A model of growth with inherited tastes

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Abstract

This research aims at studying a general equilibrium closed economy with overlapping generations and inherited tastes (aspirations), as in de la Croix (1996). It shows that the interaction between the intensity of aspirations and the elasticity of substitution of effective consumption affects the qualitative and quantitative long-term dynamics from both local and global perspectives. The related literature is extended by showing that 1) the Neimark-Sacker bifurcation found by de la Croix (1996) does not necessarily give rise to fluctuations 2) endogenous (long lasting) fluctuations occur through the emergence of period-doubling bifurcations.

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1 Introduction

Since the seminal articles of Becker and Murphy (1988), Abel (1990) and Becker (1992) endogenous preferences have become an issue of increasing importance in macroeconomics from both theoretical and empirical perspectives. One of the most important objectives for macroeconomists is to understand the reasons why output and other macroeconomic variables (e.g., employment, investments and so on) fluctuate over time both in the short term (business cycles, related essentially to exogenous stochastic shocks) and long term (economic growth and/or ever lasting cycles, often explained in deterministic models).

When the utility function of individuals depends on both their own consumption and a reference level where comparing it, consumption externalities may emerge. The macroeconomic effects of phenomena known as catching-up-with-the-Joneses (the consumption reference of an individual is represented by past average consumption at the economy-wide level) or keeping-up-with-the-Joneses (the consumption reference of an individual is represented by current average consumption at the economy-wide level) have been widely studied (for instance, Galí, 1994; Alonso-Carrera et al., 2005). There are several articles that include consumption externalities and analyse their implications at the macroeconomic level (Alonso-Carrera et al., 2004, 2005, 2007, 2008) dealing also with habits and aspirations. Habits [resp. aspirations] refer to the case in which preferences of an individual depend on both his own consumption and a benchmark level that weights the consumer's own past consumption experience [resp. the consumption experience of others]. In an overlapping generations (OLG) framework with a representative agent, the existence of internal [resp. external] habits implies that the consumption bundle of an individual when old [resp. of the current generation] is evaluated in comparison with his own consumption when young (Alonso-Carrera et al., 2007) [resp. with the consumption bundle of the past generation (de la Croix, 1996; de la Croix and Michel, 1999)].

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Focusing just on aspirations, de la Croix (1996) analysed the local properties of a Cobb-Douglas OLG economy with capital accumulation. The existence of this kind of external habits tends to increase consumption of the current generation thus leading to a reduction in saving, capital accumulation and wages, which in turn produces a reduction in the negative effects that aspirations can play on the macro-economy. This is because there will be fewer resources to be allocated to consumption. When the strength of aspirations becomes sufficiently low, this process can be reverted and capital accumulation can increase so that fluctuations are possible. With this regard, de la Croix (1996) showed that the steady-state equilibrium may undergo a Neimark-Sacker bifurcation (when the importance of aspirations in the utility function is large enough) but did not clarify whether this bifurcation was sub-critical or super-critical. Knowing whether a Neimark-Sacker bifurcation is super-critical or sub-critical is of importance in a macro-dynamic setting. In fact, only in the former case it is possible to observe persistent long-term fluctuations in income. Later, de la Croix and Michel (1999) extended de la Croix (1996) by considering a general specification for utility and production functions. Amongst other things, they provided sufficient conditions for the existence and uniqueness of a long-term equilibrium. However, they did not study the nature of the bifurcations and did not account for a global analysis of the model. Therefore, in their context nothing can be said about the existence of observable and persistent oscillations for the general class of utility and production functions used by de la Croix and Michel (1999).

This article aims at filling this gap, and the question the model can actually address is of the type: why do some countries overtake some others in the long-term, and why others are falling behind? For doing this, the work contains a thoughtful study of local and global dynamics in an OLG economy à la de la Croix (1996). This is done by assuming individuals with preferences described by a Constant Inter-temporal Elasticity of Substitution (CIES) lifetime utility function and static expectations about future prices or other key variables. The assumption of CIES preferences is aimed to keep the model as general as possible but still able to produce a formal treatment allowing to provide results from local and global perspectives. On the assumption of static expectations, we are aware that it may be subject to some criticism, especially because in deterministic contexts it resembles the paradigm in which individuals continuously make mistakes without ever correcting them. However, it allows getting closed form expressions of the two-dimensional map, whereas the case of perfect foresight (which resembles the case in which individuals have a perfect knowledge about future prices or other key variables) does not. Indeed, we pinpoint that the results of the present work hold even when the elasticity of inter-temporal substitution is close to one (log-utility), so that the expectations formation mechanisms of individuals becomes irrelevant. In addition, we stress that there exists no evidence in favour of the perfect foresight approach against the static expectations approach in deterministic contexts. The use of the former assumption is often not based on empirical reasons but simply due to the forward-looking decision-making approach made by economic agents on which modern economic theory is based upon. Instead, the difference between rational and static expectations tends to be significant in stochastic models based on random external shocks. In fact, the experimental economics literature has found evidence in favour of mechanisms different from the rational expectations approach, suggesting in particular that agents behave adaptively according to the adaptive expectations formation mechanism (see Hommes et al., 2000). One of the argument against everything different from rational expectations had much to do with the linear framework in which static or adaptive expectations were frequently used. In such a context, the dynamics of an economy is essentially characterised by systems that produce a unique stationary state equilibrium, which can be globally asymptotically stable or unstable, thus producing converging trajectories or explosive trajectories that are not economically meaningful. As often happens in economics, linear models represent an approximation of nonlinear models. Then, the study of the behaviour of a system far away from the steady state cannot be performed with the help of instruments used to approximate it linearly in the neighbourhood of the stationary state equilibrium. This analysis is of importance when a long-term equilibrium is unstable. In such a case, in fact, a dynamic system can generally produce non-regular trajectories that are difficult to be predicted, so that the argument discussed above against static or adaptive expectations is no more convincing (see Agliari et al. (2006a) for a discussion on this issue). This is the case of the deterministic OLG model under scrutiny, which is able to produce non-regular or even chaotic dynamics.

The main finding of this work is that the steady-state equilibrium of an OLG model of growth with inherited tastes à la de la Croix (1996) can either undergo a sub-critical Neimark-Sacker bifurcation or a flip bifurcation. Therefore, the system is effectively able to generate observable persistent oscillations (economic cycles) only through a mechanism not discussed in de la Croix (1996), which is the usual cascade of period-doubling bifurcations. More in detail, this article shows (1) the conditions under which a feasible region for an OLG with aspirations does actually exist (which is of importance from a global perspective), (2) the *necessary and sufficient* conditions for the existence of fixed points, whereas de la Croix and Michel (1999) only state sufficient conditions for a general class of utility and production functions; (3) for some limiting cases that the fixed point can be *locally asymptotically stable or locally unstable*, whereas de la Croix (1996) and de la Croix and Michel (1999) gave only conditions to have a *saddle point*.

The work provides some other additional results due to the assumption of CIES preferences. Specifically, (a) there exists evidence of a different route to chaos (period doubling) compared to de la Croix (1996) and de la Croix and Michel (1999), who showed that the steady-state equilibrium can lose stability only through a Neimark-Sacker bifurcation and (b) aspirations can play a stabilising role. This contrasts the existing literature on this issue that showed that aspirations are always a destabilising device. Indeed, as also stressed by de la Croix and Michel (1999), the existence of a consumption externality that causes a spillover effect from two subsequent generations implies: the existence of decreasing returns in the process that transfers resources between generations, as the stock of capital currently used in production (and then the wage of current workers) is financed by the saving of the previous generation; the existence of constant returns in the process that generates standard-of-living aspirations from the old generation to the young generation. Due to the former effect, the increasing wage rate of the young workers may not be high enough to offset the need of higher consumption due to aspirations (which is a mechanism that operates with constant returns). Therefore, saving reduces from this channel causing in turn a reduction in capital accumulation and production per workers. If this reduction is sufficiently strong, the degree of aspirations reduces as well. With a low degree of aspirations, saving starts increasing so that the process can be inverted. This cyclical behaviour may generate convergence towards the steady-state equilibrium or destabilisation through a Neimark-Sacker or flip bifurcation. However, with CIES preferences the destabilising role of aspirations can be reverted. This result depends on the relative size of the elasticity of substitution, and then on how aspirations affects the interest rate and savings. Therefore, when the elasticity of substitution of effective consumption is sufficiently high, aspirations can play an opposite role to the one found in the works of de la Croix (1996) and de la Croix and Michel (1999), i.e. aspirations can represent a stabilising device. Definitely, the main message of this article is that in an OLG framework with inherited habits, the Neimark-Sacker bifurcation shown in the previous literature seems to be subcritical. However, economic cycles or complex dynamics can emerge via period-doubling bifurcations.

In addition, there exist two other contributions of de la Croix and his co-authors showing examples where the bifurcation generated by aspirations was sub-critical. These works are de la Croix (2001) and Artige et al. (2004) that studied an endogenous growth model with human capital and wealth breeding decline, respectively. The former article showed through numerical simulations (de la Croix, 2001, Figure 4, p. 1429) that in a neighbourhood of the bifurcation point a repelling limit cycle appears. The latter one provided evidence for divergence depending on initial conditions (Artige et al., 2004, Figure 5, left panel, p. 436). The present article confirms these results but also provides numerical examples that help reconciling the existence of long-lasting fluctuations in OLG economies with aspirations via a flip (instead of Neimark-Sacker) bifurcation.

The rest of this article is organised as follows. Section 2 builds on the model. Sections $3-4-5$ focus on the study of local and global dynamics of the model and underline the main economic results. Numerical simulations were also used to support the theoretical analysis. Section 6 concludes.

2 The economy

Consider a general equilibrium OLG closed economy populated by a continuum of identical two-period lived individuals of measure one per generation ($t = 0, 1, 2, ...$). Life of the typical agent is divided into youth and old age. An individual works when he is young and then retires when he is old. The young member of generation t is endowed with one unit of labour inelastically supplied to firms in exchange for the competitive wage $w_t > 0$. The budget constraint when young is standard and reads as follows:

$$c_{1,t} + s_t = w_t. \tag{1}$$

Eq. (1) implies that working income (w_t) is divided between material consumption when young ($c_{1,t}$) and saving (s_t). When old, an individual retires and lives with the amount of resources saved when young plus the expected interest accrued from time t to time $t + 1$ at rate r^e_{t+1} (which will become the realised interest rate at time $t + 1$). The model incorporates a (perfect) market for annuities (Fanti et al., 2017), so that the budget constraint at time $t + 1$ of a young individual of generation t is the following:

$$c_{2,t+1} = \frac{R^e_{t+1}}{p} s_t, \tag{2}$$

where $c_{2,t+1}$ is consumption when old, $R^e_{t+1} := 1+r^e_{t+1}$ is the expected interest factor and $0 < p < 1$ is the constant inter-temporal subjective discount factor, which can be interpreted also as a (constant) measure of individual longevity.

The typical individual of generation t draws utility from consumption when young and consumption when old. In addition, he evaluates his own consumption when young in comparison with the level of aspirations inherited by his parent (h_t). This is in line with de la Croix (1996) and de la Croix and Michel (1999). These are bequeathed tastes for the individual born at time t representing a reference where comparing current consumption. The expected lifetime utility function of generation t is of the CIES type (de la Croix and Michel, 2002; Chen et al., 2008; Fanti and Spataro, 2008; Fanti and Gori, 2013), that is:

$$U_t = \begin{cases} \frac{(c_{1,t}-\gamma h_t)^{1-\sigma}}{1-\sigma} + p\frac{c_{2,t+1}^{1-\sigma}}{1-\sigma}, & if \quad \sigma > 0, \sigma \neq 1, \\ \ln(c_{1,t} - \gamma h_t) + p\ln(c_{2,t+1}), & if \quad \sigma = 1 \end{cases}, \tag{3}$$

where $0 < \gamma < 1$ captures the intensity of aspirations in utility and σ is (the absolute value of) the elasticity of marginal utility. From (3), the elasticity of marginal utility is the reciprocal of the inter-temporal elasticity of substitution. With this formulation, the inter-temporal elasticity of substitution is given by $\frac{1}{\sigma}$. If $0 < \sigma < 1$ (resp. $\sigma > 1$) the inter-temporal elasticity of substitution is larger (resp. smaller) than 1. Empirical research using aggregate consumption data or cross-country data (Hall, 1988; Blundell-Wignall et al., 1995; Lund and Engsted, 1996; Guvenen, 2006; Havranek et al., 2015) generally found values of the elasticity of substitution in consumption smaller than one though some analyses based on micro data (Blundell et al., 1994; Attanasio and Browning, 1995; Browning et al., 1999) obtained the opposite result. Indeed, the recent work of Havranek et al. (2015) actually pointed out that the elasticity of substitution in consumption for rich households or asset holders systematically tend to larger values than that for poorer households.[1] This is because richer "households substitute consumption across time periods more easily because necessities, which are difficult to substitute intertemporally, constitute a smaller fraction of their consumption bundle in comparison with poor households." (Havranek et al., 2015, p. 111). The use of CIES preferences is aimed for generality in this work. When there is no uncertainty this seems a reasonable choice to preserve the possibility of having closed form expressions for consumption and saving.

By taking the wage rate, the expected interest factor and the level of aspiration as given, the individual representative of generation t chooses $c_{1,t}$ and $c_{2,t+1}$ to maximise lifetime utility function (3) subject to (1), (2) and $c_{1,t} > \gamma h_t$. Then, one gets:

[1] Their quantitative survey of estimates regarding the elasticity of intertemporal substitution found that the mean elasticity of intertemporal substitution in consumption is 0.5. However, they clearly reported that estimates may vary across countries and methods. This essentially depends on income and the functioning of the asset market. They concluded that the elasticity of intertemporal substitution is larger in rich countries or in countries with high stock market participation.

$$(c_{1,t} - \gamma h_t)^{-\sigma} = \lambda_t, \tag{4}$$

and

$$c_{2,t+1}^{-\sigma} = \frac{\lambda_t}{R^e_{t+1}}, \tag{5}$$

where λ_t is the Lagrange multiplier. From (4) and (5), the first order conditions for an interior solution are the following:

$$c_{2,t+1} = (R^e_{t+1})^{\frac{1}{\sigma}}(c_{1,t} - \gamma h_t,), \tag{6}$$

$$s_t = \frac{p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}[w_t - \gamma h_t]. \tag{7}$$

Finally, by combining (6) with (1) and (2) one can get the expressions for consumption when young and consumption when old, that is:

$$c_{1,t} = \frac{w_t + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}\gamma h_t}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}, \tag{8}$$

$$c_{2,t+1} = \frac{(R^e_{t+1})^{\frac{1}{\sigma}}[w_t - \gamma h_t]}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}. \tag{9}$$

Firms are identical and act competitively on the market. The production function of the representative firm is the standard neoclassical Cobb-Douglas technology with constant returns to scale, that is $Q_t = AK_t^{\alpha}L_t^{1-\alpha}$, where Q_t, K_t and L_t are output, capital and labour input at time t respectively, $A > 0$ is a scale parameter and $0 < \alpha < 1$ is the output elasticity of capital. Defining $k_t := K_t/L_t$ and $q_t := Q_t/L_t$ as capital and output per worker, respectively, the intensive form production function is $q_t = Ak_t^{\alpha}$. By assuming that output is sold at the unit price and capital fully depreciates at the end of every period (this is a reasonable assumption given that every period t consists of 30 years in standard OLG models), profits maximisation implies that the interest factor and wage rate are equal to the marginal productivity of capital and marginal productivity of labour, respectively, that is:

$$R_t = \alpha A k_t^{\alpha-1}, \tag{10}$$

$$w_t = (1-\alpha)Ak_t^{\alpha}. \tag{11}$$

Following de la Croix (1996), aspirations depend on the standard of living of individuals of the previous generation when young. This implies that

$$h_t = c_{1,t-1}. \tag{12}$$

The market-clearing condition in the capital market is given by $k_{t+1} = s_t$. Then, the two-dimensional map that characterises the dynamics of the economy is the following:

$$\begin{cases} k_{t+1} = \frac{p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}{1+p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}[w_t - \gamma h_t] \\ h_{t+1} = \frac{w_t + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}\gamma h_t}{1+p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}} \end{cases}, \tag{13}$$

where $w_t = (1-\alpha)Ak_t^{\alpha}$ and $R^e_{t+1} = \alpha A k_t^{\alpha-1}$ if individuals have static expectations or $R^e_{t+1} = \alpha A k_{t+1}^{\alpha-1}$ if individuals are rational and have perfect foresight. Obviously, in the case of logarithmic preferences ($\sigma = 1$) it is not important to specify whether an individual has static expectations or perfect foresight about future factor prices. With regard to CIES preferences ($\sigma \neq 1$), we study local and global dynamics of the model under static expectations as it allows us defining an explicit expression for the accumulation of the stock of capital and stock of aspirations, and leave the case of perfect foresight to future research.

Define $x := k$, $y := h$, $\beta := (1-\sigma)/\sigma$ and

$$A(x_t) = m_1 x_t^{\alpha\beta}, \;\; B(x_t) = x_t^{\beta}, \;\; C(x_t) = m_2 x_t^{\alpha},$$

where $m_1 := p(\alpha A)^{\beta}$ and $m_2 := (1-\alpha)A$. If $\sigma \in (0,1)$ then $\beta > 0$, in particular if $\sigma \to 1^-$ (resp. 0^+) then $\beta \to 0^+$ (resp. $+\infty$). This replicates the case of Cobb-Douglas (resp. Leontief) preferences. If $\sigma > 1$ then $\beta \in (-1,0)$ and, in particular, if $\sigma \to 1^+$ (resp. $\sigma \to +\infty$) then $\beta \to 0^-$ (resp. $\beta \to -1^+$). In what follows, we will distinguish between the cases in which the elasticity of inter-temporal substitution is larger than 1, that is $\beta > 0$ ($\sigma \in (0,1)$) and smaller than 1, that is $\beta \in (-1,0)$ ($\sigma > 1$). The two-dimensional dynamic system described in (13) can be rewritten by resorting to the following continuous and differentiable map:

$$T : \begin{cases} x_{t+1} = f(x_t, y_t) = \frac{A(x_t)[C(x_t)-\gamma y_t]}{A(x_t)+B(x_t)} \\ y_{t+1} = g(x_t, y_t) = \frac{B(x_t)C(x_t)+A(x_t)\gamma y_t}{A(x_t)+B(x_t)} \end{cases} . \tag{14}$$

The dynamics of system T are quite difficult to be handled in a neat analytical form. Therefore, we now transform T in a simpler form by taking into account that (14) can be rewritten in a one-dimensional, second order difference equation as follows.

Let system T be given by (14). Then

$$x_{t+1} + y_{t+1} = f(x_t, y_t) + g(x_t, y_t) = C(x_t) \Rightarrow y_{t+1} = C(x_t) - x_{t+1}. \tag{15}$$

From the first equation of T we have $x_{t+2} = f(x_{t+1}, y_{t+1})$ that is, by taking into account equation (15),

$$x_{t+2} = F(x_{t+1}, x_t) = \frac{A(x_{t+1})[C(x_{t+1}) - \gamma C(x_t) + \gamma x_{t+1}]}{A(x_{t+1}) + B(x_{t+1})}, \tag{16}$$

for all initial conditions (x_0, x_1) where $x_1 = \frac{A(x_0)[C(x_0)-\gamma y_0]}{A(x_0)+B(x_0)}$. Equation (16) is a one-dimensional, second order difference equation.

Finally, let

$$z_{t+1} = G(x_t) = x_t, \tag{17}$$

then (16) can be written as follows

$$x_{t+2} = F(x_{t+1}, z_{t+1}) \Rightarrow x_{t+1} = F(x_t, z_t), \tag{18}$$

and consequently the following system of two first order difference equations is obtained

$$S : \begin{cases} x_{t+1} = F(x, z) = \frac{A(x_t)\{C(x_t)+\gamma x_t - \gamma C(z_t)\}}{A(x_t)+B(x_t)} \\ z_{t+1} = G(x_t) = x_t \end{cases} . \tag{19}$$

The dynamics of the model T can be carried out by investigating system S describing the time evolution of the capital per worker x_t, whereas the dynamics of aspirations y_t are obtained as $y_t = C(z_t) - x_t$.

In the rest of the article we will deal with the study of the dynamics generated by system S for any $\sigma > 0$, that is $\beta > -1$ holds. Knowing that $m_1 = p(\alpha A)^{\beta}$ and $m_2 = (1-\alpha)A$, we note that map S can be written as follows:

$$S^* : \begin{cases} x_{t+1} = F(x_t, z_t) = \frac{p(\alpha A)^{\beta}[(1-\alpha)A x_t^{\alpha} + \gamma x_t - \gamma(1-\alpha)A z_t^{\alpha}]}{p(\alpha A)^{\beta} + x_t^{(1-\alpha)\beta}} \\ z_{t+1} = G(x_t) = x_t \end{cases} . \tag{20}$$

Notice that an initial condition (x_0, y_0) of T corresponds to the initial condition (x_0, z_0) of S^* where, being C invertible, $z_0 = C^{-1}(x_0 + y_0)$.

3 The feasible region

Before starting with the discussion of the dynamics generated by system S^*, we observe that $\mathbb{R}^2_+$ is not a trapping set for system S^* so that any attractor at finite distance of system S^* (if it exists) cannot be globally attracting in $\mathbb{R}^2_+$. To prove this result, we observe that, when considering the evolution of the two state variables for initial conditions $(x_0, z_0) \in \mathbb{R}^2_+$, system S^* may produce trajectories that exit set $\mathbb{R}^2_+$. In fact, at each iteration the condition $C(x_t) - \gamma C(z_t) + \gamma x_t \geq 0$ must be satisfied. Of course only trajectories that do not exit a suitable (positively invariant) set $D \subset \mathbb{R}^2_+$ are meaningful from an economic point of view. We now recall the following definition.

Definition 1. *Let $(x_t, z_t) = {S^*}^t(x_0, z_0)$ denote the $t - th$ iterate of system S^* for a given initial condition (i.c.) $(x_0, z_0) \in \mathbb{R}^2_+$. Then the sequence $\psi = \{(x_t, z_t)\}_{t=0}^{\infty}$ is called trajectory. A trajectory ψ is feasible for system S^* if $(x_t, z_t) \in \mathbb{R}^2_+$ for all $t \in \mathbb{N}$, otherwise it is unfeasible.*

About the existence of unfeasible trajectories the following proposition holds.

Proposition 2. *System S^* always admits unfeasible trajectories.*

Proof. Let $(x_0, z_0) \in \mathbb{R}^2_+$ be such that $z_0 > \left(\frac{m_2 x_0^\alpha + \gamma x_0}{\gamma m_2}\right)^{\frac{1}{\alpha}}$. Then the first iteration of S^* gives a negative value of x, that is $x_1 < 0$. This means that the point $(x_1, z_1) = S^*(x_0, z_0)$ does not belong to the set $\mathbb{R}^2_+$, hence the obtained trajectory is unfeasible. □

From Proposition 2, it follows that if S^* admits feasible trajectories then set D containing all initial conditions (x_0, z_0) that generate feasible trajectories is a subset of $\mathbb{R}^2_+$. We call set D the feasible region. Furthermore, observe that $S^*(0,0) = (0,0)$ for all parameter values so that D is non-empty. In order to better characterise the structure of set D, a preliminary consideration is the following.

From the proof of Proposition 2, we observe that the function

$$z = \tilde{h}(x) = \left(\frac{m_2 x^\alpha + \gamma x}{\gamma m_2}\right)^{\frac{1}{\alpha}}, \ x \geq 0 \tag{21}$$

defines a curve in the (x, z) plane which is strictly increasing and convex and such that $\lim_{x \to 0^+} \tilde{h}(x) = 0$ and $\lim_{x \to +\infty} \tilde{h}(x) = +\infty$.

Condition $z_0 < \tilde{h}(x_0)$ gives only a necessary condition for the feasibility of S^* stating that at the initial state, given a positive initial value of the capital per worker x_0, the initial value of aspirations y_0 should not be too high.

We note that the set of initial conditions and parameter values leading to feasible trajectories cannot easily be obtained. Anyway some limiting cases can be considered and some numerical simulations can be produced.

The following proposition concerning the structure of the feasible region holds.

Proposition 3. *Let $\beta > 1$ hold and system S^* be given by (20). Then the feasible set D is bounded.*

Proof. Observe that all initial conditions $(0, z_0)$, $z_0 > 0$, generate unfeasible trajectories while $(0,0) \in D$. Hence in what follows we consider the set $(0, +\infty) \times [0, +\infty)$. Define

$$D_1 = \{(x_0, z_0) \in (0, +\infty) \times [0, +\infty) : m_2 x_0^\alpha - \gamma(m_2 z_0^\alpha - x_0) \leq 0\}$$

and recall that $m_2 x_0^\alpha - \gamma(m_2 z_0^\alpha - x_0) = 0$ defines a curve $z_0 = \tilde{h}(x_0)$ in the (x_0, z_0) plane which is strictly increasing and convex and such that $\lim_{x_0 \to 0^+} \tilde{h}(x_0) = 0$ and $\lim_{x_0 \to +\infty} \tilde{h}(x_0) = +\infty$.

Then, it is easy to observe that D_1 does not belong to the feasible region D. Define $\bar{D}_1 = (0, +\infty) \times [0, +\infty) - D_1$ then $D \subseteq \bar{D}_1 \cup \{(0,0)\}$.

Consider now all the preimages of first rank of set D_1 belonging to $\bar{D}_1$, i.e. the set

$$D_2 = \{(x_{-1}, z_{-1}) \in \bar{D}_1 : S^*(x_{-1}, z_{-1}) \in D_1\}.$$

From the first equation of system S^* we have that $x_0 = F(x_{-1}, z_{-1})$ whereas, from the second equation, we have that $z_0 = x_{-1}$. The inequality $m_2 x_0^\alpha - \gamma(m_2 z_0^\alpha - x_0) \leq 0$ may then be rewritten in terms of x_{-1} and z_{-1} thus obtaining, after some algebra, the following:

$$m_2 \left(\frac{F(x_{-1}, z_{-1})}{x_{-1}} \right)^\alpha + \gamma \frac{F(x_{-1}, z_{-1})}{x_{-1}^\alpha} \leq \gamma m_2, \forall (x_{-1}, z_{-1}) \in \bar{D}_1 \tag{22}$$

where (22) defines the set of points generating trajectories which exit $\mathbb{R}^2_+$ at the second iteration.

Define $H(x_{-1}, z_{-1}) = m_2 x_{-1}^\alpha + \gamma x_{-1} - \gamma m_2 z_{-1}^\alpha$ then (22) can be rewritten as follows:

$$\begin{cases} v(x_{-1}, z_{-1}) = m_2 \left(\frac{m_1 H(x_{-1}, z_{-1})}{m_1 x_{-1} + x_{-1}^{(1-\alpha)\beta+1}} \right)^\alpha + \gamma \left(\frac{m_1 H(x_{-1}, z_{-1})}{m_1 x_{-1}^\alpha + x_{-1}^{(1-\alpha)\beta+\alpha}} \right) \leq \gamma m_2 \\ H(x_{-1}, z_{-1}) > 0 \end{cases} . \tag{23}$$

Notice that

$$0 < H(x_{-1}, z_{-1}) \leq m_2 x_{-1}^\alpha + \gamma x_{-1}$$

and that, if $\beta > 1$,

$$\lim_{x_{-1} \to +\infty} m_2 \left(\frac{m_1 (m_2 x_{-1}^\alpha + \gamma x_{-1})}{m_1 x_{-1} + x_{-1}^{(1-\alpha)\beta+1}} \right)^\alpha + \gamma \left(\frac{m_1 (m_2 x_{-1}^\alpha + \gamma x_{-1})}{m_1 x_{-1}^\alpha + x_{-1}^{(1-\alpha)\beta+\alpha}} \right) = \lim_{x_{-1} \to +\infty} V(x_{-1}) = 0,$$

hence then $\forall \epsilon > 0$, $\exists \bar{x}$ such that $V(x_{-1}) < \epsilon$ as long as $x_{-1} > \bar{x}$. Consider $\epsilon = \gamma m_2$ then from (23) it follows that

$$0 < v(x_{-1}, z_{-1}) \leq V(x_{-1}) < \gamma m_2 \quad \forall x_{-1} > \bar{x}$$

providing that the feasible set D must result as follows:

$$D \subseteq \bar{D}_1 \cup \{(0,0)\} \cap [0, \bar{x}] \times [0, +\infty)$$

hence it is bounded.

□

From the previous Proposition it follows that if $\beta > 1$ then $\exists I(\underline{0}, r)$, where $I(\underline{0}, r)$ is a generic neighbourhood of the origin, such that all initial conditions $(x_0, z_0) \in \{\mathbb{R}^2_+ - I(\underline{0}, r)\}$ generate unfeasible trajectories (an example is in Figure 1 (b)). Several numerical simulations show that the result proved in Proposition 3 for $\beta > 1$ holds also for $\beta \in (0, 1]$, as it is shown in Figure 1 (d), while for negative values of β the feasible set can be unbounded as in Figure 1 (a).

To obtain Figure 1 we fix the key parameters of the model and depict the feasible region in white for different values of β and γ. We also represent curve $z = \tilde{h}(x)$ in yellow. Observe that the set of initial conditions that generates unfeasible trajectories is also given by the points lying below curve $\tilde{h}(x)$, representing initial conditions that generate trajectories that exit set $\mathbb{R}^2_+$ after the first iterate.

More in detail, points $(x, z) \in \mathbb{R}^2_+ : z = \tilde{h}(x), x > 0$ are mapped into the $y-$semi-axis with $z > 0$ so that, at the second iteration, they do not belong to the set $\mathbb{R}^2_+$. Points on $\mathbb{R}^2_+$ above the curve $z = \tilde{h}(x)$ generate trajectories that exit at the first iteration (the black points above the yellow curve in Figure 1). Let R_j be the region on $\mathbb{R}^2_+$ containing points leaving set $\mathbb{R}^2_+$ at the $j-$th iteration, $j = 1, 2, ..., N$, then such points are depicted in grey scale in Figure 1, so that, once fixed a sufficiently high numbers N of iterations (we fixed $N = 10000$), the white region represents the feasible region. As it can be observed in Figure 1, the feasible set can be unbounded as in panel (a) or bounded and, even, internal as in panel (c). Furthermore the white region becomes smaller as γ increases, and, then, it disappears when almost all trajectories become unfeasible. The way in which this bifurcation in the structure of the feasible region occurs will be better explained later in the article.

The results herewith obtained show that whereas the unique equilibrium in the Diamond's model is globally stable and thus all trajectories converge towards it, the model extended with aspirations is able to produce feasible trajectories only whether the initial conditions belong to an appropriate set. In particular, for any initial value of capital per worker, the initial value of aspirations, must

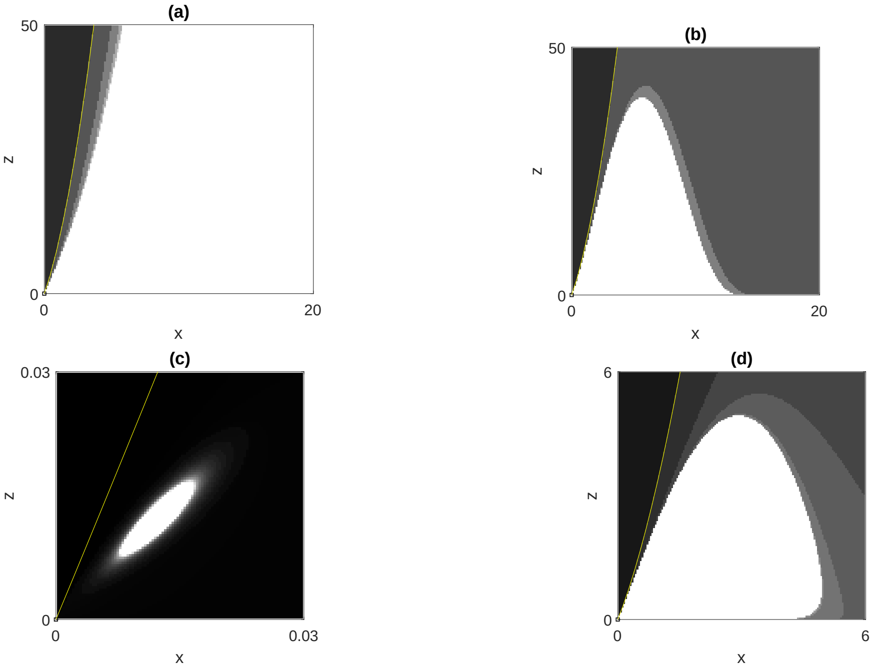

Figure 1: The feasible set is depicted in white, the greyscale and black regions represent the set of initial conditions generating unfeasible trajectories; the curve $\tilde{h}(x)$ defined in (21) is depicted in yellow. Common paramters $A = 10$, $\alpha = 0.27$ and $p = 0.5$ (a) $\beta = -0.7$ and $\gamma = 0.6$. (b) $\beta = 2$ and $\gamma = 0.6$. (c) $\beta = -0.7$ and $\gamma = 0.79$. (d) $\beta = 0.9$ and $\gamma = 0.79$.

be sufficiently low. Furthermore, numerical evidences have shown that the size of the feasible region increases when σ increases. With Cobb-Douglas preferences, set D becomes unbounded and it remains unbounded also for $\sigma > 1$. More in detail the following two cases occur.

(i) If $\sigma \in (0, 1)$ (high elasticity of substitution of effective consumption), an economy may be located in a region that generates unfeasible trajectories even if it starts with high values of the capital stock per worker (developed countries). This holds irrespective of the stock of aspirations. The standard OLG model extended with aspirations is able to produce feasible trajectories for intermediate initial values of the capital stock per worker and stock of aspirations. The economic reason for this result is twofold: (1) when the economy begins, the generation living at the initial state of the world must not have consumed too much when young to adequately save to allow the future generation to avoid to inherit a level of aspirations that generates unfeasible trajectories (negative savings); (2) however, saving should not be at too high a level to avoid unfeasible trajectories as well. The second point is relevant especially with regard to the effects of positive shocks on physical capital (i.e., capital transfers from foreign countries, capital donations from external donors), which may therefore be detrimental in an economy with aspirations because they may be a source of unfeasible trajectories. This case seems to be much more in line with the empirical research of Blundell et al. (1994), Attanasio and Browning (1995) and Browning et al. (1999) based on micro data.

(ii) If $\sigma > 1$ (low elasticity of substitution with respect to effective consumption), an initial capital stock per worker large enough always guarantees that an economy lies in a region that generates feasible trajectories. This holds even if the initial value of the stock of aspirations is small. With this kind of preferences, only economies that begin with a small stock of capital per worker (developing or underdeveloped countries) may be entrapped in a region that generates unfeasible trajectories. Whether an economy lies in a region that generates feasible or unfeasible trajectories is an empirical matter with σ. This case seems to be much more in line with the empirical research of Hall (1998), Blundell-Wignall et al. (1995), Lund and Engsted (1996), Guvenen (2006) and Havranek et al.

(2015) based on aggregate consumption data or cross-country data.

4 Existence and number of fixed points

We now consider the question of the existence and number of fixed points (or steady states) of system S^*. The steady states of system S^* are all solutions of the system $S^*(x,z) = (x,z)$ in $\mathbb{R}^2_+$.

The following Proposition holds.

Proposition 4. *System S^* admits two fixed points for all parameter values: the origin $E_0 = (0,0)$, and the interior fixed point $E^* = (x^*, x^*)$.*

Proof. Trivially, the origin is a fixed point since $S^*(0,0) = (0,0)$ and no other fixed points exist on the z-axis. Now, let $x > 0$. Then a fixed point of (20) must solve equation $x^* = F(x^*, x^*)$. After some algebra one gets

$$-(x^*)^{\beta(1-\alpha)} + m_1 m_2 (1-\gamma)(x^*)^{\alpha-1} - m_1(1-\gamma) = 0$$

so that, it must be

$$\frac{\omega^{\beta+1}}{1-\gamma} = m_1 m_2 - m_1 \omega$$

where we posed $\omega = (x^*)^{1-\alpha}$. Taking into account the geometrical properties of functions $f(\omega) = \frac{\omega^{\beta+1}}{1-\gamma}$ and $g(\omega) = m_1 m_2 - m_1 \omega$ it can easily be shown that they intersect each other only once, i.e. there exists a unique $\omega^* < m_2$ such that $f(\omega^*) = g(\omega^*)$, and consequently system S^* always admits a unique fixed point given by $E^* = (x^*, x^*)$, where $x^* = (\omega^*)^{\frac{1}{1-\alpha}}$. □

The position of the unique interior fixed point E^* on the plane depends on the parameters of the model; in particular, it depends on the two key parameters γ and β, that measure the intensity of aspirations and the inter-temporal elasticity of substitution with respect to effective consumption, respectively. By taking into account the proof of Proposition 4, we note that $g(\omega)$ is a strictly decreasing function and it does not depend on γ, whereas $f(\omega)$ is a strictly increasing function and it depends on γ. More precisely, for any given value of $\omega > 0$ and $\beta > -1$ we have that

$$\frac{\omega^{\beta+1}}{1-\gamma_1} < \frac{\omega^{\beta+1}}{1-\gamma_2}, \quad \forall\, 0 < \gamma_1 < \gamma_2 < 1.$$

As a consequence, x^* is decreasing with respect to γ. On the one hand, this implies that the steady-state capital stock per worker is lower in the economy with bequeathed tastes than in the standard Diamond economy, as in de la Croix (1996). On the other hand, the role of β on E^* can be ambiguous as it depends also on the value of x^*. The following proposition holds.

Proposition 5. *Let $E^* = (x^*, x^*)$ be the interior fixed point of system S^*. Then: (i) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$, $\frac{\partial x^*}{\partial \beta} > 0$; (ii) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > 1$, $\frac{\partial x^*}{\partial \beta} < 0$; (iii) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) = 1$, $\frac{\partial x^*}{\partial \beta} = 0$.*

Proof. From the proof of Proposition 4, we have that, at the steady state, the following equality holds

$$\omega_1^{\beta+1} = \frac{p(1-\gamma)(1-\alpha)}{\alpha} - p(1-\gamma)\omega_1$$

where $\omega_1 = \frac{x^{1-\alpha}}{\alpha A}$ and ω_1 is strictly increasing with respect to x. Notice that $f(\omega_1) = \omega_1^{\beta+1}$ depends on β whereas $g(\omega_1) = \frac{p(1-\gamma)(1-\alpha)}{\alpha} - p(1-\gamma)\omega_1$ does not depend on β. Let $\omega_1^* > 0$ such that $f(\omega_1^*) = g(\omega_1^*)$, then it can be easily observed that if $\omega_1^* < 1$ then ω_1^* is strictly increasing w.r.t. β and $\lim_{\beta\to+\infty} \omega_1^* = 1^-$, whereas if $\omega_1^* > 1$ then ω_1^* is strictly decreasing w.r.t. β and $\lim_{\beta\to+\infty} \omega_1^* = 1^+$. Finally if $\omega_1^* = 1$ then it does not change as β changes. Observe that condition $\omega_1^* < 1$ (resp. $\omega_1^* > 1$) corresponds to condition $g(1) < 1$ (resp. $g(1) > 1$) that is given by

$$p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < (resp. >)1.$$

□

According to the previous result the effect of a change in β on the position of the interior fixed point is ambiguous. Observe also that the condition $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$ corresponds to $\omega_1^* = \frac{(x^*)^{1-\alpha}}{\alpha A} < 1$ and consequently to $x^* < (\alpha A)^{\frac{1}{1-\alpha}} = x_\infty$. Hence, from Proposition 5, it follows that if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$ (resp. $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > 1$) then $x^* < x_\infty$(resp. $x^* > x_\infty$), and if β increases, then x^* increases (resp. decreases) up to the limit value x_∞ to which x^* converges when $\beta \to +\infty$, i.e. x^* is upper (resp. lower) bounded.

Now, let

$$x_{-1}^* = \left[A\frac{p(1-\gamma)(1-\alpha)-\alpha}{p(1-\gamma)}\right]^{\frac{1}{1-\alpha}}.$$

Then, according to Proposition 5 it can be observed that if x^* is increasing (resp. decreasing) with respect to β, then x^* converges to its minimum (resp. maximum) value as $\beta \to -1^+$, that is given by 0 (resp. x_{-1}^*). Finally, in the Cobb-Douglas case ($\beta = 0$) one gets

$$x^* = x_0^* = \left[\frac{(1-\gamma)p(1-\alpha)A}{1+(1-\gamma)p}\right]^{1/(1-\alpha)}.$$

The previous results can be summarised in the following remark.

Remark 6. *Let $E^* = (x^*, x^*)$ be the interior fixed point of system S^*.*
(i) If $\beta \to \infty$ (i.e. $\sigma \to 0^+$) then $x^* \to x_\infty = (\alpha A)^{1/(1-\alpha)}$ and $\forall \beta > -1$ if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < (>)1$ then $x^* < (>)x_\infty$;
(ii) if $\beta = 0$ (i.e. $\sigma = 1$) then $x^* = x_0^* = \left[\frac{(1-\gamma)p(1-\alpha)A}{1+(1-\gamma)p}\right]^{1/(1-\alpha)}$;
(iii) if $\beta \to -1^+$ and $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > (<)1$ then $x^* \to x_{-1}^*(\to 0)$.

Figure 2 (b) shows - for two different values of α (the output elasticity of capital) - that the effect of a change in β on the position of the interior fixed point is ambiguous. If α is sufficiently high (resp. low) then condition (i) (resp. (ii)) of Proposition 5 holds and when $\beta \to -1^+$ the steady-state stock of capital is the smallest (resp. largest) one with respect to other values of the individual degree of substitution of consumption over time. Observe that a sufficient condition for x^* to be increasing in β is $\alpha > 1/3$, or γ (resp. p) is sufficiently high (resp. low). This result sheds new light on the role of preference parameters (the aspiration intensity and the inter-temporal discount factor in this context) on steady-state GDP. For any given value of β, the economy may converge towards a long-term high or low income level depending on technology and preference parameters. In particular, the lower the capital share in production and aspiration intensity, and the higher the inter-temporal subjective discount factor, the more likely an economy converges towards a steady state with low income (as is shown in Panels (c) and (d) of Figure 2).

5 Stability, bifurcations and economic fluctuations

In order to study the local stability of the two fixed points of system S^*, consider the Jacobian matrix associated to S^*, representing the linearization of the dynamic system S^*, given by:

$$JS^*(x,z) = \begin{pmatrix} F_x(x,z) & F_z(x,z) \\ 1 & 0 \end{pmatrix}. \tag{24}$$

About the local stability of E_0 it can be observed that, since

$$det(JS^*(x,z)) = \frac{\gamma m_1 m_2 \alpha}{(m_1 + x^{\beta(1-\alpha)})z^{(1-\alpha)}}$$

then if $\beta > 0$ and $x \to 0^+$, $z \to 0^+$ we have that $det(JS^*(x,z)) \to +\infty$; if $\beta \in (-1, 0]$ and $x \to 0^+$, $z \to 0^+$, and, for instance, $z = kx$ $k > 0$, then $\lim_{x\to 0^+} det(JS^*(x,kx)) = +\infty$. In both cases a condition for the local stability is violated (see Medio and Lines 2001). These considerations prove the following Proposition.

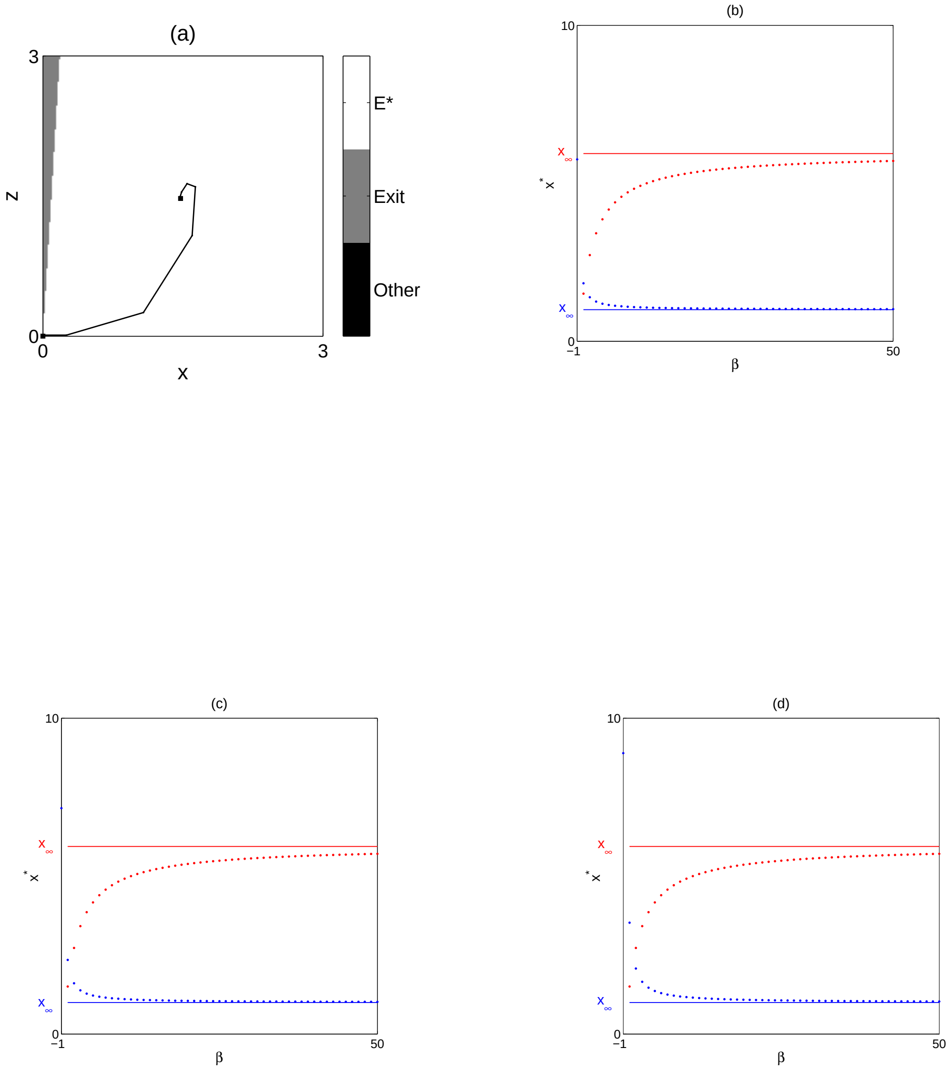

Figure 2: (a) A feasible trajectory starting from $(0.001, 0.001)$ and converging to E^* is depicted for $A = 10$, $p = 0.4$, $\gamma = 0.4$ and $\beta = 0$, $\alpha = 0.33$. (b) Long term capital per capita equilibrium value as β increases for different values of α: in red $\alpha = 0.33$ (and the sequence is increasing) while in blue $\alpha = 0.1$ (and the sequence is decreasing). (c) Long term capital per capita equilibrium value as β increases for different values of γ: in red $\gamma = 0.4$ (and the sequence is increasing) while in blue $\gamma = 0.2$ (and the sequence is decreasing). (d) Long term capital per capita equilibrium value as β increases for different values of p: in red $p = 0.4$ (and the sequence is increasing) while in blue $p = 0.9$ (and the sequence is decreasing).

Proposition 7. *The origin is a locally unstable fixed point of system S^*.*

The previous Proposition holds for all parameter values, i.e. E_0 is locally unstable also with Cobb-Douglas utility. In Figure 2 (a) we depict a feasible trajectory starting from an initial condition close to the origin and converging to E^* when $\beta = 0$.

In order to consider the Jacobian matrix evaluated at the interior fixed point E^*, observe that $x^* = z^* > 0$ and that the following relation holds (see the proof of Proposition 4):

$$(x^*)^{\beta(1-\alpha)} = m_1 m_2 (1-\gamma)(x^*)^{\alpha-1} - m_1(1-\gamma). \tag{25}$$

As a consequence, the Jacobian matrix evaluated at E^* can be written as follows:

$$JS^*(x^*, x^*) = \begin{pmatrix} F_x(x^*, x^*) & F_z(x^*, x^*) \\ 1 & 0 \end{pmatrix}$$

where

$$F_x(x^*, x^*) = \frac{m_1 \left\{ m_2[\alpha + (1-\gamma)(\alpha\beta - \beta)] x^{*(\beta-1)(1-\alpha)} + m_1 m_2 [\gamma(1-\beta+\alpha\beta)+\alpha] x^{*(\alpha-1)} + \gamma^2 m_1 \right\}}{(m_1 m_2 (1-\gamma) x^{*(\alpha-1)} + \gamma)^2}, \tag{26}$$

and

$$F_z(x^*, x^*) = -\frac{\gamma m_2 \alpha}{m_2(1-\gamma) + \gamma (x^*)^{1-\alpha}}. \tag{27}$$

From $JS^*(x^*, x^*)$ the stability conditions are:

$$(1) 1 + F_x(x^*, x^*) - F_z(x^*, x^*) > 0, (2) 1 - F_x(x^*, x^*) - F_z(x^*, x^*) > 0, (3) 1 + F_z(x^*, x^*) > 0. \tag{28}$$

Since we cannot explicitly obtain the coordinates of fixed point E^*, the local stability analysis cannot be carried out for all parameter values. However, we can find some results concerning the stability of the interior fixed point in some limit cases related to the parameters of interest γ and β. The following Proposition holds.

Proposition 8. *Consider system S^*. (i) If $\gamma \to 0^+$ and $\beta \to 0$ then E^* is locally stable; (ii) if $\gamma \to 0^+$ and $\beta \to +\infty$ then E^* is locally unstable; (iii) if $\gamma \to 1^-$ then E^* is locally unstable.*

Proof. (i) If $\gamma \to 0^+$ and $\beta \to 0$ then $x^* \to \left(\frac{pm_2}{1+p}\right)^{\frac{1}{1-\alpha}}$ and consequently $det(JS^*(E^*)) \to 0$ whereas $tr(JS^*(E^*)) \to \alpha$ hence all conditions for the local stability hold.

(ii) If $\beta \to +\infty$ then $x^* \to (\alpha A)^{\frac{1}{1-\alpha}}$. It can be also verified that if $\gamma \to 0^+$ then $det(JS^*(E^*)) \to 0$ whereas $tr(JS^*(E^*)) \to -\infty$ hence conditions for the local stability cannot hold.

(iii) If $\gamma \to 1^-$ then $x^* \to 0^+$, and consequently $det(JS^*(E^*)) \to +\infty$ hence conditions for the local stability cannot hold.

□

We now want to consider the local stability of E^* for negative values of parameter β, and the other limit case $\beta \to -1^+$ and $\gamma \to 0^+$. A preliminary consideration is that, according to the proof of Proposition 5, if $\beta = -1$ then $f(\omega_1) = 1$ and consequently the interior fixed point still exists if and only if $\frac{p(1-\gamma)(1-\alpha)}{\alpha} > 1$. This last inequality holds for $\gamma = 0$ iff $p > \frac{\alpha}{1-\alpha}$. Then, the following Proposition holds.

Proposition 9. *Let $p > \frac{\alpha}{1-\alpha}$. Then if $\gamma \to 0^+$ and $\beta \to -1^+$ E^* is locally stable.*

Proof. Observe that if $\gamma \to 0^+$ then $det(JS^*(x^*, x^*)) \to 0$. Assume also that $\beta \to -1^+$, then we have to distinguish between two cases: (i) if $\frac{\alpha}{1-\alpha} < p < \frac{\alpha}{1-2\alpha}$ then $x^* \to 0$ and $tr(JS^*(x^*, x^*)) \to \frac{\alpha}{p(1-\alpha)} \in (0,1)$; (ii) if $p \geq \frac{\alpha}{1-2\alpha}$ then $x^* \to x^*_{-1}$ and $tr(JS^*(x^*, x^*)) \to \alpha\frac{1+p}{p} \in (0,1)$. Hence E^* is locally stable. □

The results concerning the local stability of the unique interior fixed point in the limiting cases studied above are confirmed by looking at the cycle cartogram depicted in Figure 3 (a). It shows a two-parameter bifurcation diagram, where each color describes a long-term dynamic behaviour for a given combination of γ and β and for an initial condition close to E^*, and exhibits a large diversity of cycles of different order. The red region indicates parameter values producing an unfeasible trajectory.

With regard to the fixed point E^*, we now focus on the bifurcations it can undergo. Notice that $F_z(x^*, x^*) < 0$ so that condition (3) in (28) can be violated and a Neimark-Sacker bifurcation related to closed invariant curves may occur. In addition since $F_x(x^*, x^*)$ can be negative then system S^* can undergo a period doubling bifurcation, i.e. condition (1) can be violated. In what follows, we will use analytical methods combined with numerical techniques to show that either a period doubling or a Neimark-Sacker bifurcation can be produced but only the former can produce (endogenous) persistent fluctuations.

First, we focus on the occurrence of a Neimark-Sacker bifurcation. By taking into account the results of Propositions 8 and 9, and by looking at Figure 3 (a), we note that once fixed a β value, then a threshold value $\gamma_\beta \in (0,1)$ may exist such that, if γ crosses γ_β, the fixed point E^* undergoes a bifurcation, i.e. E^* loses stability and the generic trajectory becomes unfeasible. In the following Proposition, we give necessary conditions for the occurrence of a Neimark-Sacker bifurcation.

Proposition 10. *Let E^* be the interior fixed point of system S^*. Then $\epsilon > 0$ does exist such that $\forall \beta \in I(0, \epsilon)$ there exists $\gamma = \gamma_\beta \in (0,1)$ at which Neimark-Sacker bifurcation can occur.*

Proof. Let $\beta = 0$, then at the steady state,

$$x^* = x_0^* = \left(\frac{p(1-\alpha)A(1-\gamma)}{1+p(1-\gamma)} \right)^{\frac{1}{1-\alpha}}$$

and

$$det(JS^*(x_0^*, x_0^*)) = \frac{\gamma\alpha(1+p(1-\gamma))}{(1-\gamma)(1+p)}.$$

In this case, if $\gamma = \gamma_0$, where $\gamma_0 = \frac{(\alpha+1)(p+1) - \sqrt{(\alpha+1)^2(p+1)^2 - 4p\alpha(1+p)}}{2p\alpha} \in (0,1)$, then ($i$) $det(JS^*(x_0^*, x_0^*)) = 1$, ($ii$) $F_x(x_0^*, x_0^*)$ is positive and less then 2 (i.e. the trace of the Jacobian matrix belongs to the interval $(-2, 2)$), (iii) the two non-real eigenvalues cross the unit circle at a non-zero speed when γ changes and (iv) none of them may be one of the first four roots of unity (excluding cases of weak resonance). According to these conditions a Neimark-Sacker bifurcation may occur at $\gamma = \gamma_0$ when $\beta = 0$.

Consider now $x^* = x^*(\beta, \gamma)$, $\beta > -1$, $\gamma \in (0,1)$. Since

$$det(JS^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) \text{ and } F_x(x^*(\beta,\gamma), x^*(\beta,\gamma))$$

are both continuous w.r.t. β and γ then

$$det(JS^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) \to 1 \text{ if } \beta \to 0 \text{ and } \gamma \to \gamma_0.$$

Hence, $\forall \epsilon_1 > 0 \; \exists I(0, \gamma_0, \epsilon_1)$ such that if $(\beta, \gamma) \in I(0, \gamma_0, \epsilon_1)$ then

$$1 - \epsilon_1 < det(JS^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) < 1 + \epsilon_1$$

and in particular, inside this neighborhood, there exists a $\gamma_\beta < 1$ such that

$$det(JS^*(x^*(\beta,\gamma_\beta), x^*(\beta,\gamma_\beta))) = 1.$$

Furthermore, there exists $I(0, \gamma_0, \epsilon_2)$ such that if $(\beta, \gamma) \in I(0, \gamma_0, \epsilon_2)$ then

$$F_x(x^*(\beta,\gamma), x^*(\beta,\gamma)) < 2.$$

Similar arguments can be used to prove that also conditions (iii) and (iv) hold thus showing that the Neimark-Sacker bifurcation can occur at $\gamma = \gamma_\beta$ if β is close to zero. □

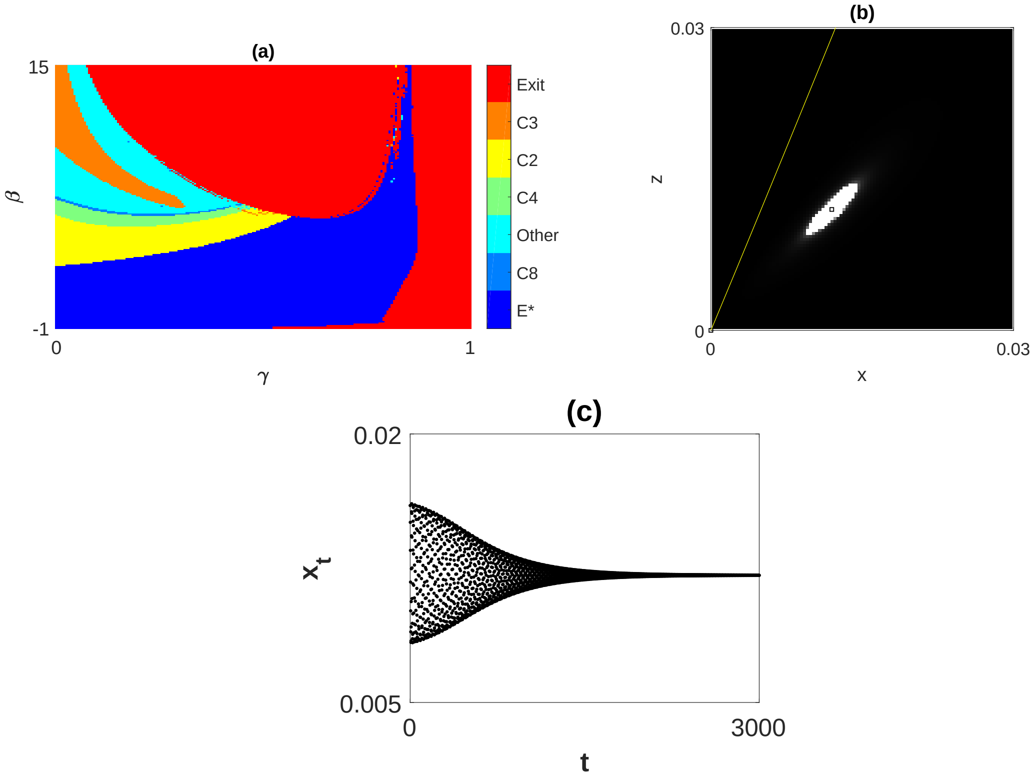

Figure 3: (a) Two dimensional bifurcation diagram of system S^* in the plane (γ, β) for the following parameter values: $A = 10$, $p = 0.5$; $\alpha = 0.27$ and the initial condition is close to E^*. (b) Locally stable fixed point and its immediate basin bounded by a closed repelling invariant curve for $\beta = -0.7$ and $\gamma = 0.7904 < \gamma_\beta \simeq 0.7907$. As γ increases the repelling closed curve shrinks. (c) The trajectory obtained for the same parameter values as in Figure 1 (c) converging to the fixed point E^* for an initial condition taken into the white region.

The previous Proposition 10 gives only necessary conditions for the occurrence of a Neimark-Sacker bifurcation as the Lyapunov coefficient has not been taken into account. In fact we recall that such a coefficient, that depends on second derivatives of the central manifold of the map, has to be different from zero to state that a Neimark-Sacker bifurcation occurs (see Guckenheimer and Holmes (1997) and Kuznetsov (2004)). Anyway, given the analytical complexity of system S^*, the sign of the Lyapunov coefficient cannot be obtained, so that in what follows we will show that the Neimark-Sacker bifurcation occurs by means of numerical simulations.

Proposition 10 shows that if β belongs to an opportune neighbourhood of the origin (thus moving away from the Cobb-Douglas case), it is possible to find a threshold value $\gamma = \gamma_\beta$ such that a Neimark-Sacker bifurcation may occur. In addition, the size of the neighbourhood of β for which the Neimark-Sacker bifurcation may take place depends on the parameter values of the model. By choosing appropriately the parameters of the model, it is possible to get the value of γ_β for different choices of $\beta \in I(0, \epsilon)$ by using numerical instruments. To this purpose, we perform an algorithm allowing us to find out that γ_β is increasing in β and that the size of the β-interval (ϵ) such that the Neimark-Sacker bifurcation can be exhibited increases if p increases.

Unfortunately, given the analytical complexity of map S^*, the type of the Neimark-Sacker bifurcation cannot be proved, so that, in order to describe the role of the closed invariant curve involved in such a bifurcation we make use of numerical simulations. Taking into account Proposition 10 and the numerical evidence in Figure 1 panel (c) it can be observed that just before the threshold bifurcation value occurring at $\gamma_\beta \simeq 0.7908$, the white region is bounded by a closed curve Ω: it can be easily checked that the fixed point E^* inside it is still attracting (see Figure 3 (c)). At the bifurcation value γ_β, the curve Ω disappears, shrinking on E^*, which then becomes repelling, and almost all initial conditions produce unfeasible trajectories.

In fact, Figure 1 (c) shows the basin of attraction of the interior fixed point E^* for $\gamma = \gamma_1 < \gamma_\beta$, while in Figure 3 panel (b) a less value $\gamma = \gamma_2 \in (\gamma_1, \gamma_\beta)$ is considered. In both cases the basin boundary of the attracting fixed point is a closed invariant curve. In fact a repelling closed curve coexists with the attracting fixed point being the boundary of its basin of attraction. Notice that, as γ increases, this basin becomes smaller while at the bifurcation value γ_β, point E^* becomes repelling merging with the repelling closed invariant curve. Summarizing, several numerical evidences show that a Neimark-Sacker bifurcation of subcritical type is exhibited, while the appearance of the repelling closed invariant curve Ω can be related to the positivity constraints, similarly to what occurs in Agliari et al. (2006b), and can be better investigated following the study there proposed.

A deeper description of the appearance of Ω is not within the goal of the present work (we leave this part to a possible future development) as we are mainly interested in understanding whether our economy is able to exhibit persistent fluctuations and the corresponding role of γ and β on the emergence of these fluctuations.

From our previous considerations it can be observed that the only way for system S^* to produce fluctuations is via period doubling bifurcations so we now move to the investigation of such local bifurcation.

From the proof of Proposition 8 it can be observed that if γ is less enough, i.e. $\gamma \to 0^+$, then as long as $\beta \to 0$ all conditions in (28) hold, while, if β is high enough, i.e. $\beta \to +\infty$ then conditions (2) and (3) of (28) are still verified while condition (1) does not hold being $1+F_x(x^*, x^+)-F_z(x^*, x^*) < 0$. Since both $F_x(x^*, x^*)$ and $F_z(x^*, x^*)$ are continuous w.r.t. β, $\forall \beta > 0$, then $\exists \bar{\beta} > 0$ at which a flip bifurcation occurs thus providing that S^* is able to produce fluctuations. The following Proposition trivially holds.

Proposition 11. *Let E^* be the locally stable interior fixed point of system S^*. Then $\epsilon > 0$ does exist such that $\forall \gamma \in I_+(0, \epsilon)$ there exists $\beta = \bar{\beta} > 0$ at which a period-doubling bifurcation occurs.*

The previous considerations together with numerical experiments can be used to describe the role of individual preferences on the asymptotic dynamics of the model. In fact it can be observed that when γ is fixed at a sufficiently low value, then a sequence of period doubling bifurcations occur as β increases as it can be observed in Figure 4 (b), where it is also shown that the long-term evolution of the capital per worker in an economy with aspirations increases in complexity as β increases (i.e. σ decreases). This result represents a new evidence of a different route to chaos with respect to de la Croix (1996) due to the presence of CIES preferences. If β is high enough, then S^* admits a chaotic attractor, as depicted in Figure 4 panel (c), i.e. fluctuations in the economy can be produced (a generic trajectory is depicted in Figure 4 panel (d)).

For what it concerns the role of aspirations, in Figure 4 (a) we fixed $\beta = 6$ and the initial condition is close to E^*: a period doubling and halving bifurcation cascade can be observed providing that the economic cycle may be produced in a different way with respect to the Neimark-Sacker bifurcation discussed in de la Croix (1996) for low values of aspirations. In sharp contrast with his work, several numerical computations have shown evidence that aspirations play a stabilising role at intermediate values as the unique interior fixed point E^* is locally stable if aspirations are not too low, as long as feasible trajectories are exhibited. Definitely, we have shown that changing the value of γ may produce a local destabilization of E^* via a period doubling bifurcation. In addition, aspirations may play an opposite role in comparison with de la Croix (1996) by acting as a stabilising device.

6 Conclusions

There exists a widespread literature on endogenous fluctuations in deterministic models designed to provide an alternative to business cycles models (driven essentially by random external shocks). In the OLG literature where individuals live for two periods, these fluctuations can be interpreted as long-term cycles. The main aim of this research was to show that the existing models in the

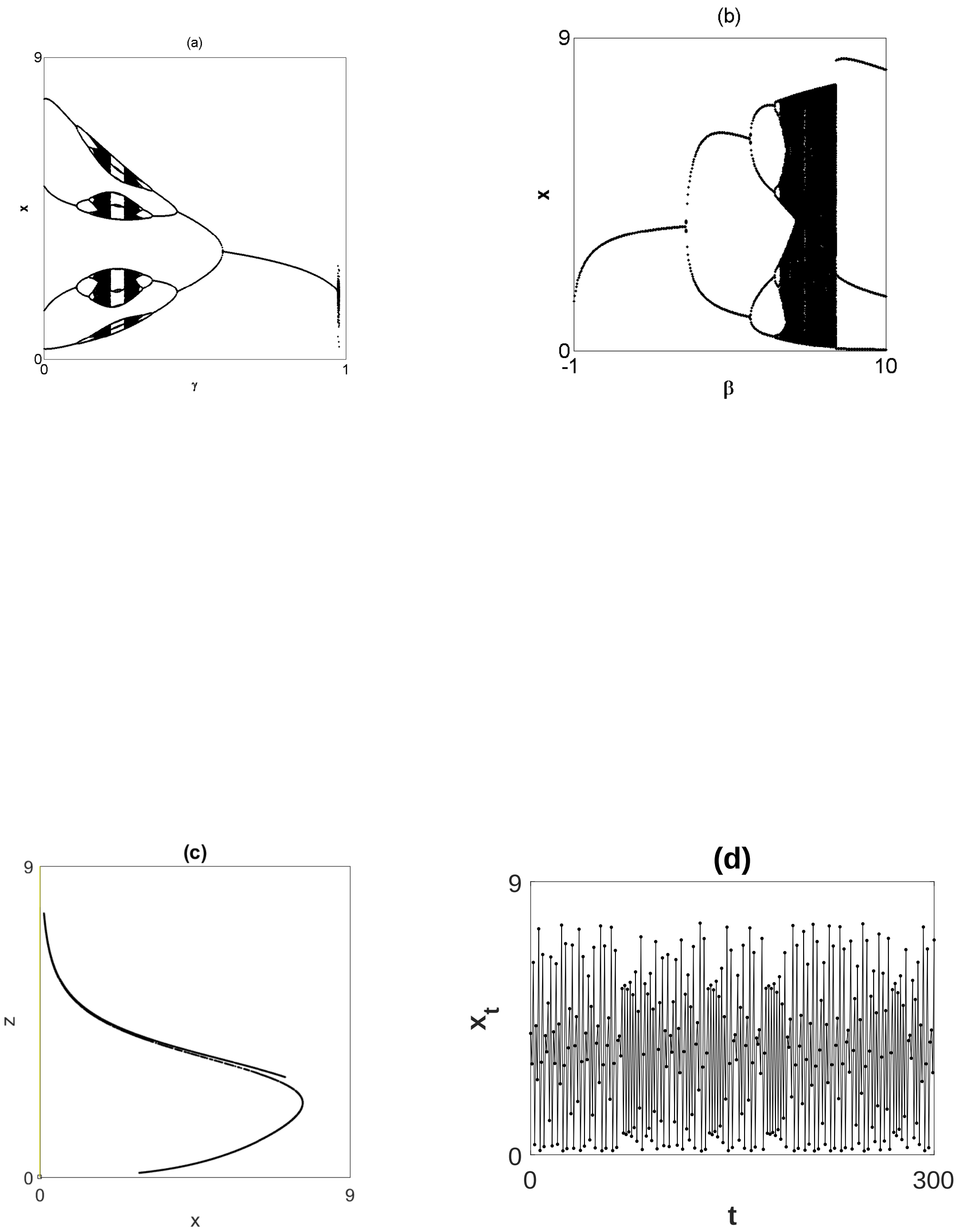

Figure 4: (a) One dimensional bifurcation diagram w.r.t. γ if $\alpha = 0.27$ and $\beta = 6$, $A = 10$ and $p = 0.5$. (b) One dimensional bifurcation diagram w.r.t. β if $\alpha = 0.27$ and $\gamma = 0.1$; $A = 10$ and $p = 0.5$. (c) One piece chaotic attractor for $\beta = 8$ and other paramters as in panel (b) and (d) its trajectory for capital per capita.

OLG literature with aspirations may actually produce endogenous and persistent fluctuations. In particular, the article has concerned with the study of a general equilibrium economy à de la Croix (1996). It has extended his work by considering a CIES utility function including the log-utility as a special case. The interaction between the intensity of aspirations and the elasticity of substitution of effective consumption affects the qualitative and quantitative long-term dynamics. First, in order to avoid unfeasible trajectories the stock of aspirations should not be fixed at too high a level and the size of the stock of capital plays a different role depending on whether the elasticity of substitution is low or high. Our findings contribute to the OLG literature on endogenous fluctuations by showing that: 1) the Neimark-Sacker bifurcation found by de la Croix (1996) and de la Croix and Michel (1999) does not necessarily produce economic fluctuations; 2) endogenous fluctuations are produced via a period doubling bifurcation; 3) the interaction between aspirations and inter-temporal preferences affects both long-term outcomes and dynamic outcomes. In particular, with non-Cobb-Douglas utility aspirations may play a stabilising role.

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