ESCAPE RATES FOR THE FAREY MAP WITH APPROXIMATED HOLES

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We study the escape rate for the Farey map, an infinite measure preserving system, with a hole including the indifferent fixed point. Due to the ergodic properties of the map, the standard theoretical approaches to this problem cannot be applied. It has been recently shown in [Knight & Munday, 2016] how to apply the standard analytical methods to a piecewise linear version of the Farey map with holes depending on the associated partition, but their results cannot be obtained in the general case we consider here. To overcome these difficulties we propose here to study approximations of the hole by means of real analytic functions. We introduce a particular family of approximations and study numerically the behavior of the escape rate for approximated holes with vanishing measure. The results suggest that the scaling of the escape rate depends on the "shape" of the approximation, and we show that this is a typical feature of systems with an indifferent fixed point, not an artifact of the particular family we consider.

Keywords: infinite measure preserving dynamical systems; transfer operators; escape rates

1. Introduction

In recent years there has been a quick growth in the number of papers dealing with statistical properties of dynamical systems with holes. The origin of these studies can be found in the paper [Pianigiani & Yorke, 1979], where questions were posed about the statistical properties of the motion of a particle inside a billiard table with a small hole. For example, if p_n is the probability that a trajectory remains on the table until time n, what is the decay rate of p_n? In general, they asked whether an initial distribution would converge, under suitable renormalization, to some limit distribution, a conditionally invariant measure. Much attention to these problems has been paid also by the physics community, see e.g. [Zyczkowski & Bollt, 1999], [Altmann *et al.*, 2013] and references therein.

In a general dynamical system with phase space X and initial conditions distributed according to a probability measure ν, the first question posed in [Pianigiani & Yorke, 1979] has become the following problem. Let H be a hole in the space X and $p_n := \nu(S_n)$, where S_n is the set of surviving points up to time n. The problem is then to study the decay rate of p_n, evaluating for example the quantity

$$\gamma_\nu := \lim_{n\to\infty} -\frac{\log(p_n)}{n},$$

which is called the *escape rate* with respect to the measure ν. The existence of the limit and its dependence on the measure ν has been studied in [Demers & Young, 2006], where it is shown that in the ideal case it is possible to compute the escape rate by using the transfer operator associated to the system. In particular it is remarked in [Demers & Young, 2006] that the escape rate with respect to two equivalent measures is the same.

Let $X = [0, 1]$ and $F : X \to X$ be a smooth map with a finite number of pre-images for each $x \in X$, then the *transfer operator* associated to F is defined to be

$$(\mathcal{P}f)(x) := \sum_{y\in F^{-1}(x)} \frac{f(y)}{|F'(y)|}.$$

The operator $\mathcal{P}$ has typically spectral radius equal to 1, and if there exists a function g such that $\mathcal{P}g = g$, then $d\nu(x) = g(x)dx$ is an F-invariant measure. We refer to [Baladi, 2000] for more properties of the transfer operators. When there is a hole H in X, one can consider the *transfer operator for the open system*

$$\mathcal{P}^{op} f := \mathcal{P}((1 - \chi_H) f),$$

where χ_H is the indicator function of the set H. That is, the transfer operator for the open system considers only pre-images of a point $x \in X$ which are not in the hole H. Then, if λ_H is the largest eigenvalue of $\mathcal{P}^{op}$ with eigenfunction g_H, the escape rate of the map F with respect to the measure $d\nu_H(x) = g_H(x)dx$ is obtained as

$$\gamma_{\nu_H} = -\log(\lambda_H) \tag{1.1}$$

where $\lambda_H \in (0, 1)$. However, as remarked above, if there exists a positive constant such that $C^{-1} \le g_H(x) \le C$, then $\gamma_{\nu_H} = \gamma$, the escape rate with respect to the Lebesgue measure on $[0, 1]$. We assume that γ can be computed by (1.1), and approximate λ_H with the largest eigenvalue of an approximation of the operator $\mathcal{P}^{op}$. This assumption is discussed in Section 4.

The escape rate has been studied mainly for hyperbolic systems and piecewise expanding maps of the interval. We refer to [Demers & Young, 2006] for references. In this paper we are interested in studying the behavior of the escape rate as the hole shrinks. Rigorous results on this asymptotic behavior are given in [Keller & Liverani, 2009] for piecewise expanding maps of the interval, for which it is found that if $|H| = \epsilon$ then γ is asymptotic to a constant times ϵ as $\epsilon \to 0^+$. Higher order corrections can be found in [Cristadoro *et al.*, 2013; Dettmann, 2013].

However, it is well known that statistical properties of dynamical systems dramatically change when we pass from uniformly hyperbolic systems to "intermittent" ones, that is, systems which have an indifferent fixed point, in particular when we consider intermittent maps on the interval which preserve only one measure which is absolutely continuous with respect to the Lebesgue measure and is infinite. The terminology "intermittent systems" is due to the fact that these systems have been introduced in the mathematical physics literature in [Pomeau & Manneville, 1980] as a simple model of the physical phenomenon of intermittency, that is the alternation of a turbulent and a laminar phase in a fluid. As dynamical systems on the unit interval $[0, 1]$, they may be represented by the family of maps $F(x) = x + x^\alpha \pmod 1$, with $\alpha > 1$, which have a fixed point at $x = 0$ with $F'(0) = 1$ and $F'(x) - 1 \approx x^{\alpha-1}$ as $x \to 0^+$. These maps admit a unique invariant measure ν which is absolutely continuous with respect to the Lebesgue measure. For $\alpha \in (1, 2)$ the measure ν is finite, whereas it is infinite for $\alpha \ge 2$. In the case of finite invariant measure, the escape rate of the system has been recently studied in [Demers & Fernandez, 2016], where it is shown that the probability p_n may decrease polynomially, so in the definition of γ one should divide by $log(n)$. However the results in [Demers & Fernandez, 2016] consider the case of a hole H which is generated by the Markov partition of the map, and such that the indifferent fixed point is far from H. Hence the polynomial escape rate is a consequence of the typical slower decay of correlations found for intermittent systems.

In this paper we are interested in the case of an intermittent dynamical system on $[0,1]$ with infinite invariant measure ν absolutely continuous with respect to the Lebesgue measure. This case has been recently studied by G. Knight and S. Munday in [Knight & Munday, 2016]. They consider piecewise linear maps of the interval and study the asymptotic behavior of the escape rate for vanishing holes, which are defined in terms of the partition associated to the map. Their methods are analytical and use the definition of the *dynamical zeta function* associated to a system, and in particular the relations between the zeroes of the zeta function and the eigenvalues of the transfer operator. To give a flavor of their results, we consider a typical example of intermittent dynamical system on the interval $[0,1]$ with infinite measure, the *Farey map.* The Farey map is defined by

$$F(x) = \begin{cases} \frac{x}{1-x}\,, \text{ if } \; 0 \le x \le \frac{1}{2} \\ \frac{1-x}{x}\,, \text{ if } \; \frac{1}{2} \le x \le 1 \end{cases} \tag{1.2}$$

and is studied in particular for its relations with the continued fractions expansion of real numbers (see e.g. [Bonanno & Isola, 2009]). Its graph is shown in Figure 1. In [Knight & Munday, 2016], the authors

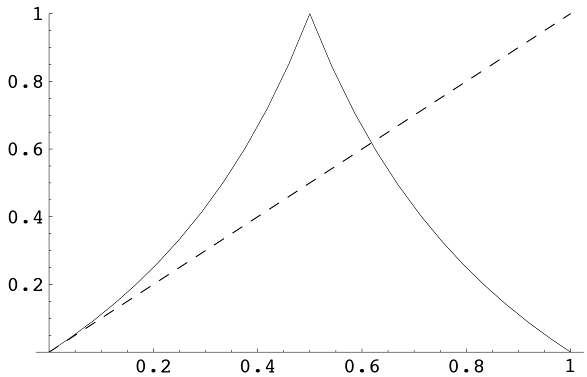


Fig. 1. The Farey map.

consider a piecewise linear version of F, which is obtained by considering the partition $A = \{(\frac{1}{n+1}, \frac{1}{n})\}$ and defining

$$F_p(x) = \begin{cases} 2-2x\,, & \text{if } \; \frac{1}{2} \le x \le 1 \\ \frac{n+1}{n-1}\, x - \frac{1}{n(n-1)}\,, & \text{if } \; \frac{1}{n+1} \le x \le \frac{1}{n} \end{cases}$$

Then for holes $H_n = [0, \frac{1}{n})$, they show that

$$\gamma \approx \frac{1}{n\, \log n} = \frac{|H_n|}{-\log |H_n|} \qquad \text{as} \quad n \to \infty\,, \tag{1.3}$$

which is a slightly faster decay than those proved for expanding maps in [Keller & Liverani, 2009]. Other behaviors are found in [Knight & Munday, 2016] by varying the partition $A = \{(a_{n+1}, a_n)\}$ with $a_n \to 0^+$, but always choosing a hole of the form $[0, a_n)$.

In principle, the analytical methods used in [Knight & Munday, 2016] apply also to the Farey map (1.2), but the difficulty lies in the possibility of obtaining estimates for the zeroes of the dynamical zeta function. Indeed in the case of piecewise linear maps with holes depending on the associated partition, the dynamical zeta function is a polynomial with coefficients depending on the partition. Hence it is possible to obtain estimates for its zeroes. However, if any of the two conditions, piecewise linearity and holes depending on the partition, are missing, the dynamical zeta function is much more difficult to study (see e.g. [Bonanno & Isola, 2014] for the Farey map).

In this paper we aim instead at studying the escape rate for the Farey map (1.2) with hole $H = (0, \epsilon)$, and its asymptotic behavior as $|H| \to 0^+$, by using (1.1) to compute the escape rate through the transfer operator $\mathcal{P}^{op}$ associated to F with a hole H. To our knowledge this is the first case where this problem is studied in such generality for an infinite measure preserving dynamical system.

We have studied the transfer operator $\mathcal{P}$ of the Farey map in [Ben Ammou *et al.*, 2015a] restricting its action to $\mathcal{H}$, a space of holomorphic functions on the disk $D := \{|z - \frac{1}{2}| < \frac{1}{2}\}$ obtained as integral transform of functions on the positive real axis (see eq. (2.2) below). First of all, it is well known that increasing the regularity of the functions on which a transfer operator acts, its spectral properties improve. This is typically used to obtain spectral gap for the operator, that is a gap between the maximal eigenvalue and the essential spectrum. A property used also in [Keller & Liverani, 2009] to study the escape rate. However, for intermittent systems it is impossible to obtain spectral gap even restricting the transfer operator to real analytic functions (see [Collet & Isola, 1991]). But in [Bonanno & Isola, 2014] we have been able to characterize the eigenfunctions of $\mathcal{P}$ with eigenvalue not embedded in the continuous spectrum $[0, 1]$, considering the restriction of $\mathcal{P}$ to holomorphic functions on D. It is proved there that all such eigenfunctions can be written as the sum of three terms, multiples of $\frac{\lambda^{1/x}}{x^2}$ and of $\frac{1}{x}$, and a function in $\mathcal{H}$. In particular, eigenfunctions in $\mathcal{H}$ with eigenvalue 1 are related to the spectral properties of the Laplace-Beltrami operator on the modular surface. A second reason for studying $\mathcal{P}$ restricted to $\mathcal{H}$ is that $\mathcal{H}$ can be given the structure of Hilbert space admitting the Laguerre polynomials as orthogonal basis. This implies that the action of $\mathcal{P}$ can be written in terms of an infinite matrix A, and in [Ben Ammou *et al.*, 2015a] we have proved that the largest eigenvalue of A can be numerically approximated by the eigenvalues of north-west corner approximations of the matrix A.

Hence we consider in this paper the action of $\mathcal{P}^{op}$ on $\mathcal{H}$. However, since the indicator function χ_H is not real analytic, we introduce an approximation of χ_H, and study the escape rate for the Farey map (1.2) with approximated holes, defined in terms of a family of two-parameter functions. In Section 2 we introduce the operator $\mathcal{P}^{op}$ and its approximations $\tilde{\mathcal{P}}^{op}$. Then in Section 3 we introduce the matrix approach to study the spectral properties of $\tilde{\mathcal{P}}^{op}$, and discuss the scaling of the escape rate as the measure of the approximated holes vanishes. The vanishing of the approximated holes is obtained by varying in different ways the two parameters of the functions defining the holes, namely by following different directions in a two-variable limit. Our main result is that the scaling of the escape rate depends on the chosen direction for the limit, and this is a feature of the Farey map as justified in Section 3.1.

2. The transfer operators

2.1. *The Farey map*

The transfer operator $\mathcal{P}$ associated to the map F acts on functions $f : (0, 1) \to \mathbb{C}$ as

$$(\mathcal{P}f)(x) := \sum_{y\,:\,F(y)=x} \frac{f(y)}{|F'(y)|}$$

which using eq. (1.2) becomes

$$(\mathcal{P}f)(x) = (\mathcal{P}_0 f + \mathcal{P}_1 f)(x)$$

with

$$(\mathcal{P}_0 f)(x) = \frac{1}{(1+x)^2}\, f\Big(\frac{x}{1+x}\Big) \quad \text{and} \quad (\mathcal{P}_1 f)(x) = \frac{1}{(1+x)^2}\, f\Big(\frac{1}{1+x}\Big) . \tag{2.1}$$

The operator $\mathcal{P}$ has been studied in [Bonanno *et al.*, 2008; Bonanno & Isola, 2014; Ben Ammou *et al.*, 2015a]. It is known from standard theory ([Collet & Isola, 1991]) that due to the presence of the indifferent fixed point of F, it is impossible to obtain spectral gap for $\mathcal{P}$ on the spaces of smooth functions C^r for all $r \in \{\mathbb{N}, \infty, \omega\}$. However it is possible to obtain some information on the eigenfunctions of $\mathcal{P}$ with eigenvalues not embedded in the continuous spectrum $[0, 1]$, when restricting $\mathcal{P}$ on the space of holomorphic functions on the disk $D := \{|z - \frac{1}{2}| < \frac{1}{2}\}$. In particular it is shown that an important role is played by eigenfunctions contained in the Hilbert space $\mathcal{H}$ defined as

$$\mathcal{H} := \big\{ f : D \to \mathbb{C} \, : \, f = \mathcal{B}[\varphi] \text{ for some } \varphi \in L^2(m) \big\} \tag{2.2}$$

where $\mathcal{B}[\cdot]$ denotes the generalized Borel transform

$$(\mathcal{B}[\varphi])(x) := \frac{1}{x^2} \int_0^\infty e^{-\frac{t}{x}}\, e^t\, \varphi(t)\, dm(t)\,, \tag{2.3}$$

and $L^2(m) := L^2(\mathbb{R}^+, m)$ where m is the measure on $\mathbb{R}^+$

$$dm(t) = te^{-t}dt.$$

The space $\mathcal{H}$ is endowed with the inner product inherited from the inner product on $L^2(m)$ through the $\mathcal{B}$-transform, that is

$$(f_1, f_2)_{\mathcal{H}} := \int_0^\infty \varphi_1(t)\, \overline{\varphi_2(t)}\, dm(t) \qquad \text{if} \quad f_i = \mathcal{B}[\varphi_i]\,.$$

The space $\mathcal{H}$ is invariant for the action of $\mathcal{P}$ and eigenfunctions of $\mathcal{P}$ in $\mathcal{H}$ can be continued to holomorphic functions on the half-plane $\{\Re(z) > 0\}$.

The Farey map has a unique invariant measure ν which is absolutely continuous with respect to the Lebesgue measure. The density of ν is the function $\bar{f}(x) = \frac{1}{x}$, which is an eigenfunction of $\mathcal{P}$ as can be easily verified applying eq. (2.1), but $\bar{f}$ is not in $\mathcal{H}$ since it is the $\mathcal{B}$-transform of $\bar{\varphi}(t) = \frac{1}{t}$, which is not in $L^2(m)$. However the spectral radius of $\mathcal{P}$ is not changed by considering its restriction to $\mathcal{H}$, as shown for example in [Bonanno *et al.*, 2008]. The role of the eigenfunction $\bar{f}$ is better understood in the contest of [Bonanno & Isola, 2014], where it is related to the eigenfunctions of the transfer operator of the Gauss map and to the zeroes of the Selberg zeta function on the full modular group.

The restriction of $\mathcal{P}$ to $\mathcal{H}$ is convenient also from a computational point of view. Indeed, using the $\mathcal{B}$-transform to read the action of $\mathcal{P}$ on $L^2(m)$, we get

$$\mathcal{P}\,\mathcal{B}[\varphi] = \mathcal{B}[(M+N)\varphi] \tag{2.4}$$

for all $\varphi \in L^2(m)$, where $M, N : L^2(m) \to L^2(m)$ are self-adjoint bounded linear operators defined by

$$(M\varphi)(t) = e^{-t}\, \varphi(t) \qquad \text{and} \qquad (N\varphi)(t) = \int_0^\infty J_1\left(2\sqrt{st}\right) \sqrt{\frac{1}{st}}\, \varphi(s)\, dm(s) \tag{2.5}$$

where J_q denotes the Bessel function of order q. Moreover $L^2(m)$ admits a countable Hilbert basis defined in terms of Laguerre polynomials, hence we can write M, N as infinite matrices. This approach has been used in [Ben Ammou *et al.*, 2015a,b], and is the fundamental tool also in this paper.

2.2. *The Farey map with a hole*

The transfer operator $\mathcal{P}^{op}$ for the map F with a hole in $H = (0, \epsilon)$ is obtained from $\mathcal{P}$ simply subtracting the contribution from the hole, that is

$$\mathcal{P}^{op} f := \mathcal{P}((1-\chi_H)f) = \mathcal{P}f - \mathcal{P}(\chi_H f)\,,$$

where $\chi_H(x)$ is the indicator function of the set H. First of all we notice that since $\frac{1}{1+x} \geq \frac{1}{2}$ for all $x \in [0,1]$, for $\epsilon < \frac{1}{2}$ we have

$$\mathcal{P}(\chi_H f) = \mathcal{P}_0(\chi_H f)\,,$$

hence by eq. (2.1)

$$(\mathcal{P}(\chi_H f))(x) = \frac{1}{(1+x)^2}\, \chi_H\Big(\frac{x}{1+x}\Big)\, f\Big(\frac{x}{1+x}\Big) = \frac{1}{(1+x)^2}\, \chi_{\tilde{H}}(x)\, f\Big(\frac{x}{1+x}\Big) = \chi_{\tilde{H}}(x)\, (\mathcal{P}_0 f)(x)$$

where $\tilde{H} = (0, \frac{\epsilon}{1-\epsilon})$. Hence

$$\mathcal{P}^{op} f = \mathcal{P}f - \chi_{\tilde{H}}\, \mathcal{P}_0 f\,. \tag{2.6}$$

However, to study the spectral properties of the operator $\mathcal{P}^{op}$ on $\mathcal{H}$, we need a real analytic approximation of the indicator function. We consider in this paper the family of approximations $\{\xi_\mu(x,a)\}_{\mu\in\mathbb{R}}$ for the indicator function of an interval $[0,a)$ given by

$$\xi_\mu(x,a) := \frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\mu\,(x-a)\right)\,, \tag{2.7}$$

where Erf is the "error function"

$$\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(2n+1)}\,. \tag{2.8}$$

In Figure 2 we have plotted the function $\xi_\mu(x,a)$ for $a = 0.4$ and $\mu = 1, 3$ and 10, against $\chi_{[0,a)}(x)$. The functions $\xi_\mu(x,a)$ converge to $\chi_{[0,a)}(x)$ as $\mu \to \infty$ for all $x \neq a$, but clearly not uniformly.

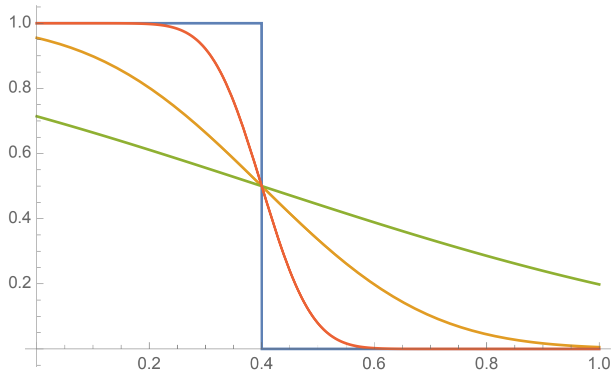


Fig. 2. The graph of the function $\xi_\mu(x,a)$ for $a = 0.4$ and $\mu = 1, 3, 10$, and of the characteristic function of $[0,a]$.

Hence we define a family of approximated transfer operators for the Farey map with hole $(0,\epsilon)$ by using ξ_μ in eq. (2.6), and let

$$(\tilde{\mathcal{P}}^{op}_\mu f)(x) := (\mathcal{P}f)(x) - \xi_\mu\left(x, \frac{\epsilon}{1-\epsilon}\right)(\mathcal{P}_0 f)(x)\,. \tag{2.9}$$

Moreover, to study the action of $\tilde{\mathcal{P}}^{op}_\mu$ on $\mathcal{H}$ as for the action of $\mathcal{P}$, we need the equivalent of eq. (2.4) for $\tilde{\mathcal{P}}^{op}_\mu$. To this aim, we first recall that by definition (2.3)

$$\mathcal{B}[\varphi(t)](x) = \frac{1}{x^2}\,\mathcal{L}[t\,\varphi(t)]\left(\frac{1}{x}\right)$$

where $\mathcal{L}[\cdot]$ denotes the standard Laplace transform, then for all $\psi, \varphi \in L^2(m)$ we can write

$$\begin{aligned} &\mathcal{B}[\psi](x) = \xi_\mu(x,a)\,\mathcal{B}[\varphi](x) \quad \Leftrightarrow \quad \mathcal{L}[t\,\psi(t)]\left(\frac{1}{x}\right) = \xi_\mu(x,a)\,\mathcal{L}[t\,\varphi(t)]\left(\frac{1}{x}\right) \quad \Leftrightarrow \\ &\Leftrightarrow \quad \mathcal{L}[t\,\psi(t)](x) = \xi_\mu\left(\frac{1}{x}, a\right)\mathcal{L}[t\,\varphi(t)](x) \quad \Leftrightarrow \quad t\,\psi(t) = \int_0^t s\,\varphi(s)\,\mathcal{L}^{-1}\left[\xi_\mu\left(\frac{1}{x}, a\right)\right](t-s)\,ds \end{aligned} \tag{2.10}$$

where in the last equivalence we have used the standard properties for the Laplace transform of convolution of functions. Using eq.(2.4) and (2.10) in eq. (2.9), we obtain

$$\begin{aligned} (\tilde{\mathcal{P}}^{op}_\mu\,\mathcal{B}[\varphi])(x) &= \mathcal{B}[(M+N)\varphi](x) - \xi_\mu\left(x, \frac{\epsilon}{1-\epsilon}\right)\mathcal{B}[M\varphi](x) = \\ &= \mathcal{B}[(M+N)\varphi](x) - \mathcal{B}\left[\frac{1}{t}\int_0^t s\,(M\varphi)(s)\,\mathcal{L}^{-1}\left[\xi_\mu\left(\frac{1}{x}, \frac{\epsilon}{1-\epsilon}\right)\right](t-s)\,ds\right](x) = \\ &= \mathcal{B}[(\tilde{M}_\mu + N)\varphi](x) \end{aligned} \tag{2.11}$$

with M, N as in (2.5), and

$$(\tilde{M}_\mu\varphi)(t) := (M\varphi)(t) - \frac{1}{t}\int_0^t s\,(M\varphi)(s)\,\mathcal{L}^{-1}\left[\xi_\mu\left(\frac{1}{x}, \frac{\epsilon}{1-\epsilon}\right)\right](t-s)\,ds\,. \tag{2.12}$$

It remains to show that our choice for the approximation of the indicator function defines an operator $\tilde{M}_\mu$ suitable for the functional study of $\tilde{\mathcal{P}}^{op}_\mu$. In particular we need to show that $\tilde{M}_\mu$ defines a bounded operator on $L^2(m)$. This is contained in the next theorem, which also uses the series expansion in eq. (2.8) to obtain an explicit expression for the inverse Laplace transform $\mathcal{L}^{-1}\left[\xi_\mu\left(\frac{1}{x}, \frac{\epsilon}{1-\epsilon}\right)\right]$, and hence for $\tilde{M}_\mu$. We remark that the possibility of obtaining the results in the next theorem is a necessary condition in the choice of the approximation functions.

Theorem 1. *For $\xi_\mu(x,a)$ as in (2.7), the operator $\tilde{M}_\mu$ is given by*

$$(\tilde{M}_\mu\varphi)(t) = \frac{1}{2}\Big(1 - \mathrm{Erf}\Big(\frac{\mu\,\epsilon}{1-\epsilon}\Big)\Big)\,(M\varphi)(t) + \\ + \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\binom{2n+1}{k}\frac{(-1)^{n+k-1}\,\mu^{2n+1}}{n!\,(k-1)!\,(2n+1)}\left(\frac{\epsilon}{1-\epsilon}\right)^{2n+1-k}\frac{1}{t}\int_0^t s(M\varphi)(s)\,(t-s)^{k-1}\,ds \tag{2.13}$$

and it is a bounded operator $\tilde{M}_\mu : L^2(m) \to L^2(m)$.

Proof. From (2.12), we have to compute the inverse Laplace transform of the function ξ. For this we use the series expansion in (2.8) and write

$$\mathcal{L}^{-1}\left[\xi_\mu\left(\frac{1}{x}, a\right)\right](t) = \mathcal{L}^{-1}\left[\frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\mu\left(\frac{1}{x} - a\right)\right)\right](t) =$$

$$= \frac{1}{2}\,\delta_0(t) - \frac{1}{2}\,\mathcal{L}^{-1}\left[\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\,\mu^{2n+1}}{n!\,(2n+1)}\left(\frac{1}{x} - a\right)^{2n+1}\right](t) =$$

$$= \frac{1}{2}\,\delta_0(t) - \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\,\mu^{2n+1}}{n!\,(2n+1)}\,\mathcal{L}^{-1}\left[\sum_{k=0}^{2n+1}\binom{2n+1}{k}\frac{1}{x^k}(-a)^{2n+1-k}\right](t) =$$

$$= \frac{1}{2}\,\delta_0(t) - \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}\,(\mu a)^{2n+1}}{n!\,(2n+1)}\,\delta_0(t) + \\ - \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\binom{2n+1}{k}\frac{(-1)^{n+k-1}\,\mu^{2n+1}\,a^{2n+1-k}}{n!\,(2n+1)}\,\frac{t^{k-1}}{(k-1)!} =$$

$$= \frac{1}{2}\Big(1 + \mathrm{Erf}(\mu a)\Big)\,\delta_0(t) - \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\binom{2n+1}{k}\frac{(-1)^{n+k-1}\,\mu^{2n+1}\,a^{2n+1-k}}{n!\,(2n+1)}\,\frac{t^{k-1}}{(k-1)!}\,,$$

where δ_0 denotes the Dirac delta at the origin, and we have used the standard results

$$\mathcal{L}^{-1}[x^{-k}](t) = \frac{t^{k-1}}{(k-1)!} \quad \text{for } t \in \mathbb{R}^+\,,\ k > 0\,, \qquad \mathcal{L}^{-1}[1](t) = \delta_0(t)\,.$$

Moreover, from the second to the third line, we have used that

$$\int_0^\infty e^{-xt}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\,\mu^{2n+1}}{n!\,(2n+1)}\left(\delta_0(t) + \sum_{k=1}^{2n+1}\binom{2n+1}{k}\frac{(-a)^{2n+1-k}}{(k-1)!}\,t^{k-1}\right)\right]dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, \mu^{2n+1}}{n! \, (2n+1)} \int_0^{\infty} e^{-xt} \left(\delta_0(t) + \sum_{k=1}^{2n+1} \binom{2n+1}{k} \frac{(-a)^{2n+1-k}}{(k-1)!} \, t^{k-1} \right) dt$$

for x big enough. Hence, by the analytical continuation of the Laplace transform, the equality is verified for all $x > 0$. It follows that the passage of the $\mathcal{L}^{-1}$ operator inside the summation, from the second to the third line, is justified.

Since

$$\int_0^t s(M\varphi)(s)\delta_0(t-s)\, ds = t(M\varphi)(t)$$

from (2.12) we obtain the first term on the right hand side of (2.13). To finish we need to show that

$$\begin{aligned} &\frac{1}{t} \int_0^t s\,(M\varphi)(s) \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \binom{2n+1}{k} \frac{(-1)^{n+k-1} \, \mu^{2n+1} \, a^{2n+1-k}}{n! \, (2n+1)} \frac{(t-s)^{k-1}}{(k-1)!} \, ds = \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \binom{2n+1}{k} \frac{(-1)^{n+k-1} \, \mu^{2n+1} \, a^{2n+1-k}}{n! \, (2n+1)} \frac{1}{t} \int_0^t s\,(M\varphi)(s) \frac{(t-s)^{k-1}}{(k-1)!} \, ds \end{aligned} \tag{2.14}$$

First of all we recall from [Olver *et al.*, 2010, eq. 13.2.2 p. 322 and eq. 13.6.19 p. 328] that for any $\varphi \in L^2(m)$ we have

$$\sum_{s=0}^{2n} \frac{(-2n)_s}{(2)_s \, s!} \Big(\frac{t}{a}\Big)^s = {}_1F_1\Big(-2n, 2; \frac{t}{a}\Big) = \frac{1}{2n+1} \, e_{2n}\Big(\frac{t}{a}\Big)$$

where $(k)_s = k(k+1)\dots(k+s-1)$ is the Pochhammer symbol, ${}_1F_1$ is the standard confluent hypergeometric function, and e_ν is the Laguerre polynomial defined in (3.1). Hence

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \binom{2n+1}{k} \frac{(-1)^{n+k-1} \, \mu^{2n+1} \, a^{2n+1-k}}{n! \, (2n+1)} \frac{(t-s)^{k-1}}{(k-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \, \mu^{2n+1} \, a^{2n}}{n! \, (2n+1)} \, e_{2n}\Big(\frac{t-s}{a}\Big)$$

We are then reduced to study the increase of the terms

$$\Big\| \frac{1}{t} \int_0^t s\,(M\varphi)(s) \, e_{2n}\Big(\frac{t-s}{a}\Big) \, ds \Big\|_{L^2(m)}^2 = \int_0^{\infty} \frac{1}{t} \, e^{-t} \left(\int_0^t s \, e^{-s} \, \varphi(s) \, e_{2n}\Big(\frac{t-s}{a}\Big) \, ds \right)^2 dt$$

Standard manipulations show that

$$\begin{aligned} &\Big\| \frac{1}{t} \int_0^t s\,(M\varphi)(s) \, e_{2n}\Big(\frac{t-s}{a}\Big) \, ds \Big\|_{L^2(m)}^2 \le \|\varphi\|_{L^2(m)}^2 \int_0^{\infty} \frac{1}{t} \, e^{-t} \int_0^t s \, e^{-s} \, e_{2n}^2\Big(\frac{t-s}{a}\Big) \, ds \, dt = \\ &= \|\varphi\|_{L^2(m)}^2 \int_0^{\infty} e^{-u} \int_0^u \Big(\frac{u-v}{u+v}\Big) \, e_{2n}^2\Big(\frac{v}{a}\Big) \, dv \, du \le a \, \|\varphi\|_{L^2(m)}^2 \int_0^{\infty} e^{-u} \int_0^{u/a} e_{2n}^2(v) \, dv \, du \, . \end{aligned}$$

By (3.1), we have

$$e_{2n}^2(v) = \sum_{i,j=0}^{2n} \binom{2n+1}{2n-i} \binom{2n+1}{2n-j} \frac{(-1)^{i+j}}{i! \, j!} \, v^{i+j}$$

hence

$$\int_0^\infty e^{-u} \int_0^{u/a} e_{2n}^2(v)\,dv\,du = \sum_{i,j=0}^{2n} \binom{2n+1}{2n-i}\binom{2n+1}{2n-j} \frac{(-1)^{i+j}}{i!\,j!} \int_0^\infty e^{-u} \int_0^{u/a} v^{i+j}\,dv\,du =$$

$$= \sum_{i,j=0}^{2n} \binom{2n+1}{2n-i}\binom{2n+1}{2n-j} \frac{(-1)^{i+j}}{a^{i+j+1}} \frac{1}{i!\,j!\,(i+j+1)} \int_0^\infty e^{-u}\,u^{i+j+1}\,du =$$

$$= \sum_{i,j=0}^{2n} \binom{2n+1}{2n-i}\binom{2n+1}{2n-j} \frac{(-1)^{i+j}}{a^{i+j+1}} \binom{i+j}{i}$$

Using the very crude estimate $\binom{k}{h} \leq 2^k$ for all $h = 0, \dots, k$, and $i + j \leq 4n$, we obtain

$$\int_0^\infty e^{-u} \int_0^{u/a} e_{2n}^2(v)\,dv\,du \leq (2n+1)^2\, 2^{4(2n+1)} \max\{a^{-s} \;:\; s = 1, \dots, 4n+1\}\,.$$

Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,\mu^{2n+1}\,a^{2n}}{n!\,(2n+1)} \left\| \frac{1}{t} \int_0^t s\,(M\varphi)(s)\,e_{2n}\Big(\frac{t-s}{a}\Big)\,ds \right\|_{L^2(m)} \leq \|\varphi\|_{L^2(m)} \sum_{n=0}^{\infty} \frac{(2n+1)\,c^{2n+1}}{n!}$$

where

$$c = \begin{cases} 16\,\frac{\mu}{a}\,, & \text{if } |a| \leq 1 \\ 16\,\mu\,a\,, & \text{if } |a| > 1 \end{cases}$$

In any case we obtain total convergence in $L^2(m)$ for the right hand side of (2.14), hence (2.14) holds in the L^2-sense.

This proves (2.13) and, together with the boundedness of M on $L^2(m)$, shows that $\tilde{M}_\mu$ is a bounded operator. This finishes the proof. ■

3. The matrix approach and the numerical results

As shown in [Bonanno *et al.*, 2008], the Hilbert space $L^2(m)$ admits a complete orthogonal system $\{e_\nu\}_{\nu \geq 0}$ given by the Laguerre polynomials defined as

$$e_\nu(t) := \sum_{m=0}^{\nu} \binom{\nu+1}{\nu-m} \frac{(-t)^m}{m!} \tag{3.1}$$

which satisfy

$$(e_\nu, e_\nu) = \frac{\Gamma(\nu+2)}{\nu!} = \nu + 1$$

for all $\nu \geq 0$. Hence, using (2.11), we can study the action of $\tilde{\mathcal{P}}_\mu^{op}$ on $\mathcal{H}$ by the action on $L^2(m)$ of an infinite matrix representing the operators $\tilde{P}_\mu^{op} := \tilde{M}_\mu + N$, defined in (2.5) and (2.13), for the basis $\{e_\nu\}_{\nu \geq 0}$. That is for any $\phi \in L^2(m)$, we can write

$$\phi(t) = \sum_{\nu=0}^{\infty} \phi_\nu e_\nu(t) \quad \text{with} \quad \phi_\nu = \frac{1}{\nu+1}\,(\phi, e_\nu)$$

hence ϕ is an eigenfunction of $\tilde{P}_\mu^{op}$ with eigenvalue λ if and only if

$$(\tilde{P}_\mu^{op}\phi, e_j) = \lambda\,(\phi, e_j) = \lambda\,(j+1)\,\phi_j \qquad \forall\, j \geq 0$$

Using the notation $c^\mu_{j\nu} := (\tilde P^{op}_\mu e_\nu, e_j)$ we obtain that

$$\tilde P^{op}_\mu \phi = \lambda\phi \quad\Leftrightarrow\quad C_\mu\phi = \lambda D\phi \quad\Leftrightarrow\quad A_\mu\phi = \lambda\phi \tag{3.2}$$

where C_μ and D are given by

$$C_\mu = (c^\mu_{j\nu})_{j,\nu\ge 0} \quad\text{and}\quad D = \operatorname{diag}(j+1)_{j\ge 0}$$

and A_μ is the infinite matrix

$$A_\mu = (a^\mu_{j\nu})_{j,\nu\ge 0} \qquad\text{with}\quad a^\mu_{j\nu} = \frac{c^\mu_{j\nu}}{j+1}\,. \tag{3.3}$$

We have then to compute the terms $c^\mu_{j\nu} := (\tilde P^{op}_r e_\nu, e_j)$. From [Ben Ammou *et al.*, 2015b, Prop. 3.1] we have

$$\frac{1}{j+1}\,(Me_\nu, e_j) = \binom{\nu+j+1}{\nu}\,\frac{1}{2^{\nu+j+2}}$$

$$\frac{1}{j+1}\,(Ne_\nu, e_j) = \sum_{\ell=0}^{\nu}(-1)^\ell\,\binom{\nu+1}{\nu-\ell}\,\binom{\ell+j+1}{l}\,\frac{1}{2^{\ell+j+2}}\,.$$

Hence

$$a^\mu_{j\nu} = \Big(1-\operatorname{Erf}\Big(\frac{\mu\,\epsilon}{1-\epsilon}\Big)\Big)\,\binom{\nu+j+1}{\nu}\,\frac{1}{2^{\nu+j+3}} + \sum_{\ell=0}^{\nu}(-1)^\ell\,\binom{\nu+1}{\nu-\ell}\,\binom{\ell+j+1}{l}\,\frac{1}{2^{\ell+j+2}} +$$
$$+\frac{1}{(j+1)\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\binom{2n+1}{k}\,\frac{(-1)^{n+k-1}\,\mu^{2n+1}}{n!\,(k-1)!\,(2n+1)}\,\Big(\frac{\epsilon}{1-\epsilon}\Big)^{2n+1-k}\,\Big(\frac{1}{t}\int_0^t s(Me_\nu)(s)\,(t-s)^{k-1}\,ds\ ,\ e_j\Big) \tag{3.4}$$

where in the last summation we have used the L^2 convergence of the series. Concerning the last terms we use [Gradshteyn & Ryzhik, 1965, eq. 7.415 p. 810] to write

$$\frac{1}{t}\int_0^t s(Me_\nu)(s)\,(t-s)^{k-1}\,ds = t^k\int_0^1 s\,(1-s)^{k-1}\,e^{-st}\,e_\nu(st)ds = \frac{\nu+1}{k(k+1)}\,t^k\,{}_1F_1(\nu+2,k+2,-t)$$

where ${}_1F_1$ is the standard confluent hypergeometric function. Moreover, by (3.1) and [Gradshteyn & Ryzhik, 1965, eq. 7.621(4) p. 822]

$$\Big(t^k\,{}_1F_1(\nu+2,k+2,-t)\ ,\ e_j(t)\Big) = \sum_{m=0}^{j}\binom{j+1}{j-m}\,\frac{(-1)^m}{m!}\int_0^\infty t^{k+m+1}\,e^{-t}\,{}_1F_1(\nu+2,k+2,-t)\,dt =$$
$$= \sum_{m=0}^{j}\binom{j+1}{j-m}\,\frac{(-1)^m\,(k+m+1)!}{m!}\,{}_2F_1(\nu+2,k+m+2;k+2;-1)$$

where ${}_2F_1$ is the hypergeometric function. Using the previous equations in (3.4), we get the following explicit expression for the general term $a^\mu_{j\nu}$ of the matrix A_μ defined in (3.2) and (3.3), which represents the transfer operator $\tilde{\mathcal{P}}^{op}_\mu$ on $L^2(m)$

$$a^\mu_{j\nu} = \Big(1-\operatorname{Erf}\Big(\frac{\mu\,\epsilon}{1-\epsilon}\Big)\Big)\,\binom{\nu+j+1}{\nu}\,\frac{1}{2^{\nu+j+3}} + \sum_{\ell=0}^{\nu}(-1)^\ell\,\binom{\nu+1}{\nu-\ell}\,\binom{\ell+j+1}{l}\,\frac{1}{2^{\ell+j+2}} +$$
$$+\frac{\nu+1}{(j+1)\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\sum_{m=0}^{j}\binom{2n+1}{k}\,\binom{j+1}{j-m}\,\binom{k+m+1}{m}\,\frac{(-1)^{n+k+m-1}\,\mu^{2n+1}}{n!\,(2n+1)}\,\Big(\frac{\epsilon}{1-\epsilon}\Big)^{2n+1-k}\cdot$$
$$\cdot\,{}_2F_1(\nu+2,k+m+2;k+2;-1) \tag{3.5}$$

Our aim is now to estimate the scaling of the escape rate γ defined in (1.1) as the measure of the hole decreases. In our case the measure of the hole has to be replaced by the integral of $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ on $[0,1]$, and more precisely on $\mathbb{R}^+$ since our approximated holes have effect on the whole real positive axis.

It is important to notice that we have defined a two-parameter family of functions, and varying the parameters ϵ and μ there are different ways of decreasing the measure of the approximated hole. The functions $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ monotonically converge to 1 as $x \to -\infty$ and to 0 as $x \to +\infty$ for all ϵ and μ, and $\xi_\mu(x, \frac{\epsilon}{1-\epsilon}) = \frac{1}{2}$ at $x = \frac{\epsilon}{1-\epsilon}$ for all μ. Varying ϵ we simply obtain a translation of the graph of $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$, whereas increasing μ the functions converge to a step function.

We would like to let $\mu \to \infty$ and then let $\epsilon \to 0^+$ (look at Figure 2). This is not compatible with the expression for the operator $\tilde{M}_\mu$, since the operator norm diverges as $\mu \to \infty$ as expected from the fact that the indicator function is not real analytic. Hence we have to vary μ and ε at the same time. However, even for μ big, to let $\epsilon \to 0^+$ is not good enough from a numerical point of view. Indeed the function $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ converges as $\epsilon \to 0^+$ to

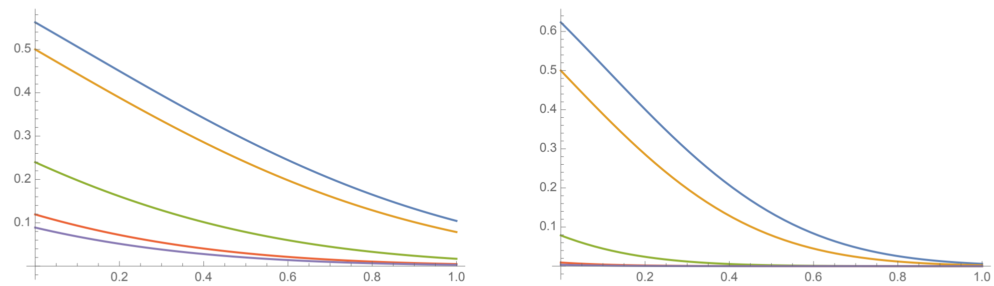
$$\xi_\mu(x, 0) = \frac{1}{2} - \frac{1}{2}\mathrm{Erf}(\mu\, x)\,,$$

whose integral on $\mathbb{R}^+$ is unfortunately slowly convergent to 0 as μ diverges. This is due to the fact that $\xi_\mu(0,0) = \frac{1}{2}$ for all μ. For $\mu = 7$, the integral is of order 10^{-2}, which implies a significant perturbation on the transfer operator. Moreover, a value of μ greater than 4 in (3.5) implies that, to have good numerical results, one should consider too many terms in the series on n.

By previous discussion, a way to rapidly decrease the integral of $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ on $\mathbb{R}^+$, is to further translate the graph of the function to the left, so that ξ_μ is smaller than $\frac{1}{2}$ at $x = 0$ and the effect of increasing μ is stronger on the integral. This is what we get by choosing small negative values for ϵ. In Figure 3 we show the behavior of $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ on $[0,1]$ for $\epsilon = 0.1, 0, -1, -5, -20$ for $\mu = 1$ on the left, and for $\mu = 2$ on the right. The functions decrease on the positive real axis more rapidly for bigger μ, only the first three cases are non-negligible for $\mu = 2$, and if we measure their integral on $\mathbb{R}^+$ we have that

$$\lim_{\epsilon \to -\infty} \int_0^{+\infty} \xi_\mu\Big(x, \frac{\epsilon}{1-\epsilon}\Big)\, dx \approx \begin{cases} 0.025 & \text{for } \mu = 1 \\ 2 \times 10^{-4} & \text{for } \mu = 2 \\ 10^{-5} & \text{for } \mu = 2.5 \\ 5 \times 10^{-7} & \text{for } \mu = 3 \\ 10^{-8} & \text{for } \mu = 3.5 \end{cases}$$

and the integral is already $\approx 10^{-6}$ for $\mu = 3$ and $\epsilon = -20$, and $\approx 10^{-7}$ for $\mu = 3.5$ and $\epsilon = -10$.



(a) (b)

Fig. 3. (a) The functions $\xi_\mu(x, \frac{\epsilon}{1-\epsilon})$ for $\mu = 1$ and $\epsilon = 0.1, 0, -1, -5, -20$. (b) The same as (a) for $\mu = 2$.

As proved in [Ben Ammou *et al.*, 2015a] for the transfer operator $\mathcal{P}$, this matrix approach is convenient since by computing the principal eigenvalue of north-west corner approximations of the infinite matrix associated to $\mathcal{P}$, we obtain an increasing sequence of eigenvalues converging to the largest eigenvalue of $\mathcal{P}$. Hence, to obtain an approximation for the escape rate γ as defined in eq. (1.1), we have computed the principal eigenvalue $\lambda_\mu(\epsilon)$ of north-west corner approximations of the matrix A_μ for small negative

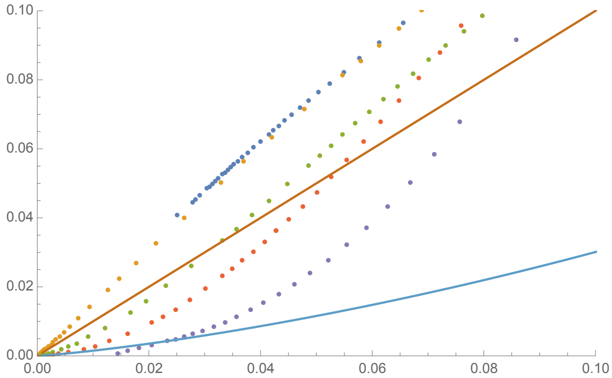


Fig. 4. The solid lines are the identity and the function $f(t) = \frac{t}{-\log t}$. The dotted lines are the points $(M_\mu(\epsilon), \gamma_\mu(\epsilon))$ for $\mu = 1, 2, 2.5, 3$ and 3.5 from the biggest to the lowest.

values of ϵ, and the principal eigenvalue λ_∞ of the same approximations of the matrix associated to the transfer operator of the Farey map without hole. To find the scaling of the escape rate, we have plotted $\gamma(\mu, \epsilon) := -\log(\lambda_\mu(\epsilon)/\lambda_\infty)$ against $M_\mu(\epsilon) := \int_0^{+\infty} \xi_\mu\left(x, \frac{\epsilon}{1-\epsilon}\right) dx$. The results are shown in Figure 4. The solid lines are the identity and the function $f(t) = \frac{t}{-\log t}$. The dotted lines are the plots of the points $(M_\mu(\epsilon), \gamma_\mu(\epsilon))$ for $\mu = 1, 2, 2.5, 3$ and 3.5 from the biggest to the lowest. Notice that for $\mu = 1$ the dots stop far from the origin because $M_1(-\infty) \approx 0.025$.

Figure 4 shows that the scaling of the escape rate for the Farey map with shrinking approximated holes is dependent on the shape of the approximation. Moreover, as μ increases we find a scaling

$$\gamma(\mu, \epsilon) \approx \frac{M_\mu(\epsilon)}{-\log M_\mu(\epsilon)} \qquad \text{as} \quad M_\mu(\epsilon) \to 0^+ ,$$

which is the same as (1.3), the theoretical result for the Markov approximation of the Farey map studied in [Knight & Munday, 2016] with holes generated by the Markov partition. Hence our conjecture is that the result in [Knight & Munday, 2016] holds for the real Farey map for general shrinking holes.

3.1. *Relations with expanding maps*

To further justify the different scalings of the escape rate shown in Figure 4, we give here an heuristic argument based on the *renewal theory* for transfer operators introduced in [Sarig, 2002], and developed for the Farey map in [Isola, 2002; Bonanno & Isola, 2014].

It well known that the Farey map is related to the famous Gauss map on the interval $[0, 1]$ by an inductive procedure. Given the Farey map $F : [0, 1] \to [0, 1]$ and the partition $I_1 := [0, \frac{1}{2}]$, $I_0 := (\frac{1}{2}, 1]$ of $[0, 1]$, let us define the function $\tau(x) := \min\{k \geq 0 \,:\, F^k(x) \in I_0\}$, and let $G : [0, 1] \to [0, 1]$ be the induced map on I_0 defined by

$$G(x) := \begin{cases} F^{\tau(x)+1}(x) \,, & \text{for } x > 0 \\ 0 \,, & \text{for } x = 0 \end{cases}$$

Then $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ for $x > 0$, which is known as the *Gauss map*. The Gauss map has transfer operator $\mathcal{M}$ given by

$$(\mathcal{M}g)(x) = \sum_{G(y)=x} \frac{g(y)}{|G'(y)|} = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}\, g\left(\frac{1}{x+n}\right) ,$$

where the terms of the sum come from the countable pre-images of any $x \in (0, 1)$, corresponding to the countable level sets of the function $\tau(\cdot)$. In particular, notice that $\tau(\frac{1}{x+n}) = n - 1$ for all $n \geq 1$.

The relation between F and G implies also a relation between the associated transfer operators. This relation has been studied in [Isola, 2002; Bonanno & Isola, 2014] for the case of signed generalized transfer operators, and it reduces in our case to the functional equation

$$(1-\mathcal{M}_z)(1-z\,\mathcal{P}_0) = (1-z\,\mathcal{P}) \tag{3.6}$$

where $z \in \mathbb{C} \setminus (1,\infty)$, the notation for $\mathcal{P}$ and $\mathcal{P}_0$ is given in eq. (2.1), and $\mathcal{M}_z$ is the operator-valued series

$$(\mathcal{M}_z g)(x) = \sum_{n=1}^{\infty} \frac{z^n}{(x+n)^2}\, g\left(\frac{1}{x+n}\right),$$

which is a generalization of the transfer operator of G. Eq. (3.6) is well defined on a space of functions studied in [Bonanno & Isola, 2014], which properly contains the Hilbert space $\mathcal{H}$, and it can formally stated in the more intuitive form

$$\mathcal{M}_z = \sum_{n=0}^{\infty} z^{n+1}\, \mathcal{P}_1 \circ \mathcal{P}_0^n = z\mathcal{P}_1\,(1-z\mathcal{P}_0)^{-1}, \tag{3.7}$$

from which it is clear the role of the inductive procedure. We now consider the Farey map and its transfer operator with an approximated hole as defined in eq. (2.9). In particular we can write

$$\tilde{\mathcal{P}}^{op}_\mu f := \tilde{\mathcal{P}}_{0,\mu} f + \mathcal{P}_1 f \qquad \text{with} \qquad \tilde{\mathcal{P}}_{0,\mu} f := \mathcal{P}_0(\psi_\mu f)$$

where $\psi_\mu(x) := 1 - \xi_\mu(x,\varepsilon)$. Hence the transfer operator-valued series for the induced map on I_0 becomes now, using eq. (3.7)

$$(\tilde{\mathcal{M}}^{op}_z g)(x) = \sum_{n=0}^{\infty} z^{n+1}\,(\mathcal{P}_1 \circ \tilde{\mathcal{P}}^n_{0,\mu} g)(x) = \sum_{n=1}^{\infty} \frac{z^n}{(x+n)^2}\, g\left(\frac{1}{x+n}\right) \prod_{k=2}^{n} \psi_\mu\left(\frac{1}{x+k}\right)$$

where we are using the standard notation $\prod_{k=2}^{1} \psi_\mu \equiv 1$, and can be written as

$$(\tilde{\mathcal{M}}^{op}_z g)(x) = \sum_{G(y)=x} e^{w_z(y)}\, g(y)$$

in terms of the potential $w_z : [0,1] \to \mathbb{R}$

$$w_z(x) := -\log|G'(x)| + (1+\tau(x))\,\log z + \sum_{k=2}^{1+\tau(x)} \log\psi_\mu\left(\frac{1}{x+k}\right)$$

where for simplicity we consider only positive real values for z.

Eq. (3.6) for the transfer operators with approximated holes, show that the principal eigenvalue $\frac{1}{z}$ of $\tilde{\mathcal{P}}^{op}_\mu$ is obtained by choosing for z the value for which $\tilde{\mathcal{M}}^{op}_z$ has principal eigenvalue equal to 1. By standard thermodynamic formalism, the value of z depends on the integral of the potential w_z, and on the escape rate of G. However G is expanding, and the standard approach based on the existence of the spectral gap for the transfer operator of expanding maps, implies that the scaling of the escape rate of G is linear with the measure of the hole.

This heuristic argument clearly implies that the scaling of the escape rate depends on the behavior of the function $\psi_\mu(x) = 1 - \xi_\mu(x,\varepsilon)$ as μ and ε vary. These relations, which is not limited to the Farey map, but to a large class of maps with an indifferent fixed point, deserve further investigations.

4. Conclusions

In this paper we have studied the escape rate for the Farey map, an infinite measure preserving system, with a hole including the indifferent fixed point. To our knowledge this is the first time this problem is studied for maps preserving an infinite measure with general holes, since previous results only considered piecewise linear maps with holes generated by the associated partition.

The problem we consider poses theoretical difficulties in the application of the standard methods for the study of the escape rate, in particular the transfer operator approach. For this reason, we propose to modify the standard approach to open systems by considering approximations of the hole, by means of a family of functions approximating the indicator function of the hole. Here we have used the family (2.7), and proved in Theorem 1 that the associated transfer operator $\tilde{\mathcal{P}}_\mu^{op}$, defined in (2.9), has the functional analytic properties necessary to pursue the approach already used in [Bonanno *et al.*, 2008; Ben Ammou *et al.*, 2015a,b] to study the spectral properties of the transfer operator of the Farey map.

Then we have used $\tilde{\mathcal{P}}_\mu^{op}$ to compute the "escape rate" γ_{app} in (1.1), using the largest eigenvalue of $\tilde{\mathcal{P}}_\mu^{op}$. Then, we have used results in [Ben Ammou *et al.*, 2015a], where it has been proved that the matrix approach developed in Section 3 provides good approximations, denoted by λ_∞ in Section 3, for the largest eigenvalue of the transfer operator of the Farey map without hole. Finally, repeating the same matrix approach for $\tilde{\mathcal{P}}_\mu^{op}$ we have computed $\lambda_\mu(\varepsilon)$, approximations to the largest eigenvalue of $\tilde{\mathcal{P}}_\mu^{op}$. The "escape rate" γ_{app} of the Farey map with approximated hole has then been approximated by $\gamma(\mu,\varepsilon) := -\log(\lambda_\mu(\varepsilon)/\lambda_\infty)$.

Our numerical results suggest that the behavior of γ_{app} as the measure of the approximated hole vanishes, is indeed dependent on the chosen approximation of the hole, but for functions "close" to the indicator function, which is obtained for fixed big values of μ, we find the same behavior proved in [Knight & Munday, 2016] for the piecewise linear approximation of the map. Moreover, we have shown by a heuristic argument that the dependence on the approximation is typical of maps with an indifferent fixed point, and is not due to the family we have chosen in this paper.

Finally, we comment on the relation with the escape rate γ of the Farey map with hole $H = (0, \varepsilon)$, with respect to the Lebesgue measure. First of all, as shown in [Demers & Young, 2006] and discussed in the Introduction, we assume that γ can be computed by (1.1) because we expect for ε small enough the eigenfunction g_H to be finite and bounded away from zero, as follows from standard perturbation arguments, since the Farey map without hole has invariant measure $\frac{1}{x}\, dx$.

What one expects is that as μ increases for fixed ε, the quantity $\gamma(\mu,\varepsilon)$ converges to the escape rate γ, that is $\lambda_\mu(\varepsilon)$ is expected to converge to the largest eigenvalue of $\mathcal{P}^{op}$ as defined in (2.6). First of all, we expect that the spectral radius of $\tilde{\mathcal{P}}_\mu^{op}$ does not change when considering its action on $L^1(0,1)$ instead of the Hilbert space $\mathcal{H}$, since typically the eigenfunction associated to the largest eigenvalue has the same regularity of the map. This happens for the Farey map without hole. Moreover, for each $f \in L^1(0,1)$ the difference $(\mathcal{P}^{op} f - \tilde{\mathcal{P}}_\mu^{op} f)$ has vanishing norm in L^1 as μ increases, by Lebesgue dominated convergence, hence $\tilde{\mathcal{P}}_\mu^{op}$ converges to $\mathcal{P}^{op}$ in the strong operator topology. This is not enough to prove the convergence of the spectrum, or at least of the spectral radius, and it's not an easy problem in particular for non-selfadjoint operators. However, the strong convergence is uniform in $L^1(X)$, with $X = (0,1) \setminus (\varepsilon - \delta, \varepsilon + \delta))$, for each positive δ, hence ensuring convergence of the spectrum in $L^1(X)$ (see for example [Kato , 1980]). Finally, the fact that the spectrum is discrete, suggests that there is convergence of the spectrum also in $L^1(0,1)$.

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