# Nonlinear dynamics and global indeterminacy in an overlapping generations model with environmental resources

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**Abstract** We analyse the dynamics of an economy formed of overlapping generations of individuals whose well-being depends on leisure, consumption of a private good and a free access environmental resource. The production activity of the private good deteriorates the environmental resource. Individuals may defend themselves from environmental degradation by increasing consumption of the private good, which may be perceived as a "substitute" for services provided by the environmental resource. However, the resulting increase in production and consumption of the private good generates a further increase in environmental deterioration leading economic agents to increase production and consumption of the private good itself. This substitution mechanism is clearly self-reinforcing and may fuel an undesirable economic growth process according to which an increase in consumption of the private good – and the resulting increase in Gross Domestic Product – is associated with a reduction in individuals' well-being. The article shows the emergence of several global phenomena, and individuals' expectations about the future evolution of the environmental quality can give rise to (local and global) indeterminacy about the growth path the economy will follow starting from a given initial position.

**Keywords** Environmental defensive expenditures; Indeterminacy; Nonlinear dynamics and chaos; OLG model; Well-being reducing economic growth

**JEL Codes** C62; O13; O41; Q22

**AMS codes** Nonlinear equations and systems, general 34A34; Numerical chaos 65P20; Bifurcations and instability 70K50.

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## 1. Introduction

In this article, we analyse the dynamics of an economy composed of a continuum of identical individuals whose well-being depends on leisure, consumption of a private good and a free access environmental resource. The production activity of the private good deteriorates the environmental resource. Individuals may defend themselves from environmental degradation by increasing the consumption of the private good, which may be perceived as a "substitute" for the environmental resource. However, the resulting increase in consumption and production of the private good generates a further increase in environmental deterioration leading agents to a further increase in production and consumption of the private good itself. This substitution mechanism is clearly self-reinforcing and may fuel an undesirable economic growth process according to which an increase in production and consumption of the private good – and the resulting increase in Gross Domestic Product (GDP) – is associated with a reduction in individuals' well-being.

In the model, individuals are assumed to have perfect foresight, i.e. they perfectly foresee the time evolution of economic variables. The state of the economy is characterised by a state variable $K_t$ – that represents the stock of productive capital of the representative firm at time $t$ – and by a choice variable $\ell_t$ – that represents the labour supplied by the representative individual in the production process of the consumption good at time $t$. The stock of capital $K_t$ is considered as a pre-determined variable as its initial value $K_0$ is determined by "history" (see the seminal article of Krugman 1991), and therefore it is considered as a given. In contrast, the initial value of the labour supply $\ell_0$ is chosen by the representative individual in order to maximise his lifetime utility. Therefore, $\ell_t$ is a jumping variable and its initial value is determined by taking into account the individual expectation about the future evolution of environmental quality, which is negatively affected by the average production of the consumption good.

As the economy is comprised of a continuum of identical agents, the impact of each single individual's choice on environmental quality is negligible. Therefore, the representative individual takes environmental quality as exogenously given. In this context, individuals may not be able to coordinate their choices and coordination failures may arise. More specifically, given the initial value $K_0$ of the pre-determined variable $K_t$, the initial value $\ell_0$ of the choice variable $\ell_t$ may not be uniquely determined and phenomena of the local and global indeterminacy may occur. The local indeterminacy phenomenon is observed when the dynamic system describing the evolution of capital and labour admits a locally attracting fixed point $(K^*, \ell^*)$. In such a case, if the (pre-determined) initial value $K_0$ is close enough to $K^*$, then there exists a continuum of initial values

$\ell_0$ such that the trajectory starting from $(K_0,\ell_0)$ approaches $(K^*,\ell^*)$. In this case, we have indeterminacy as there does not exist a unique choice of $\ell_0$ that leads the economy to approach $(K^*,\ell^*)$. The choice of $\ell_0$ by each individual depends on his expectation about the other individuals' choices of $\ell_0$.[1] In contrast, global indeterminacy occurs when, by starting from a given initial value $K_0$, the economy may converge to different fixed points (or, more in general, different invariant sets such as cycles, closed invariant curves or chaotic sets) according to the choice of the initial value $\ell_0$. In case of global indeterminacy, different initial choices of the non-predetermined variable $\ell_t$ may imply different long-term behaviours and the initial value of the state variable does not necessarily determine the set to which the economy will eventually converge.

The analysis of the linearization of a dynamic system around a fixed point gives all information required to detect local indeterminacy. The relative simplicity of the local analysis explains the reasons why a large amount of works in the literature focuses only on local indeterminacy issues (see Benhabib and Farmer, 1999). However, a fast growing number of contributions deal with dynamic systems exhibiting global indeterminacy (see, amongst others, Matsuyama, 1991; Raurich-Puigdevall, 2000; Boldrin et al., 2001; Benhabib and Eusepi, 2005; Benhabib et al., 2008; Coury and Wen, 2009; Mattana et al., 2009; Brito and Venditti, 2010; Carboni and Russu 2013; Antoci et al., 2011, 2014; Gori and Sodini 2011, 2014). Global indeterminacy is usually observed in highly nonlinear systems and is detected by global analysis techniques. The contribution of the present work to the literature on indeterminacy is the analysis of the role played by the interaction between environmental degradation and individuals' consumption/investment choices in generating nonlinearities and very complex scenarios of global dynamics. Different from the main body of the literature on global indeterminacy, where indeterminacy is generated by market imperfections in the production sector, the focus of this work is on the demand side of the economy. In particular, in the model indeterminacy is produced by choices of individuals who defend themselves against environmental degradation by consuming higher quantities of private consumption goods. The present article extends the analysis of Antoci and Sodini (2009) and Antoci et al. (2010), where the negative relationship between environmental quality and aggregate production of the private good is linear. Specifically, we consider a nonlinear dependence between these variables and obtain more complex dynamic scenarios. This article is also related to the literature on environmental defensive expenditures. In the model, the more interesting dynamic scenarios take place under the assumption of substitutability between the private consumption good and environmental quality. Economic

[1] In this kind of models, it is usually assumed that all individuals make the same initial choice of $\ell_0$.

growth in industrialised countries is often associated with a substitution process according to which the services provided by free access environmental resources are substituted by the consumption of (costly) private goods (see, amongst others, Hueting, 1980; Leipert, 1986; Leipert and Simonis, 1988; Bartolini and Bonatti, 2002; Antoci and Borghesi, 2012). The degradation of coastal areas next to urban centres can motivate costly trips to less contaminated areas by car, boat or airplane. Individuals buy mineral water when tub water is non-drinkable, double windows and medicines to protect themselves against, respectively, traffic noise and pollution-related diseases. Air conditioners provide a paradigmatic example of the self-enforcing process analysed in the article (Antoci and Bartolini, 1999). These devices cool the interior of homes and offices and then protect individuals against global warming. However, they produce an increase in the external temperature that tends to encourage their use even further.

According to the literature on environmental defensive expenditures, the GDP level of an economy is not a good proxy of individuals' well-being. This because the methodology used to measure GDP accounts for defensive expenditures but does not evaluate environmental degradation. Consequently, the positive effect due to an increase in GDP may be more than compensated by the deterioration of natural resources (see Costanza et al., 2014). We show that growth orbits along which the increase in production/consumption of the consumption good (i.e., the increase in GDP) generates a reduction in individuals' well-being may exist even if individuals are rational. The process of well-being reducing economic growth is observed if coordination failure occurs: given the initial value $K_0$ of the capital stock, individual well-being would be higher by choosing a lower initial value $\ell_0$. However, no individual has an incentive to modify his choice of $\ell_0$ if the other individuals do not do the same.

The rest of the article is organised as follows. Section 2 describes the model. Section 3 illustrates some basic properties of dynamics. Section 4 highlights some global dynamics properties through numerical simulations. Section 5 concludes.

## 2. The model

We consider an overlapping generations (OLG) economy comprised of a continuum of perfectly rational and identical individuals who live for two periods (Diamond, 1965): youth and old age. The time horizon is indexed by the discrete variable $t = 0, 1, 2, ..., \infty$. A new generation (assumed to be of size 1) is born in every period. Each generation overlaps for one period with the previous generation and then overlaps for one period with the next one. Each young individual of generation $t$ chooses how to allocate his time endowment (set to be equal to 2 for simplicity) between labour $\ell_t \in (0, 2)$

– offered to firms and remunerated at the wage rate $w_t$ – and leisure activities $2-\ell_t$. In the economy, a population of perfectly competitive firms (a continuum of size 1) produces a homogeneous consumption good by means of the labour forces provided by young individuals. Individuals buy and consume the good produced by firms only in the second period of life (see Reichlin, 1986; Galor and Weil, 1996; Grandmont et al., 1998; Antoci and Sodini, 2009; Gori and Sodini 2011; Antoci et al., 2010, 2014). As identical individuals and identical firms compose each generation and the (infinitely living) population of firms, respectively, we can focus the analysis on the choices of a "representative" individual of generation $t$ and a "representative" firm. The budget constraint of the representative member of generation $t$ can be determined as follows. He earns labour income $w_t\,\ell_t$, which is entirely saved at time $t$ i.e., $w_t\,\ell_t = s_t$. Savings $s_t$ are invested in productive capital $K_{t+1}$ ($K_{t+1} = s_t$) that the representative individual will rent to the representative firm in period $t+1$ at the interest factor $R_{t+1}$. The revenue $R_{t+1}\,s_t$ is used by the representative consumer at time $t+1$ to buy and consume the quantity $C_{t+1}$ of the good produced by the representative firm, so that $C_{t+1} = R_{t+1}\,s_t$. Therefore, the lifetime budget constraint of the individual of generation born at time $t$ is given by:

$$C_{t+1} = R_{t+1}\,w_t\ell_t\,. \tag{1}$$

Individuals have preferences over leisure when they are young and consumption when they are old. Moreover, the utility deriving from consumption $C_{t+1}$ is assumed to be affected by environmental quality $E_{t+1}$. More specifically, we assume that the lifetime utility index of the representative member of generation $t$ is determined by a constant inter-temporal elasticity of substitution (CIES) function:

$$U_t(\ell_t, C_{t+1}, E_{t+1}) = \frac{(2-\ell_t)^{1-\gamma}}{1-\gamma} + \frac{(C_{t+1}E_{t+1}^{\tau})^{1-\sigma}}{1-\sigma}, \tag{2}$$

where parameters $\sigma > 0$ ($\sigma \neq 1$) and $\gamma > 0$ ($\gamma \neq 1$) represent the (constant) elasticity of utility with respect to consumption and leisure, respectively. If $\sigma \in (0,1)$ (resp. $\sigma > 1$) then consumption $C_{t+1}$ and environmental quality $E_{t+1}$ are complements (resp. substitutes). In particular, for $\sigma \in (0,1)$ (resp. $\sigma > 1$) the following derivative holds:

$$\frac{\partial^2 U_t(\ell_t, C_{t+1}, E_{t+1})}{\partial C_{t+1}\partial E_{t+1}} > 0 \quad (\text{resp.} < 0)\,.$$

This means that the increase in $U_t$ due to an increase in $C_{t+1}$ is positively related to the value of the index of environmental quality $E_{t+1}$ if $\sigma \in (0,1)$, vice versa if $\sigma > 1$. The parameter $\tau \geq 0$ captures the relative importance of environmental quality in determining the utility of individuals. If $\tau = 0$, the model coincides with a standard OLG model in which individuals only care about the level of consumption $C_{t+1}$ and leisure $2 - \ell_t$ (that is, environmental quality does not affect their choices). Notice that the utility function (2) is always concave with respect to the control variables $\ell_t$ and $C_{t+1}$ whatever is the value of $\sigma$, whereas it is jointly concave in $\ell_t$, $C_{t+1}$ and $E_{t+1}$ if and only if either $\sigma > 1$ or $0 < \sigma < 1$ and $\tau < \sigma/(1-\sigma)$. In our decentralised economy, joint concavity is not required as $E_{t+1}$ is not a choice variable. However, the more interesting results will be obtained under the joint concavity assumption.

We will consider all prices expressed in terms of unities of the consumption good. Prices $R_{t+1}$, $w_t$ and the value of the index $E_{t+1}$ are considered as exogenously given by every individual. This because he takes the impact of his choices as negligible on these variables (see Antoci and Sodini, 2009, Antoci et al. 2011). However, we assume that at time $t$ the representative individual is able to perfectly foresee the values of such variables.

The representative individual of generation $t$ chooses labour supply $\ell_t$ in order to maximise the value of utility function (2), subject to lifetime budget constraint (1) and $\ell_t \in (0,2)$. The first order conditions for an interior solution are given by:

$$-(2-\ell_t)^{-\gamma} + \frac{(R_{t+1} w_t \ell_t E_{t+1}^{\tau})^{1-\sigma}}{\ell_t} = 0. \qquad (3)$$

The aim of the present work is to study the effects of environmental degradation (that is, of the reduction of the value of the index *E*) on individuals' utility and on their consumption choices. The focus, therefore, is on the "demand side" of the economy. Different from Grandmont et al. (1998), Cazzavillan (2001) and other related works, which adopt a constant elasticity of substitution (CES) technology or consider market imperfections (positive externalities) in production, we assume that at time $t$ the representative firm produces the quantity $Y_t$ by combining capital $K_t$ and labour $\ell_t$ through a constant returns to scale *Cobb-Douglas* technology:[2]

$$Y_t = A \cdot F(K_t, \ell_t) = A K_t^{\alpha} \ell_t^{1-\alpha},$$

where $A > 0$ is a production scaling parameter and $0 < \alpha < 1$ represents the elasticity of the production function with respect to capital. By assuming that the capital stock $K_t$ fully depreciates

[2] The stock of capital $K_t$ is financed by saving of the representative agent born at time $t-1$, while labour supply $\ell_t$ is provided by the representative agent born at time $t$.

at the end of every period and that output is sold at the unit price, profit maximisation implies that the production inputs are remunerated at their marginal products, that is:

$$R_t = \alpha A K_t^{\alpha-1} \ell_t^{1-\alpha}, \tag{4}$$

$$w_t = (1-\alpha) A K_t^{\alpha} \ell_t^{-\alpha}. \tag{5}$$

With regard to environmental resources, we assume that the value $E_{t+1}$ of the environmental quality index negatively depends on the level of the economy-wide average production at time $t$, that is:

$$E_{t+1} = \frac{\overline{E}}{1+(\overline{Y}_t)^{\rho}}, \tag{6}$$

where $\overline{E}$ is a positive parameter representing the maximum value that the index can assume (that is, the value of the index when output is zero). Variable $\overline{Y}_t$ captures the economy-wide average level of production, whereas $\rho > 0$ weights the environmental influence of production. When $\rho \le 1$, (6) is a convex decreasing function in $\overline{Y}_t$ (Figure 1(a)). When $\rho > 1$, the value of $E_{t+1}$ decreases slowly when $\overline{Y}_t$ increases if $\overline{Y}_t$ is small enough, whereas when $\overline{Y}_t$ exceeds a given threshold, a further increase in $\overline{Y}_t$ causes a sharp reduction in $E_{t+1}$ (Figure 1(b)).

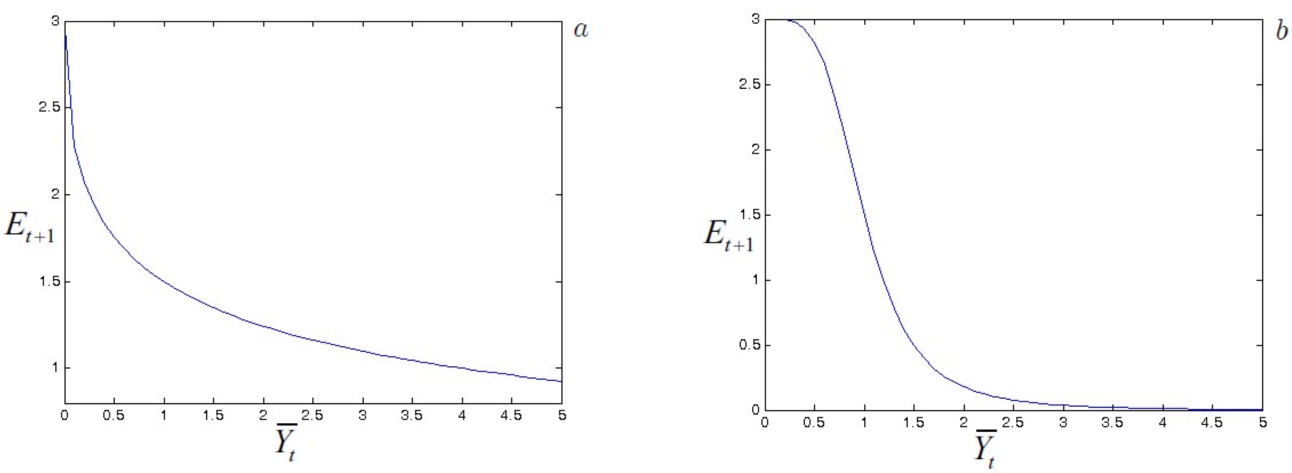


**Figure 1**. (a) Graph of $E_{t+1}$ when $\rho \le 1$. (b) Graph of $E_{t+1}$ when $\rho > 1$.

The function specified in (6) presents some advantages with respect to the functional specifications proposed by John and Pecchenino (1994) and Antoci and Sodini (2009, 2010). On the one hand, it takes into account the possible existence of nonlinear relationships between economic activity and environmental quality (as suggested by Rosser, 2001); on the other hand, it allows us to have an index that takes non-negative values for all levels of average production (this property is useful for several specifications of the utility function).

The market-clearing condition in the capital market can be expressed as follows:

$$K_{t+1} = s_t = w_t \ell_t \,. \tag{7}$$

Knowing that individuals have perfect foresight and $\overline{Y}_t = Y_t$ (i.e. the economy-wide average output coincides, ex post, with the output of the representative firm), by using equations (3)-(7) one yields the following equilibrium conditions:

$$-(2-\ell_t)^{-\gamma} + \frac{(A^2\alpha(1-\alpha)K_t^{\alpha}\ell_t^{1-\alpha}K_{t+1}^{\alpha-1}\ell_{t+1}^{1-\alpha})^{1-\sigma}}{\ell_t}\left(\frac{\overline{E}}{1+[A\cdot F(K_t,l_t)]^{\rho}}\right)^{\tau(1-\sigma)} = 0\,, \tag{8}$$

$$K_{t+1} = (1-\alpha)AK_t^{\alpha}\ell_t^{1-\alpha}\,. \tag{9}$$

The system (8)-(9) defines $K_{t+1}$ and $\ell_{t+1}$ as functions of $K_t$ and $\ell_t$, that is:

$$M:\begin{cases} K_{t+1} = V(K_t,\ell_t) := (1-\alpha)AK_t^{\alpha}\ell_t^{1-\alpha} \\ \ell_{t+1} = Z(K_t,\ell_t) := K_t^{-\frac{\alpha^2}{1-\alpha}}\left(\frac{\ell_t^{1-\alpha(1-\alpha)(1-\sigma)}}{(2-\ell_t)^{\gamma}}\right)^{\frac{1}{(1-\alpha)(1-\sigma)}}\left(\frac{[1+(AK_t^{\alpha}\ell_t^{1-\alpha})^{\rho}]^{\tau}}{\alpha(1-\alpha)^{\alpha}A^{1+\alpha}\overline{E}^{\tau}}\right)^{\frac{1}{1-\alpha}} \end{cases} \tag{10}$$

The two-dimensional map $M$ in (10) is defined on set:

$$D := \{(K_0,\ell_0)\in R^2 \;:\; K_t>0,\; 0<\ell_t<2,\, \forall t>0\}\,.$$

## 3. Local analysis: existence and stability of fixed points

Let us first consider the fixed points of map (10). According to the first equation in (10), $K/\ell = [(1-\alpha)A]^{\frac{1}{1-\alpha}}$ always holds at a fixed point of the map and the steady-state values of $\ell$ ($\ell_{ss}$, which represents a generic steady-state value of $\ell$) are solutions of $\ell = G(\ell) = Z([(1-\alpha)A]^{\frac{1}{1-\alpha}}\ell,\ell)$ or, equivalently, they are solutions of the following equation:

$$f(\ell) := \frac{G(\ell)}{\ell} = \left(\frac{\ell^{\sigma}}{(2-\ell)^{\gamma}}\right)^{\frac{1}{1-\sigma}} \frac{\left[1+(1-\alpha)^{\frac{\alpha\rho}{1-\alpha}}A^{\frac{\rho}{1-\alpha}}\ell^{\rho}\right]^{\tau}}{\alpha(1-\alpha)^{\frac{\alpha}{1-\alpha}}A^{\frac{1}{1-\alpha}}\overline{E}^{\tau}} = 1\,. \tag{11}$$

We can now state the following propositions characterising existence, multiplicity (Proposition 1) and stability of fixed points (Proposition 2).

**Proposition 1**. *Map $M$ always admits at least one fixed point on set $D$. If $0<\sigma<1$, then the fixed point is unique. If $\sigma>1$ then one fixed point exists or three fixed points exist.*

**Proof**. First, note that if $0<\sigma<1$ then $\lim_{\ell\to 0^+} f(\ell) = 0$ and $\lim_{\ell\to 2^-} f(\ell) = +\infty$ hold; if $\sigma>1$, then

$\lim_{\ell \to 0^+} f(\ell) = +\infty$ and $\lim_{\ell \to 2^-} f(\ell) = 0$ hold. In order to study the number of fixed points of map $M$ observe that $\operatorname{sgn}(f'(\ell)) = (1-\sigma)\operatorname{sgn}(h(\ell))$, where:

$$h(\ell) := (1-\alpha)^{\frac{\alpha\rho}{1-\alpha}} A^{\frac{\rho}{1-\alpha}} \ell^{\rho} \{[\gamma - \sigma - \tau\rho(1-\sigma)]\ell + 2\sigma + 2\tau\rho(1-\sigma)\} + (\gamma - \sigma)\ell + 2\sigma . \qquad (12)$$

We can now distinguish two cases:

(1) Case $0 < \sigma < 1$. We observe that $h(\ell) > 0$ always holds. In fact, the addendum $(\gamma - \sigma)\ell + 2\sigma$ in (12) is positive, whereas the expression in brackets is increasing (resp. decreasing) in $\ell$ and positive for $\ell = 0$ (resp. $\ell = 2$) if $\gamma > \sigma + \tau\rho(1-\sigma)$ (resp. $\gamma < \sigma + \tau\rho(1-\sigma)$).

(2) Case $\sigma > 1$. Note that $h'' = 0$ at most for a value of $\ell$, namely $\hat{\hat{\ell}}$, where:

$$\hat{\hat{\ell}} := \frac{2(\rho-1)[\tau\rho(\sigma-1) - \sigma]}{(\rho+1)[\tau\rho(\sigma-1) - \sigma + \gamma]} . \qquad (13)$$

If $\hat{\hat{\ell}} \notin (0,2)$ then the sign of $h''$ does not change in $(0,2)$. This implies that $h'$ has at most one zero and therefore $h$ has at most two zeros in $(0,2)$. Consequently, there exist at most three values of $\ell$ satisfying equation (11). With similar arguments, it can be checked that $h$ has at most three zeros in $(0,2)$ if $\hat{\hat{\ell}} \in (0,2)$. By taking into account of the behaviour of function $f$ close to the extrema of the interval $(0,2)$, there exist at most three values of $\ell$ satisfying equation (11). **Q.E.D.**

**Corollary 1**. *If there exists a value $0 < \tilde{\ell} < 2$ such that $f'(\tilde{\ell}) = 0$, then there exist two threshold values $\overline{E}_1$ and $\overline{E}_2$, with $\overline{E}_1 < \overline{E}_2$, such that when $\overline{E} < \overline{E}_1$ or $\overline{E} > \overline{E}_2$ a unique fixed point does exist. If $\overline{E}_1 < \overline{E} < \overline{E}_2$ then the fixed points are three.*

**Proof**. The existence of $0 < \tilde{\ell} < 2$ implies the existence of two extrema for $f(\ell)$. By observing the role of $\overline{E}$ in $f(\ell)$, we get the result. **Q.E.D.**

**Remark 1**. *We do not have studied the number of fixed points directly as solutions of the system:*

$$\begin{cases} K = V(K,\ell) \\ \ell = Z(K,\ell) \end{cases} \qquad (14)$$

*in $(K,\ell)$ plane, as the second equation in (14) does not generally define $K$ as a function of $\ell$ (i.e., $K = K(\ell)$) or $\ell$ as a function of $K$ (i.e., $\ell = \ell(K)$). This makes it difficult to study the location and the identification of intersection points of the curves in (14). Then, this approach does not allow us to get a clear-cut analysis of the problem.*

Now, it is useful to recall that fixed points correspond to the solutions of equation $\ell = G(\ell)$, that is they are determined by the values of $\ell$ at the intersection points between curves $x = \ell$ and $x = G(\ell)$. We will show that the value of $G'(\ell)$ evaluated at such intersection points will be crucial to classify the dynamic properties of the fixed points.

**Corollary 2**. *If $0 < \sigma < 1$, the graph of $x = G(\ell)$ crosses the line $x = \ell$ from below. Then, in the unique intersection point we have that $G'(\ell^*) > 1$, where $\ell^*$ is the stationary-state value of $\ell$. If $\sigma > 1$ and a unique fixed point, $(K^*, \ell^*)$, does exist such that $G(\ell^*) = \ell^*$, then $G'(\ell^*) < 1$. If $\sigma > 1$ and three fixed points of $G(\ell) = \ell$ do exist, that is $(K_1^*, \ell_1^*)$, $(K_2^*, \ell_2^*)$ and $(K_3^*, \ell_3^*)$, then $G'(\ell_1^*) < 1$, $G'(\ell_2^*) > 1$ and $G'(\ell_3^*) < 1$.*

Based on Corollary 2, it is possible to show that if the environmental good and the consumption good are complements ($0 < \sigma < 1$) the fixed point is a saddle (determinate) for all parameter values. In contrast, when the environmental good and the consumption good are substitutes ($\sigma > 1$), two cases are possible: a unique fixed point does exist; three fixed points do exist. In the former case, the unique fixed point can be a saddle or a sink. In the latter case, unfortunately (by comparing this work with Agliari and Vachadze, 2011) none of the three fixed points has dynamic properties independent of the parametrisation used. The following proposition introduces some conditions (not easy to read) that characterise the dynamic properties of fixed points. They tell us that a generic fixed point $(K_{ss}, \ell_{ss})$ is a saddle if $G'(\ell_{ss})$ is neither too large nor too small. Instead, when $G'(\ell_{ss})$ is sufficiently large (resp. small), the fixed point is a source (resp. sink).

**Proposition 2**. *If $0 < \sigma < 1$, then the fixed point is a saddle. If $\sigma > 1$, we have three cases. 1) If (a) $G'(\ell_{ss}) > \dfrac{2Z'_\ell(K_{ss}, \ell_{ss}) + 1 + \alpha}{1 - \alpha}$ and $G'(\ell_{ss}) < 1$ or (b) $G'(\ell_{ss}) < \dfrac{2Z'_\ell(K_{ss}, \ell_{ss}) + 1 + \alpha}{1 - \alpha}$ and $G'(\ell_{ss}) > 1$, then the fixed point is a saddle. 2) If $G'(\ell_{ss}) > \max\left\{1, \dfrac{2Z'_\ell(K_{ss}, \ell_{ss}) + 1 + \alpha}{1 - \alpha}\right\}$, then the fixed point is a source. 3) If $G'(\ell_{ss}) < \min\left\{1, \dfrac{2Z'_\ell(K_{ss}, \ell_{ss}) + 1 + \alpha}{1 - \alpha}\right\}$, then the fixed point is a sink.*

**Proof**. Let us consider the Jacobian matrix associated with map $M$ evaluated at a generic fixed point $(K_{ss}, \ell_{ss})$:

$$J(K_{ss},\ell_{ss})=\begin{pmatrix}\alpha & (1-\alpha)^{\frac{2-\alpha}{1-\alpha}}A^{\frac{1}{1-\alpha}}\\ Z'_{K_t}(K_{ss},\ell_{ss}) & Z'_{\ell_t}(K_{ss},\ell_{ss})\end{pmatrix}. \tag{15}$$

We recall that it is possible to investigate the nature of the fixed points by using trace and determinant of $J(K_{ss},\ell_{ss})$. By direct calculations, we get:

$$Det(J(K_{ss},\ell_{ss}))=$$
$$=\frac{\left(\dfrac{\ell_{ss}^{1-2\alpha(1-\alpha)(1-\sigma)}}{(2-\ell_{ss})^{\gamma+(1-\alpha)(1-\sigma)}}\right)^{\frac{1}{(1-\alpha)(1-\sigma)}}\left(\dfrac{[1+(AK_{ss}^{\alpha}\ell_{ss}^{1-\alpha})^{\rho}]^{\tau}\,\overline{E}^{-\tau}}{\alpha(1-\alpha)^{\alpha}A^{1+\alpha}}\right)^{\frac{1}{1-\alpha}}\alpha A(2-\ell_{ss}+\gamma\ell_{ss})}{K_{ss}^{\frac{\alpha^2+(1-\alpha)^2}{1-\alpha}}(1-\sigma)}, \tag{16}$$

which is positive for $0<\sigma<1$ and negative for $\sigma>1$, and

$$Tr(J(K_{ss},\ell_{ss}))=\alpha+Z'_{\ell_t}(K_{ss},\ell_{ss}). \tag{17}$$

To clarify the proof of the proposition, we recall that when $0<\sigma<1$ (that is, $Det(J(K_{ss},\ell_{ss}))>0$) the fixed point is a saddle if $1-Tr+Det<0$ or $1+Tr+Det<0$. Instead, when $\sigma>1$ (that is, $Det(J(K_{ss},\ell_{ss}))<0$) the fixed point is:

1) a saddle in two different cases, that is (a) when $1+Tr+Det<0$ and $1-Tr+Det>0$ or (b) when $1+Tr+Det>0$ and $1-Tr+Det<0$;

2) a source if $1+Tr+Det<0$ and $1-Tr+Det<0$;

3) a sink if $1+Tr+Det>0$ and $1-Tr+Det>0$.

In order to show the results about the stability of fixed points, we note that:

$$\begin{aligned}1-Tr+Det&=1-\alpha-Z'_{\ell_t}(K_{ss},\ell_{ss})+\alpha Z'_{\ell_t}(K_{ss},\ell_{ss})-Z'_{K_t}(K_{ss},\ell_{ss})V'_{\ell_t}(K_{ss},\ell_{ss})=\\&=(1-\alpha)[1-Z'_{\ell_t}(K_{ss},\ell_{ss})]-Z'_{K_t}(K_{ss},\ell_{ss})V'_{\ell_t}(K_{ss},\ell_{ss})\end{aligned}.$$

Thus, $1-Tr+Det<0$ if and only if $G'(\ell_{ss})=Z'_{K_t}(K_{ss},\ell_{ss})\dfrac{V'_{\ell_t}(K_{ss},\ell_{ss})}{1-\alpha}+Z'_{\ell_t}(K_{ss},\ell_{ss})>1$.

In addition,

$$1+Tr+Det=1-Tr+Det+2Tr.$$

Thus, $1+Tr+Det<0$ if and only if

$$1-Tr+Det<-2Tr \Leftrightarrow$$
$$1-Z'_{\ell_t}(K_{ss},\ell_{ss})-Z'_{K_t}(K_{ss},\ell_{ss})\frac{V'_{\ell_t}(K_{ss},\ell_{ss})}{1-\alpha}<\frac{-2\left[\alpha+Z'_{\ell_t}(K_{ss},\ell_{ss})\right]}{1-\alpha}\Leftrightarrow$$
$$-Z'_{\ell_t}(K_{ss},\ell_{ss})-Z'_{K_t}(K_{ss},\ell_{ss})\frac{V'_{\ell_t}(K_{ss},\ell_{ss})}{1-\alpha}<-1-\frac{2\left[\alpha+Z'_{\ell_t}(K_{ss},\ell_{ss})\right]}{1-\alpha}\Leftrightarrow$$
$$G'(\ell_{ss})>\frac{2Z'_{\ell}(K_{ss},\ell_{ss})+1+\alpha}{1-\alpha}$$

By combining these results and taking into account Corollary 2, we have the result. **Q.E.D.**

By taking into account Proposition 2 and Corollary 2, we have the following cases that will be highlighted in the global analysis illustrated in Section 4.

**Corollary 3**. *From Corollary 2 and Proposition 2 it follows that* $(K_1^*,\ell_1^*)$ *can be a sink or a saddle,* $(K_3^*,\ell_3^*)$ *can be a sink or a saddle, whereas* $(K_2^*,\ell_2^*)$ *can be a source or a saddle.*

## 4. Global analysis

### *4.1. Invertibility of map $M$*

We start the global analysis by showing that map $M$ is invertible. This is actually an important step in the study of the global analysis. For instance, such a property implies that the basin of attraction of any attracting set is a connected set. Furthermore, by making use of the inverse map, it is possible to get the stable manifolds of saddle points. The following proposition holds.

**Proposition 3**. *Map $M$ is invertible on set $D$.*

**Proof**. It is not possible to write the inverse map of $M$, namely $M^{-1}$, in closed-form. However, after some algebraic manipulations, we find that $M^{-1}$ is defined by the following system:

$$M^{-1}:\begin{cases}\dfrac{\ell_t}{(2-\ell_t)^{\gamma}}=\dfrac{\ell_{t+1}^{(1-\alpha)(1-\sigma)}K_{t+1}^{\alpha}}{B\{1+[K_{t+1}/(1-\alpha)]^{\rho}\}^{\tau(1-\sigma)}}\\[2ex] K_t=\left(\dfrac{K_{t+1}\ell_t^{\alpha-1}}{(1-\alpha)A}\right)^{\frac{1}{\alpha}}\end{cases}\tag{18}$$

where $B:=[\overline{E}^{\tau}\alpha(1-\alpha)^{2\alpha}A^{1+2\alpha}]^{\sigma-1}$. It is easy to verify that, given a couple $(K_{t+1},\ell_{t+1})\in D$, there always exists a unique couple $(K_t,\ell_t)\in D$ satisfying system (18). In fact, the left-hand side of the first equation in (18) represents a one-to-one correspondence between the interval $[0,2)$ and the set of positive real numbers. **Q.E.D.**

*4.2. Numerical simulations*

To carry on with the global analysis of map $M$ we first recall the definitions of the stable and unstable manifolds of a periodic point $p$ of period $z$, respectively given by:

$$W^s(p) = \{x : M^{zn}(x) \to p \; as \; n \to +\infty\}, \tag{19}$$

and

$$W^u(p) = \{x : M^{zn}(x) \to p \; as \; n \to -\infty\}. \tag{20}$$

If the periodic point $p \in R^2$ is a saddle, then the stable (resp. unstable) manifold of $p$ is a smooth curve passing through $p$, tangent at $p$ to the eigenvector of the Jacobian matrix evaluated at $p$ corresponding to the eigenvalue $\lambda$ with $|\lambda| < 1$ (resp. $|\lambda| > 1$). Outside the small neighbourhood of $p$ to which the local stability analysis refers, the stable and unstable manifolds may intersect each other with important consequences on the global dynamics of the model (see, e.g., Guckenheimer and Holmes, 1983, p. 22). However, in the present article we will concentrate especially on the role that the stable manifold plays either representing itself an equilibrium path that the economy may follow or as separatrix between basins of attraction. This last property is important as it is strongly related to global indeterminacy (see Agliari and Vachadze, 2011; Gori and Sodini, 2011).

Let us briefly recall that in the two-dimensional model studied in this article, the labour supply $\ell_t$ is a control variable and the capital stock $K_t$ is a state variable (in other words, the initial value $K_0$ is given whereas the initial value $\ell_0$ is fixed after the agent's optimisation process). Then:

1) A stationary equilibrium $(K^*, \ell^*)$ is said to be locally indeterminate if it is an attractor from a dynamic point of view (i.e., the absolute value of $|\lambda_i| < 1$, $i = 1,2$).

2) A stationary equilibrium $(K^*, \ell^*)$ is said to be locally determined if from a dynamic point of view it is a saddle, that is an eigenvalue (in modulus) is smaller than 1 and the other eigenvalue (in modulus) is larger than 1. We recall that associated with this specific dynamic study, a stationary equilibrium can be reached by the economy if, given a value of $K_0$, the optimisation process of the individual generates a value $\ell_0$ such that $(K_0, \ell_0)$ lies on the stable manifold of $(K^*, \ell^*)$. This property, often defined in the economic literature as saddle point (or saddle path) stability has a justification in infinite horizon dynamic models due to the transversality condition that must be taken into account in the optimisation problem. In other words, trajectories that do not lie on the saddle path generate unfeasible or non-optimising trajectories. However, the economic literature often considers that a saddle point is stable point even in finite horizon models with optimising

agents. Although the reasons for this may have economic justifications (given an initial condition of a macroeconomic state variable it is possible to generate the trajectory that the economy will follow from that point forward), from a mathematical point of view the justifications are almost absent. In recent years, in fact, several works in the economic literature have concentrated on the study of models where given an initial value of the state variable ($K_0$), there exist several admissible initial values of the control variable $\ell_0$ and therefore different trajectories leading towards the same stationary equilibrium.

3) A stationary equilibrium $(K^*,\ell^*)$ is said to be unstable if both the eigenvalues (in modulus) are larger than 1.

4) A model is said to be globally indeterminate if there is coexistence of attractors and/or saddles.

The aim of this section is to show some global phenomena related to map $M$. To this purpose, we will use four different parameter sets and classify some phenomena that can be observed when $\rho$ varies. Remember that $\rho$ captures how production affects the environment (see Figure 1).

*4.2.1. Example 1*

The parameter set used in the first example is the following: $A=1.25$, $\overline{E}=12.2$, $\alpha=0.2$, $\gamma=1.12$, $\sigma=3$ and $\tau=1$. In this case, when $\rho<0.873$ a unique fixed point $(K_1^*,\ell_1^*)$ does exist and it is a saddle point. This implies that given an initial value $K_0$ of the pre-determined variable $K_t$ close enough to $(K_1^*,\ell_1^*)$, there exists a unique initial value $\ell_0$ of $\ell_t$ such that the trajectory starting from $(K_0,\ell_0)$ approaches $(K_1^*,\ell_1^*)$, that is, the fixed point is locally determinate (saddle). A different choice of $\ell_0$ would give rise to a trajectory leaving set $D$ in finite time (see Figure 2(a), plotted for $\rho=0.2$). In this case, the labour supply is relatively inelastic, in that it exhibits low dependence on variations in the capital stock $K_t$.

By considering a higher value of $\rho$ ($\rho=1$), the fixed point $(K_1^*,\ell_1^*)$ becomes stable (both eigenvalues lie inside the unit circle; a reverse flip bifurcation occurs for $\rho\cong 0.873$). This implies local indeterminacy (a stable fixed point): given an initial condition $K_0$ close enough to $K_1^*$, there exists a continuum of initial values $\ell_0$ such that the trajectory starting from $(K_0,\ell_0)$ converges to $(K_1^*,\ell_1^*)$. In this context, however, a unique attractor does exist and all feasible trajectories (i.e.

those belonging to set $D$) converge towards it. Figure 2(b) represents (in grey) the basin of attraction of the fixed point $(K_1^*, \ell_1^*)$ and in white the region of unfeasible trajectories, that is the trajectories that exit from $D$ after a finite number of iterations (the same rule is used for all the basins of attraction reported in the present article). Notice that indeterminacy (about the choice of $\ell_0$) is observed for every initial value $K_0$ belonging to the range of $K$-values considered in the numerical simulation in Figure 2(b).

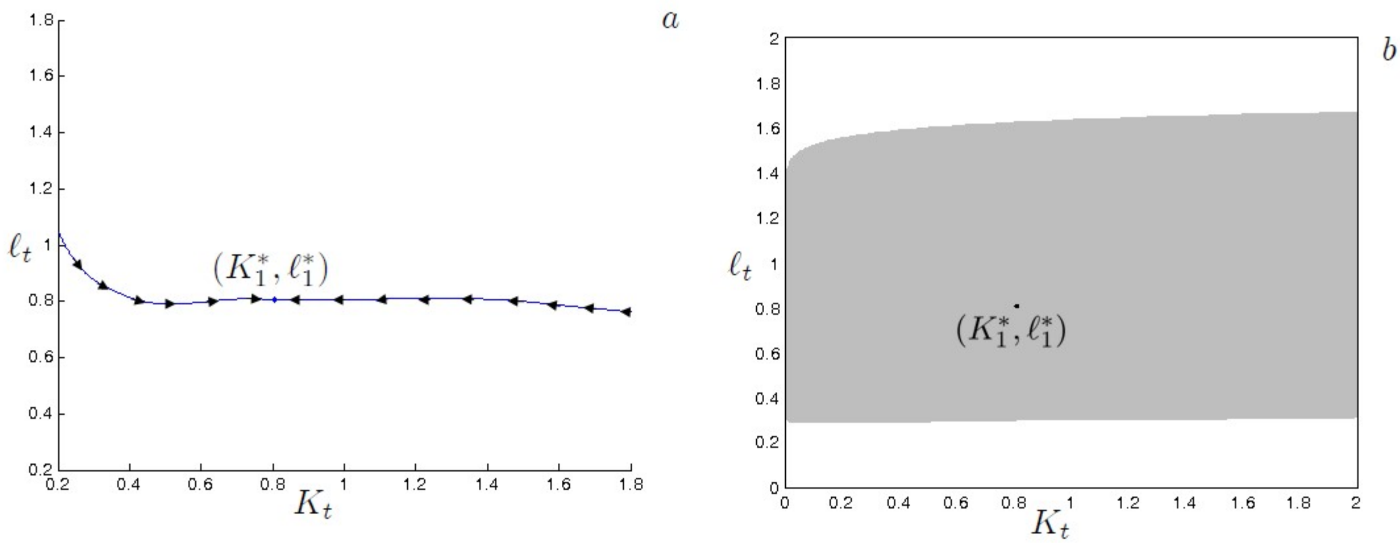


**Figure 2**. (a) Stable manifold of the locally determinate fixed point $(K_1^*, \ell_1^*)$, obtained for $\rho = 0.2$. (b) Basin of attraction (in grey) of the locally indeterminate fixed point $(K_1^*, \ell_1^*)$, obtained for $\rho = 1$.

A further increase in $\rho$ causes a saddle node bifurcation (for $\rho \cong 3.08$): two attracting fixed points, $(K_1^*, \ell_1^*)$ and $(K_3^*, \ell_3^*)$, and one saddle point $(K_2^*, \ell_2^*)$ arise (as is shown in Figure 3(a)). We note that $(K_1^*, \ell_1^*)$ is characterised by a low level of capital accumulation and a low labour supply of individuals (working time). In contrast, $(K_3^*, \ell_3^*)$ is characterised by a high level of capital accumulation and a high working time. Finally, $(K_2^*, \ell_2^*)$ represents an intermediate situation between the preceding two. However, it is important to stress that due to a high environmental quality and a large time devoted to leisure, well-being in $(K_1^*, \ell_1^*)$ is higher than in $(K_2^*, \ell_2^*)$ and $(K_3^*, \ell_3^*)$. In fact, we have that $U(K_1^*, \ell_1^*) = -8.57$, $U(K_2^*, \ell_2^*) = -8.88$ and $U(K_3^*, \ell_3^*) = -11.29$. In this context, there exist initial values $K_0$ from which it is possible to reach all coexisting fixed points by opportunely choosing the initial value $\ell_0$ of the control variable. This is an interesting example of global indeterminacy. By starting from the same initial condition $K_0$, the trajectory followed by the economy can exhibit rather different long-term behaviour according to the initial choice $\ell_0$. In Figure 3(a), the basins of attraction of the states $(K_1^*, \ell_1^*)$ and $(K_3^*, \ell_3^*)$ are separated by the stable manifold of the saddle point $(K_2^*, \ell_2^*)$ (represented by a red point in the figure).[3] The dashed (black) vertical line is traced out in correspondence of one of the initial values $K_0$ ($\overline{K}_0$ in Figure 3(a)), from which it is possible to reach all the three fixed points by opportunely choosing $\ell_0$. Note that, in contexts of global indeterminacy, the local stability analysis may be misleading (as stressed, e.g., by Cazzavillan et al., 1998; Coury and Wen, 2009; Mattana et al., 2009; Antoci et al., 2011, 2014), in that it refers to a small neighbourhood of a fixed point, whereas $\ell_0$ can be chosen in the whole interval (0,2). According to the local analysis, the saddle point $(K_2^*, \ell_2^*)$ is locally determinate. However, as Figure 3(a) shows, by starting from an initial value $K_0$ very close to (or also equal to) the $K$-coordinate of the saddle $(K_2^*, \ell_2^*)$, not only $(K_2^*, \ell_2^*)$ but also $(K_1^*, \ell_1^*)$ and $(K_3^*, \ell_3^*)$ can be reached by opportunely choosing $\ell_0$.

If $\rho$ further increases, the attracting fixed point $(K_1^*, \ell_1^*)$ approaches the saddle point $(K_2^*, \ell_2^*)$. In addition, due to the increase in $\rho$ the fixed point $(K_3^*, \ell_3^*)$ loses its stability (it becomes a saddle point) in favour of a period-2 cycle via a supercritical flip bifurcation. After the collision, $(K_1^*, \ell_1^*)$ and $(K_2^*, \ell_2^*)$ disappear and almost all trajectories belonging to $D$ are captured by the period-2 cycle arisen from $(K_3^*, \ell_3^*)$. Figure 3(b) is obtained (for $\rho = 3.28$) by considering the second iterate of map $M$, that is $M^2$. This procedure transforms the attracting cycle of map $M$ in two attracting fixed points ($P_1$ and $P_2$) of $M^2$. The basins of attraction of $P_1$ and $P_2$ are separated by the stable manifold of $(K_3^*, \ell_3^*)$, which is the separating set between the dark grey and light grey regions in the figure. As compared with the context illustrated in Figure 2(a), in this case the shape of the stable manifold is more complicated. However, this difference (that has noteworthy economic implications) cannot be pointed out by the local analysis only. In this case, in fact, starting from an initial value $K_0$ far enough from $K_3^*$, there exist several initial values $\ell_0$ such that the state $(K_0, \ell_0)$ belongs to the stable manifold of $(K_3^*, \ell_3^*)$. Consequently, starting from $K_0$ the economy may approach the locally determinate fixed point $(K_3^*, \ell_3^*)$ by following very different transition paths.[4] Furthermore, as in the scenario illustrated in Figure 3(b), given the initial value of $K_0$, there exist

[3] The saddles (when reported in figures) are always portrayed as red points throughout the article.

[4] Note that the complicated structure of the stable manifold can generate non-monotonic dynamics approaching the saddle point $(K_3^*, \ell_3^*)$.

other values (an infinite number) of $\ell_0$ leading towards the period-2 cycle, we get a case of global indeterminacy (multistability) similar to the one analysed by Coury and Wen (2009). In such a case, globally indeterminacy occurs in a context in which a unique fixed point exists and it is locally determinate. Figures 3(c) and 3(d) show the role played by an increase in $\rho$ on the time evolution of the capital stock and on generations' well-being (measured by the value of the utility function $U_t$), respectively. They illustrate the evolution of $K_t$ and $U_t$ in an economy by starting from an initial condition lying in the dark grey region of Figure 3(a), and subject to an exogenous change in $\rho$ occurring after 40 iterates. The dashed lines in such figures represent the evolution of $K_t$ and $U_t$ along the original trajectory (that is, the one the economy would follow in absence of the exogenous shock on $\rho$). Notice that the increase in $\rho$ generates an increase of capital accumulation and a reduction in well-being.

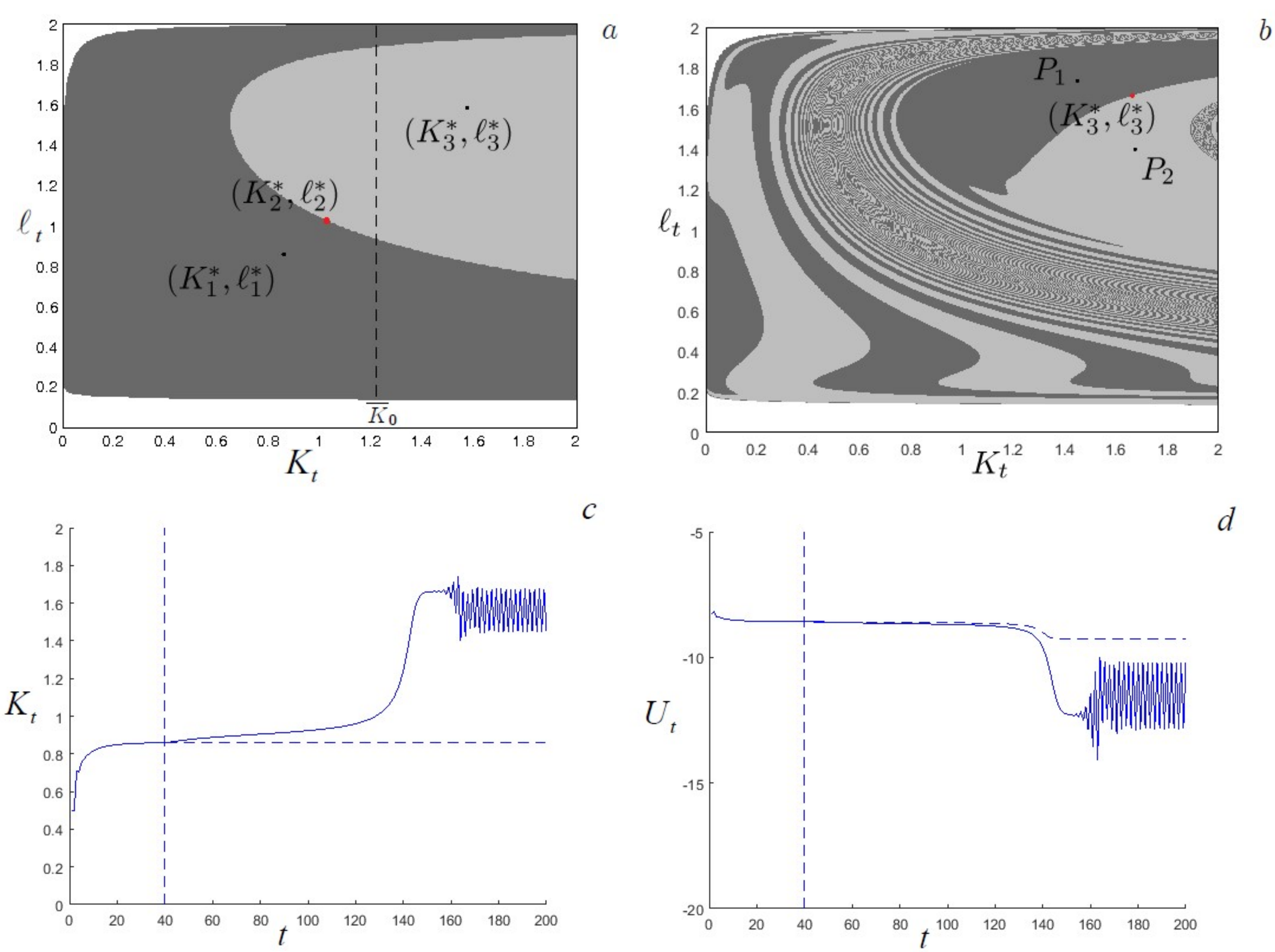

**Figure 3**. (a) Global indeterminacy ( $\rho = 3.2$ ): two attracting fixed points, $(K_1^*, \ell_1^*)$ and $(K_3^*, \ell_3^*)$, coexist with the saddle $(K_2^*, \ell_2^*)$. (b) For $\rho = 3.28$, a unique saddle point $(K_3^*, \ell_3^*)$ exists surrounded by an attracting period-2 cycle ( $P_1$ and $P_2$ ). (c) The evolution of capital accumulation over time. (d) The evolution of well-being over time. The vertical lines in Figures 3(c) and 3(d) identify the time at which the exogenous shock in $\rho$ occur.

*4.2.2. Example 2*

We now let $\sigma$ increase from 3 to 3.274 and leave the other parameters as in Example 1. Until now, the dynamics of the model – although interesting from an economic point of view – are rather trivial. However, with these parameter values we may observe and characterise very interesting dynamic phenomena. By starting from a dynamic scenario qualitatively similar to the one illustrated in Figure 3(a) (that can be obtained, for example, by posing $\rho = 3.1$), if $\rho$ increases due to an exogenous shock the dynamic system undergoes a sequence of flip bifurcations around $(K_3^*, \ell_3^*)$ leading to a chaotic attractor when $\rho$ is close to $3.2465$. Then, predicting the long-term behaviour of the economy becomes rather difficult. Small changes in the initial values of $K_0$ (*history matters*) or $\ell_0$ (*expectations matter*) may produce a change in the $\omega$-limit set that the economy can approach. Figure 4(a) shows the bifurcation diagram generated by changing $\rho$. Figure 4(b) illustrates the basins of attraction when the attracting fixed point $(K_1^*, \ell_1^*)$ coexists with a four-piece chaotic attractor $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (this scenario is obtained with $\rho = 3.2465$). The basin of attraction of the fixed point $(K_1^*, \ell_1^*)$ is depicted in dark grey, whereas the basin of $\Gamma$ is depicted in light grey. Notice that the basins of attraction of these attractors are rather tangled and therefore Figure 4(b) represents a very interesting example of global indeterminacy (multistability). This case shows how focusing only on the local analysis can actually be misleading. In fact, by the linearisation of the system around the equilibria we would observe only the coexistence of two saddle points and a sink.

The sensitivity of the dynamic system (10) to the magnitude of the environmental impact of production activity can be stressed by observing the effects of a further slight change in $\rho$. As Figure 5 illustrates, for $\rho = 3.2$, three attractors coexist and the dynamics undergo important changes. An attracting fixed point $(K_1^*, \ell_1^*)$, whose basin of attraction is depicted in dark grey, coexists with a period-2 cycle ( $P_1$ and $P_2$ ) -whose basin of attraction is depicted in light grey- and with an eight-piece chaotic attractor $\Gamma = \bigcup_{i=1}^{8} \Gamma_i$ (not labelled in the figure) -whose basin of attraction is depicted in yellow.

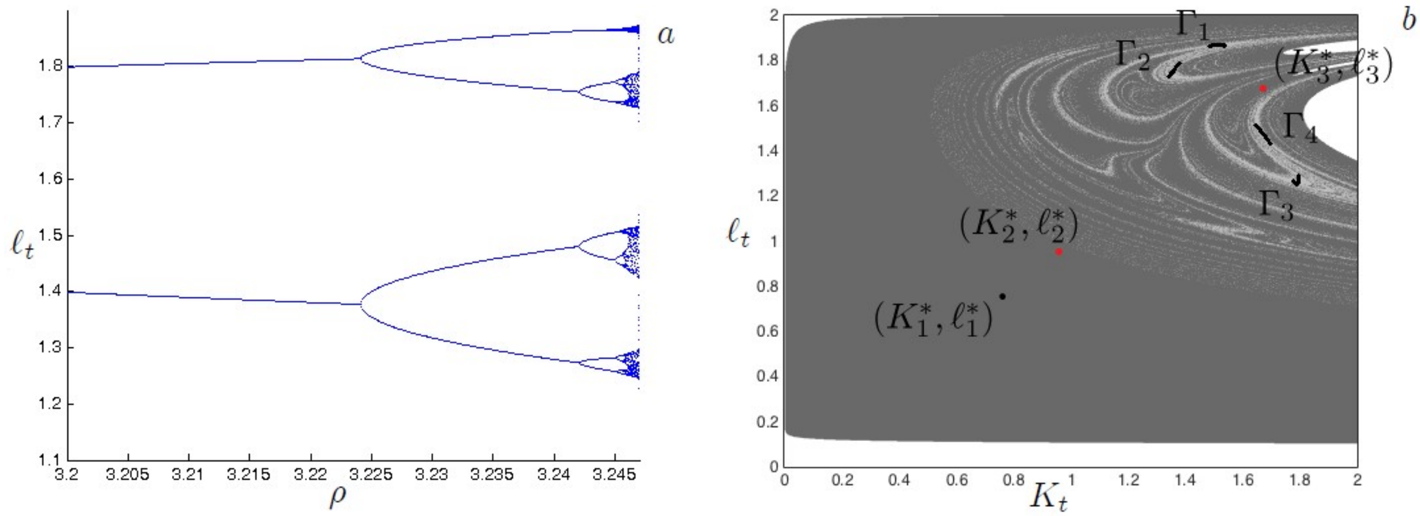


**Figure 4**. (a) Bifurcation diagram for $\rho$. Due to the rise of attractors with rather tangled basins of attraction, the initial conditions were opportunely adapted to the change in $\rho$ in order to follow the evolution of each attractor. (b) Basins of attraction when the attracting fixed point $(K_1^*, \ell_1^*)$ coexists with a four-piece chaotic attractor ( $\rho = 3.2465$).

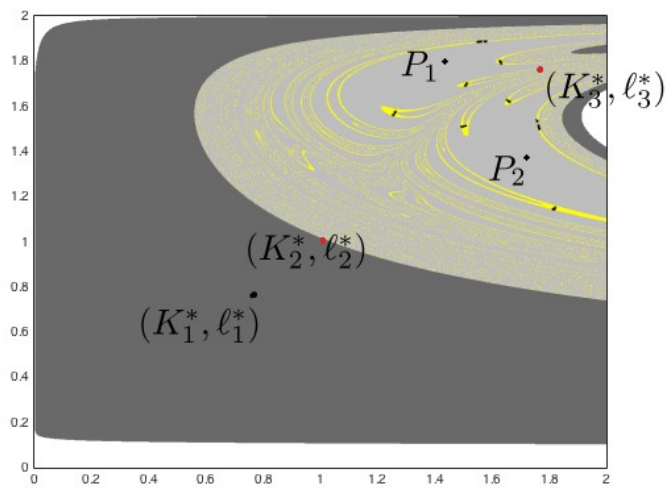


**Figure 5.** Coexistence of three attractors for $\rho = 3.2$ .

*4.2.3. Example 3*

In this example, we consider the following parameter specifications: $A = 1.5384$ , $\overline{E} = 9.509568329$ , $\alpha = 0.35$ , $\gamma = 1.7$ , $\sigma = 2.8$ and $\tau = 1$ . By starting from a dynamic regime where two attracting fixed points $(K_1^*, \ell_1^*)$ and $(K_3^*, \ell_3^*)$, and a saddle point $(K_2^*, \ell_2^*)$, with $K_1^* < K_2^* < K_3^*$, coexist (similar to the case illustrated in Figure 3(a)), Figure 6(a) shows that after a first flip bifurcation of $(K_3^*, \ell_3^*)$, the cycle of period 2 generated by such a bifurcation undergoes a super-critical Neimark-Sacker bifurcation when $\rho$ increases. This gives rise to a two-piece attracting invariant curve ($\Gamma = \Gamma_1 \cup \Gamma_2$) that generates quasi-periodic dynamics. By the use of map $M^2$, the figure shows a portion of the stable manifold of the saddle $(K_2^*, \ell_2^*)$ (represented by the boundary between the dark grey region and the region in its interior) and a portion of the stable manifold of the saddle $(K_3^*, \ell_3^*)$ (represented by the boundary between the blue and light grey regions). A further increase in $\rho$ causes a global bifurcation, that is a contact between the invariant curve $\Gamma$ and the stable manifold of the intermediate fixed point $(K_2^*, \ell_2^*)$. Such a contact implies the disappearance of $\Gamma$ so that $(K_1^*, \ell_1^*)$ remains the sole attractor of the system, whereas $(K_2^*, \ell_2^*)$ and $(K_3^*, \ell_3^*)$ remain saddles (Figure 6(b)).

From the economic point of view, the global bifurcation that marks the transition from Figure 6(a) to Figure 6(b) has an interesting interpretation. In fact, by looking at Figure 6(a) it is possible to note that before the occurrence of such a bifurcation, trajectories starting from initial conditions that fall within the light grey area are characterised by significant fluctuations both in capital and in labour supply. These fluctuations, however, occur around high levels of both the variables. This also implies that the time evolution of environmental quality is characterised by cyclical dynamics with a high degree of environmental degradation. After the occurrence of the global bifurcation, the economy converges toward a long-term equilibrium characterised by smaller production (therefore with lower levels of capital and labour), but with a higher level of environmental quality. In this case, we underline a counter-intuitive phenomenon as measured by the increase in the environmental impact $\rho$. Specifically, an increase in $\rho$ first causes a strong instability of the dynamic system, but there exists a threshold beyond which the sharp fluctuations de facto induce positive feedback for economic agents that in turn coordinate themselves on trajectories converging towards $(K_1^*, \ell_1^*)$.

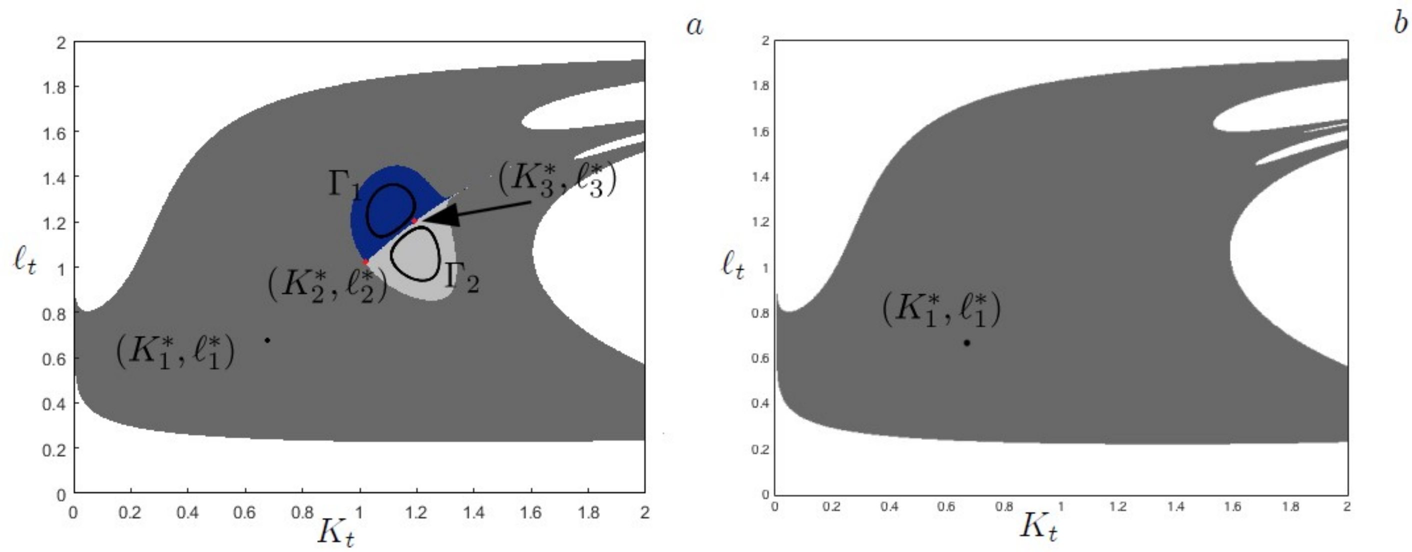


**Figure 6.** (a) Coexistence of an attracting fixed point $(K_1^*, \ell_1^*)$, whose basin of attraction is depicted in dark grey, and an attractor born after a Neimark-Sacker bifurcation around a period-2 cycle, whose basin of attraction is depicted in light grey ( $\rho = 3.28$). (b) $(K_1^*, \ell_1^*)$ is the unique attractor of the system ( $\rho = 3.3$).

*4.2.4. Example 4*

Finally, let us now consider the following parameter specification: $A = 1.5384$, $\overline{E} = 9.509568329$, $\alpha = 0.35$, $\gamma = 1.73$, $\sigma = 2.8$ and $\tau = 0.9$. In this case, when the value of $\rho$ is small enough there exists a unique saddle point $(K_1^*, \ell_1^*)$. By considering higher values of $\rho$ ($\rho \cong 2.643$), a saddle-node bifurcation changes the nature of $(K_1^*, \ell_1^*)$, that becomes attracting until it undergoes a flip bifurcation for $\rho \cong 3.11$ giving rise to an attracting period-2 cycle, which continues to exist until $\rho \cong 3.48$. The dynamic properties of the system for $\rho > 3.48$ are summarised in the bifurcation diagram plotted in Figure 7(a), where the birth (through a Neimark-Sacker bifurcation) of an attracting closed invariant curve around the period-2 cycle is illustrated. For values of $\rho$ higher than 3.487 it is possible to note the existence of several discontinuities in the bifurcation diagram. This means that the same initial condition is attracted by different $\omega$-limit sets when $\rho$ varies. In particular, Figure 7(b) illustrates (for $\rho = 3.5$) the coexistence of an attracting invariant curve, $\Gamma = \Gamma_1 \cup \Gamma_2$ (arisen from the Neimark-Sacker bifurcation) and an attracting cycle of period 14 ($P = \bigcup_{i=1}^{14} P_i$, the black points in the figure), born through a saddle-node bifurcation with a companion saddle-cycle of period 14 ($S = \bigcup_{i=1}^{14} S_i$, the red points in the figure). The stable manifold of the saddle-cycle of period 14 separates the basins of attraction of the attracting period-14 cycle and the attracting invariant curve $\Gamma$. From an economic point of view, it is important to note that at this stage, regardless of where agents coordinate themselves, the dynamics of the economy are characterised by significant fluctuations and all the economic indicators tend to fluctuate in the long term.

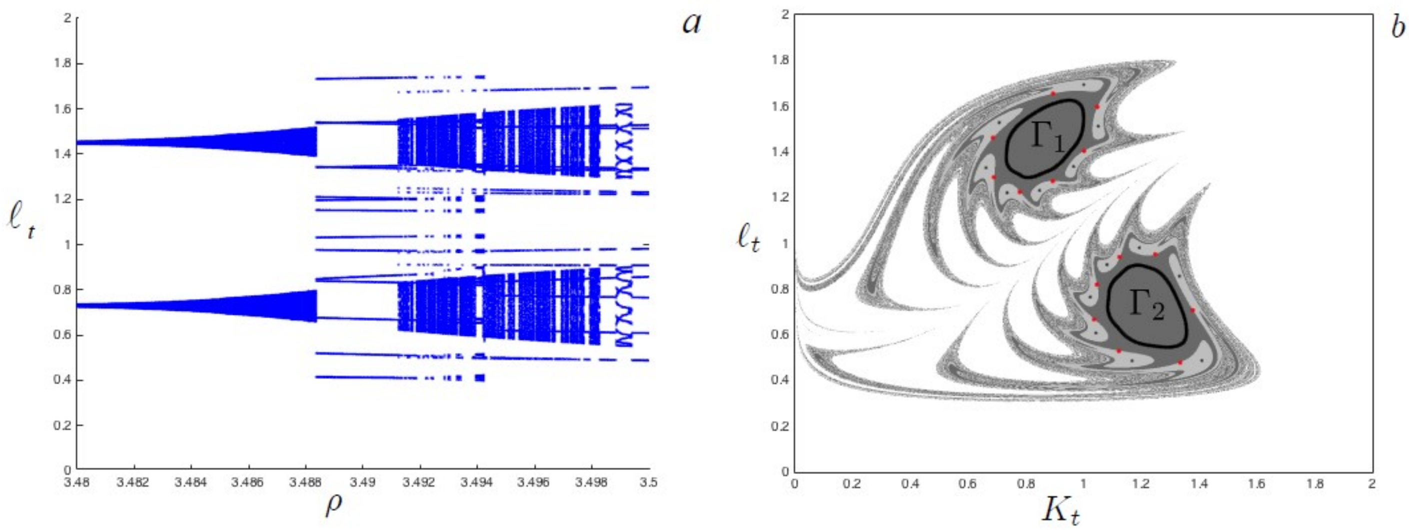


**Figure 7**. (a) Bifurcation diagram for $\rho$ . (b) Phase plane for $\rho = 3.5$ .

**5. Conclusions**

The analysis of a model with overlapping generation and natural resources showed the following results.

(1) The more interesting dynamic scenarios take place under the assumption of substitutability between environmental quality and (costly) consumption of the good produced in the economy (case $\sigma > 1$). In such a context, processes of well-being reducing economic growth may be observed (see Example 1, in Section 4). These processes are due to a coordination failure: given the initial value $K_0$ of the capital stock, individuals' well-being would be higher by choosing a lower initial value $\ell_0$ . However, no individual has an incentive to modify the choice of $\ell_0$ if the other individuals do not do the same.

(2) A coordination problem arises at every time. Given the value of the pre-determined variable $K_0$, the choice of $\ell_0$ is not uniquely determined if either the local or global indeterminacy occurs. Examples 1-4 illustrate complex scenarios of the local/global indeterminacy where locally determinate/indeterminate fixed points (that is, saddles and/or attracting fixed points) coexist with attracting cycles and chaotic attractors. In such scenarios, *expectations* (which affect the choice of

the initial value $\ell_0$) and *history* (which determines the initial value $K_0$) matter, and the prediction of long-term behaviour of the economy becomes rather difficult. Small changes in the initial values of $K_0$ or $\ell_0$ may produce a change in the $\omega$-limit set that the economy will approach.

(3) The local stability analysis of fixed points can be misleading in that it refers to a small neighbourhood of a fixed point, whereas the initial value $\ell_0$ can be chosen in the whole interval (0,2). Global indeterminacy may be observed even in the limiting case in which a unique fixed point exists and it is locally determinate (see Example 1). Coury and Wen, 2009 obtain this scenario of global indeterminacy also.

(4) According to Examples 1-4, an increase in the value of parameter $\rho$, which measures the environmental impact of production activity, initially causes a strong instability of the dynamic system: fluctuations of capital stock, labour supply, and environmental quality can be observed around high levels of the first two variables and a low level of the other one. However, a threshold value of $\rho$ may exist (Example 3) beyond which economic agents are induced to coordinate themselves on trajectories converging towards a fixed point characterised by a high environmental quality and low values of capital stock and labour supply (the fixed point $(K_1^*, \ell_1^*)$ in Figure 6(b)).

Finally, it is worth to stress that, different from Grandmont et al. (1998), Cazzavillan (2001) and other related works, which adopt a CES technology or consider market imperfections (positive externalities) in production, we assumed a constant-returns-to-scale Cobb-Douglas technology. Therefore, all the dynamic regimes showed in this article are generated by the effect of negative environmental externalities on the demand side of the economy. According to the mechanism analysed, environmental degradation leads individuals to consume larger quantities of the produced good, which allows them to substitute for the services provided by environmental resources.

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