A characterisation of duopoly dynamics with frictions in production adjustments

Luca Gori*• Luca Guerrini†• Mauro Sodini‡

June 11, 2017

Abstract

This article revisits the classical work of Puu (1991) on duopoly dynamics by gathering two distinct aspects of the functioning of markets: production of goods requires time and is subject to some gestation lags, but trading takes place continuously. Dynamics are characterised by a two-dimensional system of delay differential equations. The main aim of this work is to show that regular and non-regular fluctuations may emerge endogenously because of the existence of heterogeneous interacting agents that choose production period by period in a myopic way. Chaotic dynamics in the discrete-time model of Puu (1991) appear to be close enough to the origin of axes. In contrast, in our continuous-time version of the model with discrete delays, the dynamic system is more suitable of generating complex dynamics far enough from the origin when marginal costs vary. This is because of the role played by time delays and inertia. From a mathematical point of view, we show the existence of Hopf bifurcations and detect how time delays and inertia affect the stability of the system by using the recent techniques of stability crossing curves introduced by Gu et al. (2005) and generalized by Lin and Wang (2012). The article also provides some findings about global bifurcations and chaotic dynamics by combining analytical studies and simulation exercises.

Keywords Chaos; Cournot duopoly; Stability crossing curves; Time delays

JEL Classification C61; C62; D43; L13

*L. Gori, Department of Political Science, University of Genoa, Piazzale E. Brignole, 3a, I–16125 Genoa (GE), Italy, e-mail: luca.gori@unige.it or dr.luca.gori@gmail.com, tel.: +39 010 209 95 03, fax: +39 010 209 55 36.

†L. Guerrini (corresponding author), Department of Management, Polytechnic University of Marche, Piazza Martelli 8, I–60121, Ancona (AN), Italy, e-mail: luca.guerrini@univpm.it, tel.: +39 071 22 07 055.

‡M. Sodini, Department of Economics and Management, University of Pisa, Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy, e-mail: mauro.sodini@unipi.it, tel.: +39 050 22 16 234, fax: +39 050 22 10 603.

1 Introduction

As part of the economic dynamics literature, the work of Puu (1991) represents one of the first attempts to mimic realistic behaviors of production dynamics by focusing on the interaction between economic agents (firms) in a deterministic (quantity-setting) duopoly without exogenous external shocks. The contribution of Puu was to provide a tractable special case (with unit-elastic market demand and constant marginal costs) of the Cournot model in which a simple adjustment rule can produce complex dynamics, as Rand (1978) foresaw. Even at the time of writing, and despite some simplifying assumptions on market knowledge and expectations formation mechanisms of each competitor, the model of Puu is still a base where comparing research on nonlinear oligopolies. For instance, Cánovas et al. (2008) provided an important extension of Puu by taking into account non-negativity constraints on prices, quantities and profits.

The model of Puu and its subsequent extensions have been developed in a discrete-time set up. However, in recent years, a burgeoning body of research has focused on the study and formalization of nonlinear oligopolies described by hybrid dynamic systems, that is systems that exhibit both continuous and discrete time dynamic behaviors (Matsumoto and Szidarovszky, 2010a, 2010b, 2015; Matsumoto et al., 2011; Gori et al., 2015a). The use of hybrid dynamic systems is actually receiving in depth attention in several disciplines, ranging from the analysis of problems concerning the diffusion of infectious diseases (Monica and Pitchaimani, 2016) to questions related to predator-prey models (Chen and Chen, 2011), reaction-diffusion systems (Chen and Shi, 2013) and cobweb models (Gori et al., 2015b). By turning on to the analysis of economic models with imperfect competition and time delays, Matsumoto and Szidarovszky (2010a) provide a study of a duopoly with small information delays in the reaction curves of firms. That work has been generalized by Matsumoto and Szidarovszky (2010b) and Matsumoto et al. (2011). In the former model, the authors study the case in which the dynamics of the economy are characterized by arbitrary values of time delays in the reaction curves of firms, whereas, in the latter, it is assumed that each firm has an information lag even in comparison with its own production. More recently, Matsumoto and Szidarovszky (2015) recounted the issue studied by Matsumoto et al. (2011) and introduced an adjustment mechanism (based on marginal profits) similar to the one proposed in discrete-time by Bischi et al. (1998). Finally, Gori et al. (2015a) have proposed a different way in which time delays can affect duopoly dynamics. In particular, by following Berezowski (2001), the authors have shown that the existence of frictions in production adjustments and the assumption of markets in which trading takes place continuously induces a sharp change in the structure of the dynamic system of a duopoly à la Bischi et al. (1998).

The present article adopts the approach proposed by Berezowski (2001) in nonlinear duopolies where the adjustment mechanism is based on best reply dynamics or adaptive dynamics. In comparison with the companion article of Gori et al. (2015a), this work introduces the recent techniques developed by Gu et al. (2005) and Lin and Wang (2012), useful to characterize the local stability properties of a dynamic system. It also provides some findings about global bifurcations and chaotic dynamics by using both analytical results and simulation exercises.

The rest of the article proceeds as follows. Section 2 briefly sets up the basic (static) model. Section 3 describes the best reply dynamic setting in a continuous time framework with discrete delays. Section 4 extends the model to the case of adaptive dynamics and applies some recent techniques (stability crossing curves) proposed by Gu et al. (2005) and Lin and Wang (2012). Section 5 outlines the conclusions and briefly discusses future research.

2 The static model

By following Puu (1991), we posit the reaction functions

$$x = \sqrt{\frac{y}{a}} - y, \tag{1}$$

and

$$y = \sqrt{\frac{x}{b}} - x, \tag{2}$$

which can be derived from a market in which two quantity-setting (profit-maximizing) firms (namely, firm x and firm y) produce a homogeneous good with constant marginal costs of production $a > 0$ and $b > 0$, respectively, facing a unit elastic demand curve $p = 1/Q$, where $Q = x + y$ (x and y are non-negative quantities) is the total supply and $p > 0$ is the marginal willingness to pay of consumers. This market has a unique locally stable Cournot-Nash equilibrium, that is:

$$x_* = \frac{b}{(a+b)^2} \text{ and } y_* = \frac{a}{(a+b)^2}. \tag{3}$$

3 Best reply dynamics

One of the most important characteristics in the approach proposed by Puu (1991)[1] is the no simultaneous occurrence between production decisions and their implementation (time to build assumption) in situations that do not necessarily start from the Cournot equilibrium. These kind of models have an important limitation as they all consider that trading takes place based on the same time schedule of production. However, this becomes an unlikely assumption in the cases in which the technology requires a long time to bring production to completion. In other words, this hypothesis implies that, within each time interval between two different productions, new products are not brought to the market. To overcome this problem, a good

[1]This approach is also used in other works with different assumptions about the degree of knowledge of economic agents (Bischi et al., 1998).

compromise is to build on a model including some characteristics of discrete-time models (time lags) together with characteristics of continuous-time models (i.e., new production activities begin - and trading takes place - continuously). There exist several ways to translate a model originally expressed in a discrete time set up in a continuous time framework (with delays) by preserving some assumptions characterizing discrete time models, such as, for instance, the non-coexistence between the benchmark economic processes (e.g., production, trading and so on). To this purpose, in the rest of the article we will use the approach proposed by Berezowski (2001), formerly adopted to describe a "physical process of defined inertia" (Berezowski, 2001, p. 84) and applied to economic models by Matsumoto and Szidarovszky (2014) and Gori et al. (2015a).

First of all, we now generalize the model of Puu. Firms x and y have a global knowledge of the market (that is, they know the market demand) but they are naïve players. This implies that, at every time t, each firm expects that rival's production is equal to the quantity produced in the last period. In a continuous time model, this implies that $\Pi_x(x(t), y^e(t)) = \Pi_x(x(t), y(t-\tau_1))$ and $\Pi_y(y(t), x^e(t)) = \Pi_y(y(t), x(t-\tau_2))$, where $\tau_1 \geq 0$ and $\tau_2 \geq 0$ are discrete delays. These delays are assumed to be different to capture heterogeneities related to the specific technology used by every firm (gestation lags). In this context, the model of Puu with best replies becomes the following:

$$\begin{cases} x(t) = \sqrt{\dfrac{y(t-\tau_1)}{a}} - y(t-\tau_1), \\ y(t) = \sqrt{\dfrac{x(t-\tau_2)}{b}} - x(t-\tau_2), \end{cases} . \tag{4}$$

By applying the method proposed by Berezowski (2001) to (4), the two-dimensional dynamic system becomes the following:

$$\begin{cases} \sigma_1 \dot{x}(t) + x(t) = \sqrt{\dfrac{y(t-\tau_1)}{a}} - y(t-\tau_1), \\ \sigma_2 \dot{y}(t) + y(t) = \sqrt{\dfrac{x(t-\tau_2)}{b}} - x(t-\tau_2), \end{cases} , \tag{5}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are a measure of the inertia in the adjustment mechanism, $\dot{x}(t) = \partial x/\partial t$ and $\dot{y}(t) = \partial y/\partial t$. Some clarifications about this method are now in order. From a mathematical point of view, we note that for $\sigma_1 = \sigma_2 = 0$ and $\tau_1 = \tau_2 = 1$ system (5) replicates the two-dimensional discrete time map given by Eqs. (12) and (13) of Puu (1991). This implies that, for values of σ_i $(i = \{1, 2\})$ close enough to zero, the dynamic properties of (5) resemble the dynamic properties of the model of the discrete time model of Puu in the case of best replies. In particular, at every time $t - \tau_i$ (with τ_i given) firm i plans the production that will be available at time t by using the best reply rule with static expectations (that is, it maximizes profits by assuming that the rival does not modify its production plan). At time t, each firm is not able to realize perfectly the production plan arranged at time $t-\tau_i$ (for instance, because of frictions due to the long time required for production).[2] Specifically, we assume that, in a phase of output growth, $\dot{x}(t) > 0$, firms are not able to realize a sufficiently large amount of products, meaning that realized production is smaller than planned production. The opposite holds in a phase of recession ($\dot{x}(t) < 0$).[3] Consequently, each firm adjusts its quantity in the direction of fixing mistakes with respect to the target. The reciprocals of σ_1 and σ_2 capture the intensity of the instantaneous change in quantities of firm x and firm y, respectively, related to the mismatch between planned and realized production, as is shown in the following system:[4]

$$\begin{cases} \dot{x}(t) & = \dfrac{1}{\sigma_1}\left(\sqrt{\dfrac{y(t-\tau_1)}{a}} - y(t-\tau_1) - x(t)\right) \\ \dot{y}(t) & = \dfrac{1}{\sigma_2}\left(\sqrt{\dfrac{x(t-\tau_2)}{b}} - x(t-\tau_2) - y(t)\right) \end{cases} \qquad a > 0,\ b > 0,\ \sigma_1 > 0,\ \sigma_2 > 0, \tag{6}$$

3.1 Case $\tau_1 \geq 0$ and $\tau_2 \geq 0$

In this section and the next ones, we will concentrate on the study of the mathematical properties of the model described in the previous section.

Lemma 1 *System* (6) *has a unique positive equilibrium* (x_*, y_*), *where*

$$x_* = \frac{b}{(a+b)^2} \text{ and } y_* = \frac{a}{(a+b)^2}.$$

Proof. Setting $\dot{x}(t) = \dot{y}(t) = 0$, $x(t-\tau_2) = x(t) = x_*$ and $y(t-\tau_1) = y(t) = y_*$ for all t, we find that an equilibrium of system (6) coincides with the Nash equilibrium of the static game defined in (3). ■

The linearization of (6) at (x_*, y_*) is

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\sigma_1}(x(t) - x_*) + \dfrac{b-a}{2a\sigma_1}(y(t-\tau_1) - y_*), \\ \dot{y}(t) = -\dfrac{1}{\sigma_2}(y(t) - y_*) + \dfrac{a-b}{2b\sigma_2}(x(t-\tau_2) - x_*). \end{cases} \tag{7}$$

Thus, the characteristic equation associated with (7) is

$$\lambda^2 + \left(\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2}\right)\lambda + \frac{1}{\sigma_1\sigma_2} + \frac{(a-b)^2}{4ab\sigma_1\sigma_2}e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{8}$$

[2] Notice that we are assuming that the model does not include problems about inventories.

[3] The existence of delays and mismatch between choices and their achievements is recognized to be a central issue in management science (Harrison and van Hoek, 2008).

[4] We note that we are considering the simplifying assumption that changing the quantities produced do not produce endogenous adjustment costs for firms. However, a sufficiently large value of σ_i (roughtly speaking, this implies that $\dot{x}(t)$ is close to zero) describes a situation in which there are strong frictions in production adjustments. In this regard, it will be interesting to study models that incorporate non-constant adjustment costs. See Bertola and Caballero (1990) for a general treatment on this issue.

Let $\tau = \tau_1 + \tau_2$. Then (8) can be written as

$$\lambda^2 + k_1\lambda + k_2 + k_3 e^{-\lambda\tau} = 0, \tag{9}$$

where

$$k_1 = \frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} > 0, \quad k_2 = \frac{1}{\sigma_1\sigma_2} > 0, \quad k_3 = \frac{(a-b)^2}{4ab\sigma_1\sigma_2} \geq 0.$$

Lemma 2 *Let* $\tau = 0$. *Then the equilibrium point of system* (6) *is locally asymptotically stable.*

Proof. In the absence of delay, Eq. (8) becomes

$$\lambda^2 + k_1\lambda + k_2 + k_3 = 0.$$

Since both coefficients are positive, the real parts of the eigenvalues are negative. Hence, the statement holds. ■

It is obvious that $\lambda = 0$ cannot be a root of Eq. (9). In order to understand the stability switches of system (6), we need to determine the critical values of the time lag at which the characteristic equation may have a pair of conjugate pure imaginary roots. If $i\omega$ is a root of the characteristic equation (9) for $\omega > 0$ then

$$-\omega^2 + i\omega k_1 + k_2 + k_3(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we have

$$\omega^2 - k_2 = k_3\cos\omega\tau, \quad k_1\omega = k_3\sin\omega\tau, \tag{10}$$

which lead to

$$\omega^4 + \left(k_1^2 - 2k_2\right)\omega^2 + k_2^2 - k_3^2 = 0, \tag{11}$$

where

$$k_1^2 - 2k_2 = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2} > 0 \quad \text{and} \quad k_2^2 - k_3^2 = \frac{16a^2b^2 - (a-b)^4}{16a^2b^2\sigma_1^2\sigma_2^2}.$$

It is clear that, if $k_2^2 - k_3^2 \geq 0$, then Eq. (11) has no positive root. Thus, the characteristic equation (9) does not have purely imaginary roots. Now, $k_2^2 - k_3^2 \geq 0$ if $b^2 - 6ab + a^2 \leq 0$, that is if $\left(3 - 2\sqrt{2}\right)a \leq b \leq \left(3 + 2\sqrt{2}\right)a$. In particular, this holds true when $a = b$. On the other hand, if $k_2^2 - k_3^2 < 0$, then Eq. (11) has only one positive root

$$\omega_+ = \sqrt{\frac{-k_1^2 + 2k_2 + \sqrt{\left(k_1^2 - 2k_2\right)^2 - 4\left(k_2^2 - k_3^2\right)}}{2}}. \tag{12}$$

Notice that $a = b$ implies $k_3 = 0$, so that $k_2^2 - k_3^2 < 0$ needs $a \neq b$. In this case, the characteristic equation (9) has purely imaginary roots when τ takes certain values. These critical values τ_j^+

($j = 0, 1, 2, ...$) of τ can be determined from (10). Since $k_1\omega_+/k_3 > 0$, we have $\sin\omega_+\tau > 0$. Consequently,

$$\tau_j^+ = \frac{1}{\omega_+}\cos^{-1}\left\{\frac{\omega_+^2 - k_2}{k_3}\right\} + \frac{2j\pi}{\omega_+}. \tag{13}$$

Furthermore, $\lambda = i\omega_+$ is a simple root of (9). If it were not simple, then differentiating (9) with respect to λ, and using (9), we would arrive to $(2 + k_1\tau_j^+)\omega_+ = 0$, which is a contradiction.

The above analysis can be summarized in the following result.

Lemma 3 *1) If $\left(3 - 2\sqrt{2}\right)a \le b \le \left(3 + 2\sqrt{2}\right)a$ holds, then all roots of equation (9) have negative real parts for all $\tau \ge 0$. 2) If $0 < b < \left(3 - 2\sqrt{2}\right)a$ or $b > \left(3 + 2\sqrt{2}\right)a$, together with $a \ne b$ hold, then Eq. (9) has a pair of simple purely imaginary roots $\pm i\omega_+$ at $\tau = \tau_j^+$, $j = 0, 1, 2, ...$*

Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ be the root of Eq. (9) satisfying $\nu(\tau_j^+) = 0$ and $\omega(\tau_j^+) = \omega_+$, with ω_+ and τ_j^+ defined in (12) and (13), respectively. Substituting $\lambda(\tau)$ into (9) and taking the derivative with respect to τ, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + k_1)\,e^{\lambda\tau}}{\lambda k_3} - \frac{\tau}{\lambda},$$

which, together with (9), leads to

$$\begin{aligned} sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau}\right|_{\tau=\tau_j^+}\right\} &= sign\left\{Re\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\tau=\tau_j^+}\right\} = sign\left\{k_1^2 - 2k_2 + 2\omega_+^2\right\} \\ &= sign\left\{\sqrt{\left(k_1^2 - 2k_2\right)^2 - 4\left(k_2^2 - k_3^2\right)}\right\} > 0. \end{aligned}$$

Hence, the crossing of the imaginary axis is from left to right as τ increases, thus resulting in the loss of stability.

Then, we have the following results about stability of the positive equilibrium of system (6) and Hopf bifurcations.

Theorem 4 *Let τ_j^+ ($j = 0, 1, 2, ...$) be defined as in (13) and $\tau = \tau_1 + \tau_2$.*

1) *If $\left(3 - 2\sqrt{2}\right)a \le b \le \left(3 + 2\sqrt{2}\right)a$ holds, then the positive equilibrium (x_*, y_*) of (6) is locally asymptotically stable for all $\tau \ge 0$.*

2) *If $0 < b < \left(3 - 2\sqrt{2}\right)a$ or $b > \left(3 + 2\sqrt{2}\right)a$, with $a \ne b$, hold, then the positive equilibrium (x_*, y_*) of (6) is locally asymptotically stable when $\tau \in [0, \tau_0^+)$ and unstable when $\tau > \tau_0^+$. Moreover, (6) undergoes Hopf bifurcations at (x_*, y_*) when $\tau = \tau_j^+$.*

The results of Theorem 4 are illustrated in Figure 1 that shows in (a, b) plane the configurations of production costs such that the stationary equilibrium is locally asymptotically stable (the region within the cone) or unstable (the region outside the cone). Therefore, a necessary condition, such that the stationary equilibrium loses stability, is that production costs of the two firms are sufficiently different. We also note that σ_1 and σ_2 play a stabilizing role in the sense that given the same delays τ_1 and τ_2, an increase in σ_1 or σ_2 enlarges the region (a, b) within the cone. From an economic point of view, a higher value of σ_i $(i = \{1, 2\})$ implies that firm i turns out to be less responsive to the actions of its rival, so that the system more likely approaches to stationary solutions.

Now, through simulations it is possible to analyze the behavior of the system for parameter values far enough from the configuration that generates the Hopf bifurcation. Specifically, we study the long-term effects of the system when τ_1 (Figure 2(a)) or σ_1 (Figure 3) changes via bifurcation diagrams. The bifurcation diagrams plotted in Figures 2(a) and 3 are depicted by taking into account a typical trajectory convergent towards the attractor of the system. The figures show (after a long enough transient) the local maximum and local minimum values when the corresponding bifurcation parameter changes. For instance, when a bifurcation diagram shows the existence of a unique line (as long as the bifurcation parameter varies), the system is stable and converges to the Nash equilibrium. In the portion of the graph showing two lines, a generic trajectory of the system (starting from close enough to the stationary equilibrium) converges to a cycle born via a Hopf bifurcation. Instead, portions of the graph in which more than two lines coexist describe situations in which the attractor projected in the pseudo phase plane $(x(t), y(t))$ changes its shape and also self-intersections can be observed (Figure 2(b)). Then the dynamics of the system are characterized by several local maximum and minimum values. The existence of discontinuities in the bifurcation diagram, when the bifurcation parameter varies, is because some local maximum and minimum values are created far enough away from the already existing ones. This is pointed out in Figures 2(c). Specifically, Figure 2(a) shows that τ_1 plays a destabilizing role. In fact, for $\tau_1 < \tau_0^+$, the stationary equilibrium is locally asymptotically stable, whereas for values of τ_1 larger than but close enough to τ_0^+, dynamics are oscillatory, showing a unique maximum value and a unique minimum value. By considering values of τ_1 larger than around 1.11, the typical trajectory convergent towards the attractor is characterized by the existence of several maximum and minimum values. We note that, for large enough values of τ_1 the ω-limit set of the system is a chaotic attractor, as is shown in Figure 4(a), Figure 4(b) and Figure 4(c).

The parameter space for which the discrete-time model of Puu with best reply adjustment generates chaotic dynamics is relatively small as compared with the values of a and b (marginal costs) generating feasible trajectories (see Puu, 1991, Figure 3, p. 579). In addition, when chaotic dynamics in Puu appear, they are close enough to the origin of axes (see Puu, 1991, Figure 4, p. 579). From an economic point of view, this last point implies that trajectories are characterized by terrific changes in the market price (i.e., there exist phases in which the price becomes very large).[5] In contrast, in the present model the dynamic system is more suitable of generating complex dynamics. In addition, an attractor far enough away from the origin can capture these dynamics. This is because this version of the model is more general including 1) heterogeneous parameters related to frictions in the adjustment mechanism of quantities, and 2) heterogeneous time delays. This result is pointed out in Figure 4(a) and Figure 4(b). The former figure (with a relatively low degree of inertia) shows a chaotic attractor looking like the chaotic attractor of Puu, which is close enough to the origin; the latter figure (with a relatively high degree of inertia) shows an example in which a chaotic attractor is far enough away from the origin.

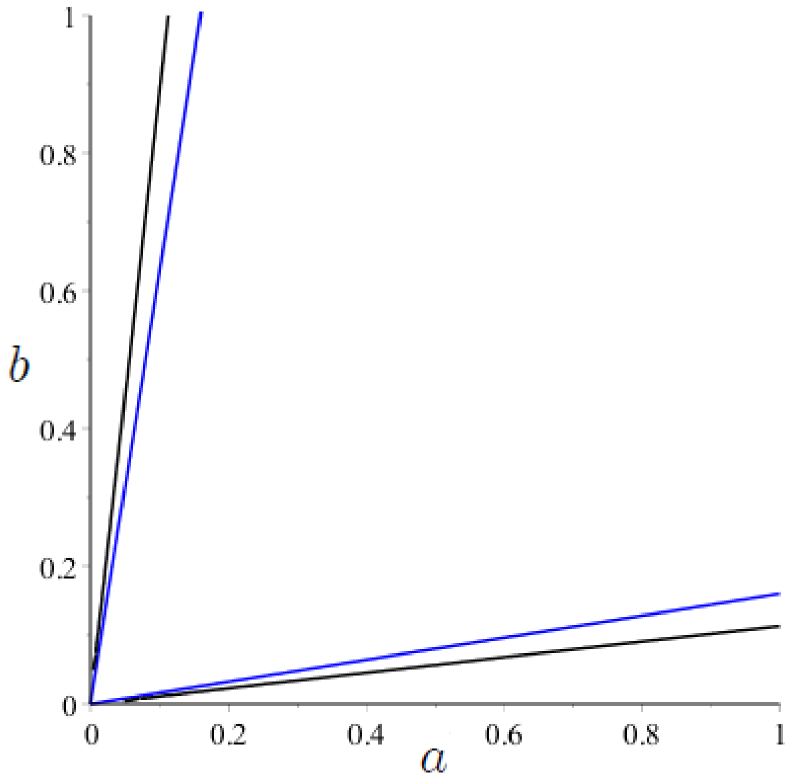
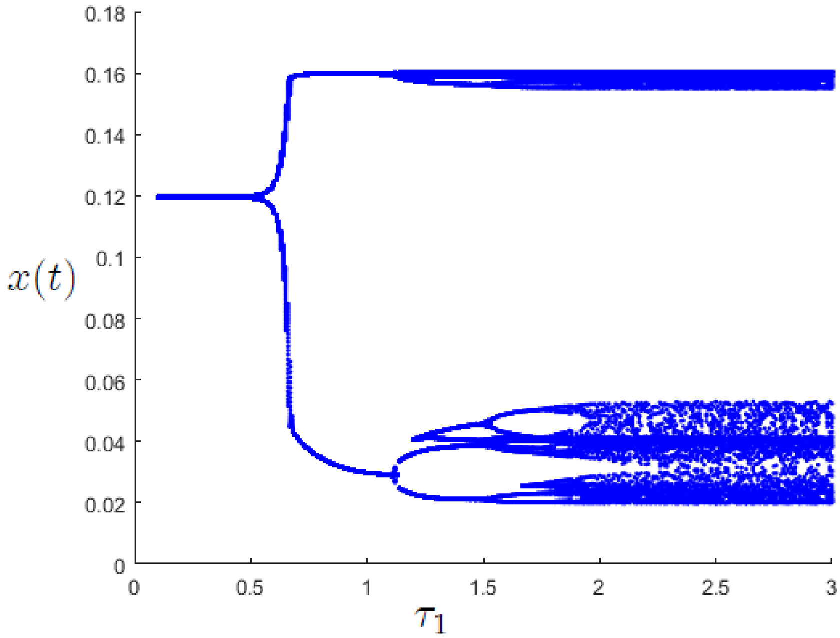
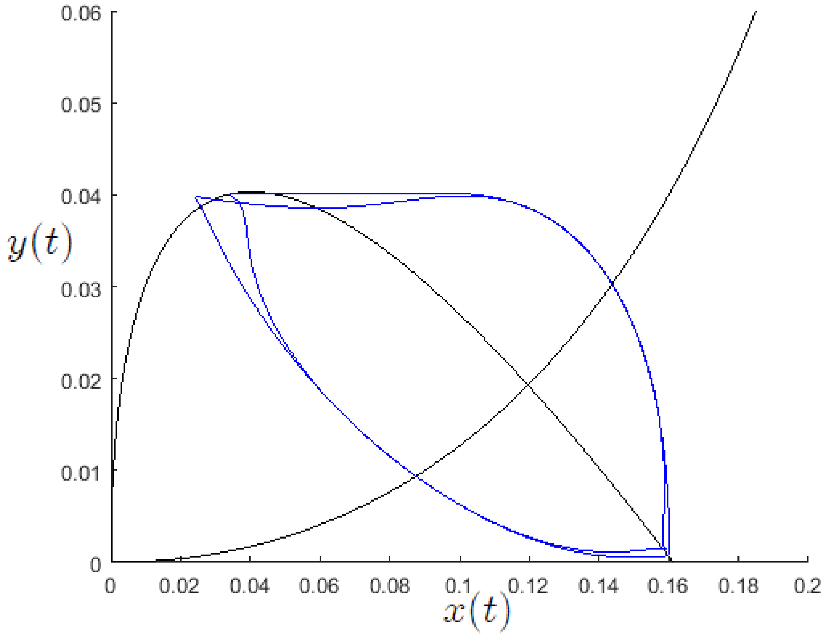


Figure 1. The two blue half lines that exit the origin define the borders of the stability region in (a, b) plane in the original model of Puu (1991), where $\tau_1 = \tau_2 = 1$ and $\sigma_1 = \sigma_2 = 0$. The stationary equilibrium is locally asymptotically stable for all the couples (a, b) within the cone. The two black half lines that exit the origin define the borders of the stability region in (a, b) plane in the model with time delays $(\tau_1 = \tau_2 = 1)$, for a given positive value of the degree of inertia $(\sigma_1 = \sigma_2 = 1)$. The figure shows that given the same values of time delays, a positive value of the degree of inertia tends to stabilize the stationary equilibrium.

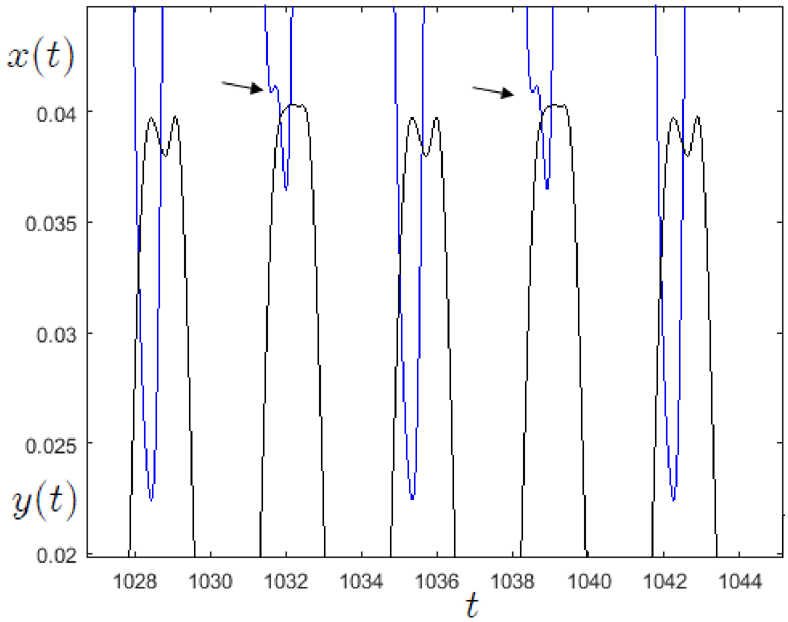
[5]The authors thank an anonymous reviewer for pointing this out.


0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
$x(t)$
0
0.5
1
1.5
2
2.5
3
τ_1

(a)


0.06
0.05
0.04
0.03
0.02
0.01
0
$y(t)$
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
$x(t)$

(b)



(c)

Figure 2. (a) Bifurcation diagram for τ_1. (b) Attractor of the system for $\tau_1 = 1.15$. (c) Time series for $\tau_1 = 1.22$. The figure shows the birth of new local maximum and minimum values not related to the already existing ones (see the arrows in the figure). Parameter values: $\sigma_1 = 0.11$, $\sigma_2 = 0.1$, $a = 1$, $b = 6.2$ and $\tau_2 = 0.3$.

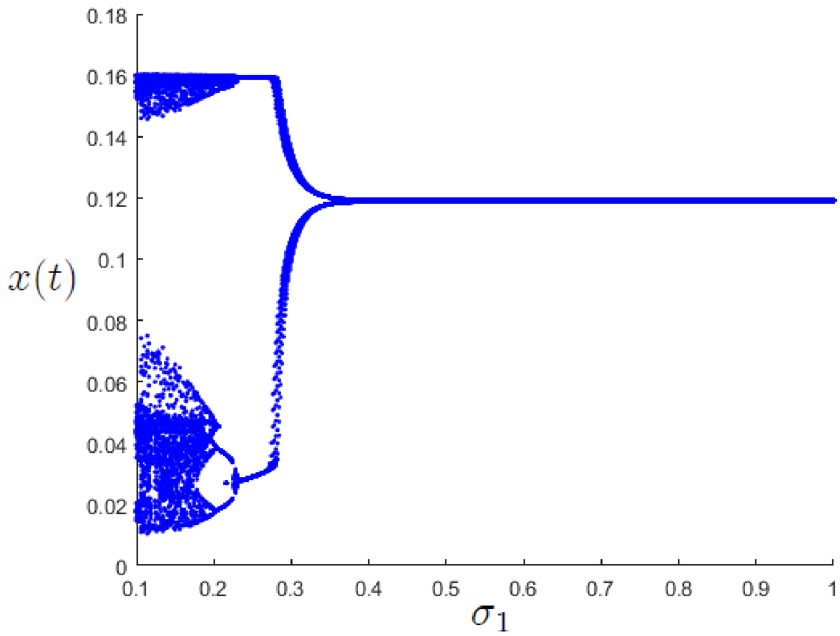
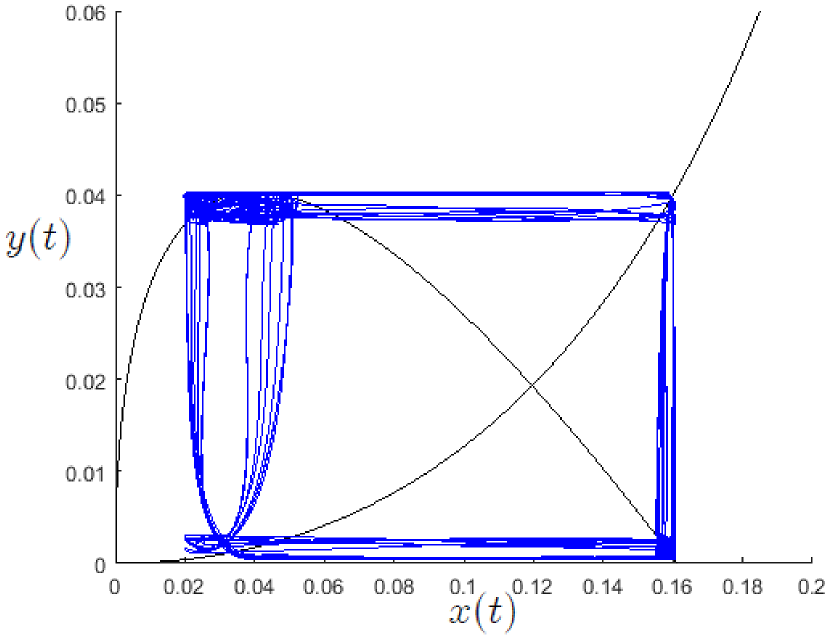
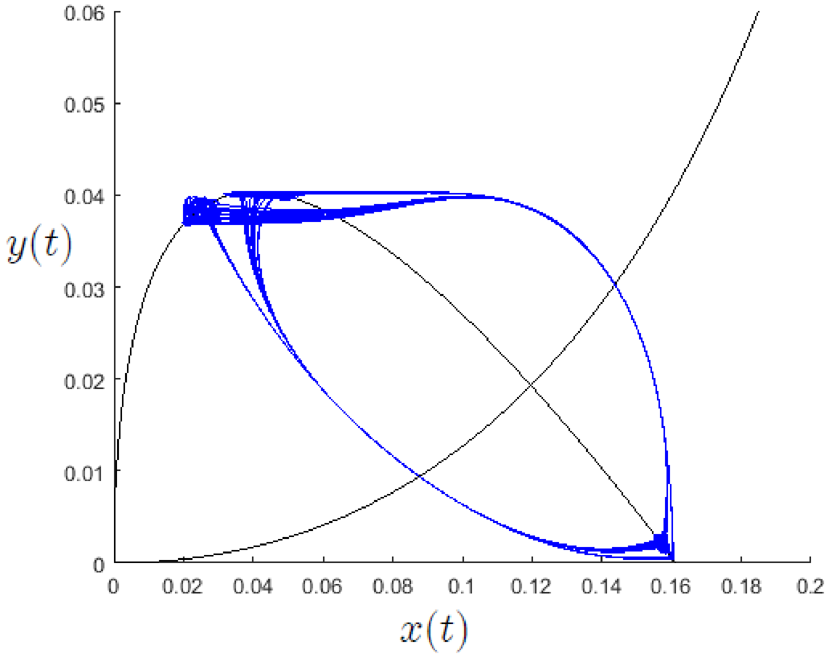


Figure 3. Bifurcation diagram for σ_1. Parameter values: $\sigma_2 = 0.1$, $a = 1$, $b = 6.24$, $\tau_1 = 1$ and $\tau_2 = 1$. The braided appearance in the lower branch almost at $\sigma_1 = 0.3$ is due to numerical approximations of maximum and minimum values.



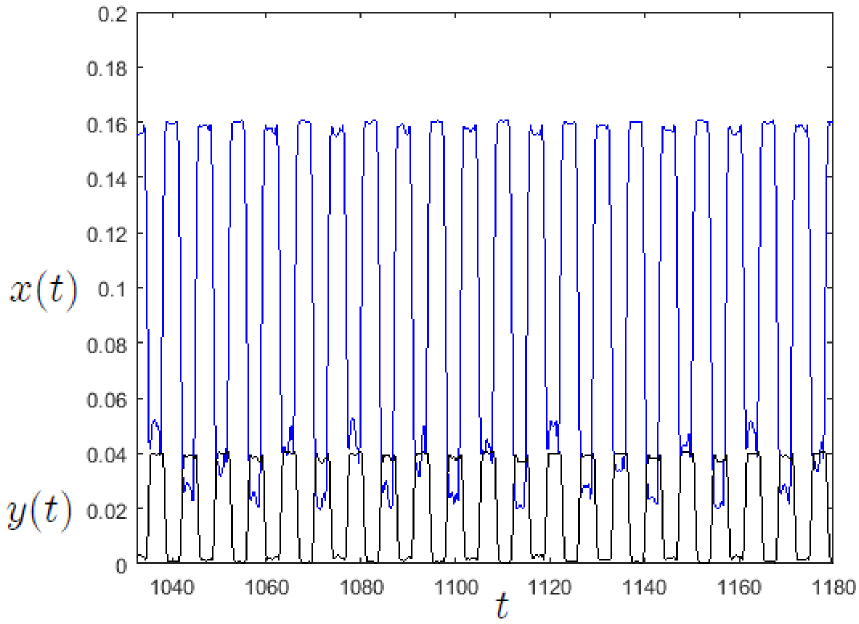
0.06
0.05
0.04
0.03
0.02
0.01
0
$y(t)$
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
$x(t)$

(a)



0.06
0.05
0.04
0.03
0.02
0.01
0
$y(t)$
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
$x(t)$

(b)



(c)

Figure 4. (a) Chaotic attractor in the pseudo phase plane $(x(t), y(t))$ with values of σ_i close to 0 ($\sigma_1 = 0.015$ and $\sigma_2 = 0.015$).[6] (b) Chaotic attractor in the pseudo phase plane $(x(t), y(t))$ with values of σ_i far enough away from 0 ($\sigma_1 = 0.11$ and $\sigma_2 = 0.1$). The dynamics are bounded in a region far enough away from the origin. (c) Time series of production ($\sigma_1 = 0.11$ and $\sigma_2 = 0.1$). Other parameter values: $a = 1$, $b = 6.2$, $\tau_1 = 2$ and $\tau_2 = 0.3$.

In the light of the results of Section 3, it is possible to infer some basic intuitions that are in line with other works that study the role of frictions in economic adjustment mechanisms (Matsumoto and Szidarovszky, 2014; Gori et al., 2015a). In particular, when inertia is introduced in an adjustment rule it is possible that there exist some market characteristics such that delays cannot destabilize the equilibrium. Instead, if other technological conditions are fulfilled, an increase in the delay tends to destabilize the equilibrium and produce complex dynamics (see Theorem 4 and Figures 2(a)-4(c)). Conversely, frictions work out in the opposite direction than time delays letting the system moving back towards the equilibrium.

4 Adaptive dynamics

This section is devoted to the study of a continuous time version with delays of the model with adaptive dynamics proposed by Puu in the second part of his work. Specifically, in addition to

[6] For values of σ_i much closer to zero, the dynamics of our delay-differential system become very similar to ones of the discrete-time model of Puu (1991), and a cluster can be obserbed close to the origin.

the best reply dynamics, Puu also proposes an adjustment mechanism of quantities with which each competitor does not immediately jump to its new optimum at every step, but gradually adjusts its own (previous) decision in the direction of the new optimum. A similar method of production adjustment has been proposed by Onozaki et al. (2003) and Bischi and Cerboni Baiardi (2015). We note that different from the model based on best replies of firms, the techniques recently proposed by Gu et al. (2005) and Lin and Wang (2012) will be useful later in this article to analyze the model with adaptive dynamics.

In a similar way to what was previously done in the present work, we now introduce the analog version of the dynamic system (23) and (24) of Puu (1991, pp. 579-580) under the assumption of time-to-build technology, existence of frictions in the ability of firms to adjust production towards the target and trading that takes place continuously in the market. Therefore, by applying the method proposed by Berezowski (2001) the adaptive dynamics model à la Puu is described by the following system:

$$\begin{cases} \sigma_1 \dot{x}(t) + x(t) = x(t-\tau_1)(1-\theta) + \theta\left[\sqrt{\dfrac{y(t-\tau_1)}{a}} - y(t-\tau_1)\right] \\ \sigma_2 \dot{y}(t) + y(t) = y(t-\tau_2)(1-\theta) + \theta\left[\sqrt{\dfrac{x(t-\tau_2)}{b}} - x(t-\tau_2)\right] \end{cases}, \tag{14}$$

where $0 < \theta < 1$ captures the weight of the production adjustment of each firm towards the new optimum.

Let us now rewrite system (14) as follows for mathematical convenience:

$$\begin{cases} \dot{x}(t) = -\dfrac{x(t)}{\sigma_1} + \dfrac{x(t-\tau_1)(1-\theta)}{\sigma_1} + \dfrac{\theta}{\sigma_1}\left[\sqrt{\dfrac{y(t-\tau_1)}{a}} - y(t-\tau_1)\right] \\ \dot{y}(t) = -\dfrac{y(t)}{\sigma_2} + \dfrac{y(t-\tau_2)(1-\theta)}{\sigma_2} + \dfrac{\theta}{\sigma_2}\left[\sqrt{\dfrac{x(t-\tau_2)}{b}} - x(t-\tau_2)\right] \end{cases}. \tag{15}$$

Then, we have the following lemma.

Lemma 5 *There exists a unique positive equilibrium* (x_*, y_*) *for model* (15), *where*

$$x_* = \frac{b}{(a+b)^2} \text{ and } y_* = \frac{a}{(a+b)^2}. \tag{16}$$

Proof. An equilibrium point for system (15) is obtained by setting $\dot{x}(t) = \dot{y}(t) = 0$, $x(t-\tau_i) = x(t) = x_*$ and $y(t-\tau_i) = y(t) = y_*$ for all t, $i = 1, 2$. This means that the equilibrium point solves the following equations:

$$\sqrt{\frac{y_*}{a}} = \sqrt{\frac{x_*}{b}} = x_* + y_*.$$

The conclusion now follows by knowing that from the previous equation we may easily find that

$$y_* = \frac{a}{b}x_* \text{ and } x_* = \frac{(a+b)^2}{b}x_*^2.$$

■

Next, we investigate the effect of time delays on the dynamics of (15). As is known, the stability of an equilibrium point and local Hopf bifurcations involve the distribution of roots of the corresponding characteristic equation. The linearization of system (15) at (x_*, y_*) is given by

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\sigma_1}\left(x(t)-x_*\right) + \dfrac{1-\theta}{\sigma_1}(x(t-\tau_1)-x_*) + \dfrac{\theta(b-a)}{2a\sigma_1}(y(t-\tau_1)-y_*), \\ \dot{y}(t) = -\dfrac{1}{\sigma_2}\left(y(t)-y_*\right) + \dfrac{\theta(a-b)}{2b\sigma_2}(x(t-\tau_2)-x_*) + \dfrac{1-\theta}{\sigma_2}(y(t-\tau_2)-y_*). \end{cases} \tag{17}$$

The associated characteristic equation of system (17) takes the form

$$\begin{vmatrix} -\dfrac{1}{\sigma_1} - \lambda + \dfrac{1-\theta}{\sigma_1}e^{-\lambda\tau_1} & \dfrac{\theta(b-a)}{2a\sigma_1}e^{-\lambda\tau_1} \\ \dfrac{\theta(a-b)}{2b\sigma_2}e^{-\lambda\tau_2} & -\dfrac{1}{\sigma_2} - \lambda + \dfrac{1-\theta}{\sigma_2}e^{-\lambda\tau_2} \end{vmatrix} = 0,$$

namely

$$P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2} + P_3(\lambda)e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{18}$$

where

$$\begin{aligned} P_0(\lambda) &:= \lambda^2 + \lambda\left(\frac{\sigma_1+\sigma_2}{\sigma_1\sigma_2}\right) + \frac{1}{\sigma_1\sigma_2}, \\ P_1(\lambda) &:= \frac{-(1-\theta)\left(1+\lambda\sigma_2\right)}{\sigma_1\sigma_2}, \\ P_2(\lambda) &:= \frac{-(1-\theta)\left(1+\lambda\sigma_1\right)}{\sigma_1\sigma_2}, \\ P_3(\lambda) &:= \frac{(a^2+b^2)\theta^2 + 2ab(\theta^2-4\theta+2)}{4ab\sigma_1\sigma_2}. \end{aligned}$$

By taking into account the results of Lemma 2 and knowing that θ is positive, it follows that:

Remark 6 *The equilibrium point* (x_*, y_*) *for model* (15) *is locally asymptotically stable in the absence of delays.*

To consider the effects of the two time delays on the stability of model (15), we need to analyze the boundary of the stability region determined by the equations $\lambda = 0$ and $\lambda = i\omega$ $(\omega > 0)$. We note that the case $\lambda = 0$ cannot occur since, in this case, we get the absurd $a + b = 0$ in (18). Therefore, only the case $\lambda = i\omega$ $(\omega > 0)$ has to be analyzed.

4.1 Case $\tau_1 = 0$ and $\tau_2 > 0$

In this section we analyze the case in which one of the two delays (τ_1) is equal to zero, whereas the other one (τ_2) is positive. We will show some results about local stability of the stationary equilibrium and the arising of the Hopf bifurcation.

Since $\tau_1 = 0$, Eq. (18) becomes

$$\lambda^2 + A\lambda + B + (C\lambda + D)\, e^{-\lambda\tau_2} = 0. \tag{19}$$

where

$$A = \frac{\sigma_1 + \theta\sigma_2}{\sigma_1\sigma_2} > 0, \quad B = \frac{\theta}{\sigma_1\sigma_2} > 0, \quad C = -\frac{1-\theta}{\sigma_2} < 0, \quad D = \frac{\theta\left[-4ab + \theta(a+b)^2\right]}{4ab\sigma_1\sigma_2}. \tag{20}$$

Assume that Eq. (19) has a purely imaginary solution of the form $\lambda = i\omega$ ($\omega > 0$). Substituting it into (19) and separating the real and imaginary parts, we have

$$-\omega^2 + iA\omega + B + (iC\omega + D)e^{-i\omega\tau_2} = 0,$$

or equivalently

$$-\omega^2 + B = -D\cos\omega\tau_2 - C\omega\sin\omega\tau_2, \qquad A\omega = -C\omega\cos\omega\tau_2 + D\sin\omega\tau_2. \tag{21}$$

Squaring each equation in (21) and taking the sum, we obtain the following equation of ω^2:

$$\omega^4 - \left(C^2 + 2B - A^2\right)\omega^2 + B^2 - D^2 = 0. \tag{22}$$

From (20), we get

$$B^2 - D^2 = \frac{(a+b)^2\theta^3\left[8ab - \theta(a+b)^2\right]}{16a^2b^2\sigma_1^2\sigma_2^2} \quad \text{and} \quad C^2 + 2B - A^2 = \frac{\theta\left[(\sigma_1^2 - \sigma_2^2)\,\theta - 2\sigma_1^2\right]}{\sigma_1^2\sigma_2^2}. \tag{23}$$

Lemma 7 *Recall that* $0 < \theta < 1$.

1) *Let* $0 < b < (3-\sqrt{2})a$ *or* $b > (3+\sqrt{2})a$. *Then* $B^2 - D^2 = 0$ *if* $\theta = 8ab/(a+b)^2$, $B^2 - D^2 > 0$ *if* $\theta < 8ab/(a+b)^2$, $B^2 - D^2 < 0$ *if* $\theta > 8ab/(a+b)^2$.

2) *Let* $(3-\sqrt{2})a \le b \le (3+\sqrt{2})a$. *Then* $B^2 - D^2 > 0$ *for all* θ.

Proof. From (23) one has $sign\,(B^2 - D^2) = sign[8ab - \theta(a+b)^2]$. The statement follows, noting that $8ab/(a+b)^2 < 1 \Leftrightarrow b^2 - 6ab + a^2 > 0$, i.e. when $0 < b < (3-\sqrt{2})a$ or $b > (3+\sqrt{2})a$.
■

Lemma 8 $C^2 + 2B - A^2 < 0$.

Proof. From (23) we can see that $sign\,(C^2+2B-A^2) = sign\,[(\sigma_1^2-\sigma_2^2)\,\theta - 2\sigma_1^2]$. If $\sigma_1^2-\sigma_2^2 \le 0$, then it is immediate that $C^2+2B-A < 0$. If $\sigma_1^2-\sigma_2^2 > 0$, then $(\sigma_1^2-\sigma_2^2)\,\theta - 2\sigma_1^2 < (\sigma_1^2-\sigma_2^2) - 2\sigma_1^2 = -\sigma_1^2-\sigma_2^2 < 0$. The conclusion holds. ■

From (22) we find that if $B^2-D^2 \ge 0$ holds, then Eq. (22) has no positive solutions. Thus, all the solutions of (19) have negative real parts when $\tau_2 \ge 0$. On the other hand, if the conditions $B^2-D^2<0$ hold, then Eq. (22) has a unique positive solution such that the characteristic equation (19) has a pair of purely imaginary roots $\pm i\omega_+$ at $\tau_2=\tau_{2,j}^+$, where

$$\omega_\pm = \sqrt{\frac{C^2+2B-A^2 \pm \sqrt{\left(C^2+2B-A^2\right)^2 - 4\left(B^2-D^2\right)^2}}{2}}. \tag{24}$$

The critical values $\tau_{2,j}^+$ $(j=0,1,2,...)$ of the delay τ_2 corresponding to ω_+ are obtained solving equations in (21) for $\sin(\omega_+\tau_{2,j}^+)$ and $\cos(\omega_+\tau_{2,j}^+)$, and getting

$$\sin(\omega_+\tau_{2,j}^+) = \frac{\omega_+\left(C\omega_+^2+AD-BC\right)}{C^2\omega_+^2+D^2}, \qquad \cos(\omega_+\tau_{2,j}^+) = \frac{(D-AC)\omega_+^2-BD}{C^2\omega_+^2+D^2}.$$

We have

$$\tau_{2,j}^+ = \begin{cases} \dfrac{1}{\omega_+}\left\{2j\pi + \cos^{-1}\left[\dfrac{(D-AC)\,\omega_+^2-BD}{C^2\omega_+^2+D^2}\right]\right\}, & \text{if } C\omega_+^2+AD-BC>0, \\[2ex] \dfrac{1}{\omega_+}\left\{(2j+1)\,\pi - \cos^{-1}\left[\dfrac{(D-AC)\,\omega_+^2-BD}{C^2\omega_+^2+D^2}\right]\right\}, & \text{if } C\omega_+^2+AD-BC\le 0. \end{cases} \tag{25}$$

The next step is to determine the sign of the derivative of $\mathrm{Re}(\lambda)$ at the points where λ is purely imaginary root of (19).

Proposition 9 *Let $\lambda(\tau_2)$ be the root of (19) near $\tau_2=\tau_{2,j}^+$ such that $\mathrm{Re}(\lambda(\tau_{2,j}^+))=0$ and $\mathrm{Im}(\lambda(\tau_{2,j}^+))=\omega_+$. Then*

$$\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^+,\omega=\omega_+} > 0.$$

Proof. Substituting $\lambda(\tau_2)$ into the left hand side of (19), differentiating with respect to τ_2, we get

$$\left\{2\lambda + A + \left[C-\tau_2\,(C\lambda+D)\right]e^{-\lambda\tau_2}\right\}\frac{d\lambda}{d\tau_2} = \lambda(C\lambda+D)e^{-\lambda\tau_2}. \tag{26}$$

Hence,

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{2\lambda+A+\left[C-\tau_2\,(C\lambda+D)\right]e^{-\lambda\tau_2}}{\lambda(C\lambda+D)e^{-\lambda\tau_2}} = \frac{(2\lambda+A)\,e^{\lambda\tau_2}+C}{\lambda(C\lambda+D)} - \frac{\tau_2}{\lambda}.$$

From (19), $e^{\lambda\tau_2} = -\,(C\lambda+D)\,/(\lambda^2+A\lambda+B)$, so that we obtain

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = -\frac{2\lambda+A}{\lambda(\lambda^2+A\lambda+B)} + \frac{C}{\lambda(C\lambda+D)} - \frac{\tau_2}{\lambda}.$$

Then

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^+,\omega=\omega_+} = -\frac{2i\omega_+ + A}{i\omega_+(B-\omega_+^2) - A\omega_+^2} - \frac{C}{C\omega_+^2 - iD\omega_+} - \frac{\tau_{2,j}^+}{i\omega_+}.$$

Furthermore, we have

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^+,\omega=\omega_+} = \frac{2\omega_+^2 + A^2 - 2B}{(B-\omega_+^2)^2 + A^2\omega_+^2} - \frac{C^2}{C^2\omega_+^2 + D^2}$$

Notice that (22) yields $C^2\omega_+^2 + D^2 = (B-\omega_+^2)^2 + A^2\omega_+^2$. Consequently,

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^+,\omega=\omega_+} = \frac{2\omega_+^2 + A^2 - 2B - C^2}{C^2\omega_+^2 + D^2}.$$

Therefore using (24) we find

$$sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2,j}^+,\omega=\omega_+}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau_2=\tau_{2,j}^+,\omega=\omega_+}\right\} = sign\left\{2\omega_+^2 + A^2 - 2B - C^2\right\}$$

$$= sign\left\{\sqrt{(C^2+2B-A^2)^2 - 4\left(B^2-D^2\right)^2}\right\}.$$

This completes the proof. ■

Based on the above result, when $\lambda = i\omega_+$, the only crossing of the imaginary axis is from left to right as τ_2 increases. Thus, the stability of the equilibrium point (x_*, y_*) can only be lost and not regained.

Lemma 10 *$\lambda = i\omega_+$ is a simple root of the characteristic equation* (19)*.*

Proof. If we suppose by contradiction that $\lambda = i\omega_+$ is a repeated root, then differentiating (19) with respect to λ, and inserting $\lambda = i\omega_+$, leads to

$$2i\omega_+ + A + \left[C - \tau_{2,j}^+(iC\omega_+ + D)\right] e^{-i\omega_+\tau_{2,j}^+} = 0$$

From (26) we get

$$i\omega_+(iC\omega_+ + D)e^{-i\omega_+\tau_{2,j}^+} = 0.$$

Separating real and imaginary parts in the above equality, we have

$$D\cos\omega_+\tau_{2,j}^+ + C\omega_+\sin\omega_+\tau_{2,j}^+ = 0, \qquad -C\omega_+\cos\omega_+\tau_{2,j}^+ + D\sin\omega_+\tau_{2,j}^+ = 0,$$

so that, from (21), we derive that $A\omega = 0$ and $\omega_\pm^2 = B$. Since $A > 0$ we have the statement. ■

Then, we can state the following results.

Theorem 11 *Let $\tau_{2,j}^+$ $(j = 0, 1, 2, ...)$ be defined as in* (25)*.*

1) *If* $(3-\sqrt{2})a \le b \le (3+\sqrt{2})a$ *holds or if* $0 < b < (3-\sqrt{2})a$, $b > (3+\sqrt{2})a$ *and* $0 < \theta \le 8ab/(a+b)^2$ *hold, then the equilibrium* (x_*, y_*) *of system* (15) *is locally asymptotically stable for all* $\tau_2 \ge 0$.

2) *If* $0 < b < (3-\sqrt{2})a$ *or* $b > (3+\sqrt{2})a$ *holds, and* $8ab/(a+b)^2 < \theta < 1$, *then the equilibrium* (x_*, y_*) *of system* (15) *is locally asymptotically stable for* $\tau_2 \in [0, \tau_{2,0}^+)$ *and unstable for* $\tau_2 \in (\tau_{2,0}^+, +\infty)$. *System* (15) *undergoes a Hopf bifurcation at* (x_*, y_*) *for* $\tau_2 = \tau_{2,0}^+$.

We note that, if $\theta = 0$, the equilibrium (x_*, y_*) is marginally (not asymptotically) stable. This may be verified by considering that every initial condition $x = c_1$ and $y = c_2$ for $t \in [-\tau, 0]$ defines a stationary solution for every $t \ge 0$. In contrast, for positive values of θ, firms should mediate between stationary choices and choices driven by the mechanism described in the model with best reply dynamics. The model with adaptive dynamics replicates the one with best reply dynamics when $\theta = 1$. The role of θ is highlighted in Figure 5. As can be seen by looking at the figure, for low values of θ the stationary equilibrium (x_*, y_*) is asymptotically stable, whereas when parameters are set to get an unstable stationary equilibrium of the model with best reply dynamics, there exists a threshold value of θ beyond which the stationary equilibrium of the model with adaptive dynamics is unstable and long-term dynamics are oscillatory. This result is certainly expected and is in line with similar results obtained in a two-dimensional context by Puu in a nonlinear duopoly and Onozaki et al. (2003) in a cobweb model with heterogeneous producers.

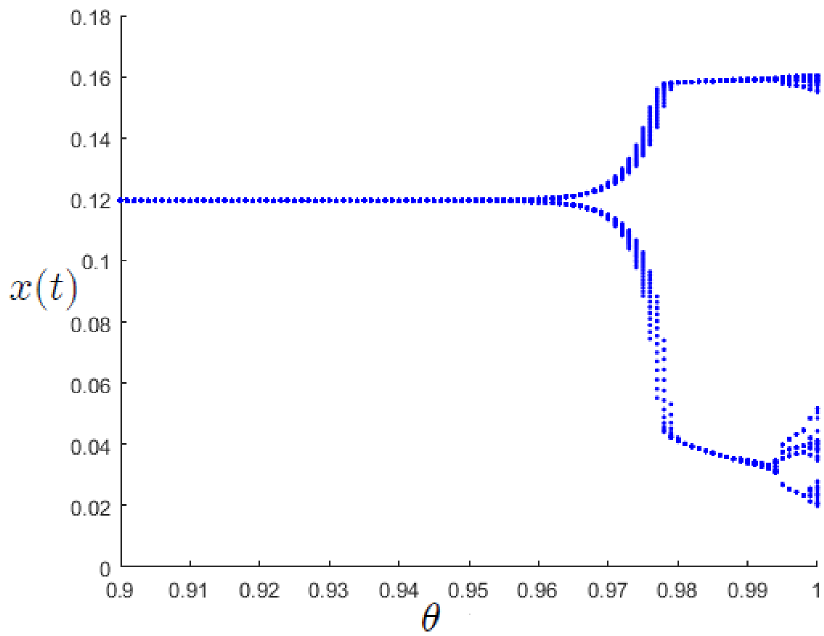


Figure 5. Bifurcation diagram for θ. Parameter set: $\sigma_1 = 0.11$, $\sigma_2 = 0.1$, $a = 1$, $b = 6.2$, $\tau_1 = 0$ and $\tau_2 = 2.2$.

4.2 Case $\tau_1 > 0$ and $\tau_2 \in [0, \tau_{2,0}^+)$

In this section, we extend the results of previous sections to the case in which τ_2 is fixed to a value belonging to the stability region, that is $\tau_2 \in [0, \tau_{2,0}^+)$, and τ_1 is positive. For convenience, we now rewrite the characteristic equation (18) as follows:

$$P(\lambda, \tau_1, \tau_2) := \lambda^2 + \widetilde{A}\lambda + \widetilde{B} + \left(\widetilde{C} + \widetilde{D}\lambda\right) e^{-\lambda\tau_1} + \left(\widetilde{C} + \widetilde{E}\lambda\right) e^{-\lambda\tau_2} + \widetilde{F}e^{-\lambda(\tau_1+\tau_2)} = 0, \quad (27)$$

where

$$\widetilde{A} = \frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} > 0, \qquad \widetilde{B} = \frac{1}{\sigma_1\sigma_2} > 0, \qquad \widetilde{C} = -\frac{1-\theta}{\sigma_1\sigma_2}, \qquad \widetilde{D} = -\frac{1-\theta}{\sigma_1}, \quad (28)$$

$$\widetilde{E} = -\frac{1-\theta}{\sigma_2}, \qquad \widetilde{F} = \frac{(1-\theta)^2}{\sigma_1\sigma_2} + \frac{\theta^2(a-b)^2}{4ab\sigma_1\sigma_2} \geq 0.$$

Now, consider $P(\lambda, \tau_1, \tau_2) = 0$ with τ_2 in its stable interval, i.e. $[0, \tau_{2,0}^+)$, and regard τ_1 as a parameter. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of $P(\lambda, \tau_1, \tau_2) = 0$. Then $P(i\omega, \tau_1, \tau_2) = 0$ gives

$$-\omega^2 + \widetilde{B} + \widetilde{C}\cos\omega\tau_2 + \widetilde{E}\omega\sin\omega\tau_2 = (\widetilde{F}\sin\omega\tau_2 - \widetilde{D}\omega)\sin\omega\tau_1 - (\widetilde{C} + \widetilde{F}\cos\omega\tau_2)\cos\omega\tau_1 \quad (29)$$

and

$$\widetilde{A}\omega + \widetilde{E}\omega\cos\omega\tau_2 - \widetilde{C}\sin\omega\tau_2 = (\widetilde{C} + \widetilde{F}\cos\omega\tau_2)\sin\omega\tau_1 + (\widetilde{F}\sin\omega\tau_2 - \widetilde{D}\omega)\cos\omega\tau_1 \quad (30)$$

Squaring and adding Eqs. (29) and (30), we get

$$g(\omega) = 0, \quad (31)$$

where

$$g(\omega) = \omega^4 - \left(2\widetilde{E}\sin\omega\tau_2\right)\omega^3 + \left[\widetilde{A}^2 + \widetilde{E}^2 - 2\widetilde{B} - \widetilde{D}^2 + 2\left(\widetilde{A}\widetilde{E} - \widetilde{C}\right)\cos\omega\tau_2\right]\omega^2$$
$$+ \left[2\left(\widetilde{D}\widetilde{F} - \widetilde{A}\widetilde{C} + \widetilde{B}\widetilde{E}\right)\sin\omega\tau_2\right]\omega + \widetilde{B}^2 - \widetilde{F}^2 + 2\widetilde{C}\left(\widetilde{B} - \widetilde{F}\right)\cos\omega\tau_2.$$

Remark 12 Notice that $\lim_{\omega\to+\infty} g(\omega) = +\infty$ and $g(0) = \widetilde{B}^2 - \widetilde{F}^2 + 2\widetilde{C}\left(\widetilde{B} - \widetilde{F}\right)$. As a result, $g(0) < 0$ when $b < (3-2\sqrt{2})a$ or $b > (3+2\sqrt{2})a$. Hence, there is at least a positive ω satisfying $g(\omega) = 0$.

Assume that Eq. (31) has finitely many positive zeros denoted by $\omega_1, \omega_2, ..., \omega_N$. Then for every fixed ω_l, $l = 1, 2, ..., N$, there exists a sequence $\tau_{1,l}^j > 0$ $(j = 1, 2, ...)$ satisfying (31). Let

$$\tilde{\tau}_1 = \min\left\{\tau_{1,l}^j, l = 1, 2, ..., N, j = 1, 2, ...\right\}. \quad (32)$$

When $\tau_1 = \tilde{\tau}_1$ the characteristic equation (27) has a pair of purely imaginary roots $\pm i\tilde{\omega}$ for $\tau_2 \in [0, \tau_{2,0}^{+})$. Let $\lambda(\tau_1)$ be the root of (27) near $\tau_1 = \tilde{\tau}_1$ satisfying $\mathrm{Re}(\lambda(\tilde{\tau}_1)) = 0$ and $\mathrm{Im}(\lambda(\tilde{\tau}_1)) = \tilde{\omega}$. To verify the transversality condition of Hopf bifurcation, we differentiate (27) with respect to τ_1, and get

$$\Big[2\lambda + \widetilde{A} + \widetilde{D}e^{-\lambda\tau_1} + \widetilde{E}e^{-\lambda\tau_2} - \left(\widetilde{C} + \widetilde{E}\lambda\right)\tau_2 e^{-\lambda\tau_2} - \widetilde{F}(\tau_1+\tau_2)e^{-\lambda(\tau_1+\tau_2)} \tag{33}$$

$$-\tau_1\left(\widetilde{C} + \widetilde{D}\lambda\right)e^{-\lambda\tau_1}\Big]\left(\frac{d\lambda}{d\tau_1}\right) = \lambda\left[\left(\widetilde{C} + \widetilde{D}\lambda\right)e^{-\lambda\tau_1} + \widetilde{F}e^{-\lambda(\tau_1+\tau_2)}\right].$$

Then,

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{2\lambda + \widetilde{A} + \widetilde{D}e^{-\lambda\tau_1} + \widetilde{E}e^{-\lambda\tau_2} - \left(\widetilde{C} + \widetilde{E}\lambda\right)\tau_2 e^{-\lambda\tau_2} - \widetilde{F}\tau_2 e^{-\lambda(\tau_1+\tau_2)}}{\lambda\left[\left(\widetilde{C} + \widetilde{D}\lambda\right)e^{-\lambda\tau_1} + \widetilde{F}e^{-\lambda(\tau_1+\tau_2)}\right]} - \frac{\tau_1}{\lambda}.$$

Plugging (27) into the above expression yields

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{2\lambda + \widetilde{A} + \widetilde{D}e^{-\lambda\tau_1} + \widetilde{E}e^{-\lambda\tau_2} + \tau_2\left[\lambda^2 + \widetilde{A}\lambda + \widetilde{B} + \left(\widetilde{C} + \widetilde{D}\lambda\right)e^{-\lambda\tau_1}\right]}{\lambda\left[\lambda^2 + \widetilde{A}\lambda + \widetilde{B} + \left(\widetilde{C} + \widetilde{E}\lambda\right)e^{-\lambda\tau_2}\right]} - \frac{\tau_1}{\lambda}.$$

After an elementary but somewhat tedious calculation, we can arrive at the following expression:

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tilde{\tau}_1} = \frac{a_1 + ia_2}{\tilde{\omega}(b_1 - ib_2)} - \frac{\tilde{\tau}_1}{i\tilde{\omega}},$$

where

$$a_1 = \widetilde{A} + \widetilde{D}\cos\tilde{\omega}\tilde{\tau}_1 + \widetilde{E}\cos\tilde{\omega}\tau_2 + \tau_2\left(-\tilde{\omega}^2 + \widetilde{B} + \widetilde{C}\cos\tilde{\omega}\tilde{\tau}_1 + \widetilde{D}\tilde{\omega}\sin\tilde{\omega}\tilde{\tau}_1\right), \tag{34}$$

$$a_2 = 2\tilde{\omega} - \widetilde{D}\sin\tilde{\omega}\tilde{\tau}_1 - \widetilde{E}\sin\tilde{\omega}\tau_2 + \tau_2\left(\widetilde{A}\tilde{\omega} - \widetilde{C}\sin\tilde{\omega}\tilde{\tau}_1 + \widetilde{D}\tilde{\omega}\cos\tilde{\omega}\tilde{\tau}_1\right),$$

$$b_1 = \widetilde{A}\tilde{\omega} - \widetilde{C}\sin\tilde{\omega}\tau_2 + \widetilde{E}\tilde{\omega}\cos\tilde{\omega}\tau_2,$$

$$b_2 = \widetilde{B} - \tilde{\omega}^2 + \widetilde{C}\cos\tilde{\omega}\tau_2 + \widetilde{E}\tilde{\omega}\sin\tilde{\omega}\tau_2.$$

Consequently,

$$sign\left[\frac{d\,\mathrm{Re}(\lambda)}{d\tau_1}\right]_{\tau_1=\tilde{\tau}_1} = sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right]_{\tau_1=\tilde{\tau}_1} = sign\left(a_1 b_1 - a_2 b_2\right), \tag{35}$$

where

$$a_1b_1 - a_2b_2 = \tilde{\omega}\left(2\tilde{\omega}^2 + \widetilde{A}^2 + \widetilde{E}^2 - 2\widetilde{B}\right) + \widetilde{A}\widetilde{D}\tilde{\omega}\cos\tilde{\omega}\tilde{\tau}_1 + \widetilde{D}\left(\widetilde{B} - \widetilde{D}\tilde{\omega}\right)\sin\tilde{\omega}\tilde{\tau}_1$$

$$+ 2\tilde{\omega}\left(\widetilde{A}\widetilde{E} - \widetilde{C}\right)\cos\tilde{\omega}\tau_2 + \left[-\widetilde{A}\widetilde{C} + \widetilde{E}\left(\widetilde{B} - 3\tilde{\omega}^2\right)\right]\sin\tilde{\omega}\tau_2$$

$$+ \widetilde{D}\widetilde{E}\tilde{\omega}\cos\tilde{\omega}(\tilde{\tau}_1 - \tau_2) + \widetilde{C}\widetilde{D}\sin\tilde{\omega}(\tilde{\tau}_1 - \tau_2)$$

$$+ \tau_2\left\{\tilde{\omega}\left[\widetilde{A}\widetilde{C} + \widetilde{D}\left(\tilde{\omega}^2 - \widetilde{B}\right)\right]\cos\tilde{\omega}\tilde{\tau}_1 + \left[\widetilde{B}\widetilde{C} + \tilde{\omega}^2\left(\widetilde{A}\widetilde{D} - \widetilde{C}\right)\right]\sin\tilde{\omega}\tilde{\tau}_1\right.$$

$$+ \tilde{\omega}\left[-\widetilde{A}\widetilde{C} + \widetilde{E}(\widetilde{B} - \tilde{\omega}^2)\right]\cos\tilde{\omega}\tau_2 + \left[-\widetilde{B}\widetilde{C} + \tilde{\omega}^2\left(\widetilde{C} - \widetilde{A}\widetilde{E}\right)\right]\sin\tilde{\omega}\tau_2$$

$$\left. + \left(\widetilde{C}^2 + \widetilde{D}\widetilde{E}\right)\sin\tilde{\omega}(\tilde{\tau}_1 - \tau_2) + \widetilde{C}\tilde{\omega}\left(-\widetilde{D} + \widetilde{E}\right)\cos\tilde{\omega}(\tilde{\tau}_1 - \tau_2)\right\}.$$

From (35), we derive that if $sign\,(a_1b_1 - a_2b_2) > 0$, then each crossing of the real part of characteristic roots at $\tilde{\tau}_1$ is from left to right; whereas $sign\,(a_1b_1 - a_2b_2) < 0$ indicates that the real part of a pair of conjugate roots of Eq. (27) changes from positive value to negative value when $\tilde{\tau}_1$ is crossed.

Theorem 13 *Let $\tilde{\tau}_1$ and a_1, b_1, a_2, b_2 be defined as in (32) and (34), $\tau_2 \in [0, \tau_{2,0}^+)$.*

1) *If $g(\omega)$ has no positive zero, then the equilibrium (x_*, y_*) of system (15) is locally asymptotically stable for $\tau_1 \geq 0$.*

2) *If $g(\omega)$ has at least a positive zero, then there exists $\tilde{\tau}_1 > 0$ such that equilibrium (x_*, y_*) of system (15) is locally asymptotically stable for $\tau_1 \in [0, \tilde{\tau}_1)$ and unstable for $\tau_1 > \tilde{\tau}_1$. System (15) undergoes a Hopf bifurcation at the equilibrium (x_*, y_*) for $\tau_1 = \tilde{\tau}_1$ if the corresponding root $\lambda = i\tilde{\omega}$ is simple and the expression $a_1b_1 - a_2b_2$ is positive.*

Remark 14 Notice that if $\lambda = i\tilde{\omega}$ is a repeated root of (27), then $dP(i\tilde{\omega}, \tilde{\tau}_1, \tau_2)/d\lambda = 0$. From (33) we see that it must be $\widetilde{C} + \widetilde{D}i\tilde{\omega} + \widetilde{F}e^{-i\tilde{\omega}\tau_2} = 0$. Hence, we have $\widetilde{C} + \widetilde{F}\cos\tilde{\omega}\tau_2 = 0$ and $\widetilde{D}\tilde{\omega} - \widetilde{F}\sin\tilde{\omega}\tau_2 = 0$, yielding $\widetilde{D}^2\tilde{\omega}^2 = \widetilde{F}^2 - \widetilde{C}^2$.

4.3 Stability crossing curves

In the previous section, we characterized analytically the dynamic properties of system (15). In this section, we will apply the techniques developed by Gu et al. (2005) and Lin and Wang (2012) with the aim at showing the properties of the adaptive dynamic system directly in the (τ_1, τ_2) plane. This geometric approach will allow us to get findings more readable for a non-specialist audience and to have clear economic insights. Our analysis begins by noting that hypotheses $(i) - (iv)$ of Lin and Wang (2012, p. 521) hold in the present set up, as the following basic assumptions are fulfilled.

(i) Finite number of characteristic roots on $C_+ = \{\lambda \in C : \text{Re}(\lambda) > 0\}$ under the condition

$$\deg(P_0(\lambda)) = 2 \geq \max\{\deg(P_1(\lambda)), \deg(P_2(\lambda)), \deg(P_3(\lambda))\} = 1.$$

(ii) $P_0(0) + P_1(0) + P_2(0) + P_3(0) = \frac{\theta^2(a+b)^2}{4ab\sigma_1\sigma_2} \neq 0.$

(iii) $P_0(\lambda)$, $P_1(\lambda)$, $P_2(\lambda)$ and $P_3(\lambda)$ are coprime polynomials.

(iv)

$$\lim_{\lambda\to\infty}\left(\left|\frac{P_1(\lambda)}{P_0(\lambda)}\right| + \left|\frac{P_2(\lambda)}{P_0(\lambda)}\right| + \left|\frac{P_3(\lambda)}{P_0(\lambda)}\right|\right) = 0 < 1.$$

The next step allows us to identify the set Ω of the values of ω that satisfy conditions such that complex conjugate roots do exist. To this end, we introduce the function

$$Z(\omega) := \left(|P_0(i\omega)|^2 + |P_1(i\omega)|^2 - |P_2(i\omega)|^2 - |P_3(i\omega)|^2\right)^2 - 4\left(L_1(\omega)^2 + L_2(\omega)^2\right),$$

defined on $W := \{\omega \in R : \omega > 0\}$, where

$$L_1(\omega) := \text{Re}(P_2(i\omega)\overline{P_3(i\omega)}) - \text{Re}(P_0(i\omega)\overline{P_1(i\omega)}),$$

and

$$L_2(\omega) := \text{Re}(P_1(i\omega)\overline{P_3(i\omega)}) - \text{Re}(P_0(i\omega)\overline{P_2(i\omega)}).$$

In order to have stability switchings, there must exist intervals such that $Z(\omega)$ is negative, where, in our case,

$$Z(\omega) := \omega^8 + z_6\omega^6 + z_4\omega^4 + z_2\omega^2 + z_0,$$

$$z_6 := \frac{2\theta(2-\theta)(\sigma_1^2+\sigma_2^2)}{\sigma_1^2\sigma_2^2} > 0,$$

$$z_4 := \frac{-\theta^2}{8a^2b^2\sigma_1^4\sigma_2^4} \times \left(\theta^2\sigma_1^2\sigma_2^2(a^4+b^4) - 12ab\sigma_1^2\sigma_2^2\left\{\left(\theta^2 - \frac{4}{3}\theta + \frac{2}{3}\right)(a^2+b^2) + \right.\right.$$
$$\left.\left. + \frac{2}{3}ab\left[(\theta-2)^2(\sigma_1^4+\sigma_2^4) + \frac{5}{4}\sigma_1^2\sigma_2^2\left(\theta^2 - \frac{48}{5}\theta + \frac{56}{5}\right)\right]\right\}\right),$$

$$z_2 := -\frac{\theta^3(\sigma_1^2+\sigma_2^2)[(a+b)^2\theta - 8ab][(a+b)^2\theta^2 + 2(a^2+6ab+b^2)(1-\theta)]}{8a^2b^2\sigma_1^4\sigma_2^4},$$

$$z_0 := \frac{\theta^4(a+b)^2[a^2\theta + 2b(\theta-4)a + b^2\theta]^2[\theta^2(a^2+b^2) + 2ab(\theta^2 - 8\theta + 8)]}{256a^4b^4\sigma_1^4\sigma_2^4} > 0.$$

In order to study the roots of polynomial $Z(\omega)$ it is convenient to introduce the change of variable $X := \omega^2$ and analyze the behavior of the polynomial:

$$\overline{Z}(X) = X^4 + z_6 X^3 + z_4 X^2 + z_2 X + z_0 \tag{36}$$

in the domain $Y := \{X \in R : X > 0\}$. In fact, we note that, for any positive root X_1 of $\overline{Z}(X)$, there exists a corresponding positive root $\sqrt{X_1}$ of $Z(\omega)$ and vice versa. By considering the second derivative of $\overline{Z}(X)$, we have

$$\overline{Z}''(X) = 12X^2 + 6z_6 X + 2z_4. \tag{37}$$

From the Cartesian rule it follows that there exists at most a change of concavity of $\overline{Z}(X)$ in Y. Thus, there exists at most one minimum for $Z(\omega)$, that is, $\omega_{\min}$, in W. Knowing that $\lim_{X\to 0^+} \overline{Z}(X) > 0$ and $\lim_{X\to +\infty} \overline{Z}(X) > 0$ we have that there exist at most two positive roots of $\overline{Z}(X)$ and then of $Z(\omega)$.

The previous results lead to the following theorem.

Theorem 15 *If $z_4 > 0$ then no stability switching exists and the stationary state is locally asymptotically stable for any $\tau_1 > 0$ and $\tau_2 > 0$. If $z_4 < 0$ and $Z(\omega_{\min}) < 0$ then there exist ω_1 and ω_2 with $\omega_1 < \omega_2$ such that $Z(\omega) < 0$ for any $\omega \in \Omega = (\omega_1, \omega_2)$.*

For the numerical simulations plotted in Figures 6 and 7 we will use the parameter values: $\sigma_1 = 0.6$, $\sigma_2 = 1.7$, $\theta = 0.68$, $a = 2$ and $b = 0.12$. Figure 6 displays function $Z(\omega)$, whereas Figure 7 represents the stability crossing curves for the parameter set specified above. We note that this configuration of crossing curves has been found for every numerical simulation run. The grey area in Figure 7 is the stability region. It shows that for sufficiently large values of τ_1 and τ_2, the stationary equilibrium of the system is locally unstable. We note that, different from other economic models (Gori et al., 2015c), in this case there exists neither a corridor stability nor the possibility of having several stability switchings by changing one of the two delays. In this case of adaptive dynamics, we limit ourselves to the local analysis as for values of θ far enough from 1 the dynamics of the model when feasible trajectories exist are qualitatively similar for every couple of values of τ_1 and τ_2 in the instability region (i.e., an attracting cycle describes the long-term dynamics of the system, as shown in Figure 8).

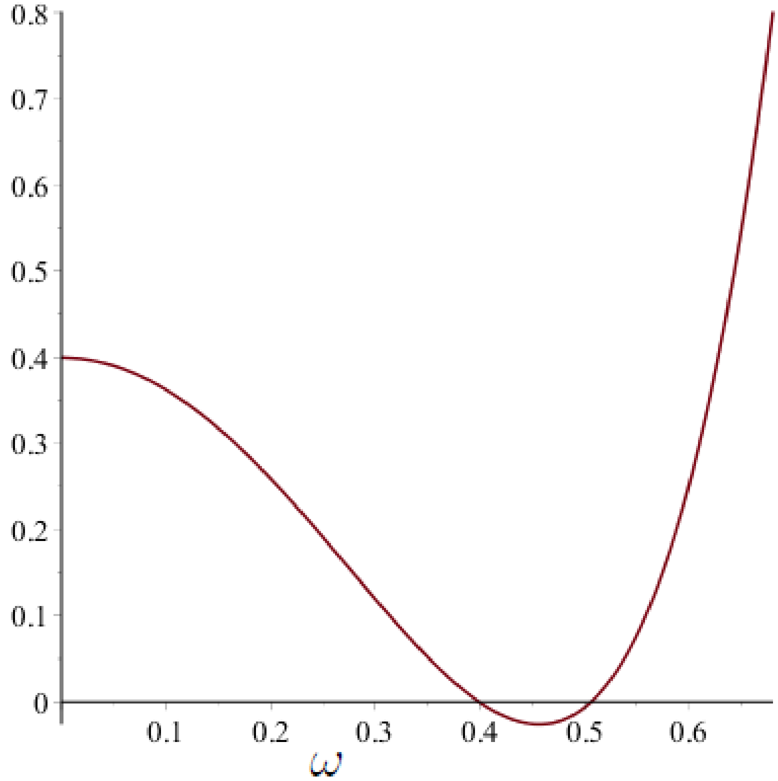


Figure 6. Graph of $Z(\omega)$. In this case, $\Omega = (0.3990553017, 0.5062414795)$.

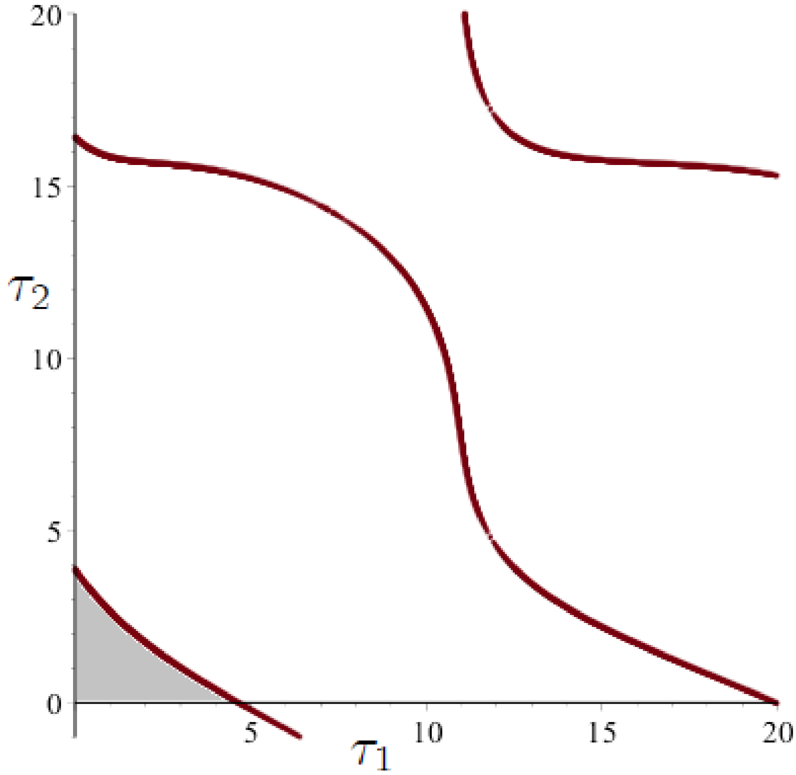


Figure 7. Some crossing curves in (τ_1, τ_2) plane. The grey area describes the stability region of the system.

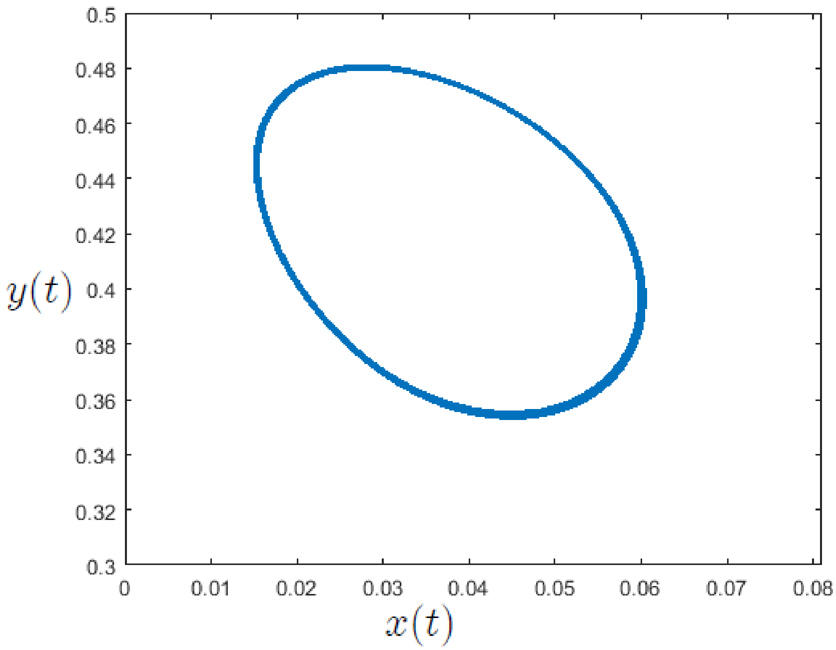


Figure 8. Limit cycle for $\sigma_1 = 0.6$, $\sigma_2 = 1.7$, $\theta = 0.7$, $a = 2$, $b = 0.17$, $\tau_1 = 1.31$ and $\tau_2 = 2.2$.

5 Conclusions

This article revisited the discrete-time dynamic duopoly of Puu (1991), which is one of the seminal works in nonlinear oligopoly theory, by using a continuous-time framework with delays in both cases of best reply dynamics and adaptive dynamics. The dynamics of the model are characterized by a hybrid system of delay differential (instead of difference) equations. The main aim of the work is to apply and study some recent mathematical techniques related to hybrid systems, such as, for instance, the stability crossing curves developed by Gu et al. (2005) and Lin and Wang (2012), to analyze the local and global properties of a Cournot duopoly with homogeneous product.

As the debate on nonlinear oligopoly theory is still high in the economic literature and there do not exist several economically coherent models described by hybrid systems (delay differential equations), transforming a discrete time model in a hybrid model may shed light in the analysis of some mathematical aspects, which will be of importance to understanding the functioning of an economic model. In addition, applying the stability crossing curves techniques may help scholars to explain better some economic phenomena in which the mixture between continuous time and discrete time is such that the use of differential equations or difference equations alone cannot capture adequately (i.e., production, trading and so on).

Some extensions of the present work can be considered by taking into account, for instance, the Cournot-like triopoly of Puu (1998) or the Cournot-like model with n competitors developed by Lampart (2012).

Conflict of Interest The authors declare that they have no conflict of interest.

Acknowledgements The authors gratefully acknowledge Tönu Puu, Ferenc Szidarovszky and participants at NED 2015 held at Chuo University, Tokyo (Japan), and AMASES 2015 held at University of Padova, Italy, for insightful comments and suggestions on an earlier draft. The authors are also indebted to two anonymous reviewers for valuable comments that have contributed to improve the work. The usual disclaimer applies.

References

[1] Berezowski M (2001) Effect of delay time on the generation of chaos in continuous systems. One-dimensional model. Two-dimensional model-tubular chemical reactor with recycle. Chaos Soliton Fract 12(1):83–89

[2] Bertola G, Caballero RJ (1990) Kinked adjustment costs and aggregate dynamics. NBER Macroeconomic Annuals 5:237–296

[3] Bischi GI, Cerboni Baiardi L (2015) A dynamic marketing model with best reply and inertia. Chaos Soliton Fract 79:145–156

[4] Bischi GI, Stefanini L, Gardini L (1998) Synchronization, intermittency and critical curves in a duopoly game. Math Comput Simulat 44(6):559–585

[5] Cánovas JS, Puu T, Ruíz M (2008) The Cournot–Theocharis problem reconsidered. Chaos Soliton Fract 37(4):1025–1039

[6] Chen L, Chen F (2011) Dynamic behaviors of the periodic predator–prey system with distributed time delays and impulsive effect. Nonlinear Anal Real 12(4):2467–2473

[7] Chen S, Shi J (2013) Global attractivity of equilibrium in Gierer–Meinhardt system with activator production saturation and gene expression time delays. Nonlinear Anal Real 14(4):1871–1886

[8] Gori L, Guerrini L, Sodini M (2015a) A continuous time Cournot duopoly with delays. Chaos Soliton Fract 79:166–177

[9] Gori L, Guerrini L, Sodini M (2015b) Equilibrium and disequilibrium dynamics in cobweb models with time delays. Int J Bifurcat Chaos 25(6):1550088

[10] Gori L, Guerrini L, Sodini M (2015c) Hopf bifurcation and stability crossing curves in a cobweb model with heterogeneous producers and time delays. Nonlinear Anal Hybrid Syst 18:117–133

[11] Gu K, Niculescu SI, Chen J (2005) On stability crossing curves for general systems with two delays. J Math Anal Appl 311(1):231–253

[12] Harrison A, van Hoek R (2008) Logistics Management and Strategy: Competing Through the Supply Chain. Third Edition. Prentice Hall

[13] Krawiec A, Szydlowski M (1999) The Kaldor-Kalecki business cycle model. Ann Oper Res 89(0):89–100

[14] Lampart M (2012) Stability of the Cournot equilibrium for a Cournot oligopoly model with n competitors. Chaos Soliton Fract 45(9-10):1081–1085

[15] Lin X, Wang H (2012) Stability analysis of delay differential equations with two discrete delays. Can Appl Math Quart 20(4):519–533

[16] Matsumoto A, Szidarovszky F (2010a) Delay differential nonlinear economic models. In: Nonlinear Dynamics in Economics, Finance and the Social Sciences. Bischi GI, Chiarella C, Gardini L, eds., Springer-Verlag, Berlin, Heidelberg, 195–214

[17] Matsumoto A, Szidarovszky F (2010b) Delayed dynamics in heterogeneous competition with product differentiation. Nonlinear Anal Real 11(2):601–611

[18] Matsumoto A, Szidarovszky F (2012) Nonlinear delay monopoly with bounded rationality. Chaos Soliton Fract 45(4):507–519

[19] Matsumoto A, Szidarovszky F (2014) Discrete and continuous dynamics in nonlinear monopolies. Appl Math Comput 232:632–642

[20] Matsumoto A, Szidarovszky F (2015) Nonlinear Cournot duopoly with implementation delays. Chaos Soliton Fract 79:157–165

[21] Matsumoto A, Szidarovszky F, Yoshida H (2011) Dynamics in linear Cournot duopolies with two time delays. Comput Econ 38:311–327

[22] Monica C, Pitchaimani M (2016) Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays. Nonlinear Anal Real 27:55–69

[23] Onozaki T, Sieg G, Yokoo M (2003) Stability, chaos and multiple attractors: a single agent makes a difference. J Econ Dyn Control 27(10):1917–1938

[24] Puu T (1991) Chaos in duopoly pricing. Chaos Soliton Fract 1(6):573–581

[25] Puu T (1998) The chaotic duopolists revisited. J Econ Behav Organ 33(3-4):385–394

[26] Rand D (1978) Exotic phenomena in games and duopoly models. J Math Econ 5(2):173–184