Stability analysis in a Bertrand duopoly with different product quality and heterogeneous expectations

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Abstract We study the local stability properties of a nonlinear Bertrand duopoly with vertical differentiation and heterogeneous players with both covered and uncovered markets. In the former case, the unique pure strategy Nash equilibrium can undergo a flip bifurcation when the extent of consumer's heterogeneity increases. In the latter, the quality differential plays a preeminent role in determining stability of prices over time. Numerical evidence is provided to show the occurrence of endogenous fluctuations.

Keywords Bifurcation; Duopoly; Heterogeneous expectations; Price competition; Vertical differentiation

JEL Classification C62; D43; L13; L15

1. Introduction

It is observed that firms often supply differentiated products on the market, so that consumers face a large domain of varieties, which can sometimes unambiguously be ranked along some quality ladders. The focus of the present study is to analyse stability properties of a nonlinear duopoly (see Bischi et al., 2010) with price competition and vertical differentiation.

There exists an established literature that deals with problems of horizontal (quantity) and vertical (quality) differentiation of goods and services in static oligopoly games, essentially to rank equilibrium outcomes in both Cournot and Bertrand competition models. Studies that deal with the former type of product differentiation date back at least to the works by Dixit (1979), Singh and Vives (1984) and Vives (1985), while examples of the latter can be found in Gabszewicz and Thisse (1979), Shaked and Sutton (1982), Motta (1993), Wauthy (1996), Häckner (2000) and Correa-López and Naylor (2004). The findings of this literature represent a cornerstone of the oligopoly theory.

Another strand of literature on nonlinear oligopolies analyse several aspects of dynamic phenomena (e.g., local and global stability of dynamic systems). This literature is of increasing importance and makes expectations formation mechanisms different from the rational expectations paradigm relevant (see, e.g., Chiarella, 1986, 1990; Puu 1991, 1998; Agliari et al., 2006). As is known, the Nash equilibrium in a dynamic duopoly with standard linear demand and cost functions is stable if expectations of every firm are "naïve" (i.e., each firm expects that the value of the strategic variable set by the rival to maximise profits in the future period is equal to the current period one), as shown by Theocharis (1960) in a duopoly with quantity

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competition à la Cournot (1838). However, if expectations of one or both firms are those of the type suggested by, e.g., Bischi et al. (1998, 1999), i.e. firms do not perfectly infer the decisions of competitors (bounded rationality) and increase/decrease their control variable in the current period depending on information given by marginal profits in the previous period (see Dixit, 1986; Bischi and Naimzada, 2000), then the equilibrium in a duopoly game with standard linear demand and cost functions may be destabilised when the reaction of every firm is large enough, as shown by Kopel (1996). In particular, the stability issue in duopoly games without vertical differentiation has been analysed, amongst others, by (*i*) Agiza and Elsadany (2003, 2004), Zhang et al. (2007), Tramontana (2010) and Fanti and Gori (2012), as regards quantity competition, and (*ii*) Zhang et al. (2009) and Fanti and Gori (2011), as regards price competition.

However, at the best of our knowledge, the stability analysis in a duopoly with price competition in which firms provide products of different (say, high and low) quality, has not been so far tackled on. In this paper we aim to fill this gap by studying two distinct cases with both covered and uncovered markets: (*i*) the high-quality firm has bounded rational expectations and the low quality firm has naïve expectations; (*ii*) the low-quality firm has bounded rational expectations and the high-quality firm has naïve expectations.[1] With regards to covered markets, we find that stability of prices in the long run depends only on the extent of consumer's heterogeneity. With regards to uncovered markets, the quality differential matters.

The remainder of the paper is organised as follows. Section 2 introduces the model. Under the hypothesis of covered market, Section 3 studies the conditions under which the unique pure strategy Nash equilibrium can loose stability in the case of heterogeneous expectations. Section 4 assumes uncovered markets and studies the dynamic properties of the two-dimensional map in such a case. Section 5 concludes.

2. The model (covered market)

Following Tarola et al. (2011), we assume that: (1) there exist two firms (H and L) in the market providing goods and services of different quality to the customers; (2) it is unanimously believed that products of firm H are of a higher quality than those of firm L; (3) the average cost of production is not affected by quality, and it is set to zero without loss of generality.

Consumers are identified by the parameter $\phi \in [a,b]$, where $0 \le a < b$, which, by following an established literature (see Tirole, 1988; Motta, 1993), can be interpreted as "the marginal rate of substitution between income and quality" (Motta, 1993, p. 115).[2] Then, ϕ measures the taste for quality of consumers, and it is assumed to be uniformly distributed with unit density (e.g., Motta, 1993; Liao, 2008), while the parameters a and b capture the extent of population heterogeneity. The larger the difference between a and b, the higher the degree of heterogeneity amongst consumers. Preferences (U) of consumer of type ϕ are described by the following expected utility function:

$$U_i(\phi, p_i) = \phi u_i - p_i, \quad i = \{H, L\}, \tag{1}$$

[1] It is usual in the literature on nonlinear oligopolies to assume firms with distinct expectations formation mechanisms (see, e.g., Leonard and Nishimura, 1999; Den-Haan, 2001).

[2] As pointed out by Motta (1993), by interpreting ϕ as a measure of the marginal rate of substitution between income and quality in a model where consumers' tastes heterogeneity is assumed, allows comparisons with models where there exists consumers' incomes heterogeneity (see, e.g., Gabszewicz and Thisse, 1979; Shaked and Sutton, 1982, 1983).

where u_H and u_L, with $u_H > u_L$, represent two indexes that capture the different quality (of products of firms H and L) perceived by consumers, and p_i is the price that consumers pay to buy product i, and then it represents the marginal willingness to pay to consume an amount of goods and services of quality i. Of course, the higher u_i, the higher utility U_i. Following Wauthy (1996), quality indexes are exogenous. Moreover, in order to guarantee that both firms set strictly positive prices at an interior equilibrium, we assume that:[3]

$$a \in \left[0, \frac{b}{2}\right). \tag{2}$$

Let $\bar{\phi}$ be an index that identifies the consumer indifferent between purchasing products of high or low quality from firms H and L at the price p_H and p_L, respectively. Such an index is obtained by equating $U_H(\phi, p_H) = U_L(\phi, p_L)$. Then, by solving the equation

$$\phi u_H - p_H = \phi u_L - p_L, \tag{3}$$

for ϕ we get:

$$\bar{\phi} = \frac{p_H - p_L}{u_H - u_L}, \tag{4}$$

which depends on both the price differential and quality differential. The higher the former (latter), the higher (lower) the type of consumers with a taste for high-quality products. Then, consumers identified by $\bar{\phi} < \phi < b$ ($a < \phi < \bar{\phi}$) purchase products of high (low) quality. Since the market is covered, $D_H(p_H, p_L) + D_L(p_H, p_L) = 1$, with $D_i(\bullet) > 0$ for $i = \{H, L\}$ (Gabszewicz and Thisse, 1979; Wauthy, 1996). Then, the demand functions to firms H and L are respectively given by:

$$D_H(p_H, p_L) = b - \bar{\phi} = b - \frac{p_H - p_L}{u_H - u_L}, \tag{5.1}$$

$$D_L(p_H, p_L) = \bar{\phi} - a = \frac{p_H - p_L}{u_H - u_L} - a. \tag{5.2}$$

Profits of the ith firm $\Pi_i(\bullet) = p_i D_i(\bullet)$, since average production costs are zero. Therefore, profits of firms H and L as a function of prices are given by:

$$\Pi_H(p_H, p_L) = p_H\left(b - \frac{p_H - p_L}{u_H - u_L}\right), \tag{6.1}$$

$$\Pi_L(p_H, p_L) = p_L\left(\frac{p_H - p_L}{u_H - u_L} - a\right). \tag{6.2}$$

The maximisation of Eqs. (6.1) and (6.2) with respect to p_H and p_L gives the following marginal profits:

$$\frac{\partial \Pi_H}{\partial p_H} = b - \frac{2p_H - p_L}{u_H - u_L}, \tag{7.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = \frac{p_H - 2p_L}{u_H - u_L} - a. \tag{7.2}$$

Therefore, the reaction- or best-reply functions of firms H and L are determined by equating Eqs. (7.1) and (7.2) to zero and solving for p_H and p_L, respectively, that is:

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[3] This will be clear from Eqs. (13) and (26) in the sequel of the paper. Note that this condition also guarantees that outputs and profits of both firms are positive at an interior equilibrium.

$$\frac{\partial \Pi_H}{\partial p_H} = 0 \Leftrightarrow p_H(p_L) = \frac{1}{2}\left[p_L + b(u_H - u_L)\right], \tag{8.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = 0 \Leftrightarrow p_L(p_H) = \frac{1}{2}\left[p_H - a(u_H - u_L)\right], \tag{8.2}$$

where $p_H - a(u_H - u_L) > 0$ should hold.

In the next section we introduce dynamic elements into the static model described above, while also assuming that firms H and L have heterogeneous (namely, bounded rational and naïve) expectations about the price should be set in the future by the rival to maximise profits. Then, we study the local stability properties of the Nash equilibrium when the high-quality and low-quality firms are alternatively bounded rational and naïve.

Before starting with the analysis, however, it could be instructive to clarify the reasons why we have assumed players with heterogeneous expectations. First, since firms are heterogeneous because vertical differentiation exists into the model and then products of firms H and L are perceived of being of different quality by customers, it is relevant (in a dynamic setting) to see how every firm reacts to a change in the price sets by the rival when different expectations formation mechanisms are in existence. This because the response of one firm to a strategy played by its competitor is different depending on the adjustment mechanism used, and this makes the study of a model where players have heterogeneous expectations interesting in a dynamic setting. Indeed, in both cases we talk about myopic expectations. However, under the naïve rule we observe that one source of uncertainty exists in such a case: the naïve firm, in fact, does not know the behaviour of the rival but knows the shape of the market demand. It can then use this information through the reaction function to behave optimally over time. In contrast, under the bounded rational rule two sources of uncertainty are in existence: the bounded rational firm does not know both the behaviour of the rival but knows and the shape of the market demand. It estimates, therefore, its marginal profits to behave optimally over time, and, in a Bertrand duopoly, the optimal price set in the future period will be equal to the last period's one plus/minus something based on current marginal profits, which can be positive or negative.[4]

Second, there exists an established branch of literature that investigates several aspects of dynamic games when firms are heterogeneous because of different expectations formation rules (see, e.g., Leonard and Nishimura, 1999; Den-Haan, 2001; Agiza et al., 2002; Agiza and Elsadany, 2003, 2004; Zhang et al., 2007; Tramontana, 2010; Fanti and Gori, 2012). Moreover, it is usual in both (static and dynamic) Cournot and Bertrand duopolies to study models with heterogeneous competitors (e.g., competitive wage versus non-competitive wage, products are substitutes or complements and so on).

Third, it is worth noting that by assuming both high-quality and low-quality firms to be bounded rational dramatically enriches the spectrum of dynamic outcomes with respect to the findings of this study. In particular, although the results as regards the effects on local stability of the parameter a (under the covered market assumption) and the parameter u_L (under the uncovered market assumption) are preserved, global bifurcations may also occur (e.g., coexistence of chaotic attractors) that cannot be observed in the case of heterogeneous expectations. In particular, the global behaviour of the noninvertible map when both players are bounded rational can be investigated through the study of critical curves, by which a two-

[4] As point out by Bischi et al. (1998, p. 561), in this class of models: "The dynamic game is based on the assumption that the two producers have not a complete knowledge of the market, hence they behave adaptively, following a bounded rationality adjustment process based on a local estimate of the marginal profit." We note that it is standard in this literature to refer to the player that expects the output/price of the competitor be equal to the last period's one as being "naïve", and to the player that uses the myopic adjustment mechanism (through marginal profits) described by Dixit (1986) and Naimzada and Bischi (2000) as being "bounded rational".

dimensional area can be defined to give a bound to the amplitude of the trajectories. This topic is included in our future research agenda, and requires a technical paper to deeply investigate the mathematical properties of the map.

3. Equilibrium and local stability with heterogeneous expectations

3.1. Covered market: case $\boxed{BH/NL}$

In this section we assume that firm H has bounded rational (BH) expectations and firm L has naïve (NL) expectations. Therefore, firm H uses information on its profit at time $t = 0,1,2,\ldots$ to increase or decrease prices at time $t+1$ according to the myopic mechanism described by Dixit (1986), that is:

$$p_H(t+1) = p_H(t) + \alpha_H p_H(t)\frac{\partial \Pi_H(t)}{\partial p_H(t)}, \tag{9}$$

where $\alpha_H > 0$ is a coefficient that captures the speed of adjustment of firm H's price with respect to a marginal change in profits when $p_H(t)$ varies, and $\alpha_H\, p_H(t)$ is the intensity of the reaction of the bounded rational player to a change in rival's price at time t. Therefore, $p_H(t+1)$ is increased or decreased depending on whether current marginal profits are positive of negative, respectively.

Using Eq. (9), and knowing that firm L has naïve expectations (i.e., the price at time $t+1$ equals the price at time t), the two-dimensional system that characterises the dynamics of this simple duopolistic market is the following:

$$\begin{cases} p_H(t+1) = p_H(t) + \alpha_H p_H(t)\dfrac{\partial \Pi_H(t)}{\partial p_H(t)} \\ p_L(t+1) = p_L(t) \end{cases}. \tag{10}$$

Using Eq. (7.1) to substitute out for $\partial \Pi_H(t)/\partial p_H(t)$ into the first equation of (10), and Eq. (8.2) to substitute out into the right-hand side of the second equation of (10), we get:

$$\begin{cases} p_H(t+1) = p_H(t) + \alpha_H p_H(t)\left(b - \dfrac{2p_H(t) - p_L(t)}{u_H - u_L}\right) \\ p_L(t+1) = \dfrac{1}{2}\left[p_H(t) - a(u_H - u_L)\right] \end{cases}. \tag{11}$$

Equilibrium implies that $p_H(t+1) = p_H(t) = p_H$ and $p_L(t+1) = p_L(t) = p_L$. Therefore, the dynamic system defined by (11) can be reduced to:

$$\begin{cases} \alpha_H p_H\left(b - \dfrac{2p_H - p_L}{u_H - u_L}\right) = 0 \\ p_L - \dfrac{1}{2}\left[p_H - a(u_H - u_L)\right] = 0 \end{cases}. \tag{12}$$

The unique interior fixed point $E_{BH/NL}^{CM} = \left(p^*{}_H, p^*{}_L\right)$ of the dynamic system defined by Eq. (11) is determined by the following non-negative solution of Eq. (12), that is:

$$E_{BH/NL}^{CM} = \left(p^*{}_H, p^*{}_L\right) = \left(\frac{1}{3}(2b-a)(u_H - u_L), \frac{1}{3}(b-2a)(u_H - u_L)\right), \tag{13.1}$$

where $E^{CM}_{BH/NL}$ stands for "covered market", which represents the pure strategy Nash equilibrium of the model. From Eq. (13.1), we note that in equilibrium $p^*_H - p^*_L = \frac{1}{3}(a+b)(u_H - u_L)$, so that $\bar{\phi} = \frac{a+b}{3}$. Then,

Corollary 1. *Under the hypothesis of covered market, the consumer type indifferent between purchasing products of high or low quality, exclusively depends on the parameters that characterise the extent of population heterogeneity.*

Proof. The proof follows immediately by looking at the equilibrium values of $\bar{\phi}$ under the hypothesis of covered market. **Q.E.D.**

Furthermore, outputs and profits at the equilibrium point are given by:

$$D^*_H = \frac{2b-a}{3}, \quad D^*_L = \frac{b-2a}{3}, \tag{13.2}$$

$$\Pi^*_H = \frac{(2b-a)^2(u_H - u_L)}{9}, \quad \Pi^*_L = \frac{(b-2a)^2(u_H - u_L)}{9}. \tag{13.3}$$

Notice that profits of both firms reduce when: (*i*) the quality differential, $u_H - u_L$, reduces, and (*ii*) for every b, the value of the parameter a increases.

In order to investigate the local stability properties of the fixed point (13.1) of the two-dimensional system (11), we build on the Jacobian matrix J evaluated at $E^{CM}_{BH/NL}$, that is:

$$J^{CM}_{BH/NL} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} 1 - \frac{2}{3}\alpha_H(2b-a) & \frac{1}{3}\alpha_H(2b-a) \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{14}$$

where $J_{ii} = \partial p_i(t+1)/\partial p_i(t)$ and $J_{ij} = \partial p_i(t+1)/\partial p_j(t)$ are evaluated at $E^{CM}_{BH/NL}$, Therefore, the trace and determinant of (14) are respectively given by:

$$T := Tr\left(J^{CM}_{BH/NL}\right) = J_{11} + J_{22} = 1 - \frac{2}{3}\alpha_H(2b-a), \tag{15}$$

$$D := Det\left(J^{CM}_{BH/NL}\right) = J_{11}J_{22} - J_{12}J_{21} = -\frac{1}{6}\alpha_H(2b-a) < 0. \tag{16}$$

The characteristic polynomial of (14) can then be written as follows:

$$G(\lambda) = \lambda^2 - T\lambda + D, \tag{17}$$

with the discriminant being determined by $Q := T^2 - 4D = \left[1 - \frac{2}{3}\alpha_H(2b-a)\right]^2 + \frac{1}{6}\alpha_H(2b-a) > 0$. Since Q is positive, the existence of complex eigenvalues of $J^{CM}_{BH/NL}$ is prevented.

As is known, bifurcation theory describes the way the topological features of a dynamic system (such as the number of stationary points or their stability) vary as some parameter values are continuously changed. In particular, for the system in two dimensions determined by (11), the stability conditions ensuring that both eigenvalues remain within the unit circle are:

$$\begin{cases} (i) \quad F := 1 + T + D = 2 - \frac{5}{6}\alpha_H(2b-a) > 0 \\ (ii) \quad TC := 1 - T + D = \frac{1}{2}\alpha_H(2b-a) > 0 \\ (iii) \quad H := 1 - D = 1 + \frac{1}{6}\alpha_H(2b-a) > 0 \end{cases} \quad . \tag{18}$$

The violation of any single inequality in (18), with the other two being simultaneously fulfilled leads to: (*i*) a flip or period-doubling bifurcation (a real eigenvalue that passes through -1) when $F=0$; (*ii*) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC=0$; (*iii*) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H=0$, namely $Det(J)=1$ and $|Tr(J)|<2$.

From (18) it is clear that conditions (*ii*) and (*iii*) are always fulfilled for any $a \in [0, b/2)$, while condition (*i*) can be violated. The existing literature on dynamic oligopolies (see, amongst others, Bischi et al., 1998, 1999; Bischi and Naimzada, 2000; Tramontana, 2010) has shown that the coefficient α (when at least one of the two players has bounded rational expectations) plays an important role in determining both the local and global stability properties of a two-dimensional map. In particular, if we refer to the dynamics around the equilibrium, a rise in α (ceteris paribus as regards marginal profits) increases the reaction of the bounded rational firm to a rise in its competitor's control variable, and then it acts as a destabilising device. This being said, the system defined by Eq. (11) that characterises the dynamics of a Bertrand duopoly with vertical differentiation, shows the same properties as regards the role played by α on local stability of a fixed point of those described by the existing literature.[5] However, at the best of our knowledge, nobody has inquired about the dynamic effects of the parameters that capture both the degree of population heterogeneity and quality differential between products H and L. This is the main feature that distinguishes the present paper with respect to the existing literature on dynamic oligopolies. In particular, in what follows we choose a as the bifurcation parameter. In other words, for any given value of both α_H and b, we let a vary within its domain if definition defined by Eq. (2), and study the stability properties of $E_{BH/NL}^{CM}$ when a reduces (i.e., the extent of consumers' heterogeneity increases).

As can be seen by looking at the map defined by Eq. (11), a reduction in a (ceteris paribus) causes an increase (of an amount equal to half the quality differential) in the price set by the naïve firm to maximise profits, and this in turn implies, as an indirect effect, a rise in marginal profits of the bounded rational firm. Then, firm H increases its price by the term $\frac{\alpha_H p_H(t)}{u_H - u_L}$, as a reaction. Therefore, a rise in the extent of consumers' heterogeneity unambiguously acts as an incentive for the bounded rational firm to increase its price, because of a twofold effect: (*i*) a rise in its competitor's price (firm L), and (*ii*) a rise its marginal profits. This can produce relevant effects on stability of the fixed point, as will be clear below.

Now, define

[5] It is important to note, by looking at Eq. (10), that when the coefficient α tends to zero, the bounded rational firm *does not* adjust its production over time, i.e. there is no strategic interaction in such a case. This implies that the bounded rational firm does not behave as if it were naïve when $\alpha = 0$. The naïve firm in fact uses the available information through the reaction function (market demand) to behave optimally over time. The bounded rational firm, instead, adjust its prices over time on the basis of its marginal profits. When both firms are naïve, the Nash equilibrium is stable (see Theocharis, 1960).

$$a_{BH/NL} = \frac{10\alpha_H b - 12}{5\alpha_H}, \tag{19}$$

and

$$\alpha_1 = \frac{6}{5b}, \tag{20}$$

$$\alpha_2 = \frac{8}{5b}, \tag{21}$$

as the flip bifurcation value of a in the case BH/NL, where $\lim_{\alpha \to +\infty} a_{BH/NL} = 2b$, and two threshold values of the speed of adjustment α, where $\alpha_2 > \alpha_1$, which are the roots for α obtained by equating Eq. (19) to zero and $b/2$, respectively, i.e., the boundaries of the domain of definition of a. Then, from (18)-(21) we have the following proposition.

Proposition 1. *(1) Let $0 < \alpha < \alpha_1$ hold. Then, the Nash equilibrium $E^{CM}_{BH/NL}$ of the two-dimensional system (11) is locally asymptotically stable for any $a \in [0, b/2)$. (2) Let $\alpha_1 \le \alpha < \alpha_2$ hold. Then, $E^{CM}_{BH/NL}$ is locally asymptotically stable for $b/2 > a > a_{BH/NL}$, it undergoes a flip bifurcation at $a = a_{BH/NL}$, while becoming locally unstable for $a_{BH/NL} > a > 0$. (3) Let $\alpha \ge \alpha_2$ hold. Then, the $E^{CM}_{BH/NL}$ is locally unstable for any $a \in [0, b/2)$.*

Proof. Since $a_{BH/NL} < 0$ for any $0 < \alpha < \alpha_1$, then $F > 0$ for any $a \in [0, b/2)$. This proves point (1). Since $0 \le a_{BH/NL} < b/2$ for any $\alpha_1 \le \alpha < \alpha_2$, then $F > 0$ for any $b/2 > a > a_{BH/NL}$, $F = 0$ if and only if $a = a_{BH/NL}$ and $F < 0$ for any $a_{BH/NL} > a > 0$. This proves point (2). Since $a_{BH/NL} \ge b/2$ for any $\alpha \ge \alpha_2$, then $F < 0$ for any $a \in [0, b/2)$. This proves point (3). **Q.E.D.**

Proposition 1 reveals (under covered market) the importance of the parameter that defines the lower bound of the consumers' type range in determining local stability outcomes, while also showing that the quality differential between products H and L does not matter for stability. This is due to the fact that the equilibrium value of the consumer's type indifferent between purchasing products of high and low quality ($\bar{\phi} = \frac{a+b}{3}$) exclusively depends on a and b. Indeed, for any given value of b the lower a, the higher the extent of consumer's heterogeneity, and the higher both the marginal profit and reaction of the bounded rational firm to a rise in its competitor's price. This, in turn, causes a destabilising effect when α is included in an intermediate range of values. Proposition 1, in fact, also shows the usual effect played by α on local stability: the higher is such a parameter, the more likely a higher degree of population heterogeneity acts as a destabilising device.

3.2. Covered market: case NH/BL

In this section we assume that firm H has naïve expectations (NH), while firm L has bounded rational expectations (BL) and then only the latter firm uses information on its current profit to increase or decrease prices at time $t+1$ according to the adjustment process:

$$p_L(t+1) = p_L(t) + \alpha_L p_L(t) \frac{\partial \Pi_L(t)}{\partial p_L(t)}, \tag{22}$$

where $\alpha_L > 0$. Therefore, the two-dimensional system that characterises the dynamics of the economy becomes the following:

$$\begin{cases} p_H(t+1)=p_H(t) \\ p_L(t+1)=p_L(t)+\alpha_L p_L(t)\dfrac{\partial \Pi_L(t)}{\partial p_L(t)} \end{cases}. \tag{23}$$

By using Eqs (7.2), (8.1) and (23) we get:

$$\begin{cases} p_H(t+1)=\dfrac{1}{2}\left[p_L(t)+b(u_H-u_L)\right] \\ p_L(t+1)=p_L(t)+\alpha_L p_L(t)\left(\dfrac{p_H(t)-2p_L(t)}{u_H-u_L}-a\right) \end{cases}. \tag{24}$$

From Eq. (24) it is clear that, different from the BH/NL case, a reduction in a (ceteris paribus) does not cause any effects on the price set by the naïve player (firm H), while determining a rise in the marginal profit of the bounded rational player (firm L), which in turn, causes an increase in its price of an amount exactly equal to $\alpha_L p_L(t)$. Therefore, analogously to the BH/NL case, under NH/BL a rise in the extent of consumers' heterogeneity unambiguously acts as an incentive for the bounded rational firm to increase its price. In this case, however, the economic reason why we observe such an increase is exclusively due to the rise in its marginal profits. Therefore, the extent of the rise in the price of the bounded rational firm under both BH/NL and NH/BL expectations is different when a reduces. This (*i*) leads us to expect different stability effects as long as a changes in the two cases, and (*ii*) makes clear the importance of the assumption heterogeneous expectations in a duopoly model with vertical differentiation and different consumers' tastes.

We now turn the equilibrium and local stability analyses. Equilibrium implies that $p_H(t+1)=p_H(t)=p_H$ and $p_L(t+1)=p_L(t)=p_L$. Then, the dynamic system defined by (24) can be reduced to:

$$\begin{cases} p_H-\dfrac{1}{2}\left[p_L+b(u_H-u_L)\right]=0 \\ \alpha_L p_L\left(\dfrac{p_H-2p_L}{u_H-u_L}-a\right)=0 \end{cases}. \tag{25}$$

The unique non-negative fixed point $E^{CM}_{NH/BL}=\left(p^*{}_H, p^*{}_L\right)$ of the dynamic system defined by Eq. (24) is determined by the following non-negative solution of Eq. (25), that is:

$$E^{CM}_{NH/BL}=\left(p^*{}_H, p^*{}_L\right)=\left(\frac{1}{3}(2b-a)(u_H-u_L), \frac{1}{3}(b-2a)(u_H-u_L)\right). \tag{26}$$

It should be noted that the different type of expectation formation mechanisms is not relevant for the equilibrium outcomes of prices, outputs and profits of both firms, while playing a crucial role for stability, as can be seen below. Indeed, by comparing Eqs. (13.1) and (26) it is easy to see that $E^{CM}_{BH/NL}=E^{CM}_{NH/BL}$.

In order to investigate the local stability properties of the fixed point (26) of the two-dimensional system (24), we build on the Jacobian matrix J evaluated at $E^{CM}_{NH/BL}$, that is:

$$J^{CM}_{NH/BL}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}=\begin{pmatrix} 0 & \dfrac{1}{2} \\ \dfrac{1}{3}\alpha_L(b-2a) & 1-\dfrac{2}{3}\alpha_L(b-2a) \end{pmatrix}. \tag{27}$$

where $J_{ii}=\partial p_i(t+1)/\partial p_i(t)$ and $J_{ij}=\partial p_i(t+1)/\partial p_j(t)$ evaluated at $E^{CM}_{NH/BL}$, whose trace and determinant are given by:

$$T := Tr\left(J^{CM}_{NH/BL}\right) = J_{11} + J_{22} = 1 - \frac{2}{3}\alpha_L(b-2a), \tag{28}$$

$$D := Det\left(J^{CM}_{NH/BL}\right) = J_{11}J_{22} - J_{12}J_{21} = -\frac{1}{6}\alpha_L(b-2a) < 0. \tag{29}$$

Therefore, the characteristic polynomial of (27) is:

$$Z(\lambda) = \lambda^2 - T\lambda + D, \tag{30}$$

with its discriminant being determined by $P := T^2 - 4D = \left[1 - \frac{2}{3}\alpha_L(b-2a)\right]^2 + \frac{1}{6}\alpha_L(b-2a) > 0$. Since the discriminant is positive, the existence of complex eigenvalues of $J^{CM}_{NH/BL}$ is prevented.

The stability conditions for the system in two-dimension (24) are the following:

$$\begin{cases} (i) \quad F := 1 + T + D = 2 - \frac{5}{6}\alpha_L(b-2a) > 0 \\ (ii) \quad TC := 1 - T + D = \frac{1}{2}\alpha_L(b-2a) > 0 \\ (iii) \quad H := 1 - D = 1 + \frac{1}{6}\alpha_L(b-2a) > 0 \end{cases}. \tag{31}$$

Now, define

$$a_{NH/BL} = \frac{5\alpha_L b - 12}{10\alpha_L}, \tag{32}$$

$$\alpha_3 = \frac{12}{5b}, \tag{33}$$

as the flip bifurcation value of a in the case NH/BL and a threshold value of the speed of adjustment α, which is the unique root for α obtained by equating Eq. (32) to zero. Note that in this case $a_{NH/BL} \to b/2$ when $\alpha \to +\infty$, and $\alpha_3 > \alpha_2 > \alpha_1$. Then, from (31)-(33) we have the following proposition.

Proposition 2. *(1) Let $0 < \alpha < \alpha_3$ hold. Then, the Nash equilibrium $E^{CM}_{NH/BL}$ of the two-dimensional system (24) is locally asymptotically stable for any $a \in [0, b/2)$. (2) Let $\alpha \geq \alpha_3$ hold. Then, $E_{NH/BL}$ is locally asymptotically stable for any $b/2 > a > a_{NH/BL}$, it undergoes flip bifurcation at $a = a_{NH/BL}$, while becoming locally unstable for any $a_{NH/BL} > a > 0$.*

Proof. Since $a_{NH/BL} < 0$ for any $0 < \alpha < \alpha_3$, then $F > 0$ for any $a \in [0, b/2)$. This proves point (1). Since $0 \leq a_{NH/BL} < b/2$ for any $\alpha \geq \alpha_3$, then $F > 0$ for any $b/2 > a > a_{NH/BL}$, $F = 0$ if and only if $a = a_{NH/BL}$ and $F < 0$ for any $a_{NH/BL} > a > 0$. This proves point (2). **Q.E.D.**

Notice that the conditions for stability in both cases of BH/NL and NH/BL (see Eqs. 18 and 31) depend on the speed of adjustment, α_L and α_H, and the parameters that reveal the difference in the degree of heterogeneity of population, a and b, while being independent of product quality differential, $u_H - u_L$. However, Proposition 2 reveals the importance of the hypothesis of begin alternatively bounded rational for firms H and L. In fact, under NH/BL expectations a reduction in a is neutral on p_H, so that the intensity of the rise in marginal profits experienced by the bounded rational firm (L) is now lower than under BH/NL, ceteris paribus as regards α. Then, the destabilising effects of a rise in consumer's heterogeneity is weaker in the NH/BL case than in the BH/NL one.

We now compare, in the case of covered market, the stability-instability regions in the BH/NL and NH/BL cases by assuming $\alpha_H = \alpha_L = \alpha$. The results are summarised in the proposition that follows.

Proposition 3. *The parametric stability region of* $E^{CM}_{BH/NL} = E^{CM}_{NH/BL}$ *is larger under NH/BL than under BH/NL when the market is covered.*

Proof. Since from Eqs. (19) and (32) we get $a_{NH/BL} < a_{BH/NL}$ for any $\alpha > \alpha_1$ ($\alpha_H = \alpha_L = \alpha$), then Proposition 3 follows immediately. **Q.E.D.**

Figure 1 shows, in a stylised way, that the stability regions (in the (α, a) plane) under NH/BL are larger than under BH/NL. It is also interesting to note, ceteris paribus as regards the speed of adjustment, that when the economy enters the unstable region in the case BH/NL, it is still "strongly" stable in the case NH/BL, ceteris paribus as regards the coefficient α that tunes the speed of adjustment of the reaction of the bounded rational firm in the dynamic game.

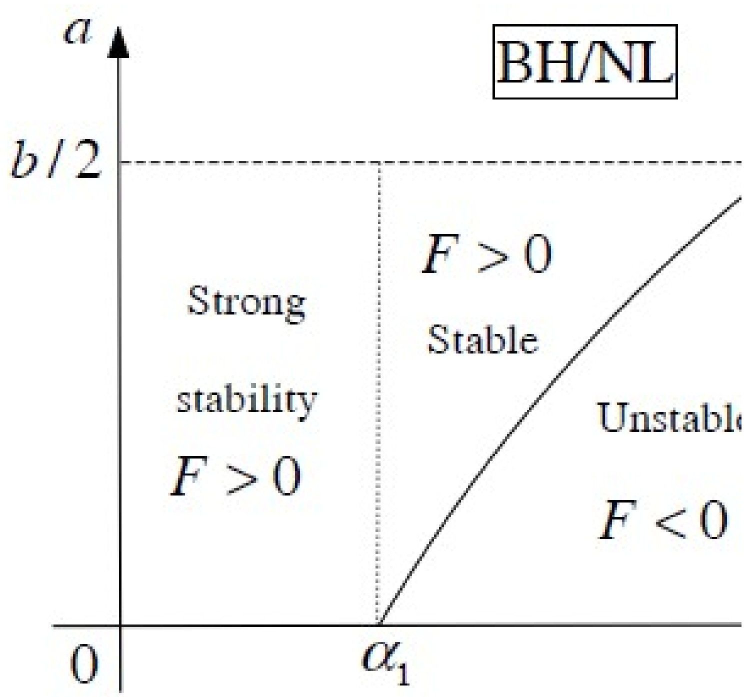

Figure 1. Covered market. Stability-instability regions in the (α, a) plane under BH/NL and NH/BL expectations. The quality differential does not matter for stability.

The following results summarise the main findings of the paper under the covered market assumption.

Result 1. *Under the hypothesis of covered market, the local stability properties of the pure strategy Nash equilibrium* $E^{CM}_{BH/NL} = E^{CM}_{NH/BL}$ *crucially depend on the size of the consumers' type (i.e., the degree of population heterogeneity), as captured by the parameter* a*, while being* *independent of the level of vertical differentiation (quality) between products H and L, as captured by the parameters u_H and u_L. The higher the degree of population heterogeneity (low values of a), the more likely $E^{CM}_{BH/NL} = E^{CM}_{NH/BL}$ is (locally) unstable.*

Result 2. *Under the hypothesis of covered market, for any given value of α, the higher the degree of population heterogeneity (low values of a), the more likely the pure strategy Nash equilibrium $E^{CM}_{BH/NL} = E^{CM}_{NH/BL}$ is (locally) unstable when firms H and L have BH/NL expectations than when they have NH/BL expectations.*

These results imply that, irrespective of the type of expectations formation mechanisms of both firms, polices that aim to increase the degree of heterogeneity amongst consumers are harmful for the (local) stability of the unique pure strategy Nash equilibrium when the market is covered, while policies that concentrate on improving or reducing the quality differential between high-quality and low-quality products are neutral on stability.

4. Uncovered market

In this section we follow an established literature (see, e.g., Gabszewicz and Thisse, 1979; Motta, 1993; Wauthy, 1996; Herguera et al., 2000, 2002; Liao, 2008) and assume that the market for products of high and low quality is uncovered, i.e. some consumers refrain from buying at prevailing prices. Then, under the hypotheses of both BH/NL and NH/BL expectations, we study the local stability properties of the pure strategy Nash equilibrium in such a case.

Let $\overline{\phi}$ be the index that identifies the consumer indifferent between purchasing products of high or low quality from firms H and L at the price p_H and p_L, respectively. Such an index is given by Eq. (4). Let $\overline{\overline{\phi}}$ be an index that identifies consumers indifferent between purchasing products of low quality or buying anything at prevailing prices. Such an index is obtained as a solution to $U_L(\phi, p_L) = 0$, i.e.

$$\phi u_L - p_L = 0\,. \tag{34}$$

Therefore, we get:

$$\overline{\overline{\phi}} = \frac{p_L}{u_L}\,. \tag{35}$$

Consumers identified by $\overline{\phi} < \phi < b$ ($\overline{\overline{\phi}} < \phi < \overline{\phi}$) purchase products of high (low) quality. Indeed, those identified by $\phi < \overline{\overline{\phi}}$ refrain from buying. $\overline{\overline{\phi}} > a$, therefore, captures the case of uncovered market because there exists a portion of consumers served neither by firm H not by firm L. Since the market is uncovered, $D_H(p_H, p_L) + D_L(p_H, p_L) < 1$, with $D_i(\bullet) > 0$ for $i = \{H, L\}$. Then, the demand functions to firms H and L as a function of prices are respectively given by:

$$D_H(p_H, p_L) = b - \overline{\phi} = b - \frac{p_H - p_L}{u_H - u_L}\,, \tag{36.1}$$

$$D_L(p_H, p_L) = \overline{\phi} - \overline{\overline{\phi}} = \frac{p_H - p_L}{u_H - u_L} - \frac{p_L}{u_L}\,. \tag{36.2}$$

Therefore, profits of firms H and L are simply given by:

$$\Pi_H(p_H, p_L) = p_H\left(b - \frac{p_H - p_L}{u_H - u_L}\right), \tag{37.1}$$

$$\Pi_L(p_H, p_L) = p_L\left(\frac{p_H - p_L}{u_H - u_L} - a\right). \tag{37.2}$$

The maximisation of Eqs. (37.1) and (37.2) with respect to p_H and p_L gives the following marginal profits:

$$\frac{\partial \Pi_H}{\partial p_H} = b - \frac{2p_H - p_L}{u_H - u_L}, \tag{38.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = \frac{p_H - 2p_L}{u_H - u_L} - \frac{2p_L}{u_L}. \tag{38.2}$$

By equating Eqs. (38.1) and (38.2) to zero and solving for p_H and p_L, we get the best reply functions of firms H and L, respectively, that is:

$$\frac{\partial \Pi_H}{\partial p_H} = 0 \Leftrightarrow p_H(p_L) = \frac{1}{2}\left[p_L + b(u_H - u_L)\right], \tag{39.1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = 0 \Leftrightarrow p_L(p_H) = \frac{u_L}{2u_H} p_H. \tag{39.2}$$

The study of the local stability properties of the unique pure strategy Nash equilibrium under BH/NL and NH/BL uses the same technique analysed in the previous sections.

4.1. Uncovered market: case $\boxed{\textit{BH/NL}}$

The two-dimensional dynamic system when the market is uncovered and firm H (L) has bounded rational (naïve) expectations is the following:

$$\begin{cases} p_H(t+1) = p_H(t) + \alpha_H p_H(t)\left(b - \dfrac{2p_H(t) - p_L(t)}{u_H - u_L}\right) \\ p_L(t+1) = \dfrac{u_L}{2u_H} p_H(t) \end{cases}. \tag{40}$$

Equilibrium implies that $p_H(t+1) = p_H(t) = p_H$ and $p_L(t+1) = p_L(t) = p_L$. In this case, the unique non-negative fixed point $E_{BH/NL}^{UM} = \left(p^*{}_H, p^*{}_L\right)$ of the dynamic system defined by Eq. (40), where UM stands for "uncovered market", is given by:

$$E_{BH/NL}^{UM} = \left(\frac{2bu_H(u_H - u_L)}{4u_H - u_L}, \frac{bu_L(u_H - u_L)}{4u_H - u_L}\right). \tag{41}$$

Given Eq. (41), we find that $\overline{\phi} = \dfrac{b(2u_H - u_L)}{4u_H - u_L}$ and $\overline{\overline{\phi}} = \dfrac{b(u_H - u_L)}{4u_H - u_L}$. Then,

Corollary 2. *Under the hypothesis of uncovered market, the consumer type indifferent between purchasing products of high or low quality,* $\overline{\phi}$*, and the consumer type indifferent between purchasing products of low quality or refraining from buying,* $\overline{\overline{\phi}}$*, depend on quality differential, while being independent of the parameter* a*.*

Proof. The proof follows immediately by looking at the equilibrium values of $\overline{\phi}$ and $\overline{\overline{\phi}}$ under the hypothesis of uncovered market. **Q.E.D.**

The stability conditions for the system in two-dimension (40) are the following:

$$\begin{cases} (i) \quad F := 1+T+D = \dfrac{4u_H(2-\alpha_H b) - u_L(2+\alpha_H b)}{4u_H - u_L} > 0 \\ (ii) \quad TC := 1-T+D = \alpha_H b > 0 \\ (iii) \quad H := 1-D = 1 + \dfrac{\alpha_H b u_L}{4u_H - u_L} > 0 \end{cases} . \tag{42}$$

From (42), it is easy to see that condition (*ii*) and (*iii*) are always fulfilled, while condition (*i*) can be violated. Now, define

$$u_L^{BH/NL} = \frac{4u_H(2-\alpha_H b)}{2+\alpha_H b}, \tag{43}$$

and

$$\alpha_4 = \alpha_1 = \frac{6}{5b}, \tag{44}$$

$$\alpha_5 = \frac{2}{b}, \tag{45}$$

as the flip bifurcation value of u_L in the case BH/NL and two threshold values of the speed of adjustment α, where $\alpha_5 > \alpha_4$. In particular, Eq. (43) is obtained by equating $F = 0$ in (42) and then solving for u_L. Eq. (44) is computed by equating Eq. (43) to u_H and then solving for α, while Eq. (45) discriminates between positive and negative values of u_L. Then, from (42)-(45) we have the following proposition.

Proposition 4. *(1) Let $0 < \alpha < \alpha_4$ hold. Then, the Nash equilibrium $E_{BH/NL}^{UM}$ of the two-dimensional system (40) is locally asymptotically stable. (2) Let $\alpha_4 < \alpha < \alpha_5$ hold. Then, $E_{BH/NL}^{UM}$ is locally asymptotically stable for any $u_L < u_L^{BH/NL}$; it undergoes a flip bifurcation at $u_L = u_L^{BH/NL}$; it is locally unstable for any $u_L > u_L^{BH/NL}$. (3) Let $\alpha > \alpha_5$ hold. Then, $E_{BH/NL}^{UM}$ is locally unstable.*

Different from the case of covered market, Proposition 4 reveals that the quality differential matters for stability when the market is uncovered. This is due to the fact the equilibrium values of the consumers' types indifferent between purchasing products of high and low quality, $\bar{\phi} = \dfrac{b(2u_H - u_L)}{4u_H - u_L}$, and purchasing products of low quality and refraining from buying, $\bar{\bar{\phi}} = \dfrac{b(u_H - u_L)}{4u_H - u_L}$, are now affected by the parameters that determine the different quality of products H and L, while being independent of a. Indeed, a rise in u_L pushes prices of firm L (naïve) up. Then, marginal profits of firm H (bounded rational) increase through two channels: the rise in its competitor's price and the rise in its competitor's product quality. However, a negative effect on marginal profits of firm H also exists when u_L becomes larger. Therefore, the final effect of a rise in u_L on prices of both firms is a priori uncertain. Proposition 4 shows , however, that (ceteris paribus) an increase in product quality of firm L unambiguously acts a destabilising device when the market is uncovered and L is included in an intermediate range of values. This holds irrespective of the degree of heterogeneity amongst consumers.

Proof. Since $u_L^{BH/NL} > u_H > 0$ for any $0 < \alpha < \alpha_4$, then $F > 0$ for any $u_L < u_H$. This proves point (1). Since $u_H > u_L^{BH/NL} > 0$ for any $\alpha_4 < \alpha < \alpha_5$, then $F > 0$ for any $u_L < u_L^{BH/NL}$, $F = 0$ if and only if $u_L = u_L^{BH/NL}$ and $F < 0$ for any $u_L > u_L^{BH/NL}$. This proves point (2). Since $u_L^{BH/NL} < 0$ for any $\alpha > \alpha_5$, then $F < 0$ for any $u_L < u_H$. This proves point (3). **Q.E.D.**

4.2. Uncovered market: case $\boxed{\textit{NH/BL}}$

Analogously to Section 4.1., the dynamic system that characterises the dynamics of the economy under NH/BL when the market is uncovered is the following:

$$\begin{cases} p_H(t+1) = \frac{1}{2}\left[p_L(t) + b(u_H - u_L)\right] \\ p_L(t+1) = p_L(t) + \alpha_L p_L(t)\left(\frac{p_H(t) - p_L(t)}{u_H - u_L} - \frac{p_L(t)}{u_L}\right) \end{cases}. \tag{46}$$

The Nash equilibrium is still given by Eq. (41). The stability conditions for the system in two-dimension (46) are therefore the following:

$$\begin{cases} (i) \quad F := 1 + T + D = \frac{4u_H(4 - \alpha_L b) - u_L(4 + \alpha_L b)}{2(4u_H - u_L)} > 0 \\ (ii) \quad TC := 1 - T + D = \frac{\alpha_L b}{2} > 0 \\ (iii) \quad H := 1 - D = 1 + \frac{\alpha_L b u_L}{2(4u_H - u_L)} > 0 \end{cases}. \tag{47}$$

From (47), it is easy to see that condition (*ii*) and (*iii*) are always fulfilled, while condition (*i*) can be violated. Now, define

$$u_L^{NH/BL} = \frac{4u_H(4 - \alpha_L b)}{4 + \alpha_L b}, \tag{48}$$

and

$$\alpha_6 = \alpha_3 = \frac{12}{5b}, \tag{49}$$

$$\alpha_7 = \frac{4}{b}, \tag{50}$$

as the flip bifurcation value of u_L in the case NH/BL and two threshold values of the speed of adjustment α, where $\alpha_7 > \alpha_6$. Eqs. (48)-(50) are the analogous of Eqs. (43)-(45). The following proposition, therefore, holds.

Proposition 5. *(1) Let* $0 < \alpha < \alpha_6$ *hold. Then, the Nash equilibrium* $E_{NH/BL}^{UM}$ *of the two-dimensional system (46) is locally asymptotically stable. (2) Let* $\alpha_6 < \alpha < \alpha_7$ *hold. Then,* $E_{NH/BL}^{UM}$ *is locally asymptotically stable for any* $u_L < u_L^{NH/BL}$*; it undergoes a flip bifurcation at* $u_L = u_L^{NH/BL}$*; it is locally unstable for any* $u_L > u_L^{NH/BL}$*. (3) Let* $\alpha > \alpha_7$ *hold. Then,* $E_{NH/BL}^{UM}$ *is locally unstable.*

Proof. Since $u_L^{NH/BL} > u_H > 0$ for any $0 < \alpha < \alpha_6$, then $F > 0$ for any $u_L < u_H$. This proves point (1). Since $u_H > u_L^{NH/BL} > 0$ for any $\alpha_6 < \alpha < \alpha_7$, then $F > 0$ for any $u_L < u_L^{NH/BL}$, $F = 0$ if and only if $u_L = u_L^{NH/BL}$ and $F < 0$ for any $u_L > u_L^{NH/BL}$. This proves point (2). Since $u_L^{NH/BL} < 0$ for any $\alpha > \alpha_7$, then $F < 0$ for any $u_L < u_H$. This proves point (3). **Q.E.D.**

Arguments similar to those used to explain Proposition 4 apply to Proposition 5, which refers to the case NH/BL. We now compare, in the case of uncovered market, the stability-instability regions under BH/NL and NH/BL expectations by assuming $\alpha_H = \alpha_L = \alpha$. The results are summarised in the following proposition.

Proposition 6. *The parametric stability region of* $E^{UM}_{BH/NL} = E^{UM}_{NH/BL}$ *is larger under NH/BL than under BH/NL when the market is uncovered.*

Proof. Since from Eqs. (19) and (32) we get $u_L^{NH/BL} < u_L^{BH/NL}$ for any configuration of economically meaningful parameter values, then Proposition 6 follows immediately. **Q.E.D.**

Figure 2 shows in a stylised way that the stability-instability regions in the (α, u_L) plane are larger under NH/BL than under BH/NL, when the market is uncovered.

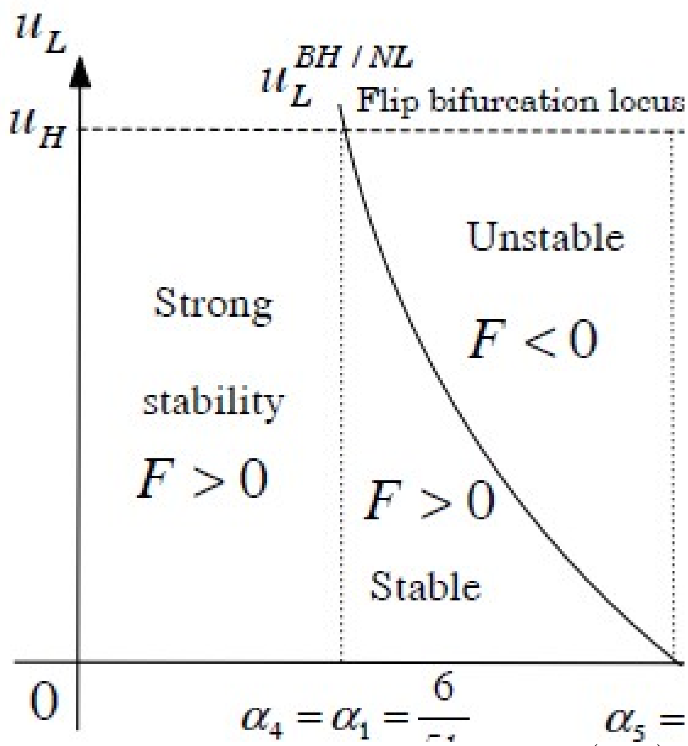

Figure 2. Uncovered market. Stability-instability regions in the (α, u_L) plane under BH/NL and NH/BL expectations. The quality differential matters for stability.

The following results summarise the main findings of the paper under the uncovered market assumption.

Result 3. *Under the hypothesis of uncovered market, the local stability properties of the pure strategy Nash equilibrium* $E^{UM}_{BH/NL} = E^{UM}_{NH/BL}$ *crucially depend on the quality differential between products* H *and* L*, while being independent of the parameter* a*. The higher* u_L*, the more likely* $E^{UM}_{BH/NL} = E^{UM}_{NH/BL}$ *is (locally) unstable.*

Result 4. *Under the hypothesis of uncovered market, for any given value of α, the higher u_L, the more likely the pure strategy Nash equilibrium $E_{BH/NL}^{UM} = E_{NH/BL}^{UM}$ is (locally) unstable when firms H and L have BH/NL expectations than when they have NH/BL expectations.*

These results imply that, irrespective of the type of expectations formation mechanisms, polices that aim to reduce the quality differential (by increasing u_L for any given value of u_H) are harmful to the local stability of prices when the market is uncovered, while policies that concentrate to improve the degree of population heterogeneity through a reduction in a are neutral.

5. Conclusions

This study originates from the increasing interest for a refined analysis in the nonlinear oligopoly literature under naïve expectations (e.g., Tramontana et al., 2009), bounded rational expectations (e.g., Tramontana, 2010) and, more in general, expectations formation mechanisms based on adaptive rules (e.g., Agliari et al., 2006). The novelty of this paper is the analysis of local stability in a duopoly game with price competition, vertical differentiation and heterogeneous expectations under the assumptions of both covered and uncovered market.

The assumption of heterogeneous expectations is usual in the literature on nonlinear oligopolies. An example is Fanti and Gori (2012), where a Cournot duopoly with horizontal differentiation is studied to show that the relative degree of substitutability may cause the appearance of flip bifurcations and complex dynamics. The purpose of the this study is to complement that paper with the analysis of a Bertrand duopoly with vertical differentiation.

We have shown that an increase (resp. reduction) in the extent of consumer's heterogeneity (resp. quality differential) acts as a destabilising device when the market is covered (resp. uncovered). Moreover, numerical experiments have revealed that complex dynamics can also be observed.

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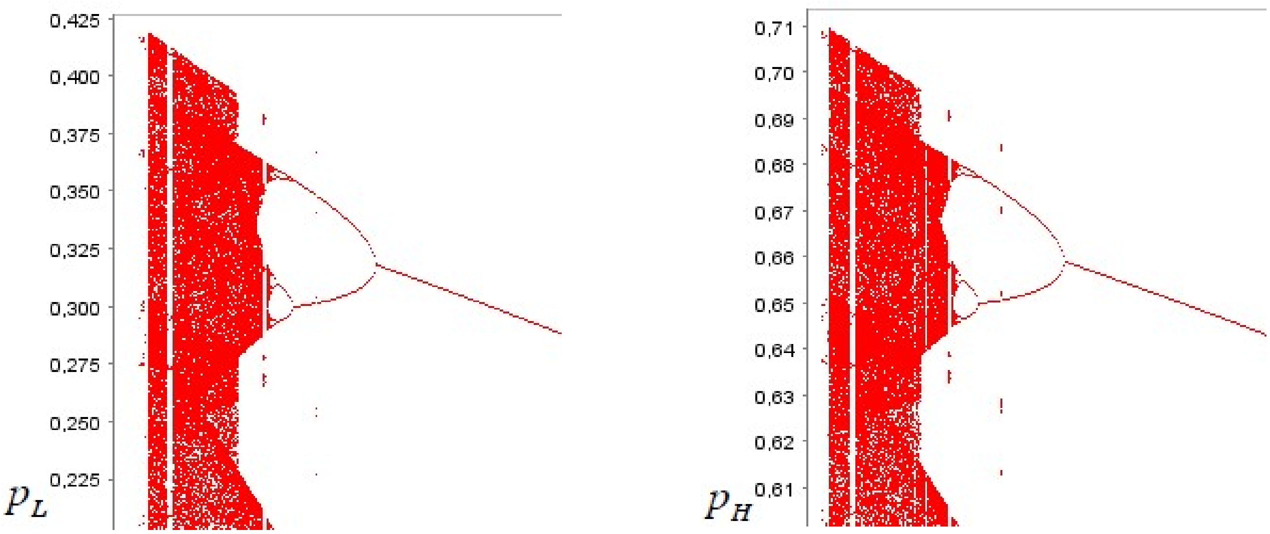
Appendix A. Endogenous fluctuations (covered market)

In this appendix we show with numerical experiments that endogenous fluctuations can occur when a reduces (covered market). The parameter values are: $b=1$, $\alpha=4$, $u_H=2$ and $u_L=1$. Then, $a_{BH/NL}=1.4$, $\alpha_1=6/5$ and $\alpha_2=8/5$ (BH/NL), and $a_{NH/BL}=0.2$ and $\alpha_3=12/5$ (NH/BL). Under this parameter constellation, we find $a_{BH/NL}>b/2$ so that the Nash equilibrium $E_{BH/NL}^{CM}=E_{NH/BL}^{CM}$ in the BH/NL model is unstable and trajectories are non-convergent for any $a\in[0,1/2)$. With regards to the NH/BL model, $4=\alpha>\alpha_3$ implying that $E_{BH/NL}^{CM}=E_{NH/BL}^{CM}$ is stable or unstable depending on the relative size of a. To this purpose, Figures A.1.a and A.1.b show the bifurcation diagrams for a under NH/BL, and depict the limit point of p_L and p_H, respectively, when the initial conditions are $p_H(0)=0.1$ and $p_L(0)=0.05$. The figures reveal that the long-run values of prices increase and they are locally asymptotically stable when a reduces from $1/2$ to 0.2. Then, a flip bifurcation occurs at $a_{NH/BL}=0.2$. Then, a two-period cycle (broken off in the range $a \in (0.05776, 0.05727)$ by more complicated dynamic events) emerges. As long as a reduces, we observe four-period cycles, eight-period cycles, and cycles of higher periodicity when a becomes lower. Figure A.2 also depicts the chaotic attractor (black-coloured) and the corresponding basin of attraction (red-coloured) for $a = 0.03$.



(a) (b)

Figure A.1. Covered market. Bifurcation diagram for a under NH/BL expectations. Initial conditions: $p_H(0) = 0.1$ and $p_L(0) = 0.05$.

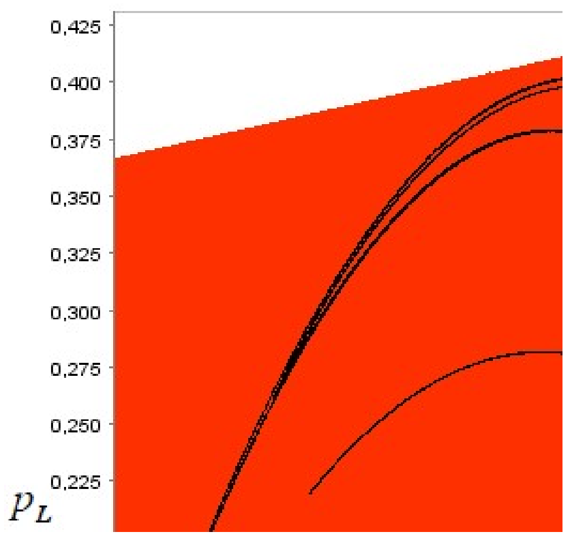


Figure A.2. Covered market. Chaotic attractor and their basic of attraction for $a = 0.03$ under NH/BL expectations.

Appendix B. Endogenous fluctuations (uncovered market)

We show here that endogenous fluctuations occur when u_L raises (uncovered market). We depict bifurcation diagrams and basins of attraction only under BH/NL expectations, since the case NH/BL shows similar dynamic events.

The parameter set is: $b=1$ (so that $\alpha_4 = 6/5$ and $\alpha_5 = 2$) and $a = 0.5$. Then, we choose $\alpha = 1.9$, $u_H = 2$ and let u_L increase from 0 to 2. These parameter values generate $u_L^{BH/NL} = 0.2051$. We note that since $\alpha_4 < \alpha < \alpha_5$, then the Nash equilibrium $E_{BH/NL}^{UM} = E_{NH/BL}^{UM}$ is locally stable or unstable depending on the relative size of u_L. Figures B.1.a and B.1.b show the bifurcation diagrams for u_L under BH/NL, and depict the limit point of p_L and p_H, respectively, when the initial conditions are $p_H(0) = 0.1$ and $p_L(0) = 0.05$. The Nash equilibrium is locally asymptotically stable when u_L is small. A flip bifurcation occurs at $u_L^{BH/NL} = 0.2051$. Then, a two-period cycle emerges and cycles of higher periodicity occur when u_L becomes larger. It is interesting to note that in the long run the trend of the high-quality price is monotonically decreasing, while the low-quality price increases (decreases) when the index u_L is small (large). Figure B.2 depicts the chaotic attractor (black-coloured) and the corresponding basin of attraction (red-coloured) for $u_L = 1.85$.

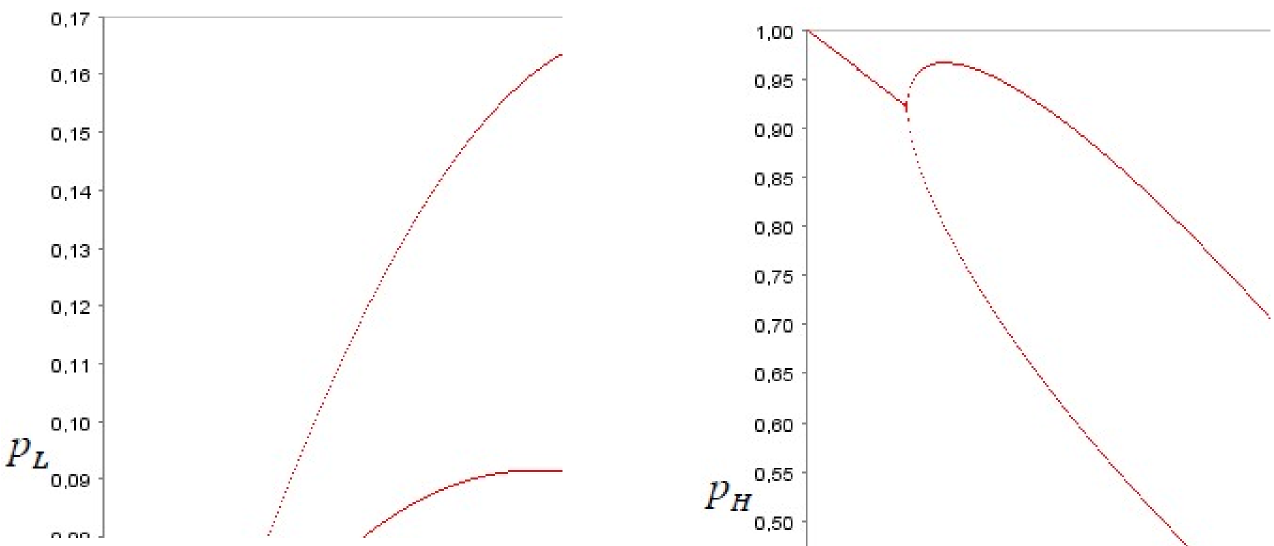


Figure B.1. Uncovered market. Bifurcation diagram for u_L under BH/NL expectations. Initial conditions: $p_H(0) = 0.1$ and $p_L(0) = 0.05$.

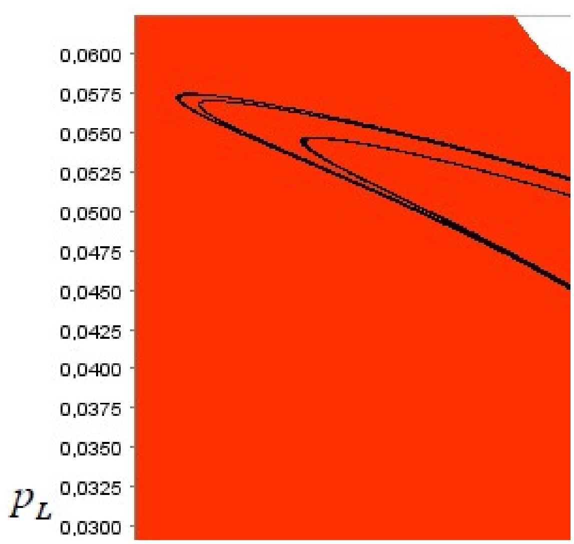


Figure B.2. Uncovered market. Chaotic attractor and their basic of attraction for $u_L = 1.85$ under BH/NL expectations.

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