Hopf bifurcation and stability crossing curves in a cobweb model with heterogeneous producers and time delays

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Abstract

We study a continuous time cobweb model with discrete time delays where heterogeneous producers behave as adapters in the market. Specifically, they partially adjust production (which is subject to some gestation lags) towards the profit-maximising quantity under static expectations. The dynamics of the economy is described by a two-dimensional system of delay differential equations. We characterise stability properties of the stationary state of the system and show the emergence of Hopf bifurcations. We also apply some recent mathematical techniques (stability crossing curves) to show how heterogeneous time delays affect the stability of the economy.

Keywords Behavioural heterogeneity; Nonlinear cobweb model; Stability crossing curves; Time delays

JEL Codes D21; E32; C61; C62

AMS Codes 34A34; 65P20; 70K50

1 Introduction

This paper analyses the behaviour of economic agents in a continuous time model with discrete delays. The economy is characterised by a single market of perishable or nonstorable goods (agricultural market), where demand is decreasing (isoelastic) in prices, supply is increasing and production of goods requires time so that suppliers must form expectations on prices that will prevail in the market. Generally speaking, alongside the problems of existence and uniqueness of general equilibrium[1] models (Arrow and Debreu, 1954), the study of stability properties of market

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[1] Roughly speaking, it is important to recall that in economic theory the term "equilibrium" is referred to a situation in which markets clear. Then, we need to distinguish the concept of equilibrium as explained above from

equilibria is one of the much debated issues in economic theory.[2] It is well known that in partial or general equilibrium models with a finite positive number of markets, where trading is allowed only at equilibrium prices, the existence of a unique stable equilibrium is guaranteed only by the introduction of some strong assumptions on the behaviour of economic agents, such as the axiom of revealed preferences in pure exchange economies and additional hypotheses related to the convexity of the production set in production economies (Mas-Colell et al., 1995). This kind of models has the negative counterpart that is not able to explain price fluctuations without introducing exogenous shocks or structural changes (in the utility function or in the production function). Such a drawback holds even in some out-of-equilibrium or disequilibrium models,[3] such as Hahn and Negishi (1962), that studies a pure exchange economy where trading is allowed at non-equilibrium prices and the equilibrium is globally asymptotically stable.

More interesting dynamic outcomes in continuous time sets up (that try to mimic the actual adjustment pattern in competitive markets) are found by considering a single market model where adjustment is driven by changes in both prices and quantities, that simultaneously react in response to disequilibrium. In this context, tâtonnement dynamics may be characterised by fluctuations in prices and quantities (limit cycles), such as in the works of Beckmann and Ryder (1969) and Mas-Colell (1986). However, it is important to stress that in their models production and trading are allowed only when demand and supply exactly balance, so that a limit cycle cannot have an adequate interpretation in "real world" economies (real time dynamics). In fact, in this kind of models it is assumed an external (Walrasian) auctioneer that progressively raises or drops the price (depending on excess demand), and fluctuations are just related to the auctioneer behaviour. Therefore, when an attracting limit cycle exists trading does not hold, while in the case the market equilibrium is stable trading holds only at infinite time. Therefore, existence and stability are surely relevant philosophical findings, but it is hard to give a meaningful interpretation of trading. An attempt of obtaining continuous time models able to generate fluctuations is the study of markets with inventories, as in Eckalbar (1985). However, strong fluctuations can actually be observed in markets with high-frequency trading where inventories are negligible (e.g., agricultural markets).

Very different findings actually exist in the related literature dealing with discrete time models. Although a modelling approach in discrete time seems able to better capture fluctuations observed in real markets, these models are subject to some criticism with regard to both the theoretical approach and empirical effectiveness. With this regard, Gandolfo (1981) pointed out that the choice between continuous time and discrete time dynamic models is not neutral because of the different analytical tools required for their analysis (differential equations versus difference equations) and the nature of stability conditions. By considering more specifically discrete time models, one of their main features is to assume that production takes time and markets work only at discrete intervals (see Evans and Honkapohja, 2001 for a review of the literature on expectations, and the literature cited therein). In this context, a key role is played by expectations formation mechanisms on prices of producers. By introducing the hypothesis of static or naïve expectations (that is, producers take the current price as an estimate of the expected one), the problem of price fluctuations has been tackled since the early seminal contributions of Kaldor (1934) and Ezekiel (1938), in which both the demand and supply were linear. Subsequently, thanks to some new mathematical results as well

the concept of equilibrium in a dynamic sense. In fact, there exist equilibrium models that generate trajectories that do not converge to the (stationary) equilibrium of the system, and disequilibrium models that, in contrast, admit stationary equilibria.

[2] A discussion on this topic started several decades ago in partial equilibrium models (Kaldor, 1934; Ezekiel, 1938).

[3] Market-clearing versus non-market-clearing dynamics does actually represent an important debate also in dynamic macroeconomic analysis, as well stressed by Flaschel et al., (1997).

as the dramatic increase in computing capacity, nonlinear versions of the cobweb model have been studied and carried out through simulation experiments, showing persistent oscillations in prices (Artstein, 1983; Jensen and Urban, 1984). A further step in this line of research was to include more realistic mechanisms with regard to expectations formations in the cobweb framework. To this purpose, Chiarella (1988) and Hommes (1994, 1998) have studied the dynamics of prices with nonlinear supply and demand curves by using the adaptive expectations hypothesis (Nerlove, 1958), with respect to which prices are revised according to prediction errors of agents. Along the same line of research, Onozaki et al. (2000) has revisited the cobweb model by considering adaptive adjustments on the quantity produced instead of price expectations.

As pointed out above, therefore, the choice between continuous time models or discrete time models does not actually represent an easy task to describe economic phenomena, especially if one wants to perform econometric studies or give policy insights, and currently there may be arguments in favour of the use of either sets up. By taking into account key factors of realism in the modelling approaches of both discrete-time models (gestation lags) and continuous-time models (trading not related to the timing of production), Gori et al. (2015) describe the behaviour of a single market (partial equilibrium) where production is subject to some gestation lags and decisions on how much to produce are taken before these goods are placed in the market, while trading takes place continuously or repeatedly, and then studies price dynamics in a continuous time model with discrete delays (one-dimensional delay differential equation). Indeed, the current paper is the dual of Gori et al. (2015), because it takes a dynamic view of quantities rather than prices in a cobweb model by also assuming heterogeneous interacting agents, thus allowing the economy to be described by a system of two delay differential equations instead of a one-dimensional system.

The present paper aims at characterising the local stability properties of the stationary equilibrium of the system with both one delay and two delays. In this last case, by applying the recent techniques developed by Ruan and Wei (2003) and Gu et al. (2005), we show the role of heterogeneity in the emergence of Hopf bifurcations. In particular, while several works including Brock and Hommes (1998), Bischi et al. (1999) and Onozaki et al. (2003) find that heterogeneity tends to have destabilising effects, in line with Bosi and Seegmuller (2008) we find that the relationship between heterogeneity and stability/instability of is more difficult to be clarified.

The rest of the paper is organised as follows. Sections 2 sets up the cobweb model with heterogeneous producers that operate as adapters of two different kinds, i.e. they move towards the profit-maximising quantity with different speeds of adjustment, but with homogeneous time delays. Section 3 characterises local stability properties and local bifurcations of equilibria of the resulting two-dimensional delay differential equation system. Section 4 extends the model to the case of heterogeneous time delays. Section 5 outlines the conclusions.

2 The model

Different from the existing literature on the topic (e.g., Hommes, 1994; Gallas and Nusse, 1996; Onozaki et al., 2000, 2003; Chiarella et al., 2006; Dieci and Westerhoff, 2009), that has focused on the study of market dynamics in a discrete time cobweb model, we set up a continuous time framework with discrete time delays. In fact, although there exists a time lag from production decisions to the time commodities are ready for sale (gestation lags), prices of perishable goods are observed frequently and are subject to (sometimes markedly significant) fluctuations. Then, a continuous time model with discrete delays (that takes into account production adjustments) may well capture the behaviour of agents in this kind of markets. Indeed, only a few papers have dealt with the study of a continuous time version of the cobweb model. For instance, Mackey (1989) has shown that the discrete time cobweb model can be viewed as a limiting case of a continuous time model with delays. Specifically, we develop a continuous time version with time delays of the cobweb model studied by Onozaki et al. (2003), where there exist heterogeneous producers that operate in a market of a perishable commodity. In addition, a Walrasian auctioneer sets the price of the commodity in order to clear the market at every moment in time. We assume the existence of two different groups of identical and competitive farmers (whose size is $0 < m < 1$ and $1 - m$, respectively) that behave as adapters. Specifically, firms that belong to group i ($i = 1, 2$) partially adjust their own production in the direction of the quantity that maximises expected profits. This because we are assuming that optimisation requires high costs related to a complete knowledge of the market, so that applying a behavioural rule (adjustment mechanism) may allow agents to overcome their informational lacunae with less effort than when firms are optimisers.

Technology requires a period of time τ to bring the production process to completion and get products to the market. The quantity that maximises the expected profit at time t is solution of the following problem referred at time $t - \tau$:

$$\max_{\{x(t)\geq 0\}} \Pi^e(t) = \max_{\{x(t)\geq 0\}} \{p^e(t)x(t) - \frac{1}{2}[x(t)]^2\}, \tag{1}$$

where $p^e(t)$, $x(t)$ and $\frac{1}{2}[x(t)]^2$ are the price expected at time t, the quantity of the nonstorable good and the quadratic cost function referred at time t, respectively. It is important to note that expectations on prices do appear in (1). This because the price that will prevail at time t is not known at the time the maximisation problem of expected profits is referred, that is $t-\tau$. Then, with static expectations ($p^e(t) = p(t - \tau)$), maximisation programme (1) gives the following solution:

$$\tilde{x}(t) = p(t - \tau). \tag{2}$$

Adapters of group i. By following the approach used by Onozaki et al. (2000), let firms be (expected) profit maximisers but use the quantity that corresponds to maximum expected profits, $\tilde{x}(t)$, as a target to adjust their production choices. In particular, if the quantity produced at time t by each single adapter, $u_i(t)$, is smaller (resp. greater) than $\tilde{x}(t)$, it will increase (resp. reduce) production. Then, we assume the following behavioural rule for production decisions of each single adapter of group i ($i = 1, 2$):

$$\dot{u}_i(t) = \alpha_i \left[\tilde{x}(t) - u_i(t)\right], \tag{3}$$

where $\dot{u}_i(t)$ is the instantaneous time variation of production of an adapter that belongs to group i and $\alpha_i > 0$ is the speed of adjustment ($\alpha_i \neq \alpha_j$). This rule can be interpreted as a precautionary behaviour with respect to the evolution of the market price.

Since there are m (resp. $1 - m$) homogeneous adapters of group 1 (resp. group 2), aggregate supply can easily be determined as follows:

$$X(t) = mu_1(t) + (1 - m)u_2(t). \tag{4}$$

With regard to consumers' side, we follow Fanti et al. (2015) and assume the existence of a continuum of identical consumers whose preferences towards both the agricultural commodity y

(whose price is p) and numeraire good w (whose price is normalised to one without loss of generality), produced by competitive firms, are represented by the following quasi-linear utility function:

$$V(y,w) = U(y) + w, \tag{5}$$

where

$$U(y) = \begin{cases} \frac{y^{1-\beta}}{1-\beta}, & \text{if } \beta > 0 \\ \ln(y), & \text{if } \beta = 1 \end{cases}, \tag{6}$$

where $1/\beta$ is the constant elasticity of demand. The representative consumer maximises utility function (6) subject to budget constraint $py + w = M$, where $M > 0$ is the exogenous nominal income of the consumer (M is assumed to be sufficiently high to avoid the existence of corner solutions). This maximisation programme implies that the isoelastic inverse demand of good y is determined as follows:

$$p = \begin{cases} y^{-\beta}, & \text{if } \beta > 0 \\ y^{-1}, & \text{if } \beta = 1 \end{cases}. \tag{7}$$

Therefore, the market demand at time t is simply given by the following equation:

$$p(t) = \frac{1}{[y(t)]^{\beta}}. \tag{8}$$

Market-clearing implies that aggregate demand equals aggregate supply, that is $X(t) = y(t)$ for any t. Then, by using (2), (3), (4) and (8) we find the following system of two delay differential equations that characterises the dynamics of the economy:

$$\dot{u}_1(t) = -\alpha_1 u_1(t) + \frac{\alpha_1}{[mu_1(t-\tau) + (1-m)u_2(t-\tau)]^{\beta}}, \tag{9}$$

$$\dot{u}_2(t) = -\alpha_2 u_2(t) + \frac{\alpha_2}{[mu_1(t-\tau) + (1-m)u_2(t-\tau)]^{\beta}}. \tag{10}$$

In what follows, we will set $\alpha_1 = \alpha > 0$ and $\alpha_2 = 1$ to be in line with the formulation of Onozaki et al. (2003).[4]

3 Existence of equilibria and local bifurcations with homogeneous time delays

By setting $\alpha_1 = \alpha$ and $\alpha_2 = 1$, the system of two delay differential equations described by (9) and (10) becomes:

$$\begin{cases} \dot{u}_1(t) = -\alpha u_1(t) + \frac{\alpha}{[mu_1(t-\tau) + (1-m)u_2(t-\tau)]^{\beta}}, \\ \dot{u}_2(t) = -u_2(t) + \frac{1}{[mu_1(t-\tau) + (1-m)u_2(t-\tau)]^{\beta}}. \end{cases} \tag{11}$$

[4] We note that in a discrete time model, Onozaki et al. (2003) distinguish between producers that belong to "cautious" adapters, that adjusts output towards the target represented by the profit-maximizing quantity with $\alpha \in (0,1)$ as the speed of adjustment, and producers that belong to optimisers, that exactly produce the quantity that maximises profits given the expectation formation mechanism ($\alpha = 1$). However, in a continuous time framework firms within each group remain adapters for any finite positive value of the speed of adjustment.

Equilibrium points of system (11) correspond to solutions of the algebraic system $\dot{u}_1(t) = \dot{u}_2(t) = 0$, with $u_1(t-\tau) = u_1(t) = u_1^*$ and $u_2(t-\tau) = u_2(t) = u_2^*$. A direct computation shows that model (11) admits the unique positive equilibrium $(u_1^*, u_2^*) = (1, 1)$. The linearised system of (11) at this steady-state equilibrium is the following:

$$\begin{cases} \dot{u}_1(t) &= -\alpha\left(u_1(t) - u_1^*\right) - \alpha\beta m\left(u_1(t-\tau) - u_1^*\right) - \alpha\beta(1-m)\left(u_2(t-\tau) - u_2^*\right), \\ \dot{u}_2(t) &= -\left(u_2(t) - u_2^*\right) - \beta m\left(u_1(t-\tau) - u_1^*\right) - \beta(1-m)\left(u_2(t-\tau) - u_2^*\right). \end{cases} \tag{12}$$

The characteristic equation associated with (12) is given by

$$\begin{vmatrix} -\alpha - \lambda - \alpha\beta m e^{-\lambda\tau} & -\alpha\beta(1-m)e^{-\lambda\tau} \\ -\beta m e^{-\lambda\tau} & -1 - \lambda - \beta(1-m)e^{-\lambda\tau} \end{vmatrix} = 0. \tag{13}$$

From (13), we obtain the following second degree exponential polynomial equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)\, e^{-\lambda\tau} = 0, \tag{14}$$

where

$$a = 1 + \alpha > 0, \qquad b = \alpha > 0, \qquad c = (1 - m + \alpha m)\beta > 0, \qquad d = \alpha\beta > 0.$$

It is known that the steady-state equilibrium is asymptotically stable if all roots of the characteristic equation (14) have negative real parts.

Lemma 1 *The stationary equilibrium* $(1, 1)$ *is locally asymptotically stable without delays.*

Proof. For $\tau = 0$, (14) becomes $\lambda^2 + (a + c)\,\lambda + b + d = 0$. By looking at this equation, we can see that it has only two negative roots since $a + c > 0$ and $b + d > 0$. Hence, the statement holds. ■

We now want to determine whether the real part of some root of (14) increases to zero and eventually becomes positive as τ varies. Clearly, $\lambda = 0$ is not a root of (14). For $\tau > 0$, if $\lambda = i\omega$ is a root of (14) with $\omega > 0$ then

$$-\omega^2 + ia\omega + b + (ic\omega + d)\left[\cos(\omega\tau) - i\sin(\omega\tau)\right] = 0.$$

Separating the real and imaginary parts in the previous equation, we get the following

$$\begin{cases} \omega^2 - b &= d\cos(\omega\tau) + c\omega\sin(\omega\tau), \\ a\omega &= d\sin(\omega\tau) - c\omega\cos(\omega\tau). \end{cases} \tag{15}$$

Squaring both sides, adding both equations and regrouping by powers of ω, we obtain that ω satisfies the following fourth degree polynomial

$$\omega^4 - \left(c^2 + 2b - a^2\right)\omega^2 + b^2 - d^2 = 0. \tag{16}$$

We remark that

$$\begin{aligned} c^2 + 2b - a^2 &= (1 - m + \alpha m)^2\beta^2 - \left(1 + \alpha^2\right), \\ b^2 - d^2 &= (1 - \beta^2)\alpha^2. \end{aligned}$$

Lemma 2 *If $\beta \leq 1$, then all roots of* (14) *have negative real parts for $\tau \geq 0$, i.e.* $(1,1)$ *is locally asymptotically stable for all $\tau \geq 0$.*

Proof. By assumption $\beta \leq 1$, so that $b^2 - d^2 \geq 0$. The statement will follow from $\omega^4 - \left(c^2 + 2b - a^2\right)\omega^2 = -(b^2 - d^2)$ showing that $c^2 + 2b - a^2 < 0$. In fact, this implies that (16) has no positive solutions. Hence, the characteristic equation (14) does not have purely imaginary roots. Now, we note that

$$c^2 + 2b - a^2 < 0 \quad \Longleftrightarrow \quad \beta < \frac{\sqrt{1+\alpha^2}}{1 - m + \alpha m}. \tag{17}$$

By recalling that $m \in (0,1)$ and noting that

$$\frac{\sqrt{1+\alpha^2}}{1-m+\alpha m} > 1 \quad \Longleftrightarrow \quad (1-\alpha)^2 m^2 - 2(1-\alpha)m - \alpha^2 \leq 0 \quad \Longleftrightarrow \quad \frac{1-\sqrt{1+\alpha^2}}{1-\alpha} < m < \frac{1+\sqrt{1+\alpha^2}}{1-\alpha},$$

we have $\sqrt{1+\alpha^2}/\left(1 - m + \alpha m\right) > 1$, i.e. (17) is always verified having assumed $\beta \leq 1$. ■

Lemma 3 *Let $\beta > 1$. Then* (16) *has only one root ω_+ which is positive, where*

$$\omega_+ = \sqrt{\frac{c^2 + 2b - a^2 + \sqrt{\Delta}}{2}}, \tag{18}$$

with

$$\Delta = \left(c^2 + 2b - a^2\right)^2 - 4\left(b^2 - d^2\right) > 0. \tag{19}$$

Proof. From (17) we derive that $c^2 + 2b - a^2 < 0$ if $1 < \beta < \sqrt{1+\alpha^2}/(1 - m + \alpha m)$, and $c^2 + 2b - a^2 \geq 0$ if $\beta \geq \sqrt{1+\alpha^2}/(1-m+\alpha m)$. In both cases, being $b^2 - d^2 < 0$ hence $\Delta > 0$, so that a direct calculation yields the conclusions. ■

Solving equations in (15) for $\sin(\omega\tau)$ and $\cos(\omega\tau)$, we get

$$\sin(\omega\tau) = \frac{c\omega^3 + (ad - bc)\omega}{c^2\omega^2 + d^2}, \tag{20}$$

and

$$\cos(\omega\tau) = \frac{(d - ac)\omega^2 - bd}{c^2\omega^2 + d^2}. \tag{21}$$

From $d - ac = -(1 - m + \alpha^2 m)\beta < 0$ and $bd > 0$, we get $\cos(\omega\tau) < 0$. Furthermore, $ad - bc > 0$ yields $\sin(\omega\tau) > 0$. The critical values τ_j^+ of τ for which the characteristic equation (14) has purely imaginary roots can be determined from (21). They are given by

$$\tau_j^+ = \frac{1}{\omega_+}\left\{\cos^{-1}\left[\frac{(d-ac)\omega_+^2 - bd}{c^2\omega_+^2 + d^2}\right] + 2j\pi\right\}, \quad j = 0, 1, 2, ... \tag{22}$$

Proposition 4 *Let $\beta > 1$.*

1. *Eq.* (14) *has a pair of simple purely imaginary roots $\pm i\omega_+$ at $\tau = \tau_j^+$.*

2. *Let* $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ *denote the root of Eq.* (14) *near* $\tau = \tau_j^+$ *satisfying* $\nu(\tau_j^+) = 0$ *and* $\omega(\tau_j^+) = \omega_+$, *with* ω_+ *and* τ_j^+ *defined in* (18) *and* (22), *respectively. Then the transversality condition reads as follows*
$$\frac{d\left[Re\lambda(\tau_j^+)\right]}{d\tau} > 0.$$

Proof. A direct calculation shows that $\lambda = i\omega_+$ is a simple root of (14). In fact, if we assume that $\lambda = i\omega_+$ is not simple, then differentiating (14) with respect to λ, using (14), and evaluating the resulting equation for $\lambda = i\omega_+$ leads to $(2d + a\tau_j^+)\omega_+ = 0$, which is an absurd. Next, substituting $\lambda(\tau)$ into (14) and taking the derivative with respect to τ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + a)\, e^{-\lambda\tau} + c}{\lambda(c\lambda + d)} - \frac{\tau}{\lambda}.$$

By using (14) and (16), a direct calculation gives

$$sign\left\{ \frac{d\,(Re\lambda)}{d\tau}\bigg|_{\tau=\tau_j^+} \right\} = sign\left\{ Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_j^+} \right\} = sign\left\{-c^2 - 2b + a^2) + 2\omega_+^2\right\} = sign\left\{\sqrt{\Delta}\right\} > 0.$$

with Δ defined in (19). ∎

Bearing the above analysis in mind, we have that the root $\lambda(\tau)$ of (14) crosses the imaginary axis from left to right at $\tau = \tau_j^+$ as τ increases. Thus, we have the following result.

Theorem 5 *Let* $\beta > 1$. *The positive equilibrium of* (11) *is locally asymptotically stable when* $\tau \in [0, \tau_0^+)$ *and unstable when* $\tau > \tau_0^+$. *Moreover,* (11) *undergoes Hopf bifurcations at the positive equilibrium when* $\tau = \tau_j^+$ $(j = 0, 1, 2, ...)$.

It is interesting to note that in the related work of Onozaki et al. (2003), where there are two groups of firms, one of the most important findings is the role played by the relative number of these two kinds of firms on the stability of the stationary equilibrium. In particular, in that model the coexistence of players of different kinds (heterogeneity) is a source of instability. This result, obtained in a discrete time model, is in line with the results on the long-term nonlinear dynamics obtained in discrete time oligopoly models with bounded rationality (Bischi et al., 1999). In the present work, by starting from equation (22) that describes the relationship between the bifurcation value of τ and the other parameters of the model, it is possible to show that heterogeneity may play a stabilising role (see Figure 1(a) and 1(b)). The following proposition summarises the result.

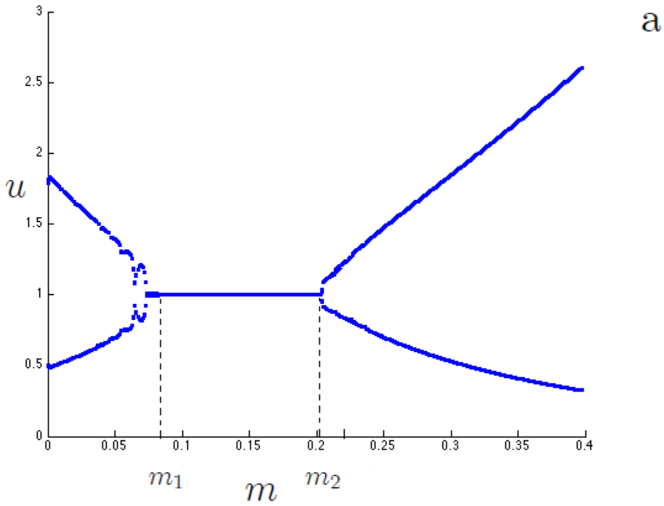
Proposition 6 *Let* $\tau_0^+(m)$ *be the function defined in (22). If* $\beta > 1$ *and* $\alpha > 1 + \frac{\beta^2}{\sqrt{\beta^2 - 1}}\left(2\pi - \arccos\frac{1}{\beta}\right)$ *for any* $\tau \in \left(\tau_0^+\mid_{m=0}, \tau_{0,\max}^+\right)$ *there exist* m_1 *and* m_2 *such that* $(1, 1)$ *is unstable for* $m < m_1$ *and* $m > m_2$, *while it is locally asymptotically stable for* $m \in (m_1, m_2)$, *where* $\tau_{0,\max}^+$ *is the global maximum point for* $\tau_0^+(m)$ *on* $(0, 1)$.

Proof. Equation (22) defines a continuous and differentiable function τ_0^+ with respect to m in the interval $[0,1]$. By direct calculation we have that

$$\left.\frac{\partial \tau_0^+}{\partial m}\right|_{m=0} = \frac{(1-\alpha)\left[(1-\alpha)\sqrt{\beta^2-1}+\beta^2\left(2\pi-\arccos\frac{1}{\beta}\right)\right]}{\left(\alpha^2+\beta^2-1\right)\sqrt{\beta^2-1}}. \tag{23}$$

Under the assumption $\beta > 1$, if $\alpha > 1 + \frac{\beta^2}{\sqrt{\beta^2-1}}\left(2\pi - \arccos\frac{1}{\beta}\right)$ then (23) is positive and $\tau_0^+(1) = \frac{\pi-\arccos\frac{1}{\beta}}{\alpha\sqrt{\beta^2-1}} < \frac{\pi-\arccos\frac{1}{\beta}}{\sqrt{\beta^2-1}} = \tau_0^+(0)$. This implies that the global maximum belongs to $(0,1)$ and the proposition follows. ■

Figure 1(a) has been obtained by using the following procedure. Let m vary in the interval $[0, 0.4]$ with a sufficiently small step-size (1000 equidistant points has been taken into account). By considering an initial condition $u(t) = \widetilde{u}$ for any $t \in [-\tau, 0]$ with $\widetilde{u}$ sufficiently close to u^*, for any m the figure depicts the local maximum and minimum values of the trajectory after a sufficiently long transient. Then, the stationary equilibrium point is locally asymptotically stable for any $m \in (m_1, m_2)$, as proved in Proposition 6. For $m = m_1$ and $m = m_2$, the system undergoes a Hopf bifurcation. In the intervals of m such that the stationary equilibrium point is unstable ($m < m_1$ or $m > m_2$), trajectories are characterised by the existence of only one local maximum and minimum values $u_{\max}$ and $u_{\min}$, with $u_{\max} \neq u_{\min}$ (which are also global). This implies that long-term dynamics tend to be captured by a limit cycle, as shown in Figure 1(b).



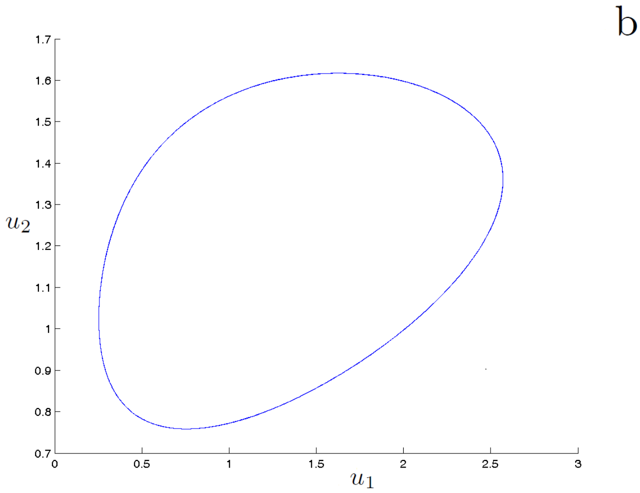


Figure 1. (a) Bifurcation diagram for m. Parameter set: $\alpha = 500$, $\beta = 2.2$ and $\tau = 1.1$. A limit cycle captures the long-term dynamics of the system for $m \in (m_1, m_2)$. We note that the parameter values used simply have an illustrative purpose. (b) Phase diagram for $m = 0.39 > m_2$.

4 Extension: heterogeneous time delays

The aim of this section is to extend the analysis developed in the previous section when firms that belong to adapters of different groups are characterised by different technologies. From a mathematical point of view, equilibrium dynamics are described by the following system of two delay differential equations:

$$\begin{cases} \dot{u}_1(t) &= -\alpha u_1(t) + \dfrac{\alpha}{[mu_1(t-\tau_1)+(1-m)u_2(t-\tau_1)]^{\beta}}, \\ \dot{u}_2(t) &= -u_2(t) + \dfrac{1}{[mu_1(t-\tau_2)+(1-m)u_2(t-\tau_2)]^{\beta}}. \end{cases} \tag{24}$$

where $\tau_1 \geq 0$, $\tau_2 \geq 0$ (with $\tau_1 \neq \tau_2$) represent the delays corresponding to adapters that belong to group 1 and group 2, respectively. In previous sections, we have assumed that firms that belong to both groups (types) have the same technology, that is they react with the same delay τ. Differently, we now assume that farmers of group 1 and group 2 use a technology for which there exist two different periods of time, τ_1 and τ_2 respectively, to bring the production process to completion and get products to the market. This to capture in a more realistic way the functioning of agricultural markets subject to different kinds of sowing and harvest times.

Obviously, $(u_1^*, u_2^*) = (1,1)$ is the unique positive equilibrium of system (24). The associated characteristic equation of the linearised system of (24) at this equilibrium point is given by

$$\begin{vmatrix} -\alpha - \lambda - \alpha\beta m e^{-\lambda\tau_1} & -\alpha\beta(1-m)e^{-\lambda\tau_1} \\ -\beta m e^{-\lambda\tau_2} & -1-\lambda-\beta(1-m)e^{-\lambda\tau_2} \end{vmatrix} = 0,$$

namely

$$\lambda^2 + (1+\alpha)\lambda + \alpha + \alpha\beta m(\lambda+1)e^{-\lambda\tau_1} + \beta(1-m)(\lambda+\alpha)e^{-\lambda\tau_2} = 0. \tag{25}$$

In order to study the emergence of a super-critical Hopf bifurcation, below we follow the approach introduced by Ruan and Wei (2003). Specifically, in Section 4.1 we will study the case in which one firm is not subject to time delays from bringing production to completion and getting products to the market while the other does. In Section 4.2 we extend this analysis to the case where both firms are characterised by (different) time delays in production.

4.1 The case $\tau_1 = 0, \tau_2 > 0$

In this case, the characteristic equation (25) reduces to

$$\lambda^2 + a\lambda + b + (c\lambda + d)\, e^{-\lambda\tau_2} = 0, \tag{26}$$

where

$$a = 1+\alpha(1+\beta m) > 0, \quad b = \alpha(1+\beta m) > 0, \quad c = \beta(1-m) > 0 \quad \text{and} \quad d = \alpha\beta(1-m) > 0.$$

The stability of the trivial equilibrium point will change when the system under consideration has zero or a pair of imaginary eigenvalues. It is immediate that the former cannot occur. Let $\lambda = i\omega$ ($\omega > 0$) be a purely imaginary root of (26). Substituting $\lambda = i\omega$ in (26), and separating the real and imaginary parts, we arrive at

$$\omega^4 - \left(c^2 + 2b - a^2\right)\omega^2 + b^2 - d^2 = 0, \tag{27}$$

where

$$c^2 + 2b - a^2 = \left[(1-\alpha^2)m^2 - 2m + 1\right]\beta^2 - 2\alpha^2\beta m - (1+\alpha^2), \tag{28}$$

and

$$b^2 - d^2 = \alpha^2\left[(2m-1)\beta^2 + 2\beta m + 1\right]. \tag{29}$$

Lemma 7 *Let*

$$\beta_2 = \frac{\alpha^2 m + \sqrt{(\alpha^4 - \alpha^2 + 1)m^2 - 2m + 1}}{(1-\alpha^2)m^2 - 2m + 1}. \tag{30}$$

Then

1. $c^2 + 2b - a^2 > 0$ *if* $0 < m < 1/(1+\alpha)$ *and* $\beta > \beta_2$ *or* $1/(1+\alpha) < m < 1$ *and* $\beta < \beta_2$;
2. $c^2 + 2b - a^2 = 0$ *if* $m = 1/(1+\alpha)$ *or* $\beta = \beta_2$;
3. $c^2 + 2b - a^2 < 0$ *if* $0 < m < 1/(1+\alpha)$ *and* $\beta < \beta_2$ *or* $1/(1+\alpha) < m < 1$ *and* $\beta > \beta_2$.

Proof. The statement follows from rewriting (28) as

$$c^2 + 2b - a^2 = \left[(1-\alpha^2)m^2 - 2m + 1\right](\beta - \beta_1)(\beta - \beta_2),$$

where $\beta_1 = \left[\alpha^2 m - \sqrt{\alpha^4 m^2 + (1-\alpha^2)m^2 - 2m + 1}\right] / \left[(1-\alpha^2)m^2 - 2m + 1\right] < 0$, and the fact that $(1-\alpha^2)m^2 - 2m + 1 = [(1-\alpha)m - 1][(1+\alpha)m - 1]$, with $(1-\alpha)m - 1 < 0$. ∎

Lemma 8

1. $b^2 - d^2 > 0$ *if* $1/2 \le m < 1$ *or if* $0 < m < 1/2$ *and* $\beta < 1/(1-2m)$.
2. $b^2 - d^2 = 0$ *if* $0 < m < 1/2$ *and* $\beta = 1/(1-2m)$.
3. $b^2 - d^2 < 0$ *if* $0 < m < 1/2$ *and* $\beta > 1/(1-2m)$.

Proof. The conclusion is immediate once (29) is written as

$$b^2 - d^2 = \alpha^2\left[(2m-1)\beta + 1\right](\beta + 1).$$

∎

Proposition 9 *Let* β_2 *be defined as in* (30).

1) *If*

$$0 < m < \frac{1}{2}: \ \beta \le \frac{1}{1-2m} \ \text{and}\ \beta \le \beta_2, \ \text{or}\ m = \frac{1}{1+\alpha} \ \text{and}\ \beta = \frac{1}{1-2m} \tag{31}$$

or

$$\frac{1}{2} \le m < 1: \ m \le \frac{1}{1+\alpha} \ \text{and}\ \beta \le \beta_2, \ \text{or}\ \frac{1}{1+\alpha} \le m < 1 \ \text{and}\ \beta \ge \beta_2 \tag{32}$$

hold, then all roots of Eq. (26) *have negative real parts for all* $\tau_2 \ge 0$.

2) *If*

$$0 < m < \frac{1}{2}: \ \beta = \frac{1}{1-2m} > \beta_2, \ \text{or}\ \beta > \frac{1}{1-2m} \tag{33}$$

hold, then (26) *has a pair of purely imaginary roots* $\pm i\omega_+$ *at* $\tau_2 = \tau_{2_j}^+$ $(j = 0, 1, 2, ...)$, *where*

$$\begin{cases} \omega_+ = \sqrt{c^2 + 2b - a^2}, & \text{if } \beta = \dfrac{1}{1-2m} > \beta_2, \\ \omega_+ = \sqrt{\dfrac{c^2 + 2b - a^2 + \sqrt{(c^2 + 2b - a^2)^2 - 4(b^2 - d^2)}}{2}}, & \text{if } \beta > \dfrac{1}{1-2m} \text{ and } \beta \ne \beta_2, \\ \omega_+ = \sqrt[4]{-(b^2 - d^2)}, & \text{if } \beta > \dfrac{1}{1-2m} \text{ and } \beta = \beta_2. \end{cases}$$

3) *If*

$$0<m<\frac{1}{2}:\ \beta<\frac{1}{1-2m}\ \text{ and } \beta>\beta_2 \tag{34}$$

or

$$\frac{1}{2}\le m<1:\ \beta>\beta_2,\ \text{ or } \frac{1}{1+\alpha}<m<1 \text{ and } \beta<\beta_2 \tag{35}$$

hold, then (26) has a pair of purely imaginary roots $\pm i\omega_+$ *(*$\pm i\omega_-$*, respectively) at* $\tau_2=\tau_{2_j}^+$ $(j=0,1,2,...)$ $(\tau_2=\tau_{2_j}^-$*, respectively), where*

$$\omega_\pm=\sqrt{\frac{c^2+2b-a^2\pm\sqrt{(c^2+2b-a^2)^2-4\left(b^2-d^2\right)}}{2}}$$

and

$$\tau_{2_j}^\pm=\begin{cases}\dfrac{1}{\omega_\pm}\left\{2j\pi+\arccos\left[\dfrac{(d-ac)\omega_\pm^2-bd}{c^2\omega_\pm^2+d^2}\right]\right\}, & \text{if } c\omega_\pm^2+ad-bc>0,\\[2ex] \dfrac{1}{\omega_\pm}\left\{(2j+1)\,\pi-\arccos\left[\dfrac{(d-ac)\omega_\pm^2-bd}{c^2\omega_\pm^2+d^2}\right]\right\}, & \text{if } c\omega_\pm^2+ad-bc\le 0.\end{cases} \tag{36}$$

Proof. If $0<m<1/2$, then one has that $b^2-d^2>0$ if $\beta<1/(1-2m)$, $b^2-d^2=0$ if $\beta=1/(1-2m)$, and $b^2-d^2<0$ if $\beta>1/(1-2m)$. On the other hand, if $m\ge 1/2$, then it is $b^2-d^2>0$. The statement follows from the previous two Lemmas and (27). In addition, notice from (21) that $\cos(\omega\tau_2)<0$ since $d-ac=-(1-m+\alpha^2m)\beta<0$ and $bd>0$, and from (20) that $\sin(\omega\tau_2)>0$ if $c\omega_\pm^2+ad-bc>0$. ■

Proposition 10 *Assume that* (33), (34) *or* (35) *holds. Then* $i\omega_\pm$ *are simple roots of* (26) *satisfying*

$$\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2_j}^+}>0 \quad \text{and} \quad \left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2_j}^-}<0.$$

Proof. From the proof of Proposition 4 we obtain

$$\begin{aligned} sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2_j}^\pm}\right\} &= sign\left\{Re\left.\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right|_{\tau_2=\tau_{2_j}^\pm}\right\}\\ &= sign\left\{-c^2-2b+a^2)+2\omega_\pm^2\right\}=sign\left\{\pm\sqrt{\Delta}\right\}.\end{aligned}$$

■

According to the previous analysis we have the following results on the stability of the positive equilibrium $(u_1^*,u_2^*)=(1,1)$.

Theorem 11 *Let* $\tau_{2_j}^\pm$ $(j=0,1,2,...)$ *be defined as in* (36).

1) *If* (31) *or* (32) *holds, then the equilibrium* (u_1^*, u_2^*) *is locally asymptotically stable for all* $\tau_2 \geq 0$.

2) *If* (33) *holds, then the equilibrium* (u_1^*, u_2^*) *is locally asymptotically stable for* $\tau_2 \in [0, \tau_{2_0}^+)$ *and unstable for* $\tau_2 > \tau_{2_0}$.

3) *If* (34) *or* (35) *holds, then there is a positive integer* m *such that the equilibrium* (u_1^*, u_2^*) *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{2_0}^+) \cup (\tau_{2_0}^-, \tau_{2_1}^+) \cup \cdots \cup (\tau_{2_{m-1}}^-, \tau_{2_m}^+)$ *and unstable when* $\tau_2 \in (\tau_{2_0}^+, \tau_{2_0}^-) \cup (\tau_{2_1}^+, \tau_{2_1}^-) \cup \cdots \cup (\tau_{2_{m-1}}^+, \tau_{2_{m-1}}^-) \cup (\tau_{2_m}^+, \infty)$. *Furthermore, system* (24) *undergoes a Hopf bifurcation at* (u_1^*, u_2^*) *when* $\tau_2 = \tau_{2_m}^{\pm}$, $m = 0, 1, 2, ...$

4.2 The case $\tau_1 > 0$ and $\tau_2 > 0$

If $\tau_1 > 0$ and $\tau_2 > 0$ we consider the characteristic equation (25) with τ_2 in its stable intervals, i.e. $\tau_2 \in [0, \tau_{2_0}^+)$ or $\tau_2 \in [0, +\infty)$. We first prove a result regarding the sign of the real parts of characteristic roots of (25).

Lemma 12 *If all roots of Eq.* (26) *have negative real parts for* $\tau_2 > 0$*, then there exists a* $\tau_1^*(\tau_2) > 0$*, such that all roots of Eq.* (25) *have negative real parts when* $0 \leq \tau_1 < \tau_1^*(\tau_2)$.

Proof. Eq. (26) having no root with nonnegative real part for $\tau_2 > 0$ implies that Eq. (25) with $\tau_1 = 0$ and $\tau_2 > 0$ has no root with nonnegative real part. Since the left hand side of Eq. (25) is analytic in λ and τ_1, following Ruan and Wei (2003), when τ_1 varies, the sum of the multiplicities of zeros of the left hand side of Eq. (25) in the open right half-plane can change only if a zero appears on or crosses the imaginary axis. Since Eq. (25) with $\tau_1 = 0$ has no root with nonnegative real part, the conclusion is immediate. ■

An application of Theorem 11 and Lemma 12 provides some conditions to ensure that all roots of the characteristic equation (26) with two delays have negative real parts, which imply the asymptotic stability of the positive equilibrium of system (24).

Theorem 13

1) *If* (31) *or* (32) *holds, then for any* $\tau_2 \geq 0$ *there exists a* $\tau_1^*(\tau_2) > 0$ *such that the equilibrium of system* (24) *is locally asymptotically stable when* $\tau_1 \in [0, \tau_1^*(\tau_2))$.

2) *If* (33) *holds, then for any* $\tau_2 \in [0, \tau_{2_0}^+)$ *there exists a* $\tau_1^*(\tau_2) > 0$ *such that the equilibrium of system* (24) *is locally asymptotically stable when* $\tau_1 \in [0, \tau_1^*(\tau_2))$.

3) *If* (34) *or* (35) *holds, then for any* $\tau_2 \in [0, \tau_{2_0}^+) \cup (\tau_{2_0}^-, \tau_{2_1}^+) \cup \cdots \cup (\tau_{2_{m-1}}^-, \tau_{2_m}^+)$ *there exists a* $\tau_1^*(\tau_2) > 0$ *such that the equilibrium of system* (24) *is locally asymptotically stable when* $\tau_1 \in [0, \tau_1^*(\tau_2))$.

Remark 14 It is clear that a Hopf bifurcation occurs at $\tau_1^*(\tau_2)$ and also that there may exist several stability switches. If we let τ_2 be in the unstable region, then there may exist no $\tau_1^*(\tau_2)$ such that when the system (24) is unstable in $0 \leq \tau_1 < \tau_1^*(\tau_2)$, it is stable in $\tau_1^*(\tau_2) < \tau_1$. Both results will be discussed later in this section by the use of the so called stability crossing curves (Gu et al., 2005).

Remark 15 If Eq. (25) for some τ_1 and τ_2 has two pairs of purely imaginary roots, say $\pm i\omega_1$ and $\pm i\omega_2$, and all the other roots of (25) have non-zero real part, and $\omega_1 : \omega_2 = l_1 : l_2$, then system (24) undergoes a double Hopf bifurcation with the ratio $l_1 : l_2$. When $l_1, l_2 \in \mathbb{Z}^+$, then it is called an $l_1 : l_2$ resonant double Hopf bifurcation; otherwise, it is called a non-resonant double Hopf bifurcation. More generally, since in the model there are several parameters in addition to τ_1 and τ_2, it is possible to observe bifurcations of co-dimension greater than 1. An interesting study on these topic can be found in the recent paper by Bi and Ruan (2013), who analyse a model with two variables, two delays (as in the present work), but in which only one variable is subject to time delays.

The results stated in Theorem 13 clearly show the different scenarios that can emerge depending on the parameters of the model. However, this approach is not completely satisfactory as it does not allow to obtain analytical results on couples (τ_1, τ_2) that generate the bifurcation. In other words, no information is given on couples (τ_1, τ_2) that generate a stable or an unstable stationary state. In order to overcome this concern, an effective approach is the one proposed by Gu et al. (2005) through the use of the stability crossing curves, which are defined as the curves that separate the stable and unstable regions in the (τ_1, τ_2) plane. Starting from the characteristic equation (25), it is possible to define the following polynomials:

$$p_0(\lambda) = \lambda^2 + (1+\alpha)\lambda + \alpha, \tag{37}$$

$$p_1(\lambda) = \alpha\beta m(1+\lambda), \tag{38}$$

$$p_2(\lambda) = \beta(1-m)(\alpha+\lambda). \tag{39}$$

The zeros of (25) coincide with the zeros of

$$A(\lambda, \tau_1, \tau_2) = 1 + A_1(\lambda)e^{-\tau_1\lambda} + A_2(\lambda)e^{-\tau_2\lambda}, \tag{40}$$

where $A_\ell = p_\ell(\lambda)/p_0(\lambda)$, $\ell = 1, 2$. Now, the procedure proposed by Gu et al. (2005) is based on the interpretation of the three addenda in the left-hand-side of $A(\lambda, \tau_1, \tau_2) = 0$ as three vectors in the complex plane. The procedure is comprised of the following steps.

The first step needs to identify the set Ω of ω that satisfies feasibility conditions such that complex conjugate roots do exist and, from a geometrical point of view, it consists of imposing conditions such that the vectors above mentioned form a triangle (Figure 2). In our model, the procedure described above allows us to characterise Ω_i (with $\cup\Omega_i = \Omega$) as the sets such that graph of the function

$$L_1(\omega) := |A_1(i\omega)| + |A_2(i\omega)| = \frac{m\beta\alpha}{\sqrt{\alpha^2+\omega^2}} + \frac{\beta(1-m)}{\sqrt{1+\omega^2}}, \tag{41}$$

belongs to region $G := \{(\omega, z) : z \in [-1, 1]\}$, while the graph of the function

$$L_2(\omega) := |A_1(i\omega)| - |A_2(i\omega)| = \frac{m\beta\alpha}{\sqrt{\alpha^2+\omega^2}} - \frac{\beta(1-m)}{\sqrt{1+\omega^2}}, \tag{42}$$

do not.

The second step consists in identifying the internal angles $\theta_1, \theta_2 \in [0, \pi]$ of the triangle in Figure 2 by using the law of the cosine, that is

$$\theta_1 = \cos^{-1}\left(\frac{1 + |A_1(i\omega)|^2 - |A_2(i\omega)|^2}{2\,|A_1(i\omega)|}\right), \tag{43}$$

and

$$\theta_2 = \cos^{-1}\left(\frac{1 + |A_2(i\omega)|^2 - |A_1(i\omega)|^2}{2\,|A_2(i\omega)|}\right). \tag{44}$$

Now, for any $\omega \in \Omega$ it is possible to identify solutions (τ_1, τ_2) of $A(\lambda, \tau_1, \tau_2) = 0$ as follows:

$$\tau_1 = \tau_1^{v_1^{\pm}}(\omega) = \frac{\arg(A_1(i\omega)) + (2v_1 - 1)\pi \pm \theta_1}{\omega} \geq 0, \quad v_1 = v_{1,0}^{\pm}, v_{1,0}^{\pm} + 1, v_{1,0}^{\pm} + 2, ..., \tag{45}$$

and

$$\tau_2 = \tau_2^{v_2^{\pm}}(\omega) = \frac{\arg(A_2(i\omega)) + (2v_2 - 1)\pi \mp \theta_2}{\omega} \geq 0, \quad v_2 = v_{2,0}^{\pm}, v_{2,0}^{\pm} + 1, v_{2,0}^{\pm} + 2, ..., \tag{46}$$

where $v_{1,0}^{+}$, $v_{1,0}^{-}$, $v_{2,0}^{+}$ and $v_{2,0}^{-}$ are the smallest possible integers (that may be negative and may depend on ω) such that the corresponding calculated values of $\tau_1^{v_{1,0}^{+}+}$, $\tau_1^{v_{1,0}^{-}-}$, $\tau_2^{v_{2,0}^{+}+}$ and $\tau_2^{v_{2,0}^{-}-}$ are non-negative.

Let

$$\Gamma_{\omega,v_1,v_2}^{\pm} = \left\{\left(\tau_1^{v_1^{\pm}}(\omega), \tau_2^{v_2^{\pm}}(\omega)\right)\right\}, \tag{47}$$

then

$$\Gamma = \bigcup_{\omega\in\Omega}\left(\left(\bigcup_{\substack{v_1\geq v_{1,0}^{+}\\ v_2\geq v_{2,0}^{+}}}\Gamma_{\omega,v_1,v_2}^{+}\right)\bigcup\left(\bigcup_{\substack{v_1\geq v_{1,0}^{-}\\ v_2\geq v_{2,0}^{-}}}\Gamma_{\omega,v_1,v_2}^{-}\right)\right), \tag{48}$$

identifies the stability crossing curves in (τ_1, τ_2) plane.

In general, as stated by Gu et al. (2005), Proposition 4.5, p. 243, we can observe three possible shapes (and generally twelve possible types) of stability crossing curves. In order to understand the possible configurations of stability crossing curves in our model, we analyse the behaviour of $L_1(\omega)$ and $L_2(\omega)$. We have that $L_1'(\omega) < 0$ for any $\omega \geq 0$ and $L_2'(\omega) = \frac{-m\beta\alpha\omega}{\sqrt[3]{\alpha^2+\omega^2}} + \frac{\beta(1-m)\omega}{\sqrt[3]{1+\omega^2}}$. According to monotonicity properties of $L_1(\omega)$ and $L_2(\omega)$, we introduce the following notation. Let ω_1 (resp. ω_2) be the unique point, if it exists, such that $L_1(\omega_1) = 1$ (resp. $L_2(\omega_2) = 1$), and let ω_3 and ω_4 be the points, if they exist, such that $L_2(\omega_i) = -1$, $i = 3, 4$. If both ω_2 and ω_3 exist, we assume that $\omega_2 < \omega_3$, while if both ω_3 and ω_4 exist, we assume that $\omega_3 < \omega_4$. Furthermore, let ω_{crit} be the unique point, if it exists, such that $L_2'(\omega_{crit}) = 0$. Then, we are able to state the following proposition.

Proposition 16 *The following cases are possible.*
If a) $\beta < 1$ then (u_1^*, u_2^*) is locally asymptotically stable for any non-negative τ_1 and τ_2.
If $\beta > 1$ we have that
1) if (a) $m < \frac{1}{2}$ and $\beta < \frac{1}{1-2m}$ and ω_{crit} does not exist (or there exists $\omega_{crit} : L_2(\omega_{crit}) > -1$) or (b) $m > \frac{1}{2}$ and $\beta < \frac{1}{2m-1}$ and ω_{crit} does not exist (or there exists $\omega_{crit} : L_2(\omega_{crit}) > -1$), then there exists an interval $\Omega_1 = (0, \omega_1)$ of type $0, 3$ and the stability crossing curves are open-ended curves;
2) if $m > \frac{1}{2}$ and $\beta > \frac{1}{2m-1}$ and ω_{crit} does not exist or there exists $\omega_{crit} : L_2(\omega_{crit}) > -1$ then there

exists an interval $\Omega_1 = (\omega_2, \omega_1)$ *of type* $1, 3$ *and the stability crossing curves are spiral-like curves with vertical axes;*
3) if $m < \frac{1}{2}$ *and* $\beta < \frac{1}{1-2m}$ *(or* $m > \frac{1}{2}$ *and* $\beta < \frac{1}{2m-1}$*) and there exists* $\omega_4 : (\omega_4 < \omega_1$ *and* $L_2(\omega_{crit}) < -1)$*, then there exists a crossing set* $\Omega = \Omega_1 \cup \Omega_2$*, where* $\Omega_1 = (0, \omega_3)$ *of type* $0, 2$ *and the stability crossing curves are open-ended curves, and* $\Omega_2 = (\omega_4, \omega_1)$ *of type* $2, 3$ *and the stability crossing curves are spiral-like curves with horizontal axes;*
4) if $m < \frac{1}{2}$ *and* $\beta > \frac{1}{1-2m}$ *then there exists an interval* $\Omega_1 = (\omega_3, \omega_1)$ *of type* $2, 3$ *and the stability crossing curves are spiral-like curves with horizontal axes;*
5) if $m > \frac{1}{2}$ *and* $\beta > \frac{1}{2m-1}$ *and there exists* $\omega_{crit} : L_2(\omega_{crit}) < -1$*, then there exists a crossing set* $\Omega = \Omega_1 \cup \Omega_2$*, where* $\Omega_1 = (\omega_2, \omega_3)$ *of type* $1, 2$ *and the stability crossing curves are spiral-like curves with axes oriented diagonally, and* $\Omega_2 = (\omega_4, \omega_1)$ *of type* $2, 3$ *and the stability crossing curves are spiral-like curves with horizontal axes.*

Proof. First of all, we note that if ω_1 does exist we have that $L_2(\omega_1) > -1$. Then, $\omega_1 > \omega_i$ when ω_i exists, with $i = 2, 3, 4$. By considering $L_1(0)$ and $L_2(0)$ and the geometric properties of the graphs of $L_1(\omega)$ and $L_2(\omega)$ we have the results. ■

The various cases listed in Proposition 16 are illustrated in Figures 3-8. In particular, Figure 3 shows that stability crossing curves do not exist when $\beta < 1$, and from Lemma 1 it follows that fixed point (u_1^*, u_2^*) is stable for any τ_1 and τ_2. In this case, high or low delays to bring the production process to completion and get products to the market do not matter for stability. Panels a from Figure 4 to Figure 8 depict $L_1(\omega)$ and $L_2(\omega)$ from Case 1 to Case 5 of Proposition 16, respectively, while the related panels b show the corresponding stability crossing curves together with the stability/instability regions in (τ_1, τ_2) plane.

The analysis of Figures 4(b)-8(b) allows us to get some interesting economic interpretations. In fact, by considering (for instance) Figure 7(b), it is relevant to point out that if the number of firms that belong to adapters of group 1 is relatively small, and the time delay associated with the technology of firms of group 2 is small too, the size of the delay related to the technology of firms of group 1 does not matter for stability. In other word, the length of time goods are produced and are effectively ready for sale in the market does not cause persistent fluctuations in that case. Similar considerations can actually be done for the cases reported in Figures 4(b), 5(b) and 6(b). In contrast, with regard to Figure 8(b), in which β is sufficiently high (that is, the elasticity of market demand is sufficiently low), m is sufficiently high (that is, the number of firms that belong to group 1 is large), and a technical condition is verified ($L_2(\omega_{crit}) < -1$), we note that by taking a small value of any of the two delays, there exists a threshold value of the other delay beyond which the steady-state equilibrium is definitely unstable. The solid line in panels b of Figures 4-8 shows that by adequately fixing one of the two delays (we recall that a delay represents the size of gestation lags in the technology of firms that belong to one of the two groups) to a given value, it is possible to observe several changes in the stability of (u_1^*, u_2^*) (stability switches) when the other delay varies. This phenomenon can be ascertained, for instance, by looking at Figure 7(b) at $\tau_2 = 0.55$.[5]

[5] There are a few applications of this technique in economics. An exmaple of the use of stability crossing curves can be found in Matsumoto and Szidarovszky (2012).

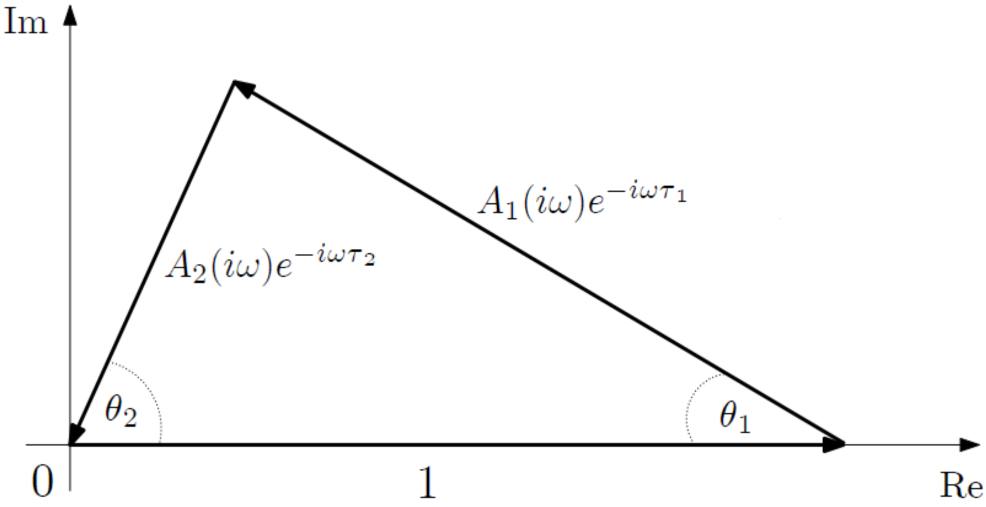


Figure 2. Triangle formed by 1, $|A_1(i\omega)|$ and $|A_2(i\omega)|$.

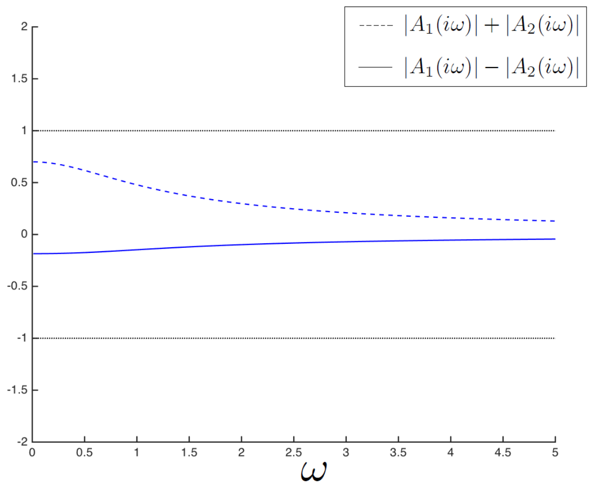


Figure 3. Case $\beta < 1$ of Proposition 16. Parameter set: $\alpha = 0.84$, $\beta = 0.70$ and $m = 0.368$. $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω.

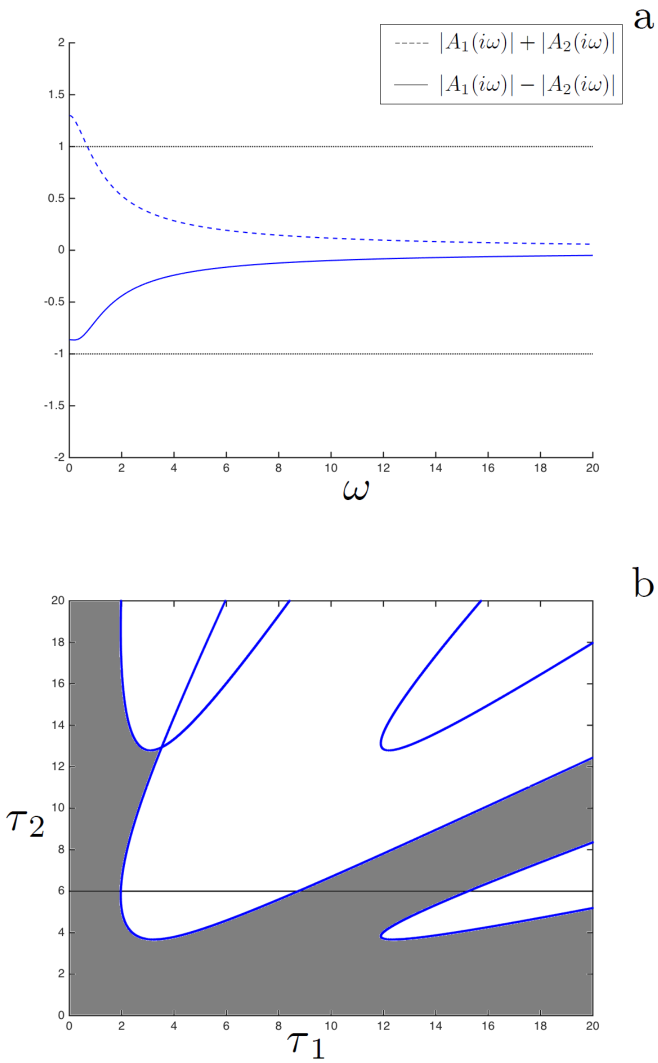


Figure 4. Case 1 of Proposition 16. Parameter set: $\alpha = 0.4$, $\beta = 1.3$ and $m = 0.168$. (a) $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω. (b) Stability crossing curves in (τ_1, τ_2) plane. Stability crossing curves are open-ended curves. The grey area shows a portion of the stability region in (τ_1, τ_2) plane. Set Ω consists of a unique interval $\Omega_1 = (0, \omega_1) = (0, 0.6896)$.

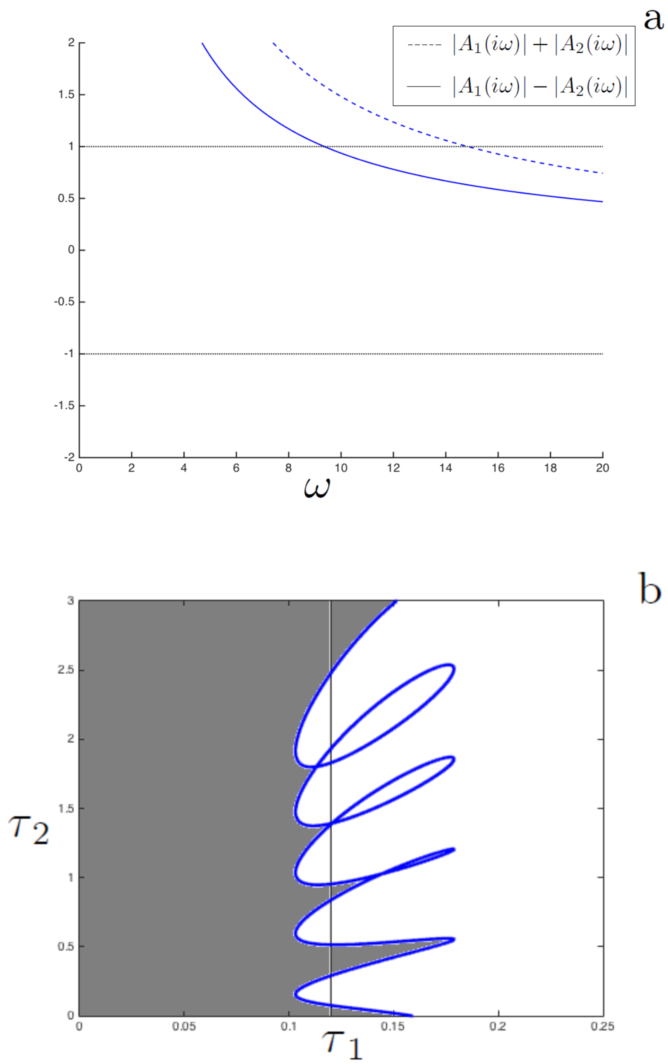


Figure 5. Case 2 of Proposition 16. Parameter set: $\alpha = 0.4$, $\beta = 33$ and $m = 0.9168$. (a) $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω. (b) Stability crossing curves in (τ_1, τ_2) plane. Stability crossing curves are spiral-like curves with vertical axes. The grey area shows a portion of the stability region in (τ_1, τ_2) plane. Set Ω consists of a unique interval $\Omega_1 = (\omega_2, \omega_1) = (9.3607, 14.8367)$.

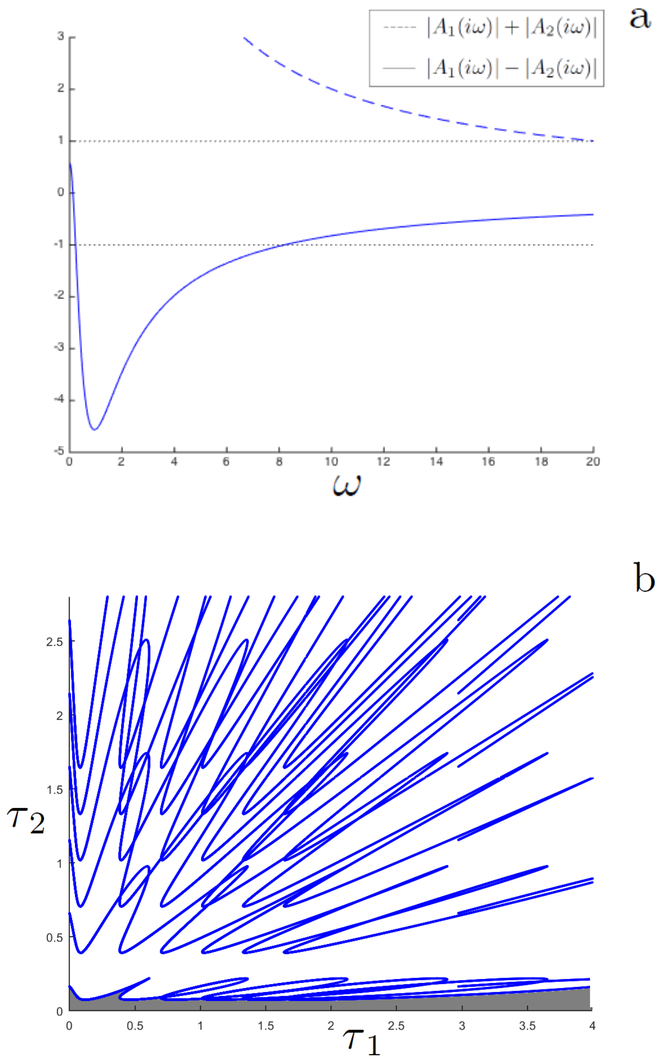


Figure 6. Case 3 of Proposition 16. Parameter set: $\alpha = 0.4$, $\beta = 29$ and $m = 0.51$. (a) $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω. (b) Stability crossing curves in the (τ_1, τ_2) plane. Stability crossing curves are open-ended curves and spiral-like curves with horizontal axes. The grey area shows a portion of the stability region in (τ_1, τ_2) plane. We do not show the open-ended curves as only spiral-like curves matter for stability. $\Omega = \Omega_1 \cup \Omega_2$ is a crossing set, where $\Omega_1 = (0, \omega_3) = (0, 0.2275)$ and $\Omega_2 = (8.1964, 20.1073)$.

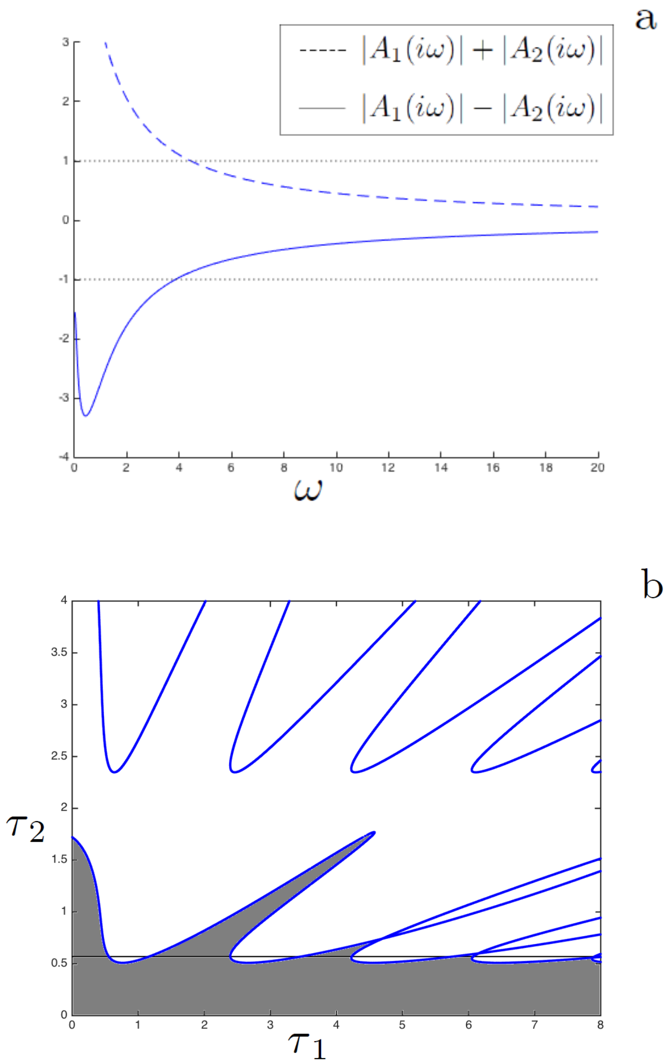


Figure 7. Case 4 of Proposition 16. Parameter set: $\alpha = 0.1$, $\beta = 7$ and $m = 0.39$. (a) $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω. (b) Stability crossing curves in the (τ_1, τ_2) plane. Stability crossing curves are spiral-like curves with horizontal axes. The grey area shows a portion of the stability region in (τ_1, τ_2) plane. Set Ω consists of a unique interval $\Omega_1 = (\omega_3, \omega_1) = (3.8607, 4.4385)$.

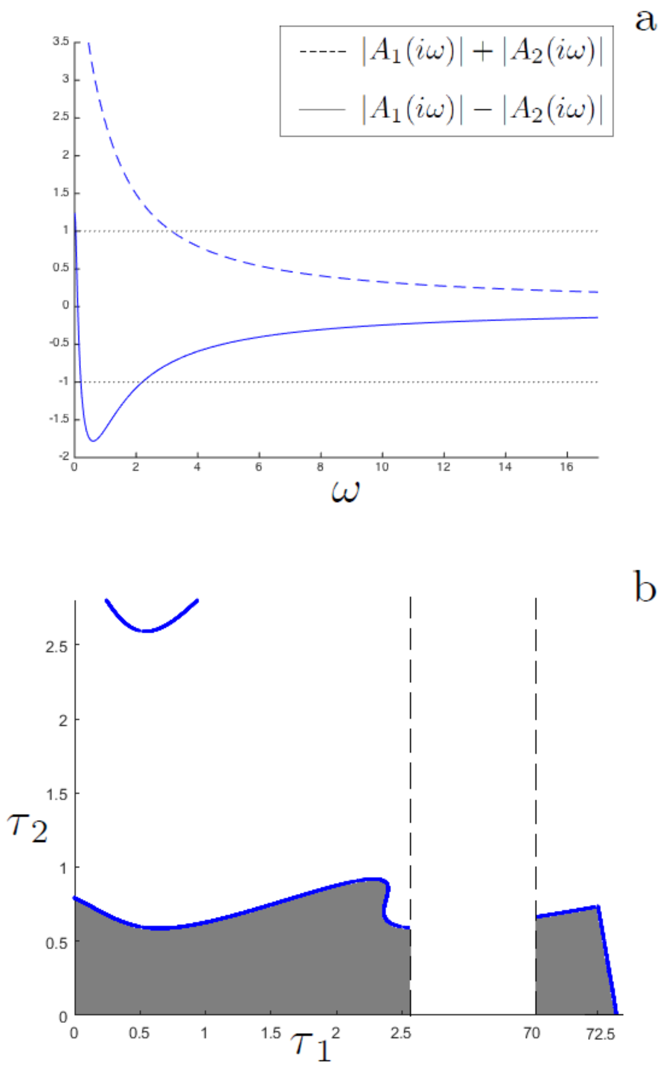


Figure 8. Case 5 of Proposition 16. Parameter set: $\alpha = 0.1$, $\beta = 7$ and $m = 0.59$. (a) $|A_1(i\omega)| \pm |A_2(i\omega)|$ versus ω. (b) Stability crossing curves in the (τ_1, τ_2) plane. Stability crossing curves are spiral-like curves with axes oriented diagonally and spiral-like curves with horizontal axes. The grey area shows the stability region in (τ_1, τ_2) plane whose borders are defined by a spiral-like curve with axes oriented diagonally and a spiral-like curves with horizontal axes. $\Omega = \Omega_1 \cup \Omega_2$ is a crossing set, where $\Omega_1 = (\omega_2, \omega_3) = (0.0374, 0.2049)$ and $\Omega_2 = (\omega_4, \omega_1) = (2.2002, 3.1481)$.

5 Conclusions

This paper analysed a continuous time version with discrete time delays of a cobweb model with heterogeneous producers and gestation lags, with quantity adjustments rather than price adjustments. It has studied conditions for which the equilibrium of the system is stable and pointed out the possibility of the emergence of Hopf bifurcations (and then cyclical dynamics) when there exist delays in production (time-to-build technology). To this purpose, the paper has used the techniques introduced by Ruan and Wei (2003) and also the geometric approach of stability crossing curves developed by Gu et al. (2005). The paper has also shown the ambiguous role of heterogeneity on the stability of the stationary equilibrium of the system. The model could be extended in several directions, for instance by considering heterogeneous expectations formation mechanisms (Brock and Hommes, 1997) or by introducing stochastic elements into the analysis (Brianzoni et al., 2008).

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