -q \\ Y \end{smallmatrix}\right] z_h \\ \underset{h\in\{1,\ldots,H\}}{(IM.h.3)} & \lambda_h \left[\begin{smallmatrix} -q \\ Y \end{smallmatrix}\right] \\ (M.x) & \sum\limits_{h=1}^H \left(x^\setminus_h - e^\setminus_h\right) \\ (M.z) & \sum\limits_{h=1}^H z_h \end{bmatrix} \tag{28}$$

$F^{IM}(\sigma, e, u, Y) = 0$ is the system of Kunh-Tucker conditions and market clearing conditions for the incomplete market model (when $A < S$) or for the complete market model (when $A = S$) associated with (e, u, Y).

Lemma 11 *There exist $u^* \in \mathcal{U}$, $Y^* \in \mathcal{Y}$, $\alpha^* \in \mathbb{R}_{++}$ and an open and nonempty set $D^* \subseteq \mathbb{R}^{GH}_{++}$ such that, for every $e \in D^*$, there exists a unique $\sigma \in \Sigma$ solution to $F^{IM}(\sigma, e, u^*, Y^*) = 0$. Moreover σ is such that $z_1^1 > \alpha^*$.*

Proof. Consider $u^* = (u_h^*)_{h\in\mathcal{H}} \in \mathcal{U}$ where, for every $h \in \mathcal{H}$, u_h^* is a Cobb-Douglas utility function and define the set $\mathcal{Y}_0 \subseteq \mathcal{Y}$ as follows:

$$\begin{array}{ll} \mathcal{Y}_0 = \left\{ Y \in \mathcal{M}_{S\times A} : Y = \left[\begin{smallmatrix} Y' \\ 0 \end{smallmatrix}\right],\ Y' \in \mathcal{M}_{A\times A},\ \det(Y') \neq 0 \right\} & \text{if} \quad A < S, \\ \mathcal{Y}_0 = \mathcal{Y} & \text{if} \quad A = S. \end{array}$$

It is immediate to verify that, for every $e \in \mathbb{R}^{GH}_{++}$ and $Y \in \mathcal{Y}_0$, there is a unique $\sigma \in \Sigma$ solution to $F^{IM}(\sigma, e, u^*, Y) = 0$. If financial markets are complete, uniqueness of equilibria is a well know fact; if financial markets are incomplete, that property immediately follows from the results on complete markets due to the special structure of the set $\mathcal{Y}_0$.

Fix now $Y^* \in \mathcal{Y}_0$. A very standard application of transversality theorem allows to prove there exists $e^* \in \mathbb{R}^{GH}_{++}$ such that, if σ^* is the unique solution to $F^{IM}(\sigma, e^*, u^*, Y^*) = 0$ then

$$D_\sigma F^{IM}(\sigma^*, e^*, u^*, Y^*) \quad \text{has full row rank}$$

and $z_1^{*1} \neq 0$. Moreover, up to replacing Y^* with $-Y^*$, we may assume $z_1^{*1} > 0$. The proof of the lemma then follows as a consequence of the implicit function theorem. ■

Let $\mathcal{O}$ be the set of economies $E^* \in \mathcal{E}$ such that there exists $(\xi^*, \delta^*) \in \Xi \times \mathbb{R}^{S+1}_{++}$ having the following properties:

$$F_\Delta(\xi^*, \delta^*, E^*) = 0, \tag{29}$$

$$\forall h \in \mathcal{H}, \forall j \in \mathcal{J} \text{ either } r_h^j(z_h, \delta \Box \overline{p}, \delta(0) q) > 0 \text{ or } \mu_h^j > 0 \text{ at } (\xi, \delta) = (\xi^*, \delta^*), \tag{30}$$

$$D_\xi F_\Delta(\xi^*, \delta^*, E^*) \text{ has full rank}, \tag{31}$$

and there exist $h^* \in \mathcal{H}$ and $j^* \in \mathcal{J}$ such that

$$r_{h^*}^{j^*}(z_{h^*}, \delta \Box \overline{p}, \delta(0) q) = 0 \text{ and } \mu_{h^*}^{j^*} > 0 \text{ at } (\xi, \delta) = (\xi^*, \delta^*) \tag{32}$$

$$D_\delta r_{h^*}^{j^*}(z_{h^*}, \delta \Box \overline{p}, \delta(0) q) \neq 0 \text{ at } (\xi, \delta) = (\xi^*, \delta^*). \tag{33}$$

We provide the proof of Theorem 4 as a consequence of the following two lemmas.

Lemma 12 *$\mathcal{O}$ is open and nonempty.*

Proof. Conditions (29) and (30) imply that F_Δ is C^1 in a neighborhood of (ξ^*, δ^*, E^*) (Lemma 9) and therefore condition (31) is well defined. That condition and Theorem 7 imply that the set $\mathcal{O}$ is open.

Define $r^* \in \mathcal{R}_{nh}$ as follows

$$r_h^{*j}(z_h, p, q) = \begin{cases} p^C(0) - z_h^1 & \text{if } h = j = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Consider D^*, u^*, Y^* and α^* described in Lemma 11 and the system

$$F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = 0, \tag{34}$$

in the unknown $\xi \in \Xi$, with $e \in D^*$ and $(\alpha^*, 1, \ldots, 1) \in \mathbb{R}^{S+1}_{++}$. Observe that from Theorem 5, system (34) admits a solution, since

$$\forall (\xi, \delta, E) \in \Xi \times \mathbb{R}^{S+1}_{++} \times \mathcal{E}, \quad F_\Delta(\xi, \delta, E) = F(\xi, E^\delta).$$

Let us prove now that if $\xi \in \Xi$ solves (34) then

$$\mu_h^j > 0 \text{ if } (h,j) = (1,1), \qquad \mu_h^j = 0 \text{ if } (h,j) \in (\mathcal{H} \times \mathcal{J}) \setminus (1,1).$$

If $(h,j) \neq (1,1)$, r_h^{*j} is identically equal to one and then $\mu_h^j = 0$. Assume now $\mu_1^1 = 0$. Then the vector $\sigma \in \Sigma$ obtained from ξ erasing multipliers μ solves $F^{IM}(\sigma, e, u^*, Y^*) = 0$ and from Lemma 11 we have $z_1^1 > \alpha^*$, contradicting that the restriction function r^* implies that $z_1^1 \leq \alpha^*$.

Then, for every $(\xi, e) \in \Xi \times D$,

$$F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = 0 \quad \Rightarrow \quad F_\Delta^{\mathbf{P}^*}(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = 0$$

where $\mathbf{P}^* = (\mathbf{P}_h^*, h \in \mathcal{H})$ and

$$\mathbf{P}_h^* = \begin{cases} \{\mathcal{J} \setminus \{1\}, \{1\}\} & \text{if} \quad h = 1, \\ \{\mathcal{J}, \varnothing\} & \text{if} \quad h \neq 1. \end{cases}$$

Then, for every $(\xi, e) \in \Xi \times D^*$ such that $F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = 0$, we have that

$$F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = F_\Delta^{\mathbf{P}^*}(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*)$$

in a neighborhood of (ξ, e). A simple rank computation shows that 0 is a regular value for

$$F_\Delta^{\mathbf{P}^*}(\,\cdot\,, (\alpha^*, 1, \ldots, 1), \,\cdot\,, u^*, Y^*, r^*) : \Xi \times D^* \to \mathbb{R}^{\dim \Xi},$$

and then there exists $e^* \in D^*$ such that 0 is regular for

$$F_\Delta^{\mathbf{P}^*}(\,\cdot\,, (\alpha^*, 1, \ldots, 1), e^*, u^*, Y^*, r^*) : \Xi \to \mathbb{R}^{\dim \Xi}.$$

In particular, for every $\xi^* \in \Xi$ such that $F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e^*, u^*, Y^*, r^*) = 0$,

$$D_\xi F_\Delta(\xi^*, (\alpha^*, 1, \ldots, 1), e^*, u^*, Y^*, r^*) \quad \text{has full row rank.}$$

It is now immediate to verify that $E^* = (e^*, u^*, Y^*, r^*)$ belongs to $\mathcal{O}$. In fact, E^* satisfies conditions defining $\mathcal{O}$ with respect to (ξ^*, δ^*), where $\delta^* = (\alpha^*, 1, \ldots, 1)$ and ξ^* is any solution to $F_\Delta(\xi, (\alpha^*, 1, \ldots, 1), e, u^*, Y^*, r^*) = 0$, and $h^* = j^* = 1$. ■

Lemma 13 *For every $E^* \in \mathcal{O}$, $X(E^*)$ contains a C^1-manifold of dimension 1. In particular $\mathcal{O} \subseteq \mathcal{E}_{nh}$.*

Proof. Given $E^* \in \mathcal{O}$ we say that $(\xi^*, \delta^*) \in \mathcal{O}(E^*)$ if it satisfies all the properties defining $\mathcal{O}$ with respect to E^*. From condition (31) and from the implicit function theorem, we can find two neighborhoods $N(\xi^*)$ and $N(\delta^*)$ of ξ^* and δ^*, respectively, and a C^1 function

$$\xi : N(\delta^*) \to N(\xi^*), \tag{35}$$

such that $\xi(\delta^*) = \xi^*$ and $F_\Delta(\xi(\delta), \delta, E^*) = 0$, for each $\delta \in N(\delta^*)$. Let us consider now the function

$$\Gamma : \Xi \to \mathbb{R}_{++}^{GH}, \quad \xi \longmapsto x.$$

First, we prove that condition

$$\operatorname{rank}\left[D_{(\xi,\delta)}(F_\Delta, \Gamma)(\xi^*, \delta^*, E^*)\right]_{(\dim \Xi + GH) \times (\dim \Xi + S + 1)} \geq \dim \Xi + 1 \tag{36}$$

implies $X(E^*)$ contains a C^1-manifold of dimension 1. Then, we show condition (36) does hold true.

Given the composition

$$(\Gamma \circ \xi) : N(\delta^*) \to \mathbb{R}_{++}^{GH}, \quad \delta \longmapsto \Gamma(\xi(\delta)), \tag{37}$$

from the implicit function theorem we get

$$D_\delta(\Gamma\circ\xi)(\delta^*) = -D_\xi\Gamma(\xi^*)\cdot[D_\xi F_\Delta(\xi^*,\delta^*,E^*)]^{-1}\cdot D_\delta F_\Delta(\xi^*,\delta^*,E^*).$$

Since

$$\left[D_{(\xi,\delta)}(F_\Delta,\Gamma)(\xi^*,\delta^*,E^*)\right] = \left[\begin{array}{cc} D_\xi F_\Delta & D_\delta F_\Delta \\ D_\xi\Gamma & 0 \end{array}\right]$$

by simple matrix algebra, we obtain,

$$\begin{aligned}\operatorname{rank}\left[D_{(\xi,\delta)}(F_\Delta,\Gamma)\right] &= \operatorname{rank}[D_\xi F_\Delta] + \operatorname{rank}\left[-D_\xi\Gamma\cdot[D_\xi F_\Delta]^{-1}\cdot D_\delta F_\Delta\right] \\ &= \operatorname{rank}[D_\xi F_\Delta] + \operatorname{rank}[D_\delta(\Gamma\circ\xi)].\end{aligned}$$

The equality $\operatorname{rank}[D_\xi F_\Delta] = \dim\Xi$ and (36) imply

$$\operatorname{rank}[D_\delta(\Gamma\circ\xi)(\delta^*)]_{GH\times(S+1)} \geq 1,$$

and we can find $s\in\mathcal{S}$ such that

$$\left[D_{\delta(s)}(\Gamma\circ\xi)(\delta^*)\right]_{GH\times 1}\neq 0.$$

Then we can find a suitable open neighborhood $N_s(\delta^*(s))$ of $\delta^*(s)$ such that the function

$$\gamma: N_s(\delta^*(s))\to\gamma(N_s(\delta^*(s)))\subseteq\mathbb{R}^{GH}_{++},\qquad \delta(s)\mapsto\gamma(\delta(s)) = (\Gamma\circ\xi)(\delta^*_{-s},\delta(s)),$$

where $\delta^*_{-s} = (\delta^*(s'), s'\neq s)$, is a diffeomorphism. Since, from (9), for all $\delta(s)\in N_s(\delta^*(s))$, $\gamma(\delta(s))\in X(E^*)$, we obtain that $X(E^*)$ contains a C^1-manifold of dimension 1.

We are left with showing that condition (36) holds true. Fix $E^*\in\mathcal{O}$, $(\xi^*,\delta^*)\in\mathcal{O}(E^*)$ and (h^*,j^*) according to the definition of $\mathcal{O}$. In a neighborhood of (ξ^*,δ^*,E^*), the function (F_Δ,Γ) has the following form

$$\left[\begin{array}{ll} \underset{h\in\mathcal{H}}{(h.1)} & D_{x_h}u_h(x_h) - \lambda_h\Phi(\overline{p}) \\ \underset{h\in\mathcal{H}}{(h.2)} & -\Phi(\overline{p})(x_h - e_h) + \left[\begin{array}{c} -q \\ Y \end{array}\right] z_h \\ \underset{h\in\mathcal{H}}{(h.3)} & \lambda_h\left[\begin{array}{c} -q \\ Y \end{array}\right] + \mu_h D_{z_h} r_h(z_h,\delta\square\overline{p},\delta(0)q) \\ \underset{h\in\mathcal{H},j\in\mathcal{P}^1_h}{(h.4.j)} & \mu^j_h \\ \underset{h\in\mathcal{H},j\in\mathcal{P}^2_h}{(h.5.j)} & r^j_h(z_h,\delta\square\overline{p},\delta(0)q) \\ (M.x) & \sum_{h=1}^{H}(x^\backslash_h - e^\backslash_h) \\ (M.z) & \sum_{h=1}^{H} z_h \\ \underset{h\in\mathcal{H}}{(h.\gamma)} & x_h \end{array}\right] \tag{38}$$

where, for every $h\in\mathcal{H}$, $\{\mathcal{P}^1_h,\mathcal{P}^2_h\}$ is a partition of $\mathcal{J}$. We know $j^*\in\mathcal{P}^2_{h^*}$ and from (33), there exists $s\in\mathcal{S}$ such that

$$D_{\delta(s)} r^{j^*}_{h^*}(z_{h^*},\delta\square\overline{p},\delta(0)q)\neq 0 \text{ at } (\xi,\delta) = (\xi^*,\delta^*).$$

We prove (36) showing that

$$\operatorname{rank}\left[D_{(\xi,\delta(s))}(F_\Delta,\Gamma)(\xi^*,\delta^*,E^*)\right]^T_{(\dim\Xi+1)\times(\dim\Xi+GH)} \tag{39}$$

has full row rank. The matrix in (39) is described in the following table

	$(h.1)$	$(h.2)$	$(h.3)$	$(h.4)$	$(h.5)$	$(M.x)$	$(M.z)$	$(h.\gamma)$	
x_h	$D^2 u_h$	$-\Phi\left(\overline{p}\right)^T$				$\widehat{I}^T$		$I*$	1
λ_h	$-\Phi\left(\overline{p}\right)*$		$\begin{bmatrix} -q \\ Y \end{bmatrix}$						2
z_h		$\begin{bmatrix} -q \\ Y \end{bmatrix}^T *$	$\circledcirc$		$D_{z_h} r_h''^T$		I		4
μ_h'			$\circledcirc$	$I*$					5
μ_h''			$D_{z_h} r_h'' *$						8
$p^\backslash$	$-\Lambda_h^T *$	$\circledcirc$	$\circledcirc$		$D_{p^\backslash} r_h''^T$				3
q		$[-z_h 0]$	$-\lambda_h(0) I + W_h^q *$		$D_q r_h''^T$				6
$\delta(s)$			$W_h^{\delta(s)}$		$D_{\delta(s)} r_h''^T *$				7

(40)

where for all $h \in \mathcal{H}$,

$$\mu_h' = \left(\mu_h^j\right)_{j \in \mathcal{P}_h^1}, \quad \mu_h'' = \left(\mu_h^j\right)_{j \in \mathcal{P}_h^2}, \quad r' = \left(r_h^j\right)_{j \in \mathcal{P}_h^1}, \quad r'' = \left(r_h^j\right)_{j \in \mathcal{P}_h^2},$$

$$\Lambda_h^T = \begin{bmatrix} \lambda_h(0) I_{C-1} & 0 & & & \\ & & \ddots & & \\ & & & \lambda_h(S) I_{C-1} & 0 \end{bmatrix}_{[G-(S+1)] \times G}$$

as in (21) the symbol $\circledcirc$ indicates a nonzero matrix whose values are insignificant for our argument, W_h^q and $W_h^{\delta(s)}$ are the transpose of the partial Jacobians of the left hand side of equations $(h.3)$ in (38) with respect to q and $\delta(s)$, respectively.

The proof of the desired rank condition is obtained following step (c) in the "three steps procedure" described in the proof of Lemma 8. The elementary column operations in the before the last super-row are performed by using Assumption r5, which implies that for every asset a there exists a household h such that the a-th columns of $D_{z_h^a} r_h$, W_h^q, $W_h^{\delta(s)}$ are zero. The last super-row can be "cleaned up" since condition (33) holds. In fact the $*$ in the last two super-rows means just that the two matrices

$$\left[\begin{array}{ccc} -\lambda_1(0) I + W_1^q & \ldots & -\lambda_H(0) I + W_H^q \end{array}\right]$$

and

$$\left[\begin{array}{ccc} D_{\delta(s)} r_1'' & \ldots & D_{\delta(s)} r_H'' \end{array}\right]$$

have full row rank.

Finally the inclusion $\mathcal{O} \subseteq \mathcal{E}_{nh}$ immediately follows from Theorem 3. ■

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