A continuous time Cournot duopoly with delays

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Abstract

This paper extends the classical repeated duopoly model with quantity-setting firms of Bischi et al. (1998) by assuming that production of goods is subject to some gestation lags but exchanges take place continuously on the market. The model is expressed in the form of differential equations with discrete delays. By using some recent mathematical techniques and numerical experiments, results show some dynamic phenomena that cannot be observed when delays are absent. In addition, depending on the extent of time delays and inertia, synchronisation failure can arise even in the event of homogeneous firms.

Keywords Chaos; Cournot duopoly; Time delays

JEL Classification C62; D43; L13

1 Introduction

The literature on nonlinear duopolies has developed models to study the behaviour of quantity-setting firms with limited information in a discrete time context (Bischi et al., 1998, 2007). The main aim of these works was essentially to question the results of stability of equilibria in dynamic models under the hypotheses of rational expectations (that require strong assumptions, such as, e.g., full information of decision makers and the efficient use of the set of available information) and homogeneous economic agents, showing that models with more realistic assumptions such as "bounded rationality" and heterogeneity may predict instability and more complex long-term dynamics. Since obtaining and using efficiently information is costly, agents may adopt some behavioural rules to overcome their informational lacunae and try to go beyond the restrictions implied by the rational expectations paradigm, for which instability of equilibria is essentially related to the existence of exogenous stochastic shocks. Then, the emergence of endogenous fluctuations and chaotic motions, and the study of the topology of the basins of attraction (to emphasise the crucial role played by initial conditions that may lead economies starting by looking very similar to end up with very different long-term outcomes), represent some of the most relevant findings.

The majority of contributions in this literature has used a discrete time framework because production processes may require gestation lags (time-to-build technology). Nevertheless, it seems reasonable to consider that markets work on a continuous time scale where trading takes

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place repeatedly. Precisely because of the coexistence of continuous time and discrete time phenomena, a modelling approach characterised by differential equations with discrete delays seems to be a good compromise to capture the essence of the behaviour of economic agents in this context. The starting point of the present paper, therefore, is the discrete time repeated Cournot duopoly of Bischi et al. (1998), with the aim of studying it in continuous time with discrete delays. To this end, in this work we use the method proposed by Berezowski (2001) - and adopted by Matsumoto and Szidarovszky (2014) in the economic literature to describe complex dynamics in a monopoly market - by introducing a form of inertia to capture in a more realistic way the existence of frictions in the production process. The paper provides some findings about local and global bifurcations and chaotic dynamics that cannot be observed in both discrete time models and continuous time models without discrete delays. The introduction of discrete time delays and inertia may cause chaotic dynamics even in the event of homogeneous firms. In addition, when a certain form of heterogeneity is introduced (e.g., by considering different degrees of inertia and/or different lengths in production processes), it is possible to observe synchronisation failures and complex dynamics.

The rest of the paper proceeds as follows. Section 2 builds on the model. Section 3 describes the dynamic setting. Section 4 studies the particular case of homogeneous degrees of both inertia and production gestation lags (dynamics on the diagonal). Section 5 analyses existence and stability of the equilibrium when 1) only one of the two firms produces with time lags, and 2) both firms have production gestation lags. Section 6 deepens the analysis of the case of homogeneous production gestation lags by applying some recent techniques proposed by Chen et al. (2013). Section 7 outlines the conclusions and briefly discusses future research.

2 The model

We consider a Cournot duopoly for a single homogeneous product with normalised linear inverse demand given by $p = 1 - X$, where p is the market price and $X < 1$ is the sum of output $x_1 \geq 0$ and output $x_2 \geq 0$ produced by firm 1 and firm 2, respectively. The average and marginal cost of producing an additional unit of output is $0 < k < 1$ for every firm. Therefore, the technology of firm $i = 1, 2$ has constant marginal returns to labour (L_i) and it is equal to $x_i = L_i$. Profits of firm i are expressed as follows:

$$\Pi_i = (p - k)\, x_i = (1 - x_1 - x_2 - k)\, x_i, \quad i = 1, 2. \tag{1}$$

3 Dynamics

A classical work in the discrete time literature on nonlinear oligopolies is Bischi et al. (1998), where it is assumed that each firm i does not have a global knowledge of the market but it is able to correctly estimate its own marginal profit at time t given by

$$\frac{\partial \Pi_i(t)}{\partial x_i(t)} = 1 - 2x_i(t) - x_j(t) - k, \tag{2}$$

and then uses (2) to choose that quantity that will be produced at time $t + 1$ by using the following behavioural rule:

$$x_i(t+1) = x_i(t) + \alpha x_i(t) \frac{\partial \Pi_i(t)}{\partial x_i(t)}, \quad i = 1, 2 \tag{3}$$

where $\alpha > 0$. This mechanism aims at capturing the "real world" evolution of economic variables (where fluctuations seem to be more plausible than the monotonic approach of the corresponding variables to stationary state values) in a context in which oscillations are not related to any exogenous stochastic influences.

We now clarify the reasons why we adopt a continuous time model with discrete delays to study the behaviour of quantity-setting firms over time. In the discrete time model of Bischi et al. (1998) there are two distinct implicit assumptions. 1) Firms perfectly realise production plans period by period through (3). This may represent a strong assumption especially when the time required to produce goods is long. 2) Trading takes place in discrete time ($t = 0, 1, 2, ...$) in the market, so that in the intervals between two subsequent periods production occurs but trading does not. The market price is in fact realised only for $t \in N$.

This last consideration is closely related to the (still open) debate about whether it is better to build on models in discrete time or continuous time to describe and explain some economic phenomena, especially in financial markets (He and Zheng, 2010; He and Li, 2012). Indeed, this debate is not only philosophical but there may produce relevant differences in both the mathematical properties (especially under bounded rationality of agents) and final outcomes of models. For instance, Dixit (1986) studies a continuous time oligopoly model where firms uses an adjustment mechanism similar to the one used by Bischi et al. (1998), and finds that dynamics converge to the stationary state that results therefore to be representative of long-term behaviours of economic variables.

A modelling approach that can somehow overcome problems related to the temporal dimension of production plans and the functioning of markets is one that includes the time-to-build technology in continuous time (this kind of models were introduced for the study of problems related, for instance, to economic growth, e.g., Asea and Zak, 1999; Matsumoto and Szidarovszky, 2011; Guerrini and Sodini, 2013; Bambi and Gori, 2014; Ferrara et al., 2014). With regard to the issue of the functioning of markets with time-to-build technology, we recall the work of Matsumoto and Szidarovszky (2012) that studies a monopoly with bounded rationality. In order to include also problems related to the difficulty of adjusting production over time, we consider the modelling approach proposed by Berezowski (2001) and introduced in the economic literature by Matsumoto and Szidarovszky (2014). Specifically, we assume the existence of a friction in the production process so that the adjustment of quantities chosen by firm i at time $t - \tau_i$ for time t is not perfectly achieved but it is subject to a feedback that acts in the opposite direction with respect to the instantaneous change in the quantity produced at the time t, that is $\dot{x}(t)$.

Definitely, a possible compromise in the debate between discrete time models and continuous time models is the use of a continuous time framework with time discrete delays (also augmented with frictions to capture the idea that production plans do not perfectly adjust from one period to another). In particular, the model is given by the following two-dimensional system with distinct time delays:

$$\begin{cases} \sigma_1 \dot{x}_1(t) + x_1(t) = x_1(t - \tau_1) + \alpha x_1(t - \tau_1) \left[1 - k - 2x_1(t - \tau_1) - x_2(t - \tau_1)\right], \\ \sigma_2 \dot{x}_2(t) + x_2(t) = x_2(t - \tau_2) + \alpha x_2(t - \tau_2) \left[1 - k - 2x_2(t - \tau_2) - x_1(t - \tau_2)\right], \end{cases} \tag{4}$$

where $\sigma_1, \sigma_2 \geq 0$ weights the inertia in the production process of firm 1 and firm 2, respectively, and $\tau_1, \tau_2 \geq 0$ are two parameters that capture time delays. We note that dynamic system (4) is the continuous time version with different discrete time delays of the discrete time two-dimensional system of Bischi et al. (1998) with normalised inverse demand. Specifically, when $\sigma_1 = \sigma_2 = 0$ and $\tau_1 = \tau_2 = 1$ (4) replicates the two-dimensional discrete time map of Bischi et al. (1998) given by (3).

4 Dynamics on the diagonal

In the particular case $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2$, and by assuming the same initial conditions (i.e., $x_1(t) = x_2(t)\ \forall t \in [-\tau, 0]$), dynamics are described by the following equation:

$$\sigma_1 \dot{x}_1(t) + x_1(t) = x_1(t-\tau) + \alpha x_1(t-\tau)\left[1 - k - 3x_1(t-\tau)\right], \tag{5}$$

and lie on the diagonal. In this case the two firms produce and sell the same quantity on the market for any t (synchronised trajectories). However, although there are some strong assumptions of symmetry long-term dynamics may show complex behaviours.

Obviously, $x_1^* = (1-k)/3$ is the unique positive equilibrium of Eq. (5). In order to study the stability properties, by setting $x = x_1 - x_1^*$ we consider the following linearization of (5) at $x = 0$:

$$\dot{x}(t) = -\frac{1}{\sigma_1}x(t) + \frac{1 - 3\alpha x_1^*}{\sigma_1}x(t-\tau). \tag{6}$$

The characteristic equation corresponding to (6) is

$$\lambda + \frac{1}{\sigma_1} - \frac{1 - 3\alpha x_1^*}{\sigma_1}e^{-\lambda\tau} = 0. \tag{7}$$

For $\tau = 0$, the only root of (7) is $\lambda = -3\alpha x_1^*/\sigma_1 < 0$. Hence, the equilibrium point of system (5) is locally asymptotically stable. Clearly, $\lambda = 0$ is not a root of (7). Suppose that $\lambda = i\omega$ ($\omega > 0$) is a root of Eq. (7). Then we get

$$\omega = -\frac{1 - 3\alpha x_1^*}{\sigma_1}\sin\omega\tau, \qquad \frac{1}{\sigma_1} = \frac{1 - 3\alpha x_1^*}{\sigma_1}\cos\omega\tau,$$

which implies

$$\omega^2 = \left[\frac{1 - 3\alpha x_1^*}{\sigma_1}\right]^2 - \left(\frac{1}{\sigma_1}\right)^2 = \frac{3\alpha x_1^*(3\alpha x_1^* - 2)}{\sigma_1^2}.$$

This is possible if and only if $\alpha > 2/(1-k)$. It is immediate that when $\alpha \leq 2/(1-k)$, all roots of (7) have negative real parts. For $\alpha > 2/(1-k)$, let

$$\tau_0 = \frac{1}{\omega_0}\left\{\tan^{-1}\left(-\sigma_1\omega_0\right) + \pi\right\}, \tag{8}$$

where

$$\omega_0 = \frac{1}{\sigma_1}\sqrt{3\alpha x_1^*(3\alpha x_1^* - 2)}.$$

A direct computation shows that $\lambda = i\omega_0$ is a simple root for (7). Let $\lambda(\tau)$ denote the root of Eq. (7) satisfying $\mathrm{Re}(\tau_0) = 0$ and $\mathrm{Im}(\tau_0) = \omega_0$. We can obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{1}{\lambda\left(\lambda + \dfrac{1}{\sigma_1}\right)} - \frac{\tau}{\lambda}.$$

Hence

$$sign\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau}\right]_{\tau=\tau_0} = sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0} = sign\left[\frac{\sigma_1^2}{\sigma_1^2\omega_0^2 + 1}\right] > 0.$$

We have the following result.

Theorem 1 *Let τ_0 be defined as in* (8).

1) *If $\alpha \leq 2/(1-k)$, then the equilibrium x_1^* is locally asymptotically stable for all $\tau \geq 0$.*

2) *If $\alpha > 2/(1-k)$, then the equilibrium x_1^* is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, Eq.* (5) *undergoes a Hopf bifurcation at x_1^* when $\tau = \tau_0$.*

By considering the parameter set $\alpha = 5.7$, $\sigma_1 = 0.3$, $k = 0.6$ Figure 1 panel a illustrates a limit cycle for $\tau = 1$ born after the Hopf bifurcation ($\tau_0 \simeq 0.92646$), while Figure 1 panel b shows stability and instability regions in the parameter space (α, σ, τ). This last figure shows that the stabilizing (resp. destabilizing) role of σ_1 (resp. α), that is for larger values of σ_1 (resp. α) the fixed point is destabilized for larger (resp. lower) values of τ.

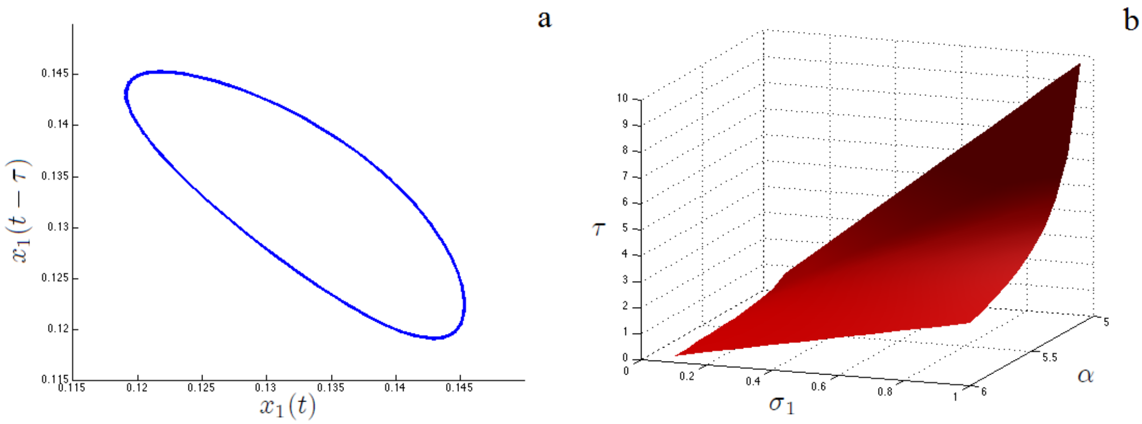

Figure 1. (a) Hopf bifurcation for τ ($\tau = 1$). (b) Stability and instability regions in the parameter space (α, σ, τ). Regions of stability (resp. instability) are below (resp. above) the red plane in Figure 1.b.

Until now we have formally characterised the local stability properties of the stationary equilibrium. In order to get some insights about the global properties of system (5), we will consider some numerical simulations that allow us to understand the role of α, σ_1 and τ.

For the first numerical exercise, we take the parameter set $\alpha = 7$, $k = 0.6$, $\tau = 1.5$ and let σ_1 vary (the initial condition is $x_1(t) = 0.15$, with $-1.5 \leq t \leq 0$). We note that by starting from a discrete time dynamic system characterised by chaos ($\sigma_1 = 0$), Figure 2 panel a shows that when the degree of inertia (σ_1) increases the system tends to converge towards a ω-limit set characterised by more and more regular dynamics until $\sigma_1 \simeq 1.0394$, after which trajectories that start from initial conditions close enough to the fixed point converge towards it.[1]

The role of α on the dynamics of system (5) is the same as that played in Bischi et al. (1998) for homogeneous firms. In particular, Figure 2 panel b - obtained for $k = 0.6$, $\sigma_1 = 0.3$ and $\tau = 1$ - shows the evolution of the dynamics of the system restricted on the diagonal when α varies. If $\alpha < 5.6243$ the system converges towards the fixed point. An increase in α causes a cascade of period-doubling bifurcations that eventually lead to chaotic dynamics (also confirmed by the calculus of the maximum Lyapunov exponent)[2] for sufficiently high values of α.

[1] Initial conditions that start too far from the stationary equilibrium generate non-feasible trajectories that involve negative values of the state variable.

[2] The maximum Lyapunov exponent can be computed by starting from the time series generated by the system, for instance through the use of the Wolf algorithm.

Consider now the behaviour of system (5) when τ varies. The Hopf bifurcation found at $\tau = \tau_0$ is the first step towards more complex and eventually chaotic dynamics, as shown in Figure 2 panel c plotted for the parameter set: $\alpha = 8$, $k = 0.6$ and $\sigma_1 = 1$ (initial condition $x_1(t) = 0.15$, with $-\tau \leq t \leq 0$). Figure 2 panel d also shows the chaotic behaviour of the system over time for this parameter set and $\tau = 2.8$.

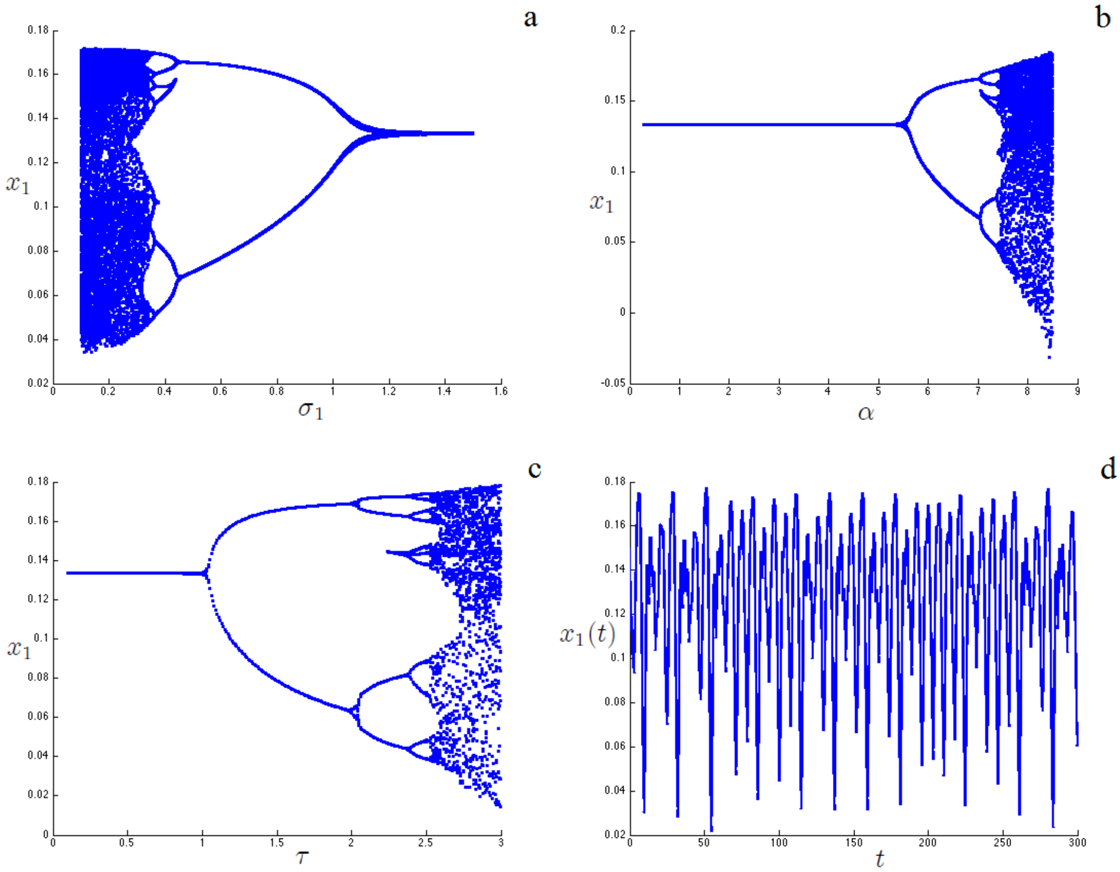

Figure 2. (a) Bifurcation diagram for σ_1. (b) Bifurcation diagram for α. (c) Bifurcation diagram for τ. (d) Time series for the parameter set as in panel c and $\tau = 2.8$. The bifurcation diagrams show the birth and evolution of local maxima and local minima when one parameter varies. By looking for instance at Figure 2.a, it is possible to deduce that by starting from a situation in which some local maxima and local minima exist for the generic trajectory, it is possible to have the birth of new local maxima and local minima that start from the inflexion points that lie between maxima and minima. This phenomenon - that can be seen by looking also at Figures 2.b and 2.c - is pointed out by the existence of interrupted branches in the figures.

5 Existence and stability of positive equilibrium

In this section, we shall study the stability of the positive fixed point and existence of Hopf bifurcation of system (4) through the study of the distribution of the eigenvalues.

Lemma 2 *System* (4) *has a unique positive equilibrium* (x_1^*, x_1^*), *where* $x_1^* = (1-k)/3$.

Proof. One can see that (x_1^*, x_2^*) is an equilibrium of system (4) if and only if (x_1^*, x_2^*) solves

$$\begin{cases} x_1^*\,(1-k-2x_1^*-x_2^*)=0, \\ x_2^*\,(1-k-2x_2^*-x_1^*)=0. \end{cases}$$

A direct computation shows that this leads to $(0,0)$, $(0,(1-k)/2)$, $((1-k)/2,0)$ and $((1-k)/3,(1-k)/3)$. This completes the proof. ■

Setting $x = x_1 - x_1^*$, $y = x_2 - x_1^*$, and linearizing the resulting system at $(0,0)$, we have

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\sigma_1}x(t) + \dfrac{(1-2\alpha x_1^*)}{\sigma_1}x(t-\tau_1) - \dfrac{\alpha x_1^*}{\sigma_1}y(t-\tau_1), \\ \dot{y}(t) = -\dfrac{1}{\sigma_2}y(t) - \dfrac{\alpha x_1^*}{\sigma_2}x(t-\tau_2) + \dfrac{(1-2\alpha x_1^*)}{\sigma_2}y(t-\tau_2). \end{cases} \tag{9}$$

The characteristic equation associated with (9) is given by

$$\begin{vmatrix} -\dfrac{1}{\sigma_1} - \lambda + \dfrac{(1-2\alpha x_1^*)}{\sigma_1}e^{-\lambda\tau_1} & -\dfrac{\alpha x_1^*}{\sigma_1}e^{-\lambda\tau_1} \\ -\dfrac{\alpha x_1^*}{\sigma_2}e^{-\lambda\tau_2} & -\dfrac{1}{\sigma_2} - \lambda + \dfrac{(1-2\alpha x_1^*)}{\sigma_2}e^{-\lambda\tau_2} \end{vmatrix} = 0,$$

namely

$$\lambda^2 + \left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)\lambda + \frac{1}{\sigma_1\sigma_2} + \left[-\frac{(1-2\alpha x_1^*)}{\sigma_1\sigma_2} - \frac{(1-2\alpha x_1^*)}{\sigma_1}\lambda\right]e^{-\lambda\tau_1}$$
$$+ \left[-\frac{(1-2\alpha x_1^*)}{\sigma_1\sigma_2} - \frac{(1-2\alpha x_1^*)}{\sigma_2}\lambda\right]e^{-\lambda\tau_2} + \left[\frac{(1-2\alpha x_1^*)^2-(\alpha x_1^*)^2}{\sigma_1\sigma_2}\right]e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{10}$$

Lemma 3 *Let $\tau_1 = \tau_2 = 0$. The equilibrium point of system* (4) *is locally asymptotically stable.*

Proof. In the absence of delay, Eq. (10) becomes

$$\lambda^2 + \left[\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)2\alpha x_1^*\right]\lambda + \frac{3(\alpha x_1^*)^2}{\sigma_1\sigma_2} = 0.$$

Clearly

$$\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)2\alpha x_1^* > 0, \quad \frac{3(\alpha x_1^*)^2}{\sigma_1\sigma_2} > 0.$$

Hence, all roots have negative real parts. The conclusion is immediate. ■

In what follows we will show that the existence of delays can destabilize the equilibrium point. In particular, we will start the analysis by letting τ_2 vary when τ_1 is fixed at zero and we will find some hypotheses such that there exists a threshold value of τ_2 (i.e., τ_{2_0}) that separates the stability and instability regions for the stationary solution. In addition, we will deepen the stability properties in the case in which $\tau_2 \in [0, \tau_{2_0})$ and let τ_1 vary.

5.1 The case $\tau_1 = 0, \tau_2 > 0$

The characteristic equation (10) takes the form

$$\lambda^2 + p\lambda + r + (s\lambda + q)\, e^{-\lambda\tau_2} = 0, \tag{11}$$

where

$$p = \frac{1}{\sigma_2} + \frac{2\alpha x_1^*}{\sigma_1}, \quad r = \frac{2\alpha x_1^*}{\sigma_1\sigma_2}, \quad s = -\frac{(1-2\alpha x_1^*)}{\sigma_2} \quad \text{and} \quad q = \frac{(3\alpha x_1^* - 2)\alpha x_1^*}{\sigma_1\sigma_2}.$$

The stability of the trivial equilibrium point will change when the system under consideration has zero or a pair of imaginary eigenvalues. The former cannot occur since it would give the contradiction $x_1^* = 0$. Let $\lambda = i\omega$ $(\omega > 0)$ be a purely imaginary root of (11). Then substituting $\lambda = i\omega$ in (11), and separating the real and imaginary parts, we have

$$\omega^2 - r = q\cos\omega\tau_2 + s\omega\sin\omega\tau_2, \qquad p\omega = -s\omega\cos\omega\tau_2 + q\sin\omega\tau_2. \tag{12}$$

Squaring and adding yields that ω satisfies

$$\omega^4 - \left(s^2 - p^2 + 2r\right)\omega^2 + r^2 - q^2 = 0, \tag{13}$$

We notice that

$$r^2 - q^2 = \frac{3(\alpha x_1^*)^3(-3\alpha x_1^* + 4)}{\sigma_1^2\sigma_2^2}, \tag{14}$$

$$s^2 - p^2 + 2r = \frac{4\alpha x_1^*\left[(\sigma_1^2 - \sigma_2^2)\alpha x_1^* - \sigma_1^2\right]}{\sigma_1^2\sigma_2^2}. \tag{15}$$

Lemma 4 *Let $\alpha = 4/(1-k)$.*

1) *If $\sigma_1 \le 2\sigma_2$, then Eq. (13) has no positive root. In particular, this holds true for $\sigma_1 = \sigma_2$.*

2) *If $\sigma_1 > 2\sigma_2$, then Eq. (13) has only one positive root ω_0, where*

$$\omega_0 = \frac{4}{3\sigma_1\sigma_2}\sqrt{\sigma_1^2 - 4\sigma_2^2}.$$

Proof. Since $-3\alpha x_1^* + 4 = 0$ we have $r^2 - q^2 = 0$. Thus, Eq. (13) yields $\omega^2 = s^2 - p^2 + 2r$. The statement follows noting that $sign(s^2 - p^2 + 2r) = sign(\sigma_1^2 - 4\sigma_2^2)$. ■

Lemma 5 *Let $\alpha \neq 4/(1-k)$.*

1) *Let $\sigma_1 = \sigma_2$.*

a) *If $\alpha < 4/(1-k)$, then Eq. (13) has no positive root.*

b) *If $\alpha > 4/(1-k)$, then Eq. (13) has only one positive root ω_0, where*

$$\omega_0 = \frac{\sqrt{\alpha x_1^*(3\alpha x_1^* - 4)}}{\sigma_1}.$$

2) *Let $\sigma_1 \neq \sigma_2$.*

a) *Let* $\alpha < 4/(1-k)$.

- *If* $\sigma_1 \le 2\sigma_2$, *or* $\sigma_1 \ge 2\sigma_2$ *and* $\alpha x_1^* \le \sigma_1^2/(\sigma_1^2-\sigma_2^2)$, *then Eq.* (13) *has no positive root.*
- *If* $\sigma_1 > 2\sigma_2$ *and* $\sigma_1^2/(\sigma_1^2-\sigma_2^2) < \alpha x_1^*$, *then Eq.* (13) *have no positive root if* $sign(\Delta) < 0$, *one positive root* ω_0 *if* $sign(\Delta) = 0$, *with*

$$\omega_0 = \frac{\sqrt{\alpha x_1^*}}{\sigma_1\sigma_2}\sqrt{2\left[(\sigma_1^2-\sigma_2^2)\alpha x_1^* - \sigma_1^2\right]},$$

and two positive roots $\omega_\pm$ *if* $sign(\Delta) > 0$, *with*

$$\omega_\pm = \frac{\sqrt{\alpha x_1^*}}{\sigma_1\sigma_2}\sqrt{2\left[(\sigma_1^2-\sigma_2^2)\alpha x_1^* - \sigma_1^2\right] \pm \sqrt{M}},$$

where

$$sign(\Delta) = sign\,(M)\,,$$

and

$$M = \left(4\sigma_1^4 + 4\sigma_2^4 + \sigma_1^2\sigma_2^2\right)(\alpha x_1^*)^2 - 4\sigma_1^4\left(2\sigma_1^4+\sigma_2^2\right)\alpha x_1^* + 4\sigma_1^4. \tag{16}$$

b) *If* $\alpha > 4/(1-k)$, *then Eq.* (13) *has only one positive root* ω_0, *where*

$$\omega_0 = \frac{\sqrt{\alpha x_1^*}}{\sigma_1\sigma_2}\sqrt{2\left[(\sigma_1^2-\sigma_2^2)\alpha x_1^* - \sigma_1^2\right] + \sqrt{M}}.$$

Proof.

1) Let $\sigma_1 = \sigma_2$. Eq. (13) becomes

$$\omega^4 + \frac{4\alpha x_1^*}{\sigma_1^2}\omega^2 + \frac{3(\alpha x_1^*)^3(-3\alpha x_1^*+4)}{\sigma_1^4} = 0. \tag{17}$$

a) The statement is immediate being the left hand side of (17) a positive number.

b) Eq. (17) gives

$$\omega^2 = \frac{-\alpha x_1^*\left[2 \pm (-3\alpha x_1^* + 2)\right]}{\sigma_1^2}$$

and so the conclusion.

2) Let $\sigma_1 \neq \sigma_2$.

a) We have $r^2 - q^2 > 0$ and $sign(-\left(s^2-p^2+2r\right)) = sign((\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2)$. If $\sigma_2 > \sigma_1$, then Eq. (13) has no positive solution since $-\left(s^2-p^2+2r\right) > 0$. If $\sigma_2 < \sigma_1$, then $-\left(s^2-p^2+2r\right) \gtreqless 0$.

i) If $\alpha x_1^* = \sigma_1^2/(\sigma_1^2-\sigma_2^2)$, then $-\left(s^2-p^2+2r\right) = 0$. If $\sigma_1 \le 2\sigma_2$, then $\sigma_1^2/(\sigma_1^2-\sigma_2^2) \ge 4/3$ and so the hypothesis $\alpha < 4/(1-k)$ implies that $(\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2 = 0$ is not possible. If $\sigma_1 > 2\sigma_2$, then $(\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2 = 0$ holds since $\alpha x_1^* = \sigma_1^2/(\sigma_1^2-\sigma_2^2) < 4/3$.

ii) If $(\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2 > 0$, then $-\left(s^2-p^2+2r\right) > 0$. Hence, $\alpha x_1^* < \sigma_1^2/(\sigma_1^2-\sigma_2^2)$. If $\sigma_1 \le 2\sigma_2$ then $\sigma_1^2/(\sigma_1^2-\sigma_2^2) \ge 4/3$ and so the hypothesis $\alpha < 4/(1-k)$ implies that $(\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2 > 0$ is always true. If $\sigma_1 > 2\sigma_2$, then $\sigma_1^2/(\sigma_1^2-\sigma_2^2) < 4/3$ and so $(\sigma_2^2-\sigma_1^2)\alpha x_1^* + \sigma_1^2 > 0$ holds if $\alpha x_1^* < \sigma_1^2/(\sigma_1^2-\sigma_2^2)$.

iii) If $(\sigma_2^2-\sigma_1^2)\alpha x_1^*+\sigma_1^2<0$, then $-\left(s^2-p^2+2r\right)<0$. Hence, $\alpha x_1^*>\sigma_1^2/(\sigma_1^2-\sigma_2^2)$. If $\sigma_1\leq 2\sigma_2$, then $\sigma_1^2/(\sigma_1^2-\sigma_2^2)\geq 4/3$ and so $(\sigma_2^2-\sigma_1^2)\alpha x_1^*+\sigma_1^2<0$ is not possible. If $\sigma_1>2\sigma_2$, then $\sigma_1^2/(\sigma_1^2-\sigma_2^2)<4/3$ and so $(\sigma_2^2-\sigma_1^2)\alpha x_1^*+\sigma_1^2<0$ holds if $\sigma_1^2/(\sigma_1^2-\sigma_2^2)<\alpha x_1^*<4/3$.

To sum up the analysis in $i)-iii)$:

I) If $\sigma_1\leq 2\sigma_2$, or $\sigma_1>2\sigma_2$ and $\alpha x_1^*<\sigma_1^2/(\sigma_1^2-\sigma_2^2)$, then $-\left(s^2-p^2+2r\right)>0$. Hence, Eq. (13) has no positive root.

II) If $\sigma_1>2\sigma_2$ and $\alpha x_1^*=\sigma_1^2/(\sigma_1^2-\sigma_2^2)$, then $-\left(s^2-p^2+2r\right)=0$. Hence, Eq. (13) has no positive root.

III) If $\sigma_1>2\sigma_2$ and $\sigma_1^2/(\sigma_1^2-\sigma_2^2)<\alpha x_1^*$, then $-\left(s^2-p^2+2r\right)<0$. Now from

$$\Delta=\left(s^2-p^2+2r\right)^2-\left(r^2-q^2\right),$$

(14) and (15), we find

$$sign(\Delta)=sign\left[\left(4\sigma_1^4+4\sigma_2^4+\sigma_1^2\sigma_2^2\right)(\alpha x_1^*)^2-4\sigma_1^4\left(2\sigma_1^4+\sigma_2^2\right)\alpha x_1^*+4\sigma_1^4\right].$$

Therefore, we can conclude that

☐ if $sign(\Delta)=0$, then Eq. (13) has only one positive root ω_0, where

$$\omega_0=\sqrt{\frac{s^2-p^2+2r}{2}};$$

☐ if $sign(\Delta)<0$, then Eq. (13) has no positive root;

☐ if $sign(\Delta)>0$, then Eq. (13) has two positive roots $\omega_\pm$, where

$$\omega_\pm=\sqrt{\frac{s^2-p^2+2r\pm\sqrt{\Delta}}{2}}.$$

The statement follows by using (14) and (15).

b) We have $r^2-q^2<0$ and $\Delta=\left(s^2-p^2+2r\right)^2-\left(r^2-q^2\right)>0$. Therefore, from (13) we get

$$\omega^2=\frac{s^2-p^2+2r\pm\sqrt{\Delta}}{2},$$

which yields

$$\omega_0=\sqrt{\frac{s^2-p^2+2r+\sqrt{\Delta}}{2}}.$$

Now use (14) and (15). If Eq. (13) has a unique positive root ω_0, then from (12) we can determine

$$\tau_{2_n}=\frac{1}{\omega_0}\cos^{-1}\left\{\frac{(q-ps)\,\omega_0^2-rq}{s^2\omega_0^2+q^2}\right\}+\frac{2n\pi}{\omega_0},\quad n=0,1,2,... \tag{18}$$

at which Eq. (11) has a pair of purely imaginary roots $\pm i\omega_0$. Similarly, if Eq. (13) has two positive root $\omega_\pm$, the characteristic equation (11) has purely imaginary roots when τ_2 takes the critical values

$$\tau_{2_j}^{\pm}=\frac{1}{\omega_\pm}\cos^{-1}\left\{\frac{(q-ps)\,\omega_\pm^2-rq}{s^2\omega_\pm^2+q^2}\right\}+\frac{2j\pi}{\omega_\pm},\quad j=0,1,2,... \tag{19}$$

■

Proposition 6 *Let $\lambda(\tau_2)$ be the root of* (11) *satisfying* $\mathrm{Re}(\tau_{2_n}) = 0$ (*resp.* $\mathrm{Re}(\tau_{2_j}^{\pm}) = 0$) *and* $\mathrm{Im}(\tau_{2_n}) = \omega_{\pm}$ (*resp.* $\mathrm{Re}(\tau_{2_j}^{\pm}) = \omega_{\pm}$). *Then*

$$\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau=\tau_{2n},\omega=\omega_0} > 0, \qquad \left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau=\tau_{2_j}^{+},\omega=\omega_+} > 0, \qquad \left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau=\tau_{2_j}^{-},\omega=\omega_-} < 0.$$

Proof. Substituting $\lambda(\tau_2)$ into (11) and taking the derivative with respect to τ_2, we get

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{(2\lambda + p)e^{\lambda\tau_2} + s}{\lambda(s\lambda + q)} - \frac{\tau_2}{\lambda}. \tag{20}$$

Now (11) yields $e^{\lambda\tau_2} = -(s\lambda + q)/(\lambda^2 + p\lambda + r)$. Hence, using this identity and (12) in (20) we arrive at

$$\begin{aligned} sign\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\lambda=i\omega} &= sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right]_{\lambda=i\omega} = sign\left[\frac{p^2 - 2(r-\omega^2)}{p^2\omega^2 + (\omega^2 - r)^2} - \frac{s^2}{s^2\omega^2 + q^2}\right] \\ &= sign\left[p^2 - 2r - s^2 + 2\omega^2\right] \end{aligned}$$

where $\omega = \omega_0$ or $\omega = \omega_{\pm}$. If $\omega = \omega_0$, then $sign\left[p^2 - 2r - s^2 + 2\omega^2\right] = +1$, while if $\omega = \omega_{\pm}$, then $sign\left[p^2 - 2r - s^2 + 2\omega^2\right] = sign\left[\pm\sqrt{\Delta}\right]$, namely the sign is positive for ω_+ and negative for ω_-. ■

The previous Proposition implies that if only one imaginary root $i\omega_0$ exists for (11), then only crossing of the imaginary axis from left to right is possible as τ_2 increases. Thus, stability of the equilibrium is lost but not regained. On the other hand, if two imaginary roots $i\omega_{\pm}$ exist for (11), then crossing from left to right with increasing τ_2 occurs whenever τ_2 assumes a value corresponding to ω_+, and crossing from right to left occurs for values of the τ_2 corresponding to ω_-.

Theorem 7 *Let $M, \tau_{2_n}, \tau_{2_j}^{\pm}$ be defined as in* (16), (18) *and* (19), *respectively.*

1) *If $\alpha x_1^* = 4/3$ and $\sigma_1 \le 2\sigma_2$, or $\alpha x_1^* < 4/3$ and $\sigma_1 = \sigma_2$, or $\alpha x_1^* < 4/3$ and $\sigma_1 \le 2\sigma_2$, or $\alpha x_1^* < 4/3$, $\sigma_1 \ge 2\sigma_2$ and $\alpha x_1^* \le \sigma_1^2/(\sigma_1^2 - \sigma_2^2)$, or $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M < 0$ hold, then the equilibrium (x_1^*, x_1^*) is locally asymptotically stable for all $\tau_2 \ge 0$.*

2) *If $\alpha x_1^* = 4/3$ and $\sigma_1 > 2\sigma_2$, or $\alpha x_1^* > 4/3$, or $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M = 0$ hold, then the equilibrium (x_1^*, x_1^*) is locally asymptotically stable for $\tau_2 < \tau_{2_0}$ and unstable for $\tau_2 > \tau_{2_0}$. Furthermore, system* (4) *undergoes a Hopf bifurcation at (x_1^*, x_1^*) when $\tau_2 = \tau_{2_0}$ if the corresponding root $\lambda = i\omega_0$ of* (11) *is simple.*

3) *If $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M > 0$ hold, then there is a positive integer m such that the equilibrium (x_1^*, x_1^*) is locally asymptotically stable when $\tau_2 \in [0, \tau_{2_0}^+) \cup (\tau_{2_0}^-, \tau_{2_1}^+) \cup \cdots \cup (\tau_{2_{m-1}}^-, \tau_{2_m}^+)$ and unstable when $\tau_2 \in (\tau_{2_0}^+, \tau_{2_0}^-) \cup (\tau_{2_1}^+, \tau_{2_1}^-) \cup \cdots \cup (\tau_{2_{m-1}}^+, \tau_{2_{m-1}}^-) \cup (\tau_{2_m}^+, \infty)$. Furthermore, system* (4) *undergoes a Hopf bifurcation at (x_1^*, x_1^*) when $\tau_2 = \tau_{2m}^{\pm}$, $m = 0, 1, 2, \ldots$, if the corresponding root $\lambda = i\omega_{\pm}$ of* (11) *is simple.*

Proof. Since Eq. (11) is stable for $\tau_2 = 0$, then necessarily $\tau_{2_0}^+ < \tau_{2_0}^-$. From $\tau_{2_{j+1}}^+ - \tau_{2_j}^+ = 2\pi/\omega_+ < 2\pi/\omega_- = \tau_{2_{j+1}}^- - \tau_{2_j}^-$, we have that there exists an integer $m > 0$ such that $0 < \tau_{2_0}^+ < \tau_{2_0}^- < \tau_{2_1}^+ < \cdots < \tau_{2_{m-1}}^- < \tau_{2_m}^+$ and there are m switches from stability to instability to stability, that is when $\tau_2 \in [0, \tau_{2_0}^+) \cup (\tau_{2_0}^-, \tau_{2_1}^+) \cup \cdots \cup (\tau_{2_{m-1}}^-, \tau_{2_m}^+)$ all root of Eq. (11) have negative real parts, and when $\tau_2 \in (\tau_{2_0}^+, \tau_{2_0}^-) \cup (\tau_{2_1}^+, \tau_{2_1}^-) \cup \cdots \cup (\tau_{2_{m-1}}^+, \tau_{2_{m-1}}^-)$ and $\tau_2 > \tau_{2_m}^+$ Eq. (11) has at least one root with positive real part. The statement follows from the previous Lemmas and Proposition. ■

Remark 8 If $\left\{d\left[\lambda^2 + p\lambda + r + (s\lambda + q)\, e^{-\lambda\tau_2}\right]/d\tau_2\right\}_{\lambda = i\hat{\omega}} \neq 0$, where $\hat{\omega} = \omega_0, \omega_\pm$, then $\lambda = i\hat{\omega}$ is a simple root of (11).

5.2 The case $\tau_1 > 0$ and τ_2 fixed in the interval $[0, \tau_{2_0})$

We consider Eq. (10) with τ_2 in its stable interval, i.e. $\tau_2 \in [0, \tau_{2_0})$, and τ_1 is regarded as a parameter. It is convenient to rewrite the characteristic equation (10) as

$$\lambda^2 + A\lambda + B + (C + D\lambda)\, e^{-\lambda\tau_1} + (C + E\lambda)\, e^{-\lambda\tau_2} + F e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{21}$$

where

$$A = \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \qquad B = \frac{1}{\sigma_1\sigma_2}, \qquad C = -\frac{(1 - 2\alpha x_1^*)}{\sigma_1\sigma_2}, \qquad D = -\frac{(1 - 2\alpha x_1^*)}{\sigma_1},$$

$$E = -\frac{(1 - 2\alpha x_1^*)}{\sigma_2}, \qquad F = \frac{(1 - 2\alpha x_1^*)^2 - (\alpha x_1^*)^2}{\sigma_1\sigma_2}.$$

The complicated form of (21) is an obstruction to predict nature of roots. For analytical reasons and in order to avoid cumbersome calculations, we now focus on the study of Eq. (21) under the following assumption.

Assumption A.1 $\alpha x_1^* > 4/3$.

In this case, the characteristic equation (21) has only a pair of purely imaginary roots $\pm i\omega_0$.

Remark 9 When $3(\alpha x_1^*)^2 - 4\alpha x_1^* + 1 = 0$, i.e. if $\alpha x_1^* = 1$ or $\alpha x_1^* = 1/3$ then $F = 0$. In this case, the term $e^{-\lambda(\tau_1+\tau_2)}$ vanishes and Eq. (21) boils down to

$$\lambda^2 + A\lambda + B + (C + D\lambda)\, e^{-\lambda\tau_1} + (C + E\lambda)\, e^{-\lambda\tau_2} = 0.$$

Furthermore, from Theorem 7 we have that the equilibrium (x_1^*, x_1^*) is locally asymptotically stable for every $\tau_2 \geq 0$. Since $\alpha x_1^* > 4/3$, the analysis of these two cases will not be not included in our discussion.

It is clear that $\lambda = 0$ is not a solution of (21). If it were, then we would have $3(\alpha x_1^*)^2 = 0$, which is impossible having this equation no real solution. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (21). Similar to what done in the previous section, by separating real and imaginary parts, after long and tedious calculations, we can get

$$g(\omega) = 0, \tag{22}$$

where

$$g(\omega)=\omega^4+\left(A^2+E^2-2B-D^2\right)\omega^2+B^2-F^2$$
$$+2\left[(DF-AC+BE)\,\omega-E\omega^3\right]\sin\omega\tau_2$$
$$+2\left[-CF+BC+(AE-C)\,\omega^2\right]\cos\omega\tau_2.$$

Since $\alpha x_1^*>4/3$ one has

$$g(0)=-\frac{3(\alpha x_1^*)^3(3\alpha x_1^*-4)}{\sigma_1^2\sigma_2^2}<0.$$

Moreover, $g(+\infty)=+\infty$. Hence, we obtain that (22) has at least one positive root. From the expression of $g(\omega)$ we have that (22) has finite positive roots $\omega_1,\omega_2,...,\omega_N$. For every fixed ω_l, $l=1,2,...,N$, there exists a sequence $\tau_{1_l}^j>0$ $(j=1,2,...)$ such that such that (22) holds. Let

$$\tau_{1_0}=\min\left\{\tau_{1_l}^j,l=1,2,...,N,j=1,2,...\right\}.$$

When $\tau_1=\tau_{1_0}$ the characteristic equation (21) has a pair of purely imaginary roots $\pm i\tilde{\omega}$ for $\tau_2\in[0,\tau_{2_0})$. Let $\lambda\left(\tau_1\right)$ be the root of (21) near $\tau_1=\tau_{1_0}$ satisfying $\mathrm{Re}(\tau_{1_0})=0$ and $\mathrm{Im}(\tau_{1_0})=\tilde{\omega}$. Differentiating (21) with respect to τ_1, we get

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=\frac{2\lambda+A+De^{-\lambda\tau_1}+Ee^{-\lambda\tau_2}-(C+E\lambda)\tau_2e^{-\lambda\tau_2}-F\tau_2e^{-\lambda(\tau_1+\tau_2)}}{\lambda\left[(C+D\lambda)e^{-\lambda\tau_1}+Fe^{-\lambda(\tau_1+\tau_2)}\right]}-\frac{\tau_1}{\lambda}.$$

Using (21), this becomes

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=-\frac{2\lambda+A+De^{-\lambda\tau_1}+Ee^{-\lambda\tau_2}+\tau_2\left[\lambda^2+A\lambda+B+(C+D\lambda)\,e^{-\lambda\tau_1}\right]}{\lambda\left[\lambda^2+A\lambda+B+(C+E\lambda)\,e^{-\lambda\tau_2}\right]}-\frac{\tau_1}{\lambda},$$

Substituting $\tau_1=\tau_{1_0}$ we get

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau_{1_0}}=\frac{a_1+ia_2}{\tilde{\omega}\,(b_1-ib_2)}-\frac{\tau_{1_0}}{i\tilde{\omega}},$$

where

$$\begin{aligned}
a_1 &= A+D\cos\tilde{\omega}\tau_{1_0}+E\cos\tilde{\omega}\tau_2+\tau_2\left(-\tilde{\omega}^2+B+C\cos\tilde{\omega}\tau_{1_0}+D\tilde{\omega}\sin\tilde{\omega}\tau_{1_0}\right),\\
a_2 &= 2\tilde{\omega}-D\sin\tilde{\omega}\tau_{1_0}-E\sin\tilde{\omega}\tau_2+\tau_2\left(A\tilde{\omega}-C\sin\tilde{\omega}\tau_{1_0}+D\tilde{\omega}\cos\tilde{\omega}\tau_{1_0}\right),\\
b_1 &= A\tilde{\omega}-C\sin\tilde{\omega}\tau_2+E\tilde{\omega}\cos\tilde{\omega}\tau_2,\\
b_2 &= B-\tilde{\omega}^2+C\cos\tilde{\omega}\tau_2+E\tilde{\omega}\sin\tilde{\omega}\tau_2.
\end{aligned}$$

Hence,

$$sign\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_1}\right]_{\tau_1=\tau_{1_0}}=sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right]_{\tau_1=\tau_{1_0}}=sign\left(a_1b_1-a_2b_2\right),\qquad(23)$$

with

$$
\begin{aligned}
a_1b_1 - a_2b_2 = {} & \tilde{\omega}\left(2\tilde{\omega}^2 + A^2 + E^2 - 2B\right) + AD\tilde{\omega}\cos\tilde{\omega}\tau_{1_0} + D(B - D\tilde{\omega})\sin\tilde{\omega}\tau_{1_0} \\
& + 2\tilde{\omega}(AE - C)\cos\tilde{\omega}\tau_2 + [-AC + E(B - 3\tilde{\omega}^2)]\sin\tilde{\omega}\tau_2 \\
& + DE\tilde{\omega}\cos\tilde{\omega}(\tau_{1_0} - \tau_2) + CD\sin\tilde{\omega}(\tau_{1_0} - \tau_2) \\
& + \tau_2\left\{\tilde{\omega}\left[AC + D\left(\tilde{\omega}^2 - B\right)\right]\cos\tilde{\omega}\tau_{1_0} + \left[BC + \tilde{\omega}^2(AD - C)\right]\sin\tilde{\omega}\tau_{1_0}\right. \\
& + \tilde{\omega}\left[-AC + E(B - \tilde{\omega}^2)\right]\cos\tilde{\omega}\tau_2 + \left[-BC + \tilde{\omega}^2(C - AE)\right]\sin\tilde{\omega}\tau_2 \\
& \left. + (C^2 + DE)\sin\tilde{\omega}(\tau_{1_0} - \tau_2) + C\tilde{\omega}(-D + E)\cos\tilde{\omega}(\tau_{1_0} - \tau_2)\right\}.
\end{aligned}
$$

Theorem 10 *Let $\tau_2 \in [0, \tau_{2_0})$.*

1) *If $g(\omega)$ has no positive zero, then the equilibrium (x_1^*, x_1^*) of system (4) is locally asymptotically stable for $\tau_1 \geq 0$.*

2) *Under Assumption A.1 there exists a positive number τ_{1_0} such that equilibrium (x_1^*, x_1^*) of system (4) is locally asymptotically stable for $\tau_1 \in [0, \tau_{1_0})$ and unstable for $\tau_1 > \tau_{1_0}$. System (4) undergoes a Hopf bifurcation at the equilibrium (x_1^*, x_1^*) for $\tau_1 = \tau_{1_0}$ if the corresponding root $\lambda = i\tilde{\omega}$ of (21) is simple and expression in (23) is positive.*

Remark 11 The critical delay τ_{1_0} depends on τ_2.

In order to show the existence of the Hopf bifurcation, in Point 2 of Theorem 10 we had to assume that the expression in (23) is positive (transversality condition). This because such an expression is impossible to be handle analytically. However, by following Krawiec and Szydlowski (1999) it is possible to get some analytical results - also related to the transversality condition - under the assumption that delays are small. According to Assumption A.1, in the following theorem we will study only the case $\alpha x_1^* > 4/3$.

Theorem 12 *Let $\tau_2 \in [0, \tau_{2_0})$. Let $4/3 < \alpha x_1^* < 2(\tau_2 + \sigma_1 + \sigma_2)/(3\tau_2)$ and $\tau_2 < \sigma_1 + \sigma_2 < 3/5$. If τ_1 is small, then there exists a positive number τ_{1_0} such the equilibrium (x_1^*, x_1^*) of system (4) is locally asymptotically stable for $\tau_1 \in [0, \tau_{1_0})$ and unstable for $\tau_1 > \tau_{1_0}$. System (4) undergoes a Hopf bifurcation at the equilibrium (x_1^*, x_1^*) for $\tau_1 = \tau_{1_0}$ since the corresponding root $\lambda = i\omega$ of (21) is simple and condition (23) holds.*

Proof. Since τ_1 and τ_2 are small, we have $e^{-\lambda\tau_1} \cong 1 - \lambda\tau_1$, $e^{-\lambda\tau_2} \cong 1 - \lambda\tau_2$ and $e^{-\lambda(\tau_1+\tau_2)} \cong 1 - \lambda(\tau_1 + \tau_2)$. In this case, (21) takes the approximate form

$$
\lambda^2 + A\lambda + B + (C + D\lambda)(1 - \lambda\tau_1) + (C + E\lambda)(1 - \lambda\tau_2) + F[1 - \lambda(\tau_1 + \tau_2)] = 0, \tag{24}
$$

Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (24). Separating real and imaginary parts leads to

$$
\begin{cases} \omega^2(1 - D\tau_1 - E\tau_2) & = & B + 2C + F, \\ A + D + E & = & (C + F)(\tau_1 + \tau_2), \end{cases} \tag{25}
$$

with

$$
B+2C+F = \frac{3(\alpha x_1^*)^2}{\sigma_1\sigma_2} > 0, \quad A+D+E = \frac{2(\alpha x_1^*)(\sigma_1 + \sigma_2)}{\sigma_1\sigma_2} > 0, \quad C+F = \frac{\alpha x_1^*(3\alpha x_1^* - 2)}{\sigma_1\sigma_2} > 0.
$$

One has $C+F>0$ since $\alpha x_1^*>4/3$. From (25), we derive

$$\omega=\sqrt{\frac{B+2C+F}{1-D\tau_1-E\tau_2}},\qquad \tau_1=\frac{2\left(\sigma_1+\sigma_2\right)}{3\alpha x_1^*-2}-\tau_2\equiv\tau_{1_0}.$$

We have that $\tau_{1_0}>0$ since $\alpha x_1^*<2\left(\tau_2+\sigma_1+\sigma_2\right)/\left(3\tau_2\right)$, with the assumption $\tau_2<\sigma_1+\sigma_2$ implying $4/3<2\left(\tau_2+\sigma_1+\sigma_2\right)/\left(3\tau_2\right)$. Finally, $\sigma_1+\sigma_2<3/5$ and (25) yield $1-D\tau_1-E\tau_2>0$. In fact,

$$\begin{aligned}1-D\tau_1-E\tau_2 &= 1+\frac{\left(1-2\alpha x_1^*\right)}{\sigma_1}\tau_1+\frac{\left(1-2\alpha x_1^*\right)}{\sigma_2}\tau_2 > 1+\left(1-2\alpha x_1^*\right)\left(\tau_1+\tau_2\right)\\ &= 1+\frac{2\left(\sigma_1+\sigma_2\right)\left(1-2\alpha x_1^*\right)}{3\alpha x_1^*-2},\end{aligned}$$

and

$$1+\frac{2\left(\sigma_1+\sigma_2\right)\left(1-2\alpha x_1^*\right)}{3\alpha x_1^*-2}>0 \iff \alpha x_1^*>\frac{2\left[1-\left(\sigma_1+\sigma_2\right)\right]}{3-4\left(\sigma_1+\sigma_2\right)},\tag{26}$$

where $\sigma_1+\sigma_2<3/5$ gives $3-4\left(\sigma_1+\sigma_2\right)>0$ and $2\left[1-\left(\sigma_1+\sigma_2\right)\right]/\left[3-4\left(\sigma_1+\sigma_2\right)\right]<4/3$. Thus, the last inequality in (26) holds true. Next, we need to prove that $\lambda=i\omega$ is a simple root of (24) when $\tau_1=\tau_{1_0}$ and verify the validity of the transversality condition. If $\lambda=i\omega$ is a repeated root of (24), then

$$2i\omega+A+D\left(1-i\omega\tau_{1_0}\right)-\left(C+Di\omega\right)\tau_{1_0}+E\left(1-i\omega\tau_2\right)-\left(C+Ei\omega\right)\tau_2-F(\tau_{1_0}+\tau_2)=0$$

holds true. By separating the real and imaginary parts yields

$$\begin{cases}1-D\tau_{1_0}-E\tau_2 = 0,\\ A+D+E = (C+F)(\tau_{1_0}+\tau_2).\end{cases}$$

Hence, it follows from (25) that $B+2C+F=0$, which is a contradiction. In order to determine the crossing direction of characteristic root through the $\lambda=i\omega$, we differentiate (24) with respect to τ_1 and get

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=\frac{2\lambda+A+D(1-\lambda\tau_1)+E(1-\lambda\tau_2)-\left(C+D\lambda\right)\tau_1-\left(C+E\lambda\right)\tau_2-F(\tau_1+\tau_2)}{F\lambda}.$$

Therefore, we have

$$\begin{aligned}sign\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau}\right]_{\tau=\tau_{1_0}} &= sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_{1_0}} = sign\left(\frac{1-D\tau_{1_0}-E\tau_2}{F}\right)\\ &= sign\left(\frac{B+2C+F}{F}\right) = sign\left(F\right).\end{aligned}$$

Now, $sign(F)>0$ if $\alpha x_1^*<1/3$ and $\alpha x_1^*>1$. Being $\alpha x_1^*>4/3$, $sign(F)$ is positive. This completes the proof. ■

Numerical explorations can also be useful in this case to observe phenomena on the dynamics of the system that go beyond those stated in Theorems 7 and 10. In particular, we will now concentrate on phenomena related to the so called synchronisation. We recall that synchronisation occurs if $\lim_{t\to+\infty}\left|x_1(t)-x_2(t)\right|=0$. In particular, it certainly occurs if the stability conditions of the stationary equilibrium shown in Theorems 7 and 10 hold true. In addition, we find that when firms are homogeneous in both inertia and time delays numerical evidences show that trajectories generated by initial conditions with $x_1(t) \neq x_2(t)$ in the interval $[-\tau, 0]$ synchronise also when a periodic or chaotic attractor exists for system (5) (see Figure 3 panel a and panel b, where - similar to the discrete time case - it is shown that convergence occurs in a sufficiently long time period).

We now continue to keep the equality $\sigma_1 = \sigma_2$ and start from the case in which there exists a chaotic attractor on the diagonal (whose birth has been described in Section 4, and for which the Hopf bifurcation occurred on the diagonal has represented the first step towards its existence); by introducing a difference between τ_1 and τ_2, we note that it is possible to have the loss of synchronisation (see Figure 3, panel c) and the chaotic attractor so generated does not lie on the diagonal anymore (see Figure 3, panel d).

Other cases of synchronisation failure can hold through a transverse Hopf bifurcation. We will show their occurrence in the next section by introducing the stronger hypothesis $\tau_1 = \tau_2 = \tau$.

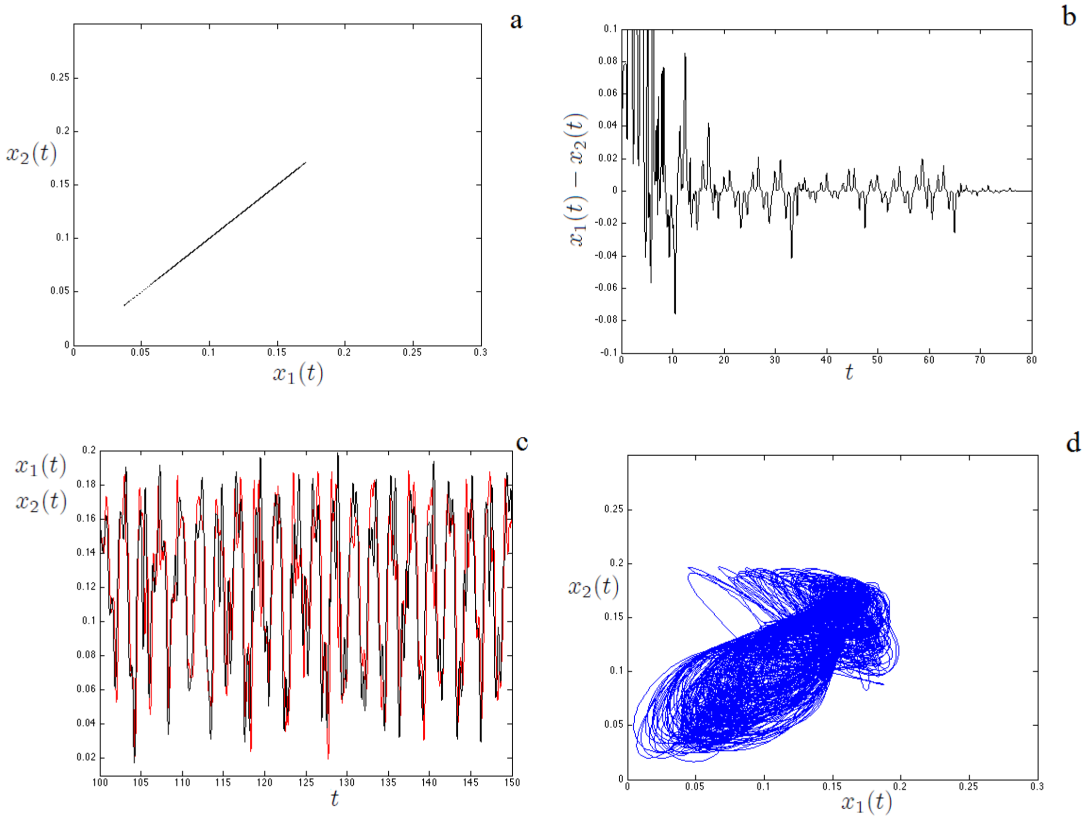

Figure 3. (a) Chaotic attractor on the diagonal. Parameter set: $\alpha = 7$, $\sigma_1 = \sigma_2 = 0.1$, $k = 0.6$ and $\tau_1 = \tau_2 = 1$. (b) Time series (synchronisation) for the parameter set as in panel a. Initial condition: $x_1(t) = 0.12$, $x_2(t) = 0.21$, $-1 \leq t \leq 0$. (c) A small mismatch in τ causes the loss of synchronisation. The figure depicts a trajectory of $x_1(t)$ and $x_2(t)$ that converges to the chaotic attractor and shows synchronisation failure between the two variables. Parameter set: $\alpha = 7$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $k = 0.6$, $\tau_1 = 1.15$ and $\tau_2 = 1$. Initial condition: $x_1(t) = 0.12$, $-1.15 \leq t \leq 0$, $x_2(t) = 0.2$, $-1 \leq t \leq 0$. (d) Chaotic attractor outside the diagonal for the parameter set as in panel c.

6 The case $\tau_1 = \tau_2 = \tau$

In this section, we will deepen some analytical results related to the particular case of equality between time delays by using some recent techniques proposed by Chen et al. (2013). By assuming $\tau_1 = \tau_2 = \tau$, the characteristic equation (10) becomes

$$\lambda^2 + a\lambda + b + (d + c\lambda)\, e^{-\lambda\tau} + he^{-2\lambda\tau} = 0, \tag{27}$$

with

$$a = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} > 0, \qquad b = \frac{1}{\sigma_1\sigma_2} > 0, \qquad c = -\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)(1 - 2\alpha x_1^*),$$

$$d = -\frac{2(1 - 2\alpha x_1^*)}{\sigma_1\sigma_2}, \qquad h = \frac{(1 - 2\alpha x_1^*)^2 - \alpha^2 x_1^{*2}}{\sigma_1\sigma_2}.$$

It is clear that the equilibrium of system (4) is locally asymptotically stable if $\tau_1 = \tau_2 = \tau = 0$, that is, all the roots of (27) with $\tau = 0$ have negative real parts.

Proposition 13 *Let $3\alpha x_1^* - 1 = 0$ or $\alpha x_1^* - 1 = 0$ or $1 - 2\alpha x_1^* = 0$. Then the equilibrium (x_1^*, x_1^*) of system (4) is locally asymptotically stable for all $\tau \geq 0$.*

Proof. If $3\alpha x_1^* - 1 = 0$ or $\alpha x_1^* - 1 = 0$, then $h = 0$. As a result, Eq. (27) takes the form $\lambda^2 + a\lambda + b + (d + c\lambda)\, e^{-\lambda\tau} = 0$. The statement now follows from Theorem 6.

If $1 - 2\alpha x_1^* = 0$, then $c = d = 0$, so that Eq. (27) is in the form $\lambda^2 + a\lambda + b + he^{-2\lambda\tau} = 0$. Suppose $\lambda = i\omega$ $(\omega > 0)$ is a root of this equation. Then we have $-\omega^2 - ai\omega + b + h\cos 2\omega\tau - ih\sin 2\omega\tau = 0$, which leads to $\omega^4 + (a^2 - 2b)\omega^2 + b^2 - h^2 = 0$. Noticing that $a^2 - 2b = \left(\sigma_1^2 + \sigma_2^2\right) / \left(\sigma_1^2\sigma_2^2\right) > 0$ and $b^2 - h^2 = 15/\left(16\sigma_1^2\sigma_2^2\right) > 0$, we obtain the statement. ■

Henceforth, we assume $3\alpha x_1^* - 1 \neq 0$, $\alpha x_1^* - 1 \neq 0$ and $1 - 2\alpha x_1^* \neq 0$, namely $c \neq 0, d \neq 0$ and $h \neq 0$.

We remark that, when at least one of c and d is not zero, and h is not zero, Chen et al. (2013) provided criteria for examining the existence of simple purely imaginary roots of (27), and the transversality at all corresponding bifurcation values. In the sequel, we use their approach.

Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (27). Then, we have

$$-\omega^2 + ai\omega + b + (d + ci\omega)\, e^{-i\omega\tau} + he^{-2i\omega\tau} = 0,$$

If $(\omega\tau)/2 \neq (\pi/2) + j\pi,\ j \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, then we have $e^{-i\omega\tau} = (1 - i\theta)/(1 + i\theta)$, with $\theta = \tan\left[(\omega\tau)/2\right]$. Separating the real and imaginary parts, one has that θ satisfies

$$\begin{cases} \left(\omega^2 - b + d - h\right)\theta^2 - 2a\omega\theta &= \omega^2 - b - d - h, \\ (c - a)\,\omega\theta^2 + \left(-2\omega^2 + 2b - 2h\right)\theta &= -(c + a)\,\omega. \end{cases} \tag{28}$$

Define

$$D(\omega) = \begin{vmatrix} \omega^2 - b + d - h & -2a\omega \\ (c - a)\,\omega & -2\omega^2 + 2b - 2h \end{vmatrix}, \tag{29}$$

$$E(\omega) = \begin{vmatrix} \omega^2 - b - d - h & -2a\omega \\ -(c + a)\,\omega & -2\omega^2 + 2b - 2h \end{vmatrix}.$$

$$F(\omega)=\left|\begin{array}{cc}\omega^2-b+d-h & \omega^2-b-d-h \\ (c-a)\,\omega & -(c+a)\,\omega\end{array}\right|.$$

Chen et al. (2013) proved that ω satisfies $D(\omega)E(\omega)=[F(\omega)]^2$, and ω^2 is a positive root of

$$z^4+s_1z^3+s_2z^2+s_3z+s_4=0, \tag{30}$$

where

$$\begin{aligned}
s_1 &= 2a^2-4b-c^2,\\
s_2 &= 6b^2-2h^2-4ba^2-d^2+a^4-a^2c^2+2c^2b+2hc^2,\\
s_3 &= 2d^2b-a^2d^2-4b^3+2b^2a^2-c^2b^2-2bc^2h+4acdh-2d^2h+4bh^2-2h^2a^2-c^2h^2,\\
s_4 &= (b-h)^2[-d^2+(b+h)^2].
\end{aligned}$$

Lemma 14 [*Chen et al.* (2013)] *If* $\pm i\omega$ $(\omega>0)$ *is a pair of purely imaginary roots of the characteristic equation* (27)*, then* ω^2 *is a positive root of the quartic polynomial equation* (30).

The next lemma gives the algorithm of solving the critical delay values for purely imaginary roots of (27).

Lemma 15 [*Chen et al.* (2013)] *If Eq.* (30) *has a positive root* ω_N^2 $(\omega_N>0)$ *and* $D(\omega_N)\neq 0$, *then system* (28) *has a unique real root*

$$\theta_N=\frac{F(\omega_N)}{D(\omega_N)},$$

when $\omega=\omega_N$. *Hence, the characteristic equation* (27) *has a pair of purely imaginary roots* $\pm i\omega_N$ *when*

$$\tau=\tau_N^j=\frac{2\tan^{-1}(\theta_N)+2j\pi}{\omega_N},\qquad j\in\mathbb{N}^0. \tag{31}$$

The following result guarantees that the condition $D(\omega_N)\neq 0$ can be verified in certain situations.

Lemma 16 *Let* $\alpha x_1^*\leq 4/3$ *or* $\alpha x_1^*>4/3$ *and* $\sigma_1\neq\sigma_2$. *Then* $D(\omega)\neq 0$. *In particular, we have* $D(\omega_N)\neq 0$.

Proof. Use Lemma 2.3 in Chen et al. (2013). If $\alpha x_1^*\leq 4/3$, then $b+h\leq (ad)/c$ holds true being equivalent to $\alpha x_1^*(3\alpha x_1^*-4)\leq 0$. If $\alpha x_1^*>4/3$, then $a\neq c$ being $\alpha x_1^*-1\neq 0$. Moreover, $(d/c)\,[2h-(ad)/c]-a\,[b+h-(ad)/c]\neq 0$ is equivalent to $\sigma_1\neq\sigma_2$. The conclusion is immediate. ■

Remark 17 Let $\alpha x_1^*>4/3$ and $\sigma_1=\sigma_2$. Then $D(\omega)\neq 0$ or $D(\omega)=0$. In case $D(\omega)=0$, then we have $a\neq c$ and $(2ahd)/c-d^2<0$. Hence, Lemma 2.6, 2), in Chen et al. (2013) implies that (27) has no purely imaginary roots.

Chen et al. (2013) provided a route of determining the purely imaginary roots of the characteristic equation (27) and the corresponding delay value τ. They also formulated for (27) the transversality condition for the roots moving across the imaginary axis. A complete formulation of their analysis would be cumbersome to be presented. Here, we only adapt to our model their main Theorem [Chen et al. (2013), Theorem 2.14].

Theorem 18 *Let $D(\omega)$ be defined as in* (29)*.*

1) *Let $\alpha x_1^* > 4/3$, $\sigma_1 = \sigma_2$ and $D(\omega) = 0$. Then the equilibrium (x_1^*, x_1^*) of system* (4) *is locally asymptotically stable for all $\tau \geq 0$.*

2) *Let $\alpha x_1^* > 4/3$, $\sigma_1 = \sigma_2$ and $D(\omega) \neq 0$, or $\alpha x_1^* > 4/3$ and $\sigma_1 \neq \sigma_2$, or $\alpha x_1^* \leq 4/3$ and $3\alpha x_1^* - 1 \neq 0$, $\alpha x_1^* - 1 \neq 0$ and $1 - 2\alpha x_1^* \neq 0$.*

i) *The quartic polynomial equation* (30) *has a root ω_N^2 for $\omega_N > 0$.*

ii) *The characteristic equation (27) has a pair of roots $\pm i\omega_N$ when $\tau = \tau_N^j$, $j \in \mathbb{N}^0$, with τ_N^j defined as in* (31)*.*

iii) *Let*

$$\mathcal{G}(\omega, \theta) = \left[d(1+\theta^2) + 2h(1-\theta^2)\right]\left[2\omega(1-\theta^2) + 2a\theta\right]$$
$$- [c\omega(1+\theta^2) - 4h\theta][a(1-\theta^2) - 4\omega\theta + c(1+\theta^2)].$$

If $\mathcal{G}(\omega_N, \theta_N) > 0$, then $i\omega_N$ is a simple root of the characteristic equation for $\tau = \tau_N^j$ and there exists $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ which is the unique root for $\tau \in (\tau_N^j - \varepsilon, \tau_N^j + \varepsilon)$ for some small $\varepsilon > 0$ satisfying $\nu(\tau_N^j) = 0$, $\omega(\tau_N^j) = \omega_N$ and $\nu'(\tau_N^j) > 0$.

iv) *If $\mathcal{G}(\omega_N, \theta_N) > 0$, then there exists $\tau_* > 0$ such that the equilibrium (x_1^*, x_1^*) of system* (4) *is locally asymptotically stable when $\tau \in [0, \tau_*)$ and it is unstable when $\tau \in (\tau_*, \tau_* + \varepsilon)$ for $\varepsilon > 0$ and small. Furthermore, a Hopf bifurcation occurs at $\tau = \tau_*$.*

By relying on Theorem 18, we now show through simulations another possible cause of synchronisation failure. Specifically, by starting from a stable equilibrium on the diagonal when $\sigma_1 = \sigma_2$, Figure 4 shows that a difference in these parameters produces a Hopf bifurcation that generates an attractor that does not lie on the diagonal (see Figure 4).

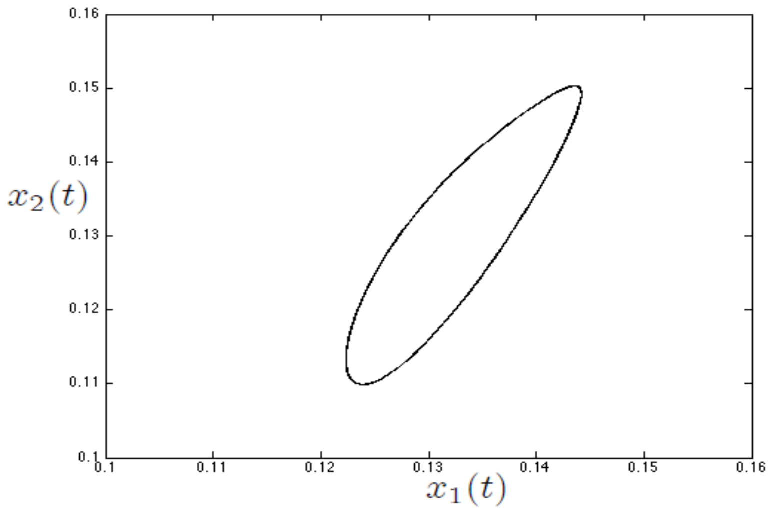

Figure 4. Limit cycle. Parameter set: $\alpha = 7$, $\sigma_1 = 0.3$, $\sigma_2 = 0.1$, $k = 0.6$ and $\tau = 0.3$.

7 Conclusions

This paper has studied a dynamic oligopoly Cournot model with two firms by extending the related discrete time literature pioneered by Bischi et al. (1998) to the case of continuous time with discrete delays. We have emphasised that in the case tradings can occur at any time, i.e. continuously (in continuous time), and there exist lags in the production process (time-to-build technology), it is possible to observe complex phenomena. Specifically, through the study of stability properties of the stationary equilibrium point, we have characterised the birth of Hopf bifurcations (cycles in production). In the particular case in which time delays in production are the same for both firms, we have also applied some recent techniques introduced by Chen et al. (2013). In order to study the occurrence of Hopf bifurcations when time delays in production are different, a possible extension of the present work may be the use of analytical and geometrical results proposed by Lin and Wang (2012). The paper has also stressed the possibility of other dynamic phenomena, such as synchronisation failures and chaotic dynamics.

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