A ROUGH CALCULUS APPROACH TO LEVEL SETS IN THE HEISENBERG GROUP

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Abstract. We introduce novel equations, in the spirit of rough path theory, that parametrize level sets of intrinsically regular maps on the Heisenberg group with values in $\mathbb{R}^2$. These equations can be seen as a sub-Riemannian counterpart to classical ODEs arising from the implicit function theorem. We show that they enjoy all the natural well-posedness properties, thus allowing for a "good calculus" on nonsmooth level sets. We apply these results to prove an area formula for the intrinsic measure of level sets, along with the corresponding coarea formula.

1. Introduction

The classical implicit function theorem asserts that regular level sets of a C^1 smooth map on a Euclidean space are C^1 smooth, with a natural parametrization which can be written in terms of differential equations involving the first derivatives of the map. This is no longer the case for maps on Carnot-Carathéodory spaces which are regular only with respect to the intrinsic geometry of their domain. In this paper, we study the simplest of such situations, where maps are defined on the first Heisenberg group $\mathbb{H}$ and take values in $\mathbb{R}^2$, so that their level sets are expected to be one-dimensional. Indeed, our results show that these level sets can be still represented by curves, that are in general only Hölder continuous, and not anymore smooth, but still solve a peculiar analogue of a classical ODE, that we call *Level Set Differential Equation* (LSDE).

At first glance, the LSDE is similar to equations driven by a rough signal appearing in the theory of *rough paths* as exposed e.g. in [9, 12, 17], but it is different, being inherently "autonomous", while the usual *rough differential equations* (RDEs) are not. However, the theory of rough paths still provides an appropriate tool, namely, the *sewing lemma*, to construct solutions to LSDEs, thus enabling some "differential" calculus for maps on the Heisenberg group, regular only in the intrinsic sense of the latter, but possibly nowhere differentiable on a set of positive measure [18]. Before presenting our results, it is worth describing the mathematical landscape motivating this study.

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Intrinsically regular level sets. A natural problem of Geometric Measure Theory in graded nilpotent Lie groups is the study of the structure of level sets of maps intrinsically (h-)differentiable in the sense of P. Pansu [22], which are known to be quite complicated [25, 24, 1], in particular not even rectifiable in the classical sense [13] and neither can be interpreted as metric currents [2, 3]. For h-differentiable maps $F : \mathbb{G} \to \mathbb{M}$ between graded nilpotent Lie groups, a convenient parametrization for level sets is available when a *semidirect factorization* of $\mathbb{G}$ with respect to $\mathbb{M}$ holds, as a consequence of a suitable implicit function theorem [20]. In case $\mathbb{G} = \mathbb{H}^n$, the n-th Heisenberg group, topologically identified with $\mathbb{R}^{2n+1}$, and $\mathbb{M} = \mathbb{R}^k$, the respective semidirect factorization is known to exist precisely when $1 \le k \le n$. If that is the case, level sets of F can be seen as naturally acting on intrinsic differential forms of the Heisenberg group forming the so-called Rumin's complex [23]. As a result, they become intrinsic Heisenberg currents and their intrinsic measure can be computed by a suitable area formula [8], leading to a coarea formula [19].

When $n < k \le 2n$ there is no general approach to the structure of level sets, which then cannot even be seen as Heisenberg currents. The simplest model of these difficult cases is $n = 1$ and $k = 2$. Here, the fact that level sets at regular points are continuous curves and cannot degenerate to a singleton was established by the first author and G.P. Leonardi [16], using an *ad hoc* method exploiting classical ODEs. Following a purely geometric approach that relies on a Reifenberg-type flatness estimate, A. Kozhevnikov [14, 15] showed that they are in fact Hölder continuous curves, furnishing in this way also a coarea formula for a large subclass of h-differentiable maps from $\mathbb{H}$ to $\mathbb{R}^2$.

A Euclidean view of the problem. In case $F\colon \mathbb{R}^3 \to \mathbb{R}^2$, $x = (x^1, x^2, x^3)$, $F(x) = (F_1(x), F_2(x))$ is a smooth (C^1) map, the differential of F at $x \in \mathbb{R}^3$ with respect to the "horizontal coordinates" x^1, x^2 is represented by the square matrix

$$\nabla_{12} F(x) := \begin{pmatrix} \partial_1 F_1(x) & \partial_2 F_1(x) \\ \partial_1 F_2(x) & \partial_2 F_2(x) \end{pmatrix}.$$

If $p \in \mathbb{R}^3$ is nondegenerate with $\nabla_{12}F(p)$ invertible, then the implicit function theorem implies that the level set $F^{-1}(F(p))$ can be parametrized, locally around p, by a C^1 curve $\gamma : I \to \mathbb{R}^3$, $t \mapsto \gamma_t$, with the parameter t, on the interval I. After a one dimensional change of variable, we may consider γ^3 as a local "vertical coordinate", i.e. $\gamma^3_t := p^3 + t$, that is $\dot\gamma^3_t = 1$, for $t \in I$. Moreover, the "horizontal components" (γ^1, γ^2) solve an ODE, as a consequence of the condition $\frac{d}{dt} F(\gamma_t) = 0$, namely

$$\begin{pmatrix} \dot\gamma^1_t \\ \dot\gamma^2_t \end{pmatrix} = - \left(\nabla_{12} F(\gamma_t)\right)^{-1} \begin{pmatrix} \partial_3 F_1(\gamma_t) \\ \partial_3 F_2(\gamma_t) \end{pmatrix} \quad \text{for } t \in I. \tag{1}$$

The nonholonomic problem. The situation radically changes in the "nonholonomic" setting, where "horizontal" directions are represented by a couple of smooth vector fields X_1, X_2, such that the classical *Lie bracket generating condition*

$$\operatorname{span}\{X_1(x), X_2(x), [X_1, X_2](x)\} = \mathbb{R}^3 \quad \text{at every } x \in \mathbb{R}^3 \tag{2}$$

holds, $[\cdot, \cdot]$ standing for the Lie bracket, and the "horizontal plane" at $x \in \mathbb{R}^3$ is defined by $\operatorname{span}\{X_1(x), X_2(x)\}$. The model situation is that of F being defined on the first Heisenberg group $\mathbb{H}$, topologically identified with $\mathbb{R}^3$, and equipped with "horizontal vector fields"

$$X_1(x^1, x^2, x^3) := \partial_1 - x^2 \partial_3 \quad \text{and} \quad X_2(x^1, x^2, x^3) := \partial_2 + x^1 \partial_3.$$

Let $F\colon \mathbb{H} \to \mathbb{R}^2$ be continuously horizontally (h-)differentiable. Even if the "horizontal differential" of F, represented by the square matrix

$$\nabla_{\mathsf{h}} F := \begin{pmatrix} X_1 F_1 & X_2 F_1 \\ X_1 F_2 & X_2 F_2 \end{pmatrix},$$

is invertible at some point $p \in \mathbb{H}$ (i.e., p is nondegenerate) the loss of Euclidean regularity may allow for highly irregular level sets in a neighbourhood of p. If F is C^1 in the Euclidean sense, then one could rewrite (1) in terms of $\nabla_{\mathsf{h}} F$ instead of $\nabla_{12} F$, getting

$$\begin{pmatrix} \dot\gamma_t^1 \\ \dot\gamma_t^2 \end{pmatrix} = -\left(\nabla_{\mathsf{h}} F(\gamma_t)\right)^{-1} \begin{pmatrix} \partial_3 F_1(\gamma_t) \\ \partial_3 F_2(\gamma_t) \end{pmatrix} \theta_{\gamma_t}(\dot\gamma_t), \quad \text{where} \tag{3}$$

$$\theta = dx^3 + x^2 dx^1 - x^1 dx^2$$

is the contact form of $\mathbb{H}$. It is then natural to consider

$$\theta_{\gamma_t}(\dot\gamma_t) = \dot\gamma_t^3 + \gamma_t^2 \dot\gamma_t^1 - \gamma_t^1 \dot\gamma_t^2 = 1 \quad \text{for } t \in I \tag{4}$$

as the condition that replaces $\dot\gamma_t^3 = 1$ occurring in the Euclidean case, closing the system (3). Here the main difficulty appears when F is only h-differentiable, since in this case the "vertical derivatives" $\partial_3 F^1$ and $\partial_3 F^2$ may not exist, so the system (3) makes no sense. In addition, the 1/2-Hölder continuity of the sub-Riemannian distance with respect to the Euclidean one leads us to a genuinely Hölder continuous curve γ (hence possibly nowhere differentiable), which makes even the definition of the term $\gamma_t^2 \dot\gamma_t^1 - \gamma_t^1 \dot\gamma_t^2$ in (4) a nonsense. From a geometric viewpoint, the inherent obstacle to this approach arises from the fact that γ cannot move along horizontal directions and in this case there is no suitable geometric notion of differentiability. If $\dot\gamma_t$ were horizontal, i.e. $t \mapsto \gamma_t$ were differentiable at $t \in I$ in the sense of [22], then the chain rule would give

$$0 = \frac{d}{dt} F(\gamma_t) = \nabla_{\mathsf{h}} F(\gamma_t) \begin{pmatrix} \dot\gamma_t^1 \\ \dot\gamma_t^2 \end{pmatrix}$$

and from the non-degeneracy of $\nabla_{\mathsf{h}} F(\gamma_t)$ we would get $\dot\gamma_t^1 = \dot\gamma_t^2 = 0$, in conflict with the natural injectivity requirements. In other words, the parametrization of the level set γ at any point must move along "vertical directions".

Description of results. In this paper, we will prove that the analogy with the Euclidean situation can be suitably extended to the nonholonomic case. If the horizontal gradient $\nabla_{\mathsf{h}} F$ is α-Hölder continuous with respect to the sub-Riemannian distance of $\mathbb{H}$ for some $\alpha \in (0, 1)$, we are able to provide a rigorous counterpart of (1), which is no longer an ODE but rather a new "differential equation', the LSDE (Definition 3.1). In fact, instead of derivatives, we consider finite differences of the unknown solution γ at "sufficiently close" points $s, t \in I$. For instance, the third component γ^3 of the solution is requested to satisfy

$$(\gamma_s^{-1} \gamma_t)^3 = t - s + o(|t - s|) \tag{5}$$

for $|t - s|$ small, with the appropriate order of "error" $o(\cdot)$. This seems to be the natural translation of (4) into our framework, where the inverse and the product are given by the group operation. Such use of finite differences allows us to circumvent the regularity problems of the naïve approach: the terms $\gamma_t^2 \dot\gamma_t^1 - \gamma_t^1 \dot\gamma_t^2$ in (4) are replaced by $\gamma_t^2(\gamma_s^1 - \gamma_t^1) - \gamma_t^1(\gamma_s^2 - \gamma_t^2)$, and the partial derivatives $\partial_3 F^1$, $\partial_3 F^2$ in (3) are replaced by an expression involving the remainder of the "horizontal" Taylor expansion of F. To "integrate" a consistent family of such local descriptions, we use an important technical tool underlying the theory of rough/controlled paths, the so-called *sewing lemma*, which in this case leads to integrals in the sense of L. C. Young [26]. As already mentioned, the LSDE is an autonomous equation and does not fit precisely in the framework of RDEs, but we may imagine that the "noise" is self-induced by (5).

Our main result (Theorem 5.6) is a version of the implicit function theorem, showing that any level set of F in a neighbourhood of a nondegenerate point $p \in \mathbb{H}$ can be parametrized by an injective continuous curve γ satisfying an LSDE . Further properties of solutions to LSDEs are proven, such as uniqueness (Theorem 6.2) and stability with respect to approximations of F (Corollary 5.7). As two applications, we provide an area formula for level sets (Theorem 7.1, Corollary 7.2) as well as a coarea formula (Theorem 8.2) for maps with Hölder continuous horizontal gradient, where the LSDE is instrumental to follow the approach of [19].

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2. Preliminaries

General notation. Throughout the paper we use the notation $|\cdot|$ for the Euclidean norm of a vector. For $\beta \in [0,1]$, given an interval I and a function $f : I \to \mathbb{R}^k$, $(k \geq 1)$, $t \mapsto f_t$, we let

$$\|f\|_\beta := \sup_{\substack{s,t\in I\\ s\neq t}} \frac{|f_t - f_s|}{|t-s|^\beta} \in [0,\infty],$$

and write $f \in C^\beta(I,\mathbb{R}^k)$ if $\|f\|_\beta < \infty$. Notice that $\|f\|_0 \leq 2\sup_{t\in I}|f_t|$.

Next, we introduce notation and basic results on the geometry and analysis of maps in the Heisenberg group. We follow in the sequel the monograph [7], other approaches can be found, e.g. in [4, 11].

Heisenberg group. We represent the Heisenberg group $\mathbb{H}$ as $\mathbb{R}^3$ equipped with the non-commutative group operation $(x, y) \mapsto xy$ defined by

$$(x^1,x^2,x^3)(y^1,y^2,y^3) = (x^1+y^1, x^2+y^2, x^3+y^3+(x^1y^2-x^2y^1)) \tag{6}$$

where $x = (x^1,x^2,x^3)$ and $y = (y^1,y^2,y^3)$.

In what follows, for simplicity of notation, we let $x^{\mathsf{h}} = (x^1,x^2) \in \mathbb{R}^2$ and $x^{\mathsf{v}} = x^3 \in \mathbb{R}$ denote the "horizontal" and "vertical" components of $x = (x^1,x^2,x^3) \in \mathbb{H}$ respectively, so that $x = (x^{\mathsf{h}}, x^{\mathsf{v}})$. We notice that $x \mapsto x^{\mathsf{h}}$ is a group homomorphism, since (6) gives $(xy)^{\mathsf{h}} = x^{\mathsf{h}} + y^{\mathsf{h}}$.

Dilations and gauges. For $r \geq 0$, we let $\delta_r : \mathbb{H} \to \mathbb{H}$ denote the intrinsic dilation (which is a group homomorphism)

$$x \mapsto \delta_r(x) = (rx^{\mathsf{h}}, r^2x^{\mathsf{v}}) = (rx^1, rx^2, r^2x^3).$$

Clearly, $\delta_r \circ \delta_{r'} = \delta_{rr'}$, for any r, $r' \geq 0$.

It is useful to introduce the following non-negative functions on $\mathbb{H}$: the "horizontal gauge", $[\cdot]^{\mathsf{h}}$, and the "vertical gauge", $[\cdot]^{\mathsf{v}}$,

$$[x]^{\mathsf{h}} := |x^{\mathsf{h}}| = \sqrt{|x^1|^2 + |x^2|^2}, \qquad [x]^{\mathsf{v}} := \sqrt{|x^{\mathsf{v}}|} = \sqrt{|x^3|}.$$

Both $[\cdot]^{\mathsf{h}}$ and $[\cdot]^{\mathsf{v}}$ are 1-homogeneous with respect to dilations, i.e.,

$$[\delta_r x]^{\mathsf{h}} = r\,[x]^{\mathsf{h}}, \quad [\delta_r x]^{\mathsf{v}} = r\,[x]^{\mathsf{v}} \quad \text{for } x \in \mathbb{H},\ r \geq 0.$$

Invariant distances. We fix from now on a distance $\mathsf{d} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, which is *left-invariant* with respect to the group operation and 1-homogeneous with respect to dilations, i.e.,

$$\mathsf{d}(x, y) = \mathsf{d}(zx, zy) \quad \text{and} \quad \mathsf{d}(\delta_r x, \delta_r y) = r\,\mathsf{d}(x, y),$$

for $x, y, z \in \mathbb{H}$ and $r \geq 0$. Closed (respectively, open) balls of center $x \in \mathbb{H}$ and radius $r \geq 0$ are denoted by $\bar{\mathrm{B}}_r(x)$ (respectively, $\mathrm{B}_r(x)$). A fundamental example of such a distance is the so-called Carnot-Carathéodory distance associated to a left-invariant horizontal distribution of vector fields. By 1-homogeneity, e.g. as in [7, Proposition 1.5], there exists some constant $\mathrm{c} = \mathrm{c}(\mathsf{d}) \geq 1$ such that

$$\mathrm{c}^{-1}\left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right) \leq \mathsf{d}(x, y) \leq \mathrm{c}\left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right) \quad \text{for } x, y \in \mathbb{H}. \tag{7}$$

Horizontally differentiable maps. We introduce the following left invariant vector fields on $\mathbb{H}$, seen as derivations,

$$X_1(x^1, x^2, x^3) = \partial_1 - x^2\partial_3, \quad X_2(x^1, x^2, x^3) = \partial_2 + x^1\partial_3,$$

where ∂_i denotes the standard partial derivative with respect to x^i, $i \in \{1, 2, 3\}$. The linear span of X_1, X_2 at any point $x \in \mathbb{H}$ defines the so-called *horizontal distribution*, which is well-known to be totally non-integrable. The Carnot-Carathéodory distance associated to X_1, X_2 yields the so-called sub-Riemannian distance on $\mathbb{H}$.

For $k \geq 1$, $\alpha \in (0, 1]$, given a function $g : \mathbb{H} \to \mathbb{R}^k$ and a subset $U \subseteq \mathbb{H}$ we let

$$\|g\|_{\alpha, U} := \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|g(x) - g(y)|}{\mathsf{d}(x, y)^{\alpha}}.$$

For $F : \mathbb{H} \to \mathbb{R}^k$, we write $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$ if both derivatives $X_1F(x)$, $X_2F(x)$ exist at every $x \in \mathbb{H}$ and the *horizontal Jacobian matrix* $\nabla_{\mathsf{h}}F(x) := (X_1F(x), X_2F(x))$ satisfies

$$\|\nabla_{\mathsf{h}}F\|_{\alpha, U} < \infty, \quad \text{for every bounded } U \subseteq \mathbb{H}.$$

We say that a sequence $(F^n)_{n \geq 1} \subseteq C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$ converge to $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$ if, for every bounded set $U \subseteq \mathbb{H}$, we have $F^n \to F$ uniformly in U and $\|\nabla_{\mathsf{h}}F - \nabla_{\mathsf{h}}F^n\|_{\alpha, U} \to 0$ as $n \to \infty$.

Remark 2.1. It would be more appropriate (but heavier) to use the notation $C^{1,\alpha}_{\mathsf{h},loc}(\mathbb{H}, \mathbb{R}^k)$ instead of $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$, because maps $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$ have only *locally* Hölder continuous derivatives.

Definition 2.2. *A function $F : \mathbb{H} \to \mathbb{R}^k$ is called* h-differentiable *at $x \in \mathbb{H}$, if there exists a group homomorphism $\mathrm{d}_{\mathsf{h}}F(x) : \mathbb{H} \to \mathbb{R}^k$ such that*

$$\lim_{y \to x} \frac{\left|F(y) - F(x) - \mathrm{d}_{\mathsf{h}}F(x)\left(x^{-1}y\right)\right|}{\mathsf{d}(x, y)} = 0.$$

We say that $x \in \mathbb{H}$ is nondegenerate *for F if $\mathrm{d}_{\mathsf{h}} F(x)$ is surjective.*

In all what follows, for $x,\, y \in \mathbb{H}$, we write

$$\mathrm{R}(x,y) := F(y) - F(x) - \mathrm{d}_{\mathsf{h}} F(x)\left(x^{-1}y\right) \tag{8}$$

for the first-order horizontal Taylor expansion in x, evaluated at y. The implicit dependence upon F in such notation will be always clear from the context.

If $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^k)$, the stratified Taylor inequality [7, Theorem 1.42] ensures that F is h-differentiable at every $x \in \mathbb{H}$, with

$$\mathrm{d}_{\mathsf{h}} F(x)(y) = \nabla_{\mathsf{h}} F(x)\, y^{\mathsf{h}} = X_1 F(x)\, y^1 + X_2 F(x)\, y^2. \tag{9}$$

The same result guarantees that there exists some constant $\mathrm{c} = \mathrm{c}(\mathsf{d}) \geq 1$ such that

$$|\mathrm{R}(x,y)| \leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{\mathrm{c}r}(x)}\, \mathsf{d}(x,y)^{1+\alpha}, \quad \text{for any } x,\, y \in \mathbb{H}, \text{ with } \mathsf{d}(x,y) \leq r. \tag{10}$$

Let us stress the fact that $\|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{\mathrm{c}r}(x)}$ above is on the ball of center x and radius $\mathrm{c}\, r$.

For technical reasons, it will be useful to use horizontal Taylor expansions at a fixed point $p \in \mathbb{H}$, relying on the algebraic identity

$$\mathrm{R}(p,y) - \mathrm{R}(p,x) = F(y) - F(x) - \mathrm{d}_{\mathsf{h}} F(p)(x^{-1}y) \quad \text{for } x,\, y \in \mathbb{H}, \tag{11}$$

following from (8) and the fact that $\mathrm{d}_{\mathsf{h}} F(p)$ is a homomorphism.

For $x, y \in \bar{\mathrm{B}}_r(p)$, $r \geq 0$, one has

$$\begin{aligned}
|\mathrm{R}(p,y) - \mathrm{R}(p,x)| &\leq \left|F(y) - F(x) - \mathrm{d}_{\mathsf{h}} F(x)(x^{-1}y)\right| + \left|\left(\mathrm{d}_h F(x) - \mathrm{d}_h F(p)\right)(x^{-1}y)\right| \\
&= |\mathrm{R}(x,y)| + \left|\left(\nabla_{\mathsf{h}} F(x) - \nabla_{\mathsf{h}} F(p)\right)(x^{-1}y)^{\mathsf{h}}\right| \quad \text{by (8), (9)} \\
&\leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2\mathrm{c}r}(p)} \left(\mathsf{d}(x,y)^{1+\alpha} + \mathsf{d}(p,x)^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}}\right) \quad \text{by (10)} \\
&\leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2\mathrm{c}r}(p)} \left(\mathsf{d}(x,y)^{1+\alpha} + r^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}}\right) \\
&\leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2\mathrm{c}r}(p)} \left(\mathrm{c}^{1+\alpha}\left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right)^{1+\alpha} + r^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}}\right),
\end{aligned}$$

the latter inequality coming from (7). Applying $|a+b|^{1+\alpha} \leq 2^{\alpha}\left(a^{1+\alpha} + b^{1+\alpha}\right)$ with $a = \left[x^{-1}y\right]^{\mathsf{h}}$, $b = \left[x^{-1}y\right]^{\mathsf{v}}$ we get, for some constant $\mathrm{c} = \mathrm{c}(\mathsf{d}, \alpha) \geq 1$, and $x,\, y \in \bar{\mathrm{B}}_r(p)$, the inequality

$$|\mathrm{R}(p,y) - \mathrm{R}(p,x)| \leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2\mathrm{c}r}(p)} \left(r^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}} + \left(\left[x^{-1}y\right]^{\mathsf{v}}\right)^{1+\alpha}\right), \tag{12}$$

where we also used (7) to estimate the term

$$\left(\left[x^{-1}y\right]^{\mathsf{h}}\right)^{1+\alpha} \leq \left(\mathrm{c}\, \mathsf{d}(x,y)\right)^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}} \leq \left(2\, \mathrm{c}^2\right)^{\alpha} r^{\alpha} \left[x^{-1}y\right]^{\mathsf{h}}.$$

We also mention the weaker version of (12), which follows from it arguing as above with $\left[x^{-1}y\right]^{\mathsf{v}}$ in place of $\left[x^{-1}y\right]^{\mathsf{h}}$:

$$|\mathrm{R}(p,y) - \mathrm{R}(p,x)| \leq \mathrm{c}\, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2\mathrm{c}r}(p)}\, r^{\alpha} \left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right) \text{ for } x,\, y \in \bar{\mathrm{B}}_r(p), \tag{13}$$

for some $\mathrm{c} = \mathrm{c}(\mathsf{d}, \alpha) \geq 1$.

3. The level set differential equation

We introduce our main objects of study, i.e., suitable "differential equations" which provide parametrizations of level sets of a function $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$, in a neighbourhood of a nondegenerate point $p \in \mathbb{H}$, i.e. when the matrix $\nabla_{\mathsf{h}} F(p)$ invertible. In what follows, we fix $\alpha \in (0, 1]$.

Definition 3.1 (Level set differential equation)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Given an interval $I \subseteq \mathbb{R}$, we say that $\gamma : I \to \mathbb{H}$, $t \mapsto \gamma_t$, is a solution to the level set differential equation (LSDE) if γ is continuous and*

$$\begin{cases} \left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{h}} & = -\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \gamma_t) - \mathrm{R}(p, \gamma_s)\right) \\ \left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{v}} & = t - s + \mathrm{E}_{st} \end{cases} \qquad \text{for every } s, t \in I, \tag{14}$$

with $\mathrm{E} : I^2 \to \mathbb{R}^2$ satisfying

$$\|\mathrm{E}\| := \sup_{\substack{s,t \in I \\ s \neq t}} \frac{|\mathrm{E}_{st}|}{|t - s|^{1+\alpha}} < \infty. \tag{15}$$

Remark 3.2 (Concentration on level sets)*.* Any solution γ to the LSDE is concentrated on a level set of F, i.e. $t \mapsto F(\gamma_t)$ is constant. Actually, this follows from the "horizontal" (i.e. first) equation in (14) only, for

$$\begin{aligned} F(\gamma_t) - F(\gamma_s) &= \mathrm{d}_{\mathsf{h}} F(p) \left(\gamma_s^{-1}\gamma_t\right) + \mathrm{R}(p, \gamma_t) - \mathrm{R}(p, \gamma_s) \\ &= \nabla_{\mathsf{h}} F(p) \left(\left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{h}} + \nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \gamma_t) - \mathrm{R}(p, \gamma_s)\right) \right) = 0. \end{aligned}$$

Remark 3.3 (On the vertical equation)*.* Using the group operation (6), we can rewrite the "vertical" (i.e. second) equation in (14) as

$$\gamma_t^{\mathsf{v}} - \gamma_s^{\mathsf{v}} = t - s + (\gamma_s^1 \gamma_t^2 - \gamma_t^1 \gamma_s^2) + \mathrm{E}_{st} \quad \text{for } s, t \in I.$$

Remark 3.4 (On "errors")*.* The term E_{st} should be regarded as a natural "error" arising from the fact that the LSDE is in fact a difference equation, rather than a differential one, in the spirit of controlled equations as developed e.g. in [12]. Note that $\|E\|$ depends on $\alpha \in (0, 1]$, which is fixed throughout the paper. Condition (15) becomes crucial to make sure that the contributions of E_{st} are infinitesimal, in some sense. In principle, one could allow as well for an error term $\mathrm{E}^{\mathsf{h}}_{st}$ also in the "horizontal" equation in (14) but then, under an assumption on $\mathrm{E}^{\mathsf{h}}_{st}$ similar to (15), necessarily one has $\mathrm{E}^{\mathsf{h}}_{st} = 0$. Indeed, the first equation of (14) with the error term becomes

$$\left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{h}} = -\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \gamma_t) - \mathrm{R}(p, \gamma_s)\right) + \mathrm{E}^{\mathsf{h}}_{st}.$$

The additivity of the left hand side and of the first addend in the previous equality allow us to write these terms over a sequence of partitions of $[s, t]$ made by intervals whose lengths uniformly converge to zero. The corresponding sum of the errors $\mathrm{E}^{\mathsf{h}}_{st}$ is infinitesimal due to the condition $|\mathrm{E}^{\mathsf{h}}_{st}| \leq C\, |t - s|^{1+\alpha}$, hence leading to the first equation of (14).

Remark 3.5 (On the role of p)*.* Taking into account (11), one could also think of replacing p with γ_s, possibly allowing for an additional error $\mathrm{E}^{\mathsf{h}}_{st}$. However, for our technique to work, namely for the sewing lemma (Lemma 4.1 below) to be applicable, such a choice seems to restrict the validity of our arguments only to $\alpha > 1/2$. Moreover, let us notice that we are not requiring $\gamma_t = p$ for some t: actually, we let $I = [-\delta, \delta]$, for some $\delta > 0$, and choose γ_0 sufficiently close (but not necessarily equal) to p.

We end this section with a basic result showing that the term $t - s$ in the "vertical" equation in (14) prevents γ from being constant, actually forcing its local injectivity.

Lemma 3.6 (Local injectivity). *Let $I \subseteq \mathbb{R}$ be an interval, $\gamma : I \to \mathbb{H}$ be a solution to the LSDE associated to F, with $p \in \mathbb{H}$ nondegenerate. Then, there exist $\delta > 0$ and $\varrho > 0$ such that there holds*

$$|t-s|^{1/2} \le \varrho\, \mathsf{d}(\gamma_s, \gamma_t) \quad \text{for } s,\, t \in I,\ |t-s| \le 2\delta. \tag{16}$$

Proof. The "vertical" equation in (14) gives

$$t - s = (\gamma_s^{-1}\gamma_t)^{\mathsf{v}} - \mathrm{E}_{st},$$

hence, if $s, t \in I$ satisfy $|t-s| \le 2\delta$, then

$$|t-s| \le \left|(\gamma_s^{-1}\gamma_t)^{\mathsf{v}}\right| + |\mathrm{E}_{st}| \le (\mathrm{c}\, \mathsf{d}(\gamma_s, \gamma_t))^2 + (2\delta)^\alpha \|\mathrm{E}\| \, |t-s|\,, \tag{17}$$

by (7) and (15). If we choose $\delta > 0$, $\varrho > 0$ such that

$$(2\delta)^\alpha \|\mathrm{E}\| \le \frac{1}{2} \quad \text{and} \quad \varrho^2 \ge 2\,\mathrm{c}^2, \tag{18}$$

we obtain, from (17),

$$|t-s| \le 2\,(\mathrm{c}\, \mathsf{d}(\gamma_s, \gamma_t))^2 = 2\,\mathrm{c}^2 \mathsf{d}(\gamma_s, \gamma_t)^2 \le (\varrho \mathsf{d}(\gamma_s, \gamma_t))^2\,,$$

hence the thesis. □

4. Existence of solutions

To provide existence of some solution to the LSDE we rely on the fundamental tool of the theory of controlled paths, sometimes called *sewing lemma*, which allows us to cast the differential equations into an "integral" form, and perform a Schauder fixed point argument.

Lemma 4.1 (Sewing lemma). *For $\alpha \in (0,1]$, $k \ge 1$, there exists some constant $\kappa > 0$ such that the following holds. For any interval I and continuous $\mathrm{A} : I^2 \to \mathbb{R}^k$ that satisfies*

$$|\mathrm{A}_{st} - \mathrm{A}_{su} - \mathrm{A}_{ut}| \le \|\mathrm{A}\| \, |t-s|^{1+\alpha} \quad \text{for } s,\, u,\, t \in I \text{ with } s \le u \le t,$$

for some constant $\|\mathrm{A}\|$, then there exists a continuous function $f : I \to \mathbb{R}$ such that

$$|f_t - f_s - \mathrm{A}_{st}| \le \kappa \, \|\mathrm{A}\| \, |t-s|^{1+\alpha} \quad \text{for } s,\, t \in I. \tag{19}$$

For a proof, we refer e.g. to [5, Lemma 2.1] (see also [6, Theorem 2] for more general moduli of continuity).

Remark 4.2 (Young integrals). The theory of integration in the sense of Young, introduced in the seminal paper [26], can be recovered as an instance of the sewing lemma. Actually, the LSDE could be stated as well as an integral Young equation, but we chose to adopt the modern point of view as in [12]. Indeed, for $g^1 : I \to \mathbb{R}$, $g^2 : I \to \mathbb{R}$, define

$$\mathrm{A}_{st} := g^1_s \left(g^2_t - g^2_s\right) \quad \text{for } s,\, t \in I. \tag{20}$$

Then, for s, t, $u \in I$, with $s \le u \le t$,

$$\begin{aligned} \mathrm{A}_{st} - \mathrm{A}_{su} - \mathrm{A}_{ut} &= g^1_s \left(g^2_t - g^2_s\right) - g^1_s \left(g^2_u - g^2_s\right) - g^1_u \left(g^2_t - g^2_u\right) \\ &= \left(g^1_s - g^1_u\right)\left(g^2_t - g^2_u\right). \end{aligned} \tag{21}$$

Therefore, if $g^1 \in C^{\beta_1}(I;\mathbb{R})$, $g^2 \in C^{\beta_2}(I;\mathbb{R})$, with $\beta_1+\beta_2>1$,

$$|\mathrm{A}_{st} - \mathrm{A}_{su} - \mathrm{A}_{ut}| \le \left\|g^1\right\|_{\beta_1} \left\|g^2\right\|_{\beta_2} |t-s|^{\beta_1+\beta_2}\,, \tag{22}$$

and the sewing lemma applies, yielding a function f, which one could show [12] that satisfies $f_t = f_0 + \int_0^t g^1_s dg^2_s$, where integration is in the sense of Young.

Remark 4.3 (Uniqueness). Clearly, if f satisfies (19) and we add to f a constant function, the sum still satisfies (19), hence we may always additionally prescribe the value of f at some (but only one) $t \in I$. Moreover, we have uniqueness up to additive constants, in the following sense: if $g : I \to \mathbb{R}$ satisfies

$$\limsup_{\substack{s,t\in I\\ |s-t|\to 0}} \frac{|g_t - g_s - \mathrm{A}_{st}|}{|t-s|} = 0,$$

then $h := f - g$ is a constant function. Indeed, the triangle inequality gives

$$\limsup_{\substack{s,t\in I\\ |s-t|\to 0}} |h_t - h_s| / |t-s| = 0,$$

i.e., the derivative of h exists and it vanishes at every point of I.

Theorem 4.4 (Existence)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H},\mathbb{R}^2)$. Then, there exist positive δ_0, ε_0, ϱ_0 such that, for any $q \in \bar{\mathrm{B}}_{\varepsilon_0}(p)$, there is an injective solution γ to the LSDE on the interval $[-\delta_0,\delta_0]$, with $\gamma_0 = q$,*

$$\|\gamma^{\mathsf{h}}\|_{\frac{1+\alpha}{2}} \le \varrho_0 \quad \textit{and} \quad \|\mathrm{E}\| \le \kappa \|\gamma^{\mathsf{h}}\|^2_{\frac{1+\alpha}{2}} \le \kappa\varrho_0^2. \tag{23}$$

The proof relies on an application of the Schauder fixed point theorem, i.e. we find a convex invariant set K for a map Φ naturally defined by the LSDE. The main technical difficulty, however, is in showing that Φ is continuous in an appropriate topology: here we deal with an integral, defined implicitly by the sewing lemma, which does not allow us to move the absolute value inside, as the in the case of Lebesgue integral, which makes the argument much more involved than the standard one working for ODE's.

Proof. For simplicity, write in what follows positive δ, ε, ϱ to be chosen sufficiently small, yielding δ_0, ε_0 and ϱ_0 as in the thesis. Write $I := [-\delta,\delta]$ and fix $q \in \bar{\mathrm{B}}_\varepsilon(p)$.

Introduction of space and map. We introduce the following compact, convex subset K of $C^{\frac{1+\alpha}{2}}(I;\mathbb{R}^2)$,

$$\mathrm{K} = \left\{\eta = (\eta^1,\eta^2) : I \to \mathbb{R}^2 \,\middle|\, \eta_0 = q^{\mathsf{h}},\ \|\eta\|_{\frac{1+\alpha}{2}} \le \varrho\right\}.$$

We define the map Φ, on K, $\eta \mapsto \Phi(\eta)$, in two steps. First, for $\eta \in \mathrm{K}$, we apply Lemma 4.1 with

$$\mathrm{A}_{st} := t - s - (\eta^1_t\eta^2_s - \eta^1_s\eta^2_t) \quad \text{for } s,t \in I \tag{24}$$

to obtain a (unique) function $f : I \to \mathbb{R}$, such that (19) and $f_0 = q^{\mathsf{v}}$ hold. Then, we define

$$\bar{\eta}_t := (\eta_t, f_t) \in \mathbb{H} \quad \text{for } t \in I, \tag{25}$$

and finally set

$$\Phi(\eta)_t := q^{\mathsf{h}} + \nabla_{\mathsf{h}} F(p)^{-1}\left(R(p,\bar{\eta}_t) - R(p,q)\right), \qquad \text{for } t \in I. \tag{26}$$

The map Φ *is well defined:* it suffices to show that the sewing lemma can be applied to (24). Adding and subtracting the quantity $\eta^2_s\eta^1_s$, we rewrite the right hand side in (24) as

$$\mathrm{A}_{st} = t - s - \eta^2_s(\eta^1_t - \eta^1_s) + \eta^1_s(\eta^2_t - \eta^2_s), \tag{27}$$

which shows that A_{st} is a sum of $t-s$ and two other terms of Young type, i.e. as in (20). Therefore, arguing as in (21), we obtain

$$\mathrm{A}_{st} - \mathrm{A}_{su} - \mathrm{A}_{ut} = -\left(\eta^2_s - \eta^2_u\right)\left(\eta^1_t - \eta^1_u\right) + \left(\eta^1_s - \eta^1_u\right)\left(\eta^2_t - \eta^2_u\right),$$

hence, as in (22), for $s \le u \le t$,

$$|\mathrm{A}_{st} - \mathrm{A}_{su} - \mathrm{A}_{ut}| \le \|\eta\|^2_{\frac{1+\alpha}{2}} |t-s|^{1+\alpha}.$$

Thus, we are in a position to apply Lemma 4.1, obtaining $f : I \to \mathbb{R}$ with $f_0 = q^{\mathsf{v}}$ and

$$|f_t - f_s - \mathrm{A}_{st}| \le \kappa \|\eta\|^2_{\frac{1+\alpha}{2}} |t-s|^{1+\alpha} \quad \text{for } s, t \in I. \tag{28}$$

Claim: $\bar\eta_t \in \bar{\mathrm{B}}_{2\varepsilon}(p)$. We provide conditions on δ, ε, ϱ which ensure the claim. By definition (25) of $\bar\eta$ and (24),

$$(\bar\eta_s^{-1}\bar\eta_t)^{\mathsf{v}} = \bar\eta^{\mathsf{v}}_t - \bar\eta^{\mathsf{v}}_s + (\eta^1_t\eta^2_s - \eta^1_s\eta^2_t) = f_t - f_s - \mathrm{A}_{st} + t - s,$$

hence (28) and the conditions $|t-s| \le 2\delta$, $\|\eta\|_{\frac{1+\alpha}{2}} \le \varrho$ imply

$$\left|(\bar\eta_s^{-1}\bar\eta_t)^{\mathsf{v}}\right| \le \kappa\varrho^2 |t-s|^{1+\alpha} + |t-s| \le \left(1 + \kappa\varrho^2(2\delta)^\alpha\right)|t-s|$$

so that

$$\left[\bar\eta_s^{-1}\bar\eta_t\right]^{\mathsf{v}} \le \sqrt{1+\kappa\varrho^2(2\delta)^\alpha}\, |t-s|^{1/2}. \tag{29}$$

Since

$$\left[\bar\eta_s^{-1}\bar\eta_t\right]^{\mathsf{h}} = \left|\left(\eta^1_t - \eta^1_s, \eta^2_t - \eta^2_s\right)\right| \le \|\eta\|_{\frac{1+\alpha}{2}} |t-s|^{\frac{1+\alpha}{2}} \le \varrho\,(2\delta)^{\alpha/2}|t-s|^{1/2}, \tag{30}$$

the bound between d and the sum of the horizontal and vertical gauges (7) yields

$$\mathsf{d}(\bar\eta_s, \bar\eta_t) \le \mathrm{c}\left(\varrho(2\delta)^{\alpha/2} + \sqrt{1+\kappa\varrho^2(2\delta)^\alpha}\right)|t-s|^{1/2}.$$

If ϱ and δ satisfy

$$\varrho(2\delta)^{\alpha/2} \le 1, \tag{31}$$

it follows that

$$\mathsf{d}(\bar\eta_t, \bar\eta_s) \le \mathrm{c}(1 + \sqrt{1+\kappa})\,|t-s|^{1/2} \quad \text{for } s,\ t \in I.$$

For $s = 0$, $\bar\eta_0 = q$, so that $\mathsf{d}(\bar\eta_t, q) \le \mathrm{c}(1+\sqrt{1+\kappa})\delta^{1/2}$. If δ and ε satisfy

$$\mathrm{c}(1+\sqrt{1+\kappa})\delta^{1/2} \le \varepsilon, \tag{32}$$

then $\mathsf{d}(\bar\eta_t, p) \le \mathsf{d}(\bar\eta_t, q) + \mathsf{d}(q,p)$ gives $\bar\eta_t \in \bar{\mathrm{B}}_{2\varepsilon}(p)$, for $t \in I$, i.e. the claim.

Φ maps K *into itself.* We give further conditions on δ, ε, ϱ to ensure that $\Phi(\mathrm{K}) \subseteq \mathrm{K}$. Since (26) for $t = 0$ gives $\Phi(\eta)_0 = q^{\mathsf{h}}$, we only have to prove that $\|\Phi(\eta)\|_{\frac{1+\alpha}{2}} \le \varrho$. To this aim, we estimate

$$\begin{aligned}
\|\Phi(\eta)_t - \Phi(\eta)_s\| &= \left|\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \bar\eta_t) - \mathrm{R}(p, \bar\eta_s)\right)\right| \quad \text{by (26)}\\
&\le \left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \left|\mathrm{R}(p, \bar\eta_t) - \mathrm{R}(p, \bar\eta_s)\right|\\
&\le \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon}(p)} \left((2\varepsilon)^{\alpha} \left[\eta_s^{-1}\eta_t\right]^{\mathsf{h}} + \left(\left[\eta_s^{-1}\eta_t\right]^{\mathsf{v}}\right)^{1+\alpha}\right)\\
&\qquad \text{by the previous claim and (12) with } x = \bar\eta_s,\ y = \bar\eta_t,\ r = 2\varepsilon\\
&\le \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon}(p)} \left((2\varepsilon)^{\alpha} \|\eta\|_{\frac{1+\alpha}{2}} + \left(1 + \kappa\varrho^2 (2\delta)^{\alpha}\right)^{\frac{1+\alpha}{2}}\right) |t-s|^{\frac{1+\alpha}{2}}\\
&\qquad \text{by (30) and (29)}\\
&\le \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon}(p)} \left[(2\varepsilon)^{\alpha}\varrho + (1+\kappa)^{\frac{1+\alpha}{2}}\right] |t-s|^{\frac{1+\alpha}{2}} \quad \text{by (31).}
\end{aligned}$$

We conclude that, if δ, ε and ϱ, satisfy (31), (32) and

$$\mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon}(p)} \left((2\varepsilon)^{\alpha}\varrho + (1+\kappa)^{\frac{1+\alpha}{2}}\right) \le \varrho, \tag{33}$$

then $\Phi(\mathrm{K}) \subseteq \mathrm{K}$. Let us then fix $\delta = \delta_0$, $\varepsilon = \varepsilon_0$ and $\varrho = \varrho_0 > 0$ such that these conditions are satisfied: this can be achieved e.g. choosing first $\varepsilon_0 > 0$ small enough and then $\varrho_0 > 0$ large enough such that (33) holds, and finally choosing $\delta_0 > 0$ small enough so that both (31) and (32) holds. To ensure that any solution (to be obtained) γ be injective, we also require that (18) in Lemma 3.6 hold with $\delta = \delta_0$ and with $\varrho = \varrho_0$, with $\kappa\varrho^2$ in place of $\|E\|$.

Existence and properties of fixed points. Let us fix $\beta \in (0, \alpha)$. Taking for granted continuity of $\Phi : \mathrm{K} \to \mathrm{K}$ in the topology of $C^{\frac{1+\beta}{2}}(I; \mathbb{R}^2)$, which will be proven in a further technical step, by compactness of the embedding $C^{\frac{1+\alpha}{2}}(I; \mathbb{R}^2)$ into $C^{\frac{1+\beta}{2}}(I; \mathbb{R}^2)$, we apply Schauder fixed point theorem, see e.g. [10, Theorem 11.1], obtaining some $\eta \in \mathrm{K}$ such that $\Phi(\eta) = \eta$.

Let us show that $\gamma = \bar\eta$ defined by (25) solves the LSDE with $\gamma_0 = q$ and (23). Indeed, by definition of $\bar\eta$, we have immediately that $\bar\eta_0 = q$ and the first inequality in (23) holds because $\eta \in \mathrm{K}$. The "horizontal" equation in (14) holds for s, $t \in I$, because

$$\left(\bar\eta_s^{-1}\bar\eta_t\right)^{\mathsf{h}} = \eta_t - \eta_s = \Phi(\eta)_t - \Phi(\eta)_s = -\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \bar\eta_t) - \mathrm{R}(p, \bar\eta_s)\right),$$

by (26). For the "vertical" equation in (14), we notice that, by (24),

$$\mathrm{E}_{st} = (\gamma_s^{-1}\gamma_t)^{\mathsf{v}} - (t-s) = \eta_t^{\mathsf{v}} - \eta_s^{\mathsf{v}} - \mathrm{A}_{st},$$

hence the second inequality in (23) follows from (28), recalling that $f = \eta^{\mathsf{v}}$ therein. Finally, γ is injective on I by Lemma 3.6 in view of the choice of δ_0 and ϱ_0.

Continuity of Φ. We see that conditions (31), (32) and (33) imply that $\Phi : \mathrm{K} \to \mathrm{K}$ is continuous with respect to the topology induced by the norm $\|\cdot\|_{\frac{1+\beta}{2}}$, for any $\beta \in (0, \alpha)$. The argument, relying on a real interpolation, is close to that in [12, Proposition 5]. For η, $\zeta \in \mathrm{K}$ and $t \in I$, by (26), we have

$$\begin{aligned}
|\Phi(\eta)_t - \Phi(\zeta)_t| &= \left|\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, \bar\eta_t) - \mathrm{R}(p, \bar\zeta_t)\right)\right|\\
&\le \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon}(p)} (2\varepsilon)^{\alpha} \left(\left[\bar\zeta_t^{-1}\bar\eta_t\right]^{\mathsf{h}} + \left[\bar\zeta_t^{-1}\bar\eta_t\right]^{\mathsf{v}}\right),
\end{aligned}$$

the inequality following from (13), applied with $x = \bar{\zeta}_t$, $y = \bar{\eta}_t$ (recall that $\bar{\eta}_t, \bar{\zeta}_t \in \bar{\mathrm{B}}_{2\varepsilon}(p)$).

Denote $\xi_t := \eta_t - \zeta_t$. Since $\zeta_0 = \eta_0 = q^{\mathsf{h}}$, we have $\xi_0 = 0$ and

$$\left[\bar{\zeta}_t^{-1}\bar{\eta}_t\right]^{\mathsf{h}} = |\xi_t| \leq \|\xi\|_0 \leq \delta^{\frac{1+\beta}{2}} \|\xi\|_{\frac{1+\beta}{2}} . \tag{34}$$

To estimate the term $\left[\bar{\zeta}_t^{-1}\bar{\eta}_t\right]^{\mathsf{v}}$, we notice first that the group operation (6) yields

$$(\bar{\zeta}_t^{-1}\bar{\eta}_t)^{\mathsf{v}} = (\bar{\eta}_t^{\mathsf{v}} - \bar{\zeta}_t^{\mathsf{v}}) + (\eta_t^1\zeta_t^2 - \zeta_t^1\eta_t^2).$$

The second term in the above sum is easily estimated, since

$$\begin{aligned}\left|\eta_t^1\zeta_t^2 - \zeta_t^1\eta_t^2\right| &= \left|\left(\eta_t^1 - \zeta_t^1\right)\zeta_t^2 - \zeta_t^1\left(\eta_t^2 - \zeta_t^2\right)\right| = \left|\xi_t^1\zeta_t^2 - \zeta^1\xi_t^2\right| \leq |\zeta_t|\,|\xi_t| \\ &\leq \left(|q^{\mathsf{h}}| + \varrho\delta^{\frac{1+\alpha}{2}}\right)\delta^{\frac{1+\beta}{2}} \|\xi\|_{\frac{1+\beta}{2}} ,\end{aligned}$$

using the last inequality of (34) and

$$\sup_{t\in I}|\zeta_t| \leq |q^{\mathsf{h}}| + \|\zeta\|_{\frac{1+\alpha}{2}} t^{\frac{1+\alpha}{2}} \leq |q^{\mathsf{h}}| + \varrho\delta^{\frac{1+\alpha}{2}}. \tag{35}$$

Hence, we are reduced to find a bound on $\bar{\eta}_t^{\mathsf{v}} - \bar{\zeta}_t^{\mathsf{v}}$. Below, we prove that for some constant $\mathrm{c} > 0$ independent of η and ζ, one has

$$\left|\bar{\eta}_t^{\mathsf{v}} - \bar{\zeta}_t^{\mathsf{v}}\right| \leq \mathrm{c}\,\|\xi\|_{\frac{1+\beta}{2}} . \tag{36}$$

Once (36) is proved, we conclude that

$$\|\Phi(\eta) - \Phi(\zeta)\|_0 = \sup_{t\in I}|\Phi(\eta)_t - \Phi(\zeta)_t| \leq \mathrm{c}\,\|\xi\|_{\frac{1+\beta}{2}} ,$$

for some (different) $\mathrm{c} > 0$ independent of η and ζ. Using the bound

$$\|\Phi(\eta) - \Phi(\zeta)\|_{\frac{1+\alpha}{2}} \leq \|\Phi(\eta)\|_{\frac{1+\alpha}{2}} + \|\Phi(\zeta)\|_{\frac{1+\alpha}{2}} \leq 2\varrho,$$

together with the interpolation inequality

$$\|\cdot\|_{\frac{1+\beta}{2}} \leq \|\cdot\|_0^{\frac{\alpha-\beta}{1+\alpha}} \|\cdot\|_{\frac{1+\alpha}{2}}^{\frac{1+\beta}{1+\alpha}}$$

finally gives, again for some (different) $\mathrm{c} > 0$,

$$\|\Phi(\eta) - \Phi(\zeta)\|_{\frac{1+\beta}{2}} \leq \mathrm{c}\,\|\xi\|_{\frac{1+\beta}{2}}^{\frac{\alpha-\beta}{1+\alpha}} = \mathrm{c}\,\|\eta - \zeta\|_{\frac{1+\beta}{2}}^{\frac{\alpha-\beta}{1+\alpha}} ,$$

which yields continuity of Φ.

Proof of (36). This follows from another application of the sewing lemma, but using its uniqueness part. Indeed, we denote $\mathrm{A}_{st}(\eta)$ (resp. $\mathrm{A}_{st}(\zeta)$) the right hand side of (24) (resp. with ζ instead of η) and define (only) here

$$\mathrm{A}_{st} := \mathrm{A}_{st}(\eta) - \mathrm{A}_{st}(\zeta).$$

Since $\bar{\eta}^{\mathsf{v}}$ and $\bar{\zeta}^{\mathsf{v}}$ are both built via sewing lemma, we have

$$\frac{\left|(\bar{\eta}_t^{\mathsf{v}} - \bar{\zeta}_t^{\mathsf{v}}) - (\bar{\eta}_s^{\mathsf{v}} - \bar{\zeta}_s^{\mathsf{v}}) - \mathrm{A}_{st}\right|}{|t-s|} \leq \frac{|\bar{\eta}_t^{\mathsf{v}} - \bar{\eta}_s^{\mathsf{v}} - \mathrm{A}_{st}(\eta)|}{|t-s|} + \frac{\left|\bar{\zeta}_t^{\mathsf{v}} - \bar{\zeta}_s^{\mathsf{v}} - \mathrm{A}_{st}(\zeta)\right|}{|t-s|} \to 0$$

as $|t-s| \to 0$. Next, we check that the sewing lemma applies to A_{st}, but looking for a quantitative bound in terms of ξ. Uniqueness will imply the required estimate. To this aim, recalling that $\mathrm{A}_{st}(\eta)$, $\mathrm{A}_{st}(\zeta)$ can be rewritten as in (27), recollecting all the terms that appear, we see that A_{st} is given by the difference between

$$\zeta_s^2\left(\zeta_t^1-\zeta_s^1\right)-\eta_s^2\left(\eta_t^1-\eta_s^1\right) \tag{37}$$

and an analogous term,

$$\zeta_s^1\left(\zeta_t^2-\zeta_s^2\right)-\eta_s^1\left(\eta_t^2-\eta_s^2\right). \tag{38}$$

Adding and subtracting the quantity $\left(\zeta_t^1-\zeta_s^1\right)\eta_s^2$ in (37), we transform it into

$$-\xi_s^2\left(\zeta_t^1-\zeta_s^1\right)-\eta_s^2\left(\xi_t^1-\xi_s^1\right). \tag{39}$$

The absolute value of the latter expression can be bounded from above by

$$\begin{aligned}
&\sup_{u\in I}|\xi_u|\,\|\zeta\|_{\frac{1+\alpha}{2}}\,|t-s|^{\frac{1+\alpha}{2}}+\sup_{u\in I}|\eta_u|\,\|\xi\|_{\frac{1+\beta}{2}}\,|t-s|^{\frac{1+\beta}{2}}\le\\
&\qquad\le\left(\varrho(2\delta)^{\frac{1+\alpha}{2}}+\left(|q^{\mathsf{h}}|+\varrho\delta^{\frac{1+\alpha}{2}}\right)\right)(2\delta)^{\frac{1+\beta}{2}}\,\|\xi\|_{\frac{1+\beta}{2}}\quad\text{by (34), (35) with }\eta\text{ instead of }\zeta\\
&\qquad\le\left(|q^{\mathsf{h}}|+2\varrho(2\delta)^{\frac{1+\alpha}{2}}\right)(2\delta)^{\frac{1+\beta}{2}}\,\|\xi\|_{\frac{1+\beta}{2}}\,.
\end{aligned}$$

Arguing with (38) in a similar way, we conclude that for $s,\,t\in I$,

$$|\mathrm{A}_{st}|\le 2\left(|q^{\mathsf{h}}|+2\varrho(2\delta)^{\frac{1+\alpha}{2}}\right)(2\delta)^{\frac{1+\beta}{2}}\,\|\xi\|_{\frac{1+\beta}{2}}\,. \tag{40}$$

Now we estimate $\mathrm{A}_{st}-\mathrm{A}_{su}-\mathrm{A}_{ut}$, for $s,\,t,\,u\in I$, with $s\le u\le t$. Arguing as in (20), (21) of Remark 4.2, we see that (39) yields a contribution to this quantity equal to

$$-\left(\xi_s^2-\xi_u^2\right)\left(\zeta_t^1-\zeta_u^1\right)-\left(\eta_s^2-\eta_u^2\right)\left(\xi_t^1-\xi_u^1\right),$$

the absolute value of which is estimated from above by

$$\left(\|\zeta\|_{\frac{1+\alpha}{2}}+\|\eta\|_{\frac{1+\alpha}{2}}\right)\|\xi\|_{\frac{1+\beta}{2}}\,|t-s|^{1+(\alpha+\beta)/2}\le 2\varrho\,\|\xi\|_{\frac{1+\beta}{2}}\,|t-s|^{1+(\alpha+\beta)/2}\,.$$

If we argue with the contribution of (38) in a similar way, we conclude that

$$|\mathrm{A}_{st}-\mathrm{A}_{su}-\mathrm{A}_{ut}|\le 4\rho\,\|\xi\|_{\frac{1+\beta}{2}}\,|t-s|^{1+(\alpha+\beta)/2}\,.$$

Thus, we are in a position to apply Lemma 4.1, obtaining a (unique) function $f:I\to\mathbb{R}$ such that $f_0=0$ and

$$|f_t-f_s-\mathrm{A}_{st}|\le 4\kappa\rho\,\|\xi\|_{\frac{1+\beta}{2}}\,|t-s|^{1+(\alpha+\beta)/2}\quad\text{for } s,\,t\in I. \tag{41}$$

with $\kappa=\kappa((\alpha+\beta)/2)$. From Remark 4.3 it follows that $f=\bar\eta^{\mathsf{v}}-\bar\zeta^{\mathsf{v}}$. By (40) and (41), with $s=0$, $t\in I$, we conclude that

$$\left|\bar\eta_t^{\mathsf{v}}-\bar\zeta_t^{\mathsf{v}}\right|\le\left[4\kappa\varrho\delta^{1+(\alpha+\beta)/2}+2\left(|q^{\mathsf{h}}|+2(2\delta)^{\frac{1+\alpha}{2}}\varrho\right)(2\delta)^{\frac{1+\beta}{2}}\right]\|\xi\|_{\frac{1+\beta}{2}}\,,$$

as claimed. □

The proof of Theorem 4.4 yields the following sensitivity result. Actually, one could provide an alternative proof of Theorem 4.4 by proving first the following result and then approximating F in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H},\mathbb{R}^2)$ with a sequence of smooth functions $\{F^n\}_{n\ge1}$.

Corollary 4.5 (Sensitivity). *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ and let $\{F^n\}_{n\geq 1} \subseteq C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ converge to F in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Then, there exist $\bar{n} \geq 1$ and positive δ_0, ε_0, ϱ_0 such that, for any $n \geq \bar{n}$ and $q^n \in \bar{\mathrm{B}}_{\varepsilon_0}(p)$, there is an injective solution $\gamma^n : [-\delta_0, \delta_0] \to \mathbb{H}$ to the LSDE associated to F^n, with $\gamma^n_0 = q^n$ and*

$$\| (\gamma^n)^{\mathsf{h}} \|_{\frac{1+\alpha}{2}} \leq \varrho_0 \quad \text{and} \quad \|\mathrm{E}^n\| \leq \kappa \| (\gamma^n)^{\mathsf{h}} \|^2_{\frac{1+\alpha}{2}}. \tag{42}$$

Moreover, the family $(\gamma^n)_{n\geq 1}$ is compact (with respect to uniform convergence) and any limit point is an injective solution γ to the LSDE associated to F, such that (23) holds.

Proof. First, we notice that the constant(s) c appearing in (32) and (33) are independent of F. As $n \to \infty$, since $\|\nabla_{\mathsf{h}} F^n - \nabla_{\mathsf{h}} F\|_{\alpha,U} \to 0$ for every bounded $U \subseteq \mathbb{H}$, choosing $U = \bar{\mathrm{B}}_{4\mathrm{c}}(p)$, for n large enough, $\nabla_{\mathsf{h}} F^n(p)^{-1}$ exists and

$$\nabla_{\mathsf{h}} F^n(p)^{-1} \to \nabla_{\mathsf{h}} F^n(p)^{-1} \quad \text{and} \quad \|\nabla_{\mathsf{h}} F^n\|_{\alpha,\bar{\mathrm{B}}_{4\mathrm{c}}(p)} \to \|\nabla_{\mathsf{h}} F\|_{\alpha,\bar{\mathrm{B}}_{4\mathrm{c}}(p)}.$$

Therefore, if we choose δ_0, ε_0, ϱ_0 such that (31), (32) and (33) hold true for F as strict inequalities, as well as (18), with $\kappa\varrho_0^2$ instead of $\|E\|$, then there exists $\bar{n} \geq 1$ such that for any $n \geq \bar{n}$, strict inequalities hold in the analogues of (31), (32) and (33), as well as (18), with F^n instead of F.

Then, the arguments in the proof of Theorem 4.4 apply for F^n and provide existence of some injective solution $\gamma^n : [-\delta_0, \delta_0] \to \mathbb{H}$ of the LSDE associated to F^n, with $\gamma^n_0 = q^n \in \bar{\mathrm{B}}_{\varepsilon_0}(p)$. Moreover, the analogues of (23) with F^n and E^n (instead of F and E) hold uniformly in $n \geq \bar{n}$. This yields compactness for $\gamma^n : [-\delta_0, \delta_0] \to \mathbb{H}$, with respect to uniform convergence. Indeed, given s, $t \in [-\delta_0, \delta_0]$ we have $\gamma^n_t \in \bar{\mathrm{B}}_{4\mathrm{c}\varepsilon_0}(p)$ and, by (7),

$$\begin{aligned} \mathsf{d}(\gamma^n_s, \gamma^n_t) &\leq \mathrm{c}\left(\left[(\gamma^n_s)^{-1}\gamma^n_t\right]^{\mathsf{h}} + \left[(\gamma^n_s)^{-1}\gamma^n_t\right]^{\mathsf{v}} \right) \\ &\leq \mathrm{c}\left(\varrho_0 \, |t-s|^{\frac{1+\alpha}{2}} + \sqrt{|t-s| + \kappa\varrho_0^2 \, |t-s|^{1+\alpha}} \right), \end{aligned}$$

using the "vertical" equation and the uniform bound on $\|\mathrm{E}^n\|$.

Finally, to show that any limit point γ of $\{\gamma^n\}_{n\geq 1}$ solves the LSDE associated to F, we recall that $\nabla_{\mathsf{h}} F^n(p)^{-1} \to \nabla_{\mathsf{h}} F^n(p)^{-1}$ and notice that $\mathrm{R}^n(p, x)$, defined by (8) with F^n in place of F converge to $\mathrm{R}(p, x)$ uniformly in $\bar{\mathrm{B}}_{4\mathrm{c}\varepsilon_0}(p)$, hence both equations in (14) pass to the limit along any converging subsequence $\{\gamma^{n_k}\}_k$. Moreover, (15) immediately yields that $\mathrm{E} \mapsto \|\mathrm{E}\|$ is lower semicontinuous with respect to uniform convergence of γ, hence in the limit $\|\mathrm{E}\| \leq \kappa\varrho_0^2$. Finally, γ is injective on $[-\delta_0, \delta_0]$ because of Lemma 3.6 and our choice of δ_0 and ϱ_0, so that (18) holds. □

5. Parametrization of level sets

In this section, we prove that any solution to the LSDE provides a local parametrization of the level set of F, where it is concentrated, see Remark 3.2. The argument relies on the following two lemmas.

Lemma 5.1 ("Horizontal" injectivity). *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Then, there exists $\varepsilon_1 > 0$ such that, whenever $x, y \in \bar{\mathrm{B}}_{\varepsilon_1}(p)$ satisfy*

$$F(x) = F(y) \quad \text{and} \quad \left[x^{-1}y\right]^{\mathsf{v}} \leq \left[x^{-1}y\right]^{\mathsf{h}},$$

we must have $x = y$. In particular, the second condition holds if $[x^{-1}y]^{\mathsf{v}} = 0$.

Proof. For simplicity, write throughout the proof $\varepsilon > 0$ to be specified below, yielding ε_1 as in the thesis. Using the condition $F(x) = F(y)$ in (11), we obtain

$$\left(x^{-1}y\right)^{\mathsf{h}} = -\nabla_{\mathsf{h}} F(p)^{-1} \left(\mathrm{R}(p, y) - \mathrm{R}(p, x)\right).$$

From (13), with ε instead of r, we have

$$\left[x^{-1}y\right]^{\mathsf{h}} \leq \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2c\varepsilon}(p)}\, \varepsilon^{\alpha} \left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right).$$

If $\varepsilon_1 = \varepsilon > 0$ is chosen so that

$$\mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{2c\varepsilon}(p)}\, \varepsilon^{\alpha} \leq \frac{1}{4}, \tag{43}$$

then

$$\left[x^{-1}y\right]^{\mathsf{h}} \leq \frac{1}{4}\left(\left[x^{-1}y\right]^{\mathsf{h}} + \left[x^{-1}y\right]^{\mathsf{v}}\right) \leq \frac{1}{2}\left[x^{-1}y\right]^{\mathsf{h}}$$

using $\left[x^{-1}y\right]^{\mathsf{v}} \leq \left[x^{-1}y\right]^{\mathsf{h}}$, which implies $\left[x^{-1}y\right]^{\mathsf{h}} = 0$ hence also $\left[x^{-1}y\right]^{\mathsf{v}} = 0$, i.e. $x = y$. □

Lemma 5.2. *Let $I \subseteq \mathbb{R}$ be an interval whose interior contains 0 and let $\gamma : I \to \mathbb{H}$ be continuous, with*

$$\left[\gamma_0^{-1}\gamma_t\right]^{\mathsf{h}} \leq \varrho\, |t|^{\frac{1+\alpha}{2}} \quad \textit{and} \quad \left|\left(\gamma_0^{-1}\gamma_t\right)^{\mathsf{v}} - t\right| \leq \varrho^2\, |t|^{1+\alpha} \quad \textit{for } t \in I, \tag{44}$$

for some $\varrho > 0$. Then, there exists $\delta_2 > 0$ such that $[-\delta_2, \delta_2] \subseteq I$ and the following holds: for any $\delta \in (0, \delta_2]$, there is $\varepsilon_2 = \varepsilon_2(\delta) > 0$ such that, if $x \in \bar{\mathrm{B}}_{\varepsilon_2}(\gamma_0)$, then

$$\left[\gamma_t^{-1}x\right]^{\mathsf{v}} \leq \left[\gamma_t^{-1}x\right]^{\mathsf{h}} \quad \textit{for some } t = t(x) \in [-\delta, \delta].$$

Remark 5.3. Solutions to the LSDE satisfy (44), up to restricting their interval of definition, by Lemma 6.1 below.

Remark 5.4 (Comparison with the Euclidean case). The above lemma, which seems intricated, becomes obvious if formulated for classical situation of a continuously differentiable curve in Euclidean space $\mathbb{R}^3$ (instead of $\mathbb{H}$), which means replacing the second condition in (44) by requiring $\dot{\gamma}_0^3 = 1$ (the first condition in (44) is then unnecessary). In fact, one can obtain $t = t(x)$ such that $x^3 = \gamma_t^3$.

Proof. For simplicity, write throughout the proof δ, ε to be specified below, yielding δ_2, ε_2 as in the thesis. Without any loss of generality, we assume $\gamma_0 = 0$: the general case follows from reducing to the curve $\gamma_0^{-1}\gamma_t$. Using the group operation (6), we write for $t \in I$,

$$(\gamma_t^{-1}x)^{\mathsf{v}} = x^{\mathsf{v}} - \gamma_t^{\mathsf{v}} - \gamma_t^1 x^2 + \gamma_t^2 x^1 = x^{\mathsf{v}} - \gamma_t^{\mathsf{v}} - \gamma_t^1(x^2 - \gamma_t^2) + \gamma_t^2(x^1 - \gamma_t^1).$$

The inequality $|ab| \leq a^2/4 + b^2$ yields

$$\begin{aligned}
(\gamma_t^{-1}x)^{\mathsf{v}} &\leq x^{\mathsf{v}} - \gamma_t^{\mathsf{v}} + \frac{\left|\gamma_t^1\right|^2}{4} + (x^2 - \gamma_t^2)^2 + \frac{\left|\gamma_t^2\right|^2}{4} + (x^1 - \gamma_t^1)^2 \\
&= x^{\mathsf{v}} - \gamma_t^{\mathsf{v}} + \frac{\left([\gamma_t]^{\mathsf{h}}\right)^2}{4} + \left(\left[\gamma_t^{-1}x\right]^{\mathsf{h}}\right)^2 \\
&\leq x^{\mathsf{v}} - t + 2\varrho^2\, |t|^{1+\alpha} + \left(\left[\gamma_t^{-1}x\right]^{\mathsf{h}}\right)^2 \quad \text{by (44),}
\end{aligned} \tag{45}$$

and in the same way

$$(\gamma_t^{-1}x)^{\mathsf{v}} \geq x^{\mathsf{v}} - t - 2\varrho^2 |t|^{1+\alpha} - \left([\gamma_t^{-1}x]^{\mathsf{h}} \right)^2. \tag{46}$$

If $[x]^{\mathsf{v}} \leq [x]^{\mathsf{h}}$, the thesis follows choosing $t = t(x) := 0$, since $\gamma_0 = 0$. Otherwise, i.e. if $[x]^{\mathsf{v}} > [x]^{\mathsf{h}}$, we distinguish between the case $x^{\mathsf{v}} > \left([x]^{\mathsf{h}}\right)^2$ and $x^{\mathsf{v}} < -\left([x]^{\mathsf{h}}\right)^2$, taking into account the sign of x^{v}. In the former case, we introduce the continuous function

$$G(t) := (\gamma_t^{-1}x)^{\mathsf{v}} - \left([\gamma_t^{-1}x]^{\mathsf{h}} \right)^2 \quad \text{for } t \in I.$$

We have $G(0) = x^{\mathsf{v}} - \left([x]^{\mathsf{h}}\right)^2 > 0$. By (7) and the condition $x \in \bar{\mathrm{B}}_\varepsilon(0)$ we deduce $0 \leq x^{\mathsf{v}} \leq \mathrm{c}^2\varepsilon^2$, where c is the constant in (7). Therefore, if δ, ε are chosen so that

$$[-\delta, \delta] \subseteq I \quad \text{and} \quad \mathrm{c}^2\varepsilon^2 \leq \frac{\delta}{2}, \tag{47}$$

then we have $2x^{\mathsf{v}} \in I$ and we estimate from above, using (45),

$$G(2x^{\mathsf{v}}) \leq x^{\mathsf{v}} - (2x^{\mathsf{v}}) + 2\varrho^2 |2x^{\mathsf{v}}|^{1+\alpha} = -x^{\mathsf{v}} + 4\varrho^2\delta^\alpha x^{\mathsf{v}} = (4\varrho^2\delta^\alpha - 1)x^{\mathsf{v}}.$$

If $\delta > 0$ is such that additionally

$$4\varrho^2\delta^\alpha - 1 \leq 0, \tag{48}$$

then $G(2x^{\mathsf{v}}) \leq 0$ and by continuity we deduce that, for some $t = t(x) \in (0, 2x^{\mathsf{v}}]$, $G(t) = 0$, from which the thesis follows.

Arguing in a symmetric way in the case $x^{\mathsf{v}} < -\left([x]^{\mathsf{h}}\right)^2$, i.e. by considering instead

$$G(t) := (\gamma_t^{-1}x)^{\mathsf{v}} + \left([\gamma_t^{-1}x]^{\mathsf{h}} \right)^2 \quad \text{for } t \in I,$$

and using (46) instead of (45) we deduce that, if δ, ε satisfy (47) and (48), there exists $t = t(x) \in [2x^{\mathsf{v}}, 0)$ such that $G(t) = 0$. In conclusion, to obtain the thesis, it is enough to fix δ_2 such that (48) holds (with δ_2 instead of δ) and then, for $\delta \in (0, \delta_2]$, let $\varepsilon = \varepsilon_2(\delta) > 0$ satisfy (47). □

Proposition 5.5 (Surjectivity). *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. If $I \subseteq \mathbb{R}$ is an interval whose interior contains 0 and*

(i) *$\gamma : I \to \mathbb{H}$ is continuous and (44) holds, with*
(ii) *$F(\gamma_t) = F(\gamma_0)$, for $t \in I$,*

then there exists $\delta_3 > 0$, such that $[-\delta_3, \delta_3] \subseteq I$ and the following holds: for any $\delta \in (0, \delta_3]$, there is $\varepsilon_3 = \varepsilon_3(\delta) > 0$ such that, if $\gamma_0 \in \bar{\mathrm{B}}_{\varepsilon_3}(p)$, we have

$$\gamma([-\delta, \delta]) \cap \bar{\mathrm{B}}_{\varepsilon_3}(p) = F^{-1}(F(\gamma_0)) \cap \bar{\mathrm{B}}_{\varepsilon_3}(p). \tag{49}$$

Proof. As usual, we write throughout the proof δ, ε, to be specified below, yielding δ_3, ε_3 such that the thesis holds. Let $\varepsilon_1 > 0$ be provided in Lemma 5.1 and $\delta_2 > 0$ be provided by Lemma 5.2. Then, if $\delta \in (0, \delta_2]$, let $\varepsilon_2 = \varepsilon_2(\delta) > 0$ as provided by Lemma 5.2. Given any $\delta \in (0, \delta_2]$, from (7) and (44), we have, for $t \in [-\delta, \delta]$,

$$\begin{aligned}\mathsf{d}(\gamma_t, \gamma_0) &\le \mathrm{c}\left(\left[\gamma_0^{-1}\gamma_t\right]^{\mathsf{h}} + \left[\gamma_0^{-1}\gamma_t\right]^{\mathsf{v}} \right) \le \mathrm{c}\left(\varrho\, |t|^{\frac{1+\alpha}{2}} + \sqrt{|t| + \varrho^2\, |t|^{1+\alpha}} \right) \\ &\le \mathrm{c}\left(\varrho\delta^{\frac{1+\alpha}{2}} + \sqrt{\delta + \varrho^2\delta^{1+\alpha}} \right)\end{aligned}$$

Therefore, if δ satisfies

$$\delta \in (0, \delta_2] \quad \text{and} \quad \mathrm{c}\left(\varrho\delta^{\frac{1+\alpha}{2}} + \sqrt{\delta + \varrho^2\delta^{1+\alpha}} \right) \le \frac{\varepsilon_1}{2}, \tag{50}$$

we deduce that $\mathsf{d}(\gamma_t, \gamma_0) \le \varepsilon_1/2$. Hence, if ε satisfies

$$\varepsilon \le \varepsilon_1/2 \quad \text{and} \quad \varepsilon \le \varepsilon_2(\delta), \tag{51}$$

for $x \in F^{-1}(F(\gamma_0)) \cap \bar{\mathrm{B}}_\varepsilon(p)$, Lemma 5.2 provides a $t = t(x) \in [-\delta, \delta]$ such that

$$\left[\gamma_t^{-1}x\right]^{\mathsf{v}} \le \left[\gamma_t^{-1}x\right]^{\mathsf{h}}.$$

Moreover, if $\gamma_0 \in \bar{\mathrm{B}}_\varepsilon(p)$, then $\mathsf{d}(\gamma_t, p) \le \mathsf{d}(\gamma_t, \gamma_0) + \mathsf{d}(\gamma_0, p) \le \varepsilon_1$, and Lemma 5.1 with x in place of y and γ_t in place of x therein, we deduce $\gamma_t = x$. Hence, if we choose δ_3 such that (50) hold (with δ_3 instead of δ) and then, for $\delta \in (0, \delta_3]$, we choose $\varepsilon_3(\delta)$ such that (51) holds, we have the inclusion $\supseteq$ in (49), while the converse inclusion is assumption (ii). □

Putting together Theorem 4.4 and Proposition 5.5, we have the following result concerning the local parametrization of level sets of maps $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ at nondegenerate points.

Theorem 5.6 (Parametrization of level sets)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Then, there exists $\delta_4 > 0$ such that the following condition holds: for any $\delta \in (0, \delta_4]$, there is an $\varepsilon_4 = \varepsilon_4(\delta)$ such that, for any $q \in \bar{\mathrm{B}}_{\varepsilon_4}(p)$ there is an injective solution to the LSDE on $I = [-\delta, \delta]$ with $\gamma_0 = q$ and*

$$\gamma(I) \cap \bar{\mathrm{B}}_{\varepsilon_4}(p) = F^{-1}(F(q)) \cap \bar{\mathrm{B}}_{\varepsilon_4}(p)\,. \tag{52}$$

Proof. As usual, write δ, ε, throughout the proof, to be specified below, yielding δ_4, $\varepsilon_4 > 0$ such that the thesis holds. Let δ_0, ε_0, ϱ_0 be as in Theorem 4.4. If $\varepsilon > 0$ satisfies

$$\varepsilon \le \varepsilon_0 \tag{53}$$

then Theorem 4.4 provides $\gamma : [-\delta_0, \delta_0] \to \mathbb{H}$ that solve the LSDE, with $\gamma_0 = q$ and (23) holds. Since γ is concentrated on the level set $F^{-1}(F(q))$, in order to apply Proposition 5.5 with such γ and $I = [-\delta_0, \delta_0]$, we have to ensure condition (44), for some $\varrho > 0$. The first inequality in (44) follows immediately from the first bound in (23), with

$$\varrho \ge \rho_0. \tag{54}$$

The second inequality in (44) follows from the second bound in (23): indeed it is sufficient to recall the definition of E_{st} in the "vertical" equation of (14), and the fact that $|\mathrm{E}_{0t}| \le \|\mathrm{E}\|\, |t|^{1+\alpha}$, for $t \in [-\delta_0, \delta_0]$. In particular, we find for ϱ the additional condition

$$\varrho^2 \ge \kappa\rho_0^2. \tag{55}$$

Hence, choosing ϱ to satisfy (54) and (55), Proposition 5.5 applies, providing a $\delta_3 > 0$ (with $\delta_3 \le \delta_0$) such that, for $\delta \in (0, \delta_3]$, if $q \in \bar{\mathrm{B}}_{\varepsilon_3(\delta)}(p)$, then (49) holds. Hence, if we let $\delta_4 := \delta_3$ and for $\delta \in (0, \delta_4]$ choose an ε_4 satisfying (53) and $\varepsilon_4 \le \varepsilon_3(\delta)$, the thesis follows. □

We end this section with the following "stability" version of Theorem 5.6. It is interesting to notice that here we do not use uniqueness of solutions to the LSDE.

Corollary 5.7 (Stability)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ and let $\{F^n\}_{n\geq 1} \subseteq C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ converge to F in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Then, there exist $\bar{n} \geq 1$, positive δ_4, ϱ_4 such that the following holds. For any $\delta \in (0, \delta_4]$, there is $\varepsilon_4 = \varepsilon_4(\delta) > 0$, such that, for any $n \geq n_0$ and $q^n \in \bar{\mathrm{B}}_{\varepsilon_4}(p)$, there is an injective solution $\gamma^n : [-\delta, \delta] \to \mathbb{H}$ to the LSDE associated to F^n, with $\gamma^n_0 = q^n$, (42) and*

$$\gamma^n(I) \cap \bar{\mathrm{B}}_{\varepsilon_4}(p) = (F^n)^{-1}(F^n(q^n)) \cap \bar{\mathrm{B}}_{\varepsilon_4}(p)$$

Moreover, the family $\{\gamma^n\}_{n\geq 1}$ is compact with respect to uniform convergence and any limit point is an injective solution γ to the LSDE associated to F, which satisfies $\gamma_0 = q$ for some $q \in \bar{\mathrm{B}}_{\varepsilon_4}(p)$, (23) and (52).

Proof. The argument is a combination of Corollary 4.5 and a simple constants-chasing throughout the results of this section. Indeed, let $\bar{n}$, δ_0 ε_0, ϱ_0 be as in Corollary 4.5, and let $\varrho_4 = \varrho_0$. Moreover, notice that if ε_1 is chosen so as to have strict inequality in (43), then by convergence of F^n to F in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ we have that, for n large enough (without loss of generality $n \geq \bar{n}$), the thesis of Lemma 3.6 holds for F^n, with such ε_1 (as well as for F).

With such choices of δ_0, ε_0, ϱ_0 and ε_1, if we follow throughout the proof of Theorem 5.6 with F^n in place of F, we see that the thesis still holds, using (42) instead of (23), provided that δ_3 and $\varepsilon_3(\delta)$ can be made independent of n, for $n \geq \bar{n}$ (as well as for F). To show this fact, we notice first that choosing $\rho > 0$ such that (54) and (55) hold, we have that (44) holds with γ^n in place of γ, for $n \geq \bar{n}$. As a consequence, in the proofs of Lemma 5.2 and Proposition 5.5, the conditions on δ_2, $\varepsilon_2(\delta)$, i.e. (47) and (48), as well as those on δ_3 and $\varepsilon_3(\delta)$, i.e. (50) and (51) can be satisfied uniformly in n, for $n \geq \bar{n}$ (as well as for F). □

6. Uniqueness of solutions

In this section, we prove that solutions to the LSDE are unique, for small times, i.e. until the first time they leave a sufficiently small neighbourhood of p. First, we give a basic result on the modulus of continuity of any solution to the LSDE, showing in particular that (44) above holds (for small times).

Lemma 6.1 (Local modulus of continuity)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Given a solution $\gamma : I \to \mathbb{H}$ to the LSDE, there is an $\varepsilon > 0$ such that, if $\gamma_t \in \bar{\mathrm{B}}_{\varepsilon}(p)$, for all $t \in I$, then*

$$\limsup_{\substack{s,t\in I\\ |t-s|\to 0}} \frac{\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}}}{|t-s|^{\frac{1+\alpha}{2}}} < \infty \quad \text{and} \quad \limsup_{\substack{s,t\in I\\ |t-s|\to 0}} \frac{\mathsf{d}(\gamma_s, \gamma_t)}{|t-s|^{1/2}} < \infty. \tag{56}$$

In particular, there exists a $\varrho > 0$ such that (44) holds, up to replacing I therein with some smaller interval $J \subseteq I$ (but still 0 is in the interior of J).

Proof. From the "horizontal" equation in (14) and inequality (12) applied to $x = \gamma_s$, $y = \gamma_t$, we have

$$\begin{aligned}\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}} &= \left|\nabla_{\mathsf{h}} F(p)^{-1}\left(\mathrm{R}(p,\gamma_t) - \mathrm{R}(p,\gamma_s)\right)\right| \\ &\le \left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \left|\mathrm{R}(p,\gamma_t) - \mathrm{R}(p,\gamma_s)\right| \\ &\le \mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha,\bar{\mathrm{B}}_{2\mathrm{c}\varepsilon}(p)} \left[r^{\alpha}\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}} + \left(\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{v}}\right)^{1+\alpha}\right]\end{aligned}$$

If ε satisfies

$$\mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha,\bar{\mathrm{B}}_{2\mathrm{c}\varepsilon}(p)}\, \varepsilon^{\alpha} \le \frac{1}{2},$$

we deduce

$$\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}} \le 2\,\mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha,\bar{\mathrm{B}}_{2\mathrm{c}\varepsilon}(p)} \left(\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{v}}\right)^{1+\alpha}. \tag{57}$$

Since we are interested in the limit as $|t-s| \to 0$, we assume that $|t-s| \le 1$. By (14) and (15), we have

$$\left|\left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{v}}\right| \le |t-s| + \|\mathrm{E}\|\,|t-s|^{1+\alpha} \le (1+\|\mathrm{E}\|)\,|t-s|\,, \tag{58}$$

i.e., $\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{v}} \le \sqrt{1+\|\mathrm{E}\|}\,|t-s|^{1/2}$, which together with (57) yields

$$\limsup_{\substack{s,t\in I\\ |t-s|\to 0}} \frac{\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}}}{|t-s|^{\frac{1+\alpha}{2}}} \le 2\,\mathrm{c}\left|\nabla_{\mathsf{h}} F(p)^{-1}\right| \|\nabla_{\mathsf{h}} F\|_{\alpha,\bar{\mathrm{B}}_{2\mathrm{c}\varepsilon}(p)} (1+\|\mathrm{E}\|)^{(1+\alpha)/2}\,,$$

which is the first bound in (56). The second one follows then from (58) and (7). Finally, the claim on the validity of the two inequalities in (44) follows respectively from the first inequality in (56), with $s=0$, and from the definition of E_{0t} and $\|\mathrm{E}\|$. □

Theorem 6.2 (Local uniqueness)**.** *Let $p \in \mathbb{H}$ be a nondegenerate point for $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H},\mathbb{R}^2)$. Given solutions $\gamma, \bar{\gamma} : I \to \mathbb{H}$ to the LSDE , there is $\varepsilon > 0$ such that if $\gamma_{t_0} = \bar{\gamma}_{t_0} \in \bar{\mathrm{B}}_{\varepsilon}(p)$ for some $t_0 \in I$, then the set*

$$\{t \in I \,:\, \gamma_t = \bar{\gamma}_t\} \tag{59}$$

contains the connected component of t_0 in $\left\{t \in I \,:\, \gamma_t \in \bar{\mathrm{B}}_{\varepsilon}(p)\right\}$.

Proof. There is no loss in generality if we prove the thesis for $t_0 = 0$, the general case following reducing to solutions of the LSDE $t \mapsto \gamma_{t_0+t}$, $t \mapsto \bar{\gamma}_{t_0+t}$. Lemma 3.6 and Lemma 6.1 give that, possibly up to replacing I with a smaller neighbourhood of $t_0 = 0$, both γ and $\bar{\gamma}$ satisfy (16) as well as (44) (without loss of generality, with the same constants δ and ϱ).

Proposition 5.5 applied to γ (respectively, to $\bar{\gamma}$) provide some δ_3 and $\varepsilon_3(\delta)$ (respectively, $\bar{\delta}_3$ and $\bar{\varepsilon}_3(\delta) > 0$). If $\delta > 0$ satisfies

$$\delta \le \delta_3 \quad \text{and} \quad \delta \le \bar{\delta}_3 \tag{60}$$

and $\varepsilon > 0$ satisfies

$$\varepsilon \le \varepsilon_3(\delta) \quad \text{and} \quad \varepsilon \le \bar{\varepsilon}_3(\delta), \tag{61}$$

then Proposition 5.5 gives

$$\gamma([-\delta,\delta]) \cap \bar{\mathrm{B}}_{\varepsilon}(p) = F^{-1}\left(F(\gamma_0)\right) \cap \bar{\mathrm{B}}_{\varepsilon}(p) = \bar{\gamma}([-\delta,\delta]) \cap \bar{\mathrm{B}}_{\varepsilon}(p)\,.$$

In particular, for any $t \in [-\delta, \delta]$, there is $\bar{t} \in [-\delta, \delta]$ such that $\bar{\gamma}_{\bar{t}} = \gamma_t$. Such $\bar{t}$ is unique by injectivity of $\bar{\gamma}$, hence the function $t \mapsto \bar{t}$ is well defined on $[-\delta, \delta]$, with $\bar{t} = 0$ for $t = 0$.

By the "vertical" equation in (14), for $s, t \in [-\delta, \delta]$, we have

$$t - s + \mathrm{E}_{st} = (\gamma_s^{-1}\gamma_t)^{\mathsf{v}} = (\bar{\gamma}_{\bar{s}}^{-1}\bar{\gamma}_{\bar{t}})^{\mathsf{v}} = \bar{t} - \bar{s} + \bar{\mathrm{E}}_{\bar{s}\bar{t}}, \tag{62}$$

hence

$$|\bar{t} - \bar{s}| \leq |t - s| + |\mathrm{E}_{st}| + \left|\bar{\mathrm{E}}_{\bar{s}\bar{t}}\right| \tag{63}$$

From (15), we estimate from above

$$\left|\bar{\mathrm{E}}_{\bar{s}\bar{t}}\right| \leq \left\|\bar{\mathrm{E}}\right\| |\bar{t} - \bar{s}|^{1+\alpha} \leq \left\|\bar{\mathrm{E}}\right\| (2\delta)^{\alpha} |\bar{t} - \bar{s}| \,. \tag{64}$$

If $\delta > 0$ satisfies additionally

$$\left\|\bar{\mathrm{E}}\right\| (2\delta)^{\alpha} \leq 1/2, \tag{65}$$

from (63) and (64) we deduce

$$|\bar{t} - \bar{s}| \leq 2\left(|t - s| + |\mathrm{E}_{st}|\right) \leq 2\left(1 + (2\delta)^{\alpha} \|\mathrm{E}\|\right) |t - s| \,, \tag{66}$$

using the bound $|\mathrm{E}_{st}| \leq \|\mathrm{E}\| (2\delta)^{\alpha} |t - s|$. Combining (64) and (66), we find

$$\left|\bar{\mathrm{E}}_{\bar{s}\bar{t}}\right| \leq \left\|\bar{\mathrm{E}}\right\| |\bar{t} - \bar{s}|^{1+\alpha} \leq \left\|\bar{\mathrm{E}}\right\| \left[2\left(1 + (2\delta)^{\alpha} \|\mathrm{E}\|\right)\right]^{1+\alpha} |t - s|^{1+\alpha} \,,$$

thus

$$\limsup_{\substack{s,t \in I \\ |t-s| \to 0}} \frac{\left|\bar{\mathrm{E}}_{\bar{s}\bar{t}}\right|}{|t - s|} = 0. \tag{67}$$

Dividing by $t - s$ the leftmost and the rightmost sides in (62), we find

$$\frac{\bar{t} - \bar{s}}{t - s} = 1 + \frac{\mathrm{E}_{st}}{t - s} - \frac{\bar{\mathrm{E}}_{\bar{s}\bar{t}}}{t - s}.$$

Letting $s \to t$, using (67), we deduce that $t \mapsto \bar{t}$ is differentiable at any $t \in (-\delta, \delta)$, with derivative identically 1: it follows that $t = \bar{t}$.

Hence, we have that if δ satisfies (60), (65) and ε satisfies (61), with $\gamma_0 = \gamma_0 \in \bar{\mathrm{B}}_{\varepsilon}(p)$, then $I \supseteq [-\delta, \delta]$. In particular, in view of (16), we have that $\gamma_t = \bar{\gamma}_t$ coincide up to the first time they leave $\bar{\mathrm{B}}_{\delta^{1/2}/\varrho}(\gamma_0)$. Hence, if we further assume $\varepsilon \leq \delta^{1/2}/\left(2\varrho\right)$, then $\bar{\mathrm{B}}_{\varepsilon}(p) \subseteq \bar{\mathrm{B}}_{\delta^{1/2}/\varrho}(\gamma_0)$, so that (59) must contain all the connected component of $t = 0$ in $\left\{t \in I \,:\, \gamma_t \in \bar{\mathrm{B}}_{\varepsilon}(p)\right\}$. □

7. Area formula

In this section, we establish an integral formula for the spherical Hausdorff measure of "vertical" curves satisfying conditions akin to those of the LSDE. Then, we obtain the corresponding area formula for level sets of $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$, in a neighbourhood of a nondegenerate point.

We recall the definition of the 2-dimensional spherical Hausdorff measure of a set $U \subseteq \mathbb{H}$. For an $\varepsilon > 0$, set

$$\begin{aligned} \mathcal{S}^2_{\mathsf{d},\varepsilon}(U) \;&:=\; \inf\left\{ \sum_{i \geq 1} \beta_{\mathsf{d}}\, r_i^2 \,:\, U \subseteq \bigcup_{i \geq 1} \bar{\mathrm{B}}_{r_i}(p_i)\,, \text{ and } r_i \leq \varepsilon, \text{ for every } i \geq 1 \right\}, \quad \text{where} \\ \beta_{\mathsf{d}} \;&:=\; \sup_{\mathsf{d}(0,y) \leq 1} \mathcal{L}^1\left(\left\{\sigma \in \mathbb{R} : (0,0,\sigma) \in \bar{\mathrm{B}}_1(y)\right\}\right), \end{aligned}$$

the inf running among all families of balls covering U and $\mathcal{L}^1$ denoting Lebesgue measure. Then, define the 2-dimensional spherical Hausdorff measure as

$$\mathcal{S}^2_{\mathsf{d}}(U) := \sup_{\varepsilon>0} \mathcal{S}^2_{\mathsf{d},\varepsilon}(U). \tag{68}$$

Theorem 7.1 (Area formula)**.** *Let $I \subseteq \mathbb{R}$ be an interval, let $\gamma : I \to \mathbb{H}$ be injective and such that*

$$\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}} \leq \sqrt{|t-s|\omega(|t-s|)} \quad \text{and} \quad \left|(\gamma_s^{-1}\gamma_t)^{\mathsf{v}} - \int_s^t \vartheta(\tau)\,\mathrm{d}\tau\right| \leq |t-s|\omega(|t-s|) \tag{69}$$

for every $s, t \in I$, where $\vartheta : I \to \mathbb{R}$ is continuous, $\omega : [0,\infty) \to [0,\infty)$ is non-decreasing and $\omega(0^+) = 0$. Then, for every Borel $U \subseteq \mathbb{H}$ and bounded Borel function $u : \mathbb{H} \to \mathbb{R}$, we have

$$\mathcal{S}^2_{\mathsf{d}}\left(\gamma(I) \cap U\right) = \gamma_\sharp(|\vartheta|\, \mathcal{L}^1 \llcorner I)(U) \quad \text{and} \quad \int_{\gamma(I)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}} = \int_I u\left(\gamma_\tau\right)|\vartheta(\tau)|\,\mathrm{d}\tau.$$

Proof. Without loss of generality, we assume that I is compact (otherwise, the argument follows by covering I with compact intervals). The proof relies on the measure-theoretic area formula [21, Theorem 11] on the metric space $(\mathbb{H}, \mathsf{d})$, which reads

$$\gamma_\sharp(|\vartheta|\, \mathcal{L}^1 \llcorner I)(U) = \int_U \theta_{\mathcal{S}^2_{\mathsf{d}}}(x)\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}(x) \quad \text{for } U \subseteq \mathbb{H} \text{ Borel}, \tag{70}$$

where $\theta_{\mathcal{S}^2_{\mathsf{d}}}(x)$ is the (upper) spherical Federer density of $\gamma_\sharp(|\vartheta|\, \mathcal{L}^1 \llcorner I)$ at $x \in \mathbb{H}$. This density, introduced in [21], can be equivalently defined as follows

$$\theta_{\mathcal{S}^2_{\mathsf{d}}}(x) := \sup\left\{\limsup_k \frac{\gamma_\sharp(|\vartheta|\, \mathcal{L}^1 \llcorner I)(\bar{\mathrm{B}}_{\varrho_k}(y_k))}{\beta_{\mathsf{d}}\varrho_k^2} : \mathsf{d}(y_k, x) \leq \rho_k \to 0\right\}. \tag{71}$$

Once we prove that the assumptions in [21, Theorem 11] are satisfied, it will suffice to show that $\theta_{\mathcal{S}^2_{\mathsf{d}}}(x) = 1$, for $\mathcal{S}^2_{\mathsf{d}}$-a.e. $x \in \gamma(I)$.

Claim: $\gamma(I)$ has finite $\mathcal{S}^2_{\mathsf{d}}$ measure. We prove the inequality

$$\mathcal{S}^2_{\mathsf{d}}\left(\gamma(I)\right) \leq \mathrm{c}\int_I |\vartheta(\tau)|\,\mathrm{d}\tau, \tag{72}$$

for some constant $\mathrm{c} > 0$. To show (72), from (69), (7) and the inequality $(a+b)^2 \leq 2\left(a^2+b^2\right)$, we obtain that, for some constant $\mathrm{c} > 0$,

$$\mathsf{d}(\gamma_t, \gamma_s)^2 \leq \mathrm{c}\left(\left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{h}} + \left[\gamma_s^{-1}\gamma_t\right]^{\mathsf{v}}\right)^2 \leq 2\,\mathrm{c}\left(2\,|t-s|\,\omega(|t-s|) + \int_s^t |\vartheta(\tau)|\,\mathrm{d}\tau\right)$$

For $\delta > 0$, choose any partition $t_0 < \ldots < t_n$ of I with $\sup_{i=1,\ldots,n}|t_i - t_{i-1}| \leq \delta$. Since ω is non-decreasing, we have from the above inequality

$$\begin{aligned}\operatorname{diam}(\gamma([t_{i-1}, t_i]))^2 &\leq 2\,\mathrm{c}\left(2|t_i - t_{i-1}|\omega(\delta) + \int_{t_{i-1}}^{t_i} |\vartheta(\tau)|\,\mathrm{d}\tau\right)\\ &\leq 2\,\mathrm{c}\delta\left(2\omega(\delta) + \sup_{\tau\in I}|\vartheta(\tau)|\right).\end{aligned} \tag{73}$$

Hence, if we let $\varepsilon(\delta)$ denote the square root of the term in the last line in (73), considering the covering $\gamma(I) \subseteq \bigcup_{i=1}^n \bar{\mathrm{B}}_{\mathrm{diam}(\gamma([t_{i-1},t_i]))}(\gamma_{t_i})$, we obtain the bound from above

$$\begin{aligned}\mathcal{S}^2_{\mathsf{d},\varepsilon(\delta)}(\gamma(I)) &\le 2\beta_{\mathsf{d}}\,\mathrm{c}\sum_{i=1}^n \left(2|t_i-t_{i-1}|\omega(\delta) + \int_{t_{i-1}}^{t_i} |\vartheta(\tau)|\,\mathrm{d}\tau\right)\\ &\le 2\beta_{\mathsf{d}}\,\mathrm{c}\left(2\mathcal{L}^1(I)\omega(\delta) + \int_I |\vartheta(\tau)|\,\mathrm{d}\tau\right),\end{aligned}$$

As $\delta \to 0^+$, since $\varepsilon(\delta) \to 0^+$, we obtain (72) (with $2\beta_{\mathsf{d}}\,\mathrm{c}$ instead of c).

Claim: $\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I) \ll \mathcal{S}^2_{\mathsf{d}} \llcorner \gamma(I)$. To this aim, we decompose

$$\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I) := \sum_{n\in\mathbb{Z}} \gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I_n^+) + \gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I_n^-), \tag{74}$$

where $I_n^+ := \{t \in I : \vartheta(t) \in [2^n, 2^{n+1})\}$ and similarly $I_n^- := \{t \in I : \vartheta(t) \in (-2^{n+1}, -2^n]\}$. Notice that the set $\{|\vartheta| = 0\}$ is negligible with respect to $|\vartheta|\,\mathcal{L}^1 \llcorner I$.

We fix $n \in \mathbb{Z}$ and prove that $\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I_n^+) \ll \mathcal{S}^2_{\mathsf{d}} \llcorner \gamma(I)$ (the case I_n^- is analogous). This follows from the following quantitative injectivity inequality

$$|t-s| \le \mathrm{c}\,\mathsf{d}(\gamma_s,\gamma_t)^2, \quad \text{for } s \in I_n^+,\ t \in I \text{ with } |t-s| \le \delta \tag{75}$$

for some positive $\mathrm{c} = \mathrm{c}(n)$, $\delta = \delta(n)$. Taking it for the moment for granted, we deduce the claim. Indeed, we can write I_n^+ as the countable union of inervals $I_{n,k}^+ = I_n^+ \cap [k\delta, (k+1)\delta]$ with $k \in \mathbb{Z}$. Thus, condition (75) leads us to the estimates

$$\mathrm{diam}\left(\gamma^{-1}(B) \cap I_{n,k}^+\right) \le \mathrm{c}\,\mathrm{diam}\left(\gamma\left(\gamma^{-1}(B) \cap I_{n,k}^+\right)\right)^2 \le \mathrm{c}\,\mathrm{diam}\,(B)^2, \tag{76}$$

for $B \subset \mathbb{H}$. Let now $U \subseteq \mathbb{H}$ be such that $\mathcal{S}^2_{\mathsf{d}}(U \cap \gamma(I)) = 0$. By definition (68), for any $\varrho > 0$ and $\kappa > 0$, we can find $(\bar{\mathrm{B}}_{r_i}(p_i))_{i\ge1}$ such that $U \cap \gamma(I) \subseteq \bigcup_{i\ge1} \bar{\mathrm{B}}_{r_i}(p_i)$, $r_i < \sqrt{\frac{\kappa}{4\,\mathrm{c}}}$ and

$$\sum_{i\ge1} \beta_{\mathsf{d}} r_i^2 \le \varrho.$$

The family $\gamma^{-1}\left(\bar{\mathrm{B}}_{r_i}(p_i)\right) \cap I_{n,k}^+$, for $i \ge 1$ provides a covering of $\gamma^{-1}(U) \cap I_{n,k}^+$, and (76) implies that each diameter of this family is smaller than κ and

$$\sum_{i\ge1} \mathrm{diam}\left(\gamma^{-1}\left(\bar{\mathrm{B}}_{r_i}(p_i)\right) \cap I_{n,k}^+\right) \le \mathrm{c}\sum_{i\ge1}(2r_i)^2 \le \frac{4\,\mathrm{c}}{\beta_{\mathsf{d}}}\sum_{i\ge1}\beta_{\mathsf{d}} r_i^2 \le \frac{4\,\mathrm{c}}{\beta_{\mathsf{d}}}\varrho.$$

Since $\varrho > 0$ is arbitrary, we deduce that $\mathcal{L}^1(\gamma^{-1}(U) \cap I_{n,k}^+) = 0$ and so also $\gamma^{-1}(U) \cap I_n^+$ is $\mathcal{L}^1$-negligible. We have proved that

$$\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I_n^+)(U) \le 2^{n+1}\gamma_\sharp(\mathcal{L}^1 \llcorner I_n^+)(U) = 0.$$

Proof of (75)*:* let $\sigma : [0,\infty) \to [0,\infty)$ be a non-decreasing modulus of continuity for ϑ (recall that $\vartheta : I \to \mathbb{R}$ is continuous and I is compact). If $\delta > 0$ satisfies $\sigma(\delta) \le 2^n/2$, then for $s \in I_n^+$, $t \in I$ with $|t-s| \le \delta$, from

$$\int_s^t |\vartheta(\tau) - \vartheta(s)|\,\mathrm{d}\tau \le \sigma(\delta)\,|t-s|$$

we deduce,

$$\int_s^t \vartheta(\tau)\,\mathrm{d}\tau \geq \vartheta(s)\,|t-s| - \sigma(\delta)\,|t-s| \geq 2^n\,|t-s|/2,$$

since $\vartheta(s) \geq 2^n$. By the second inequality in (69) and (7),

$$2^n\,|t-s|/2 \leq \int_s^t \vartheta(\tau)\,\mathrm{d}\tau - \left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{v}} + \left(\gamma_s^{-1}\gamma_t\right)^{\mathsf{v}} \leq |t-s|\,\omega(|t-s|) + \mathrm{c}\,\mathsf{d}(\gamma_s,\gamma_t)^2,$$

hence if $\delta > 0$ satisfies also $\omega(\delta) \leq 2^n/4$, we obtain (75) with $2^{2-n}\,\mathrm{c}$ instead of c.

Measure-theoretic area formula. Up to considering the Borel regular extension of the measure $\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I)$, i.e. letting

$$E \mapsto \inf_{\substack{A \supset E \\ A \text{ Borel}}} \int_{\gamma^{-1}(A)} |\vartheta(\tau)|\,\mathrm{d}\tau \quad \text{for } E \subseteq \mathbb{H},$$

we are now in a position to apply the measure-theoretic area formula [21, Theorem 11], so that (70) holds. To conclude, we show that the spherical Federer density $\theta_{\mathcal{S}^2_{\mathsf{d}}}(x)$ in (71) is 1, for $\mathcal{S}^2_{\mathsf{d}}$-a.e $x \in \gamma(I)$. In view of (74), it is enough to assume $x = \gamma_s$, with $s \in I_n^+$, for some $n \in \mathbb{Z}$ (the case $s \in I_n^-$ being analogous). To simplify notation, we let in what follows $s = 0$ and assume that $x = \gamma_0 = 0$ (the general case can be reduced to this situation by considering the curve $t \mapsto \gamma_s^{-1}\gamma_{s+t}$). Then, we have $\vartheta(0) \in [2^n, 2^{n+1})$, in particular $\vartheta(0) > 0$. For $\varepsilon > 0$, given $\{y_k\}_{k\geq 1} \in \mathbb{H}$, $\{r_k\}_{k\geq 1}$ with $r_k > 0$, $d(0,y_k) \leq \varepsilon r_k$ and $r_k \to 0$, as $k \to \infty$, we compute

$$\limsup_{k\to\infty} \frac{\gamma_\sharp(|\vartheta|\,\mathcal{L}^1 \llcorner I)(\bar{\mathrm{B}}_{\varepsilon r_k}(y_k))}{(\varepsilon r_k)^2} = \limsup_{k\to\infty} \frac{1}{(\varepsilon r_k)^2} \int_I |\vartheta(\tau)|\,\mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_\tau)\,\mathrm{d}\tau.$$

By compactness, we assume

$$\lim_{k\to\infty} \delta_{r_k^{-1}} y_k = y \in \bar{\mathrm{B}}_\varepsilon(0)\,. \tag{77}$$

Recall that from (75) with $s = 0$, $\gamma_0 = 0 \in I_n^+$,

$$|\tau| \leq \mathrm{c}\,\mathsf{d}(0,\gamma_\tau)^2 \quad \text{for } \tau \in I \text{ with } |\tau| \leq \delta, \tag{78}$$

for some positive δ, c (without loss of generality, c can be chosen arbitrarily large). Since γ is injective and continuous, there exists an $\eta > 0$ such that, for every $\tau \in I$, $|\tau| \geq \delta$, then $\mathsf{d}(0,\gamma_\tau) > \eta$ (otherwise, by compactness, one would obtain a point $\tau \in I$ with $|\tau| \geq \delta$ and $\gamma_\tau = 0$). In view of the inclusion $\bar{\mathrm{B}}_{\varepsilon r_k}(y_k) \subseteq \bar{\mathrm{B}}_{2\varepsilon r_k}(0)$, if k is sufficiently large, then $2\varepsilon r_k \leq \eta$ and we obtain, for such k, the identity

$$\begin{aligned} \int_I |\vartheta(\tau)|\,\mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_\tau)\,\mathrm{d}\tau &= \int_{-\delta}^{\delta} |\vartheta(\tau)|\,\mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_\tau)\,\mathrm{d}\tau \\ &= \int_{\{|\tau|\leq 4\,\mathrm{c}\varepsilon^2 r_k^2\}} |\vartheta(\tau)|\,\mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_\tau)\,\mathrm{d}\tau \quad \text{by (78) with } \mathsf{d}(0,\gamma_\tau) \leq 2\varepsilon r_k,\ \tau \leq \delta. \\ &= r_k^2 \int_{\{|\sigma|\leq 4\,\mathrm{c}\varepsilon^2\}} \left|\vartheta(r_k^2\sigma)\right|\,\mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_{r_k^2\sigma})\,\mathrm{d}\sigma \quad \text{by substitution } \sigma := \tau/r_k^2. \end{aligned} \tag{79}$$

We have $\gamma_{r_k^2\sigma} \in \bar{\mathrm{B}}_{\varepsilon r_k}(y_k)$ if and only if $\delta_{r_k^{-1}}\gamma_{r_k^2\sigma} \in \bar{\mathrm{B}}_\varepsilon\left(\delta_{r_k^{-1}} y_k\right)$, and

$$\lim_{k\to\infty} \delta_{r_k^{-1}}\gamma_{r_k^2\sigma} = \lim_{k\to\infty} (\gamma^{\mathsf{h}}_{r_k^2\sigma}/r_k, \gamma^{\mathsf{v}}_{r_k^2\sigma}/r_k^2) = (0,0,\vartheta(0)\sigma) \in \mathbb{H}$$

by (69) with $s=0$, $\gamma_0=0$, $t=r_k^2\sigma$. Therefore, as $k\to\infty$, the functions $\sigma\mapsto \mathbb{1}_{\bar{\mathrm{B}}_{\varepsilon r_k}(y_k)}(\gamma_{r_k^2\sigma})$ converge pointwise to the characteristic function of the set,

$$\left\{\sigma\in\mathbb{R} : (0,0,\vartheta(0)\sigma)\in\bar{\mathrm{B}}_\varepsilon(y)\right\},$$

with the possible exception of the points σ such that $(0,0,\vartheta(0)\sigma)\in\partial\bar{\mathrm{B}}_\varepsilon(y)$. By Fatou lemma, from (79), we have

$$\begin{aligned}\limsup_{k\to\infty}\frac{\gamma_\sharp(|\vartheta|\,\mathcal{L}^1\llcorner I)(\bar{\mathrm{B}}_{\varepsilon r_k}(y_k))}{r_k^2} &\le \int_{\{|\sigma|\le 4\,c\varepsilon\}}|\vartheta(0)|\,\mathbb{1}_{\bar{\mathrm{B}}_\varepsilon(y)}((0,0,\vartheta(0)\sigma))\,\mathrm{d}\sigma\\ &= \int_{\mathbb{R}}|\vartheta(0)|\,\mathbb{1}_{\bar{\mathrm{B}}_\varepsilon(y)}((0,0,\vartheta(0)\sigma))\,\mathrm{d}\sigma\\ &= \mathcal{L}^1\left(\left\{\sigma\in\mathbb{R} : (0,0,\sigma)\in\bar{\mathrm{B}}_\varepsilon(y)\right\}\right)\le\beta_{\mathsf{d}}.\end{aligned}$$

Dividing by $\varepsilon^2\beta_{\mathsf{d}}$ and letting $\varepsilon=1$, we deduce $\theta_{\mathcal{S}^2_{\mathsf{d}}}(0)\le 1$. To show the converse inequality, let $\varepsilon>1$, choose a maximizing sequence $\{y_n\}_{n\ge1}$ for β_{d} and any infinitesimal sequence $\{r_k\}_{k\ge1}$. For fixed $n\ge1$, let $y_k:=\delta_{r_k}y_n$, so that $\mathsf{d}(0,y_k)\le r_k\le\varepsilon r_k$. From Fatou lemma, using (77), there holds

$$\begin{aligned}\liminf_{k\to\infty}\frac{\gamma_\sharp(|\vartheta|\,\mathcal{L}^1\llcorner I)(\bar{\mathrm{B}}_{\varepsilon r_k}(y_k))}{r_k^2} &\ge \int_{\{|\sigma|\le 4\,c\varepsilon\}}|\vartheta(0)|\,\mathbb{1}_{\mathrm{B}_\varepsilon(y)}((0,0,\vartheta(0)\sigma))\,\mathrm{d}\sigma\\ &\ge \mathcal{L}^1\left(\left\{\sigma\in\mathbb{R} : (0,0,\sigma)\in\bar{\mathrm{B}}_1(y)\right\}\right)\ge\beta_{\mathsf{d}}-o(1)\end{aligned}\tag{80}$$

with $o(1)$ infinitesimal as $n\to\infty$. Taking $\varepsilon_n\to1^+$ and dividing (80) by $\varepsilon^2\beta_{\mathsf{d}}$ with $\varepsilon=\varepsilon_n$, we see that $\theta_{\mathcal{S}^2_{\mathsf{d}}}(0)\ge 1-o(1)$ as $n\to\infty$, hence the thesis. □

Corollary 7.2 (Area formula for LSDE)**.** *Let us fix a nondegenerate point $p\in\mathbb{H}$ for $F\in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H},\mathbb{R}^2)$, and let $\gamma: I\to\mathbb{H}$ be an injective solution to the LSDE . If* $\operatorname{dist}(p,\gamma(I)) := \sup_{t\in I}\mathsf{d}(p,\gamma_t)$ *is sufficiently small, then we have*

$$\mathcal{S}^2_{\mathsf{d}}(\gamma(J))=\mathcal{L}^1(J),\quad \textit{for every closed } J\subseteq I.$$

Moreover, if for some $\varepsilon>0$ and for $q\in\gamma(I)$ one has

$$\gamma(I)\cap\bar{\mathrm{B}}_\varepsilon(p)=F^{-1}(F(q))\cap\bar{\mathrm{B}}_\varepsilon(p),$$

then for every Borel set $U\subseteq\bar{\mathrm{B}}_\varepsilon(p)$ and for every bounded Borel function $u:\bar{\mathrm{B}}_\varepsilon(p)\to\mathbb{R}$,

$$\mathcal{S}^2_{\mathsf{d}}\left(F^{-1}(F(q))\cap U\right)=\int_I\mathbb{1}_U(\gamma_\tau)\,\mathrm{d}\tau\quad \textit{and}\quad \int_{F^{-1}(F(q))}u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}=\int_I u(\gamma_\tau)\,\mathrm{d}\tau.$$

Proof. Since the second inequality in (69) always holds for γ solution to the LSDE with $\vartheta=1$ and $\omega(s)=\|\mathrm{E}\|\,s^\alpha$, we have to ensure that the first inequality in (69) holds (possibly with a different ω). This follows e.g. from Lemma 6.1, if $\operatorname{dist}(p,\gamma(I))$ is sufficiently small. □

Remark 7.3. If γ satisfies (23) (or a sequence γ^n satisfies (42)) there is no need to ensure that $\operatorname{dist}(p,\gamma(I))$ is sufficiently small, for (69) immediately follows from (23).

8. Coarea formula

In this section, we prove a coarea formula for maps $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$.

Definition 8.1. *If $F : \mathbb{H} \to \mathbb{R}^2$ is h-differentiable at $x \in \mathbb{H}$, we define the* horizontal Jacobian *$J_{\mathsf{h}}F(x)$ of F at x setting $J_{\mathsf{h}}F(x) := |\det \nabla_{\mathsf{h}} F(x)|$.*

Theorem 8.2 (Coarea formula). *Let $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$. Then, for every $U \subseteq \mathbb{H}$ Borel,*

$$\int_U J_{\mathsf{h}} F \, \mathrm{d}\mathcal{L}^3 = \int_{\mathbb{R}^2} \mathcal{S}^2_{\mathsf{d}} \left(U \cap F^{-1}(z) \right) \mathrm{d}\mathcal{L}^2(z), \tag{81}$$

and for every Borel function $u : \mathbb{H} \to [0, +\infty]$,

$$\int_{\mathbb{H}} u \, J_{\mathsf{h}} F \, \mathrm{d}\mathcal{L}^3 = \int_{\mathbb{R}^2} \int_{F^{-1}(z)} u \, \mathrm{d}\mathcal{S}^2_{\mathsf{d}} \, \mathrm{d}\mathcal{L}^2(z). \tag{82}$$

Our proof follows the approach introduced in [19], in particular a blow-up argument akin to [19, Theorem 4.1], but here we rely on the parametrization provided by the LSDE. Given $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$, for $p \in \mathbb{H}$ and $r > 0$, define the map

$$F_{p,r} : \mathbb{H} \to \mathbb{R}^2, \quad q \mapsto F_{p,r}(q) := \frac{F\left(p\delta_r(q)\right) - F(p)}{r} \quad \text{and set} \quad F_r := F_{0,r}.$$

Lemma 8.3. *As $r \to 0^+$, the maps $\{F_{p,r}\}_{r>0}$ converge in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$ to the group homomorphism $\mathrm{d}_{\mathsf{h}}F(p)$. Moreover, (81) and (82) hold with the map $\mathrm{d}_{\mathsf{h}}F(p)$ in place of F.*

Proof. Without loss of generality, we prove the thesis for $p = 0$ (the general case follows from considering $q \mapsto F(pq)$). Let $\varrho > 0$ and $q \in \bar{\mathrm{B}}_{\varrho}(0)$. Then, from (10) with $x = 0$, $y = \delta_r(q)$, we deduce

$$|F_r(q) - \mathrm{d}_{\mathsf{h}}F(0)(q)| \le \mathsf{c} \, \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{\mathsf{c}r\varrho}(0)} \, \varrho r^{\alpha} \to 0, \quad \text{as } r \to 0^+.$$

To show convergence of the horizontal gradients, we notice first that the definition of ∇_{h} in terms of left-invariant fields X_1, X_2 yields the identity

$$\left(\nabla_{\mathsf{h}} F_r\right) q^{\mathsf{h}} = \nabla_{\mathsf{h}} F \left(\delta_r(q)\right)^{\mathsf{h}},$$

and that $\nabla_{\mathsf{h}}(\mathrm{d}_{\mathsf{h}}F(0)) = \nabla_{\mathsf{h}}F(0)$ is constant. Then,

$$\begin{aligned} \|\nabla_{\mathsf{h}} F_r - \nabla_{\mathsf{h}} F(0)\|_{\alpha, \bar{\mathrm{B}}_{\varrho}(0)} &= \|\nabla_{\mathsf{h}} F\left(\delta_r(\cdot)\right) - \nabla_{\mathsf{h}} F(0)\|_{\alpha, \bar{\mathrm{B}}_{\varrho}(0)} \\ &\le \|\nabla_{\mathsf{h}} F\|_{\alpha, \bar{\mathrm{B}}_{r\varrho}(0)} \, r^{\alpha} \to 0 \quad \text{as } r \to 0^+. \end{aligned}$$

Finally, to show (81) and (82), we consider first the case $\mathrm{d}_{\mathsf{h}}F(0)(q) = q^{\mathsf{h}}$, i.e. $\nabla_{\mathsf{h}}F(0) = \mathrm{Id}$ the identity matrix. For $z \in \mathbb{R}^2$, the level set of z is precisely $\{(z,t) : t \in \mathbb{R}\}$, hence coarea formula follows from Fubini's theorem and Theorem 7.1 with $\gamma_t = (z,t)$, to obtain

$$\mathcal{S}^2_{\mathsf{d}}(\{(z,t) : \, t \in \mathbb{R}\} \cap U) = \mathcal{L}^1(\{t \in \mathbb{R} : \, (z,t) \in U\}).$$

If the matrix $\nabla_{\mathsf{h}}F(0) = M$ is not the identity (but invertible) we consider the map $M^{-1} \, \mathrm{d}_{\mathsf{h}}F(0)$ and we reduce to the previous case, using $J_{\mathsf{h}}(M^{-1}F) = \left|\det M^{-1}\right| J_{\mathsf{h}}(F)$ and $(M^{-1}F)^{-1}(z) = F^{-1}(Mz)$. When $\nabla_{\mathsf{h}}F(0)$ is not invertible both integrands are zero (the one in the right hand side for a.e. $z \in \mathbb{R}^2$). □

In the next lemma, we use the homotopic invariance of the degree of a continuous map.

Lemma 8.4 (Convergence of images)**.** *If $p \in \mathbb{H}$ is a nondegenerate point for the map $F \in C^{1,\alpha}_{\mathsf{h}}(\mathbb{H}, \mathbb{R}^2)$, then for every $\varepsilon > 0$ and every compact set $K \subseteq \mathrm{d}_{\mathsf{h}} F(p)(\mathrm{B}_\varepsilon(0))$, there is a $\bar r > 0$ such that, for every $r \in [0, \bar r]$, $K \subseteq F_{p,r}(\mathrm{B}_\varepsilon(0))$.*

Proof. Without loss of generality, we let $p = 0$. We let $D_\varepsilon := \mathrm{B}_\varepsilon(0) \cap \{x^{\mathsf{v}} = 0\}$ which is a bounded open set in $\mathbb{R}^2$, identified with $\{x^{\mathsf{v}} = 0\}$, and set $\bar D_\varepsilon := \bar{\mathrm{B}}_\varepsilon(0) \cap \{x^{\mathsf{v}} = 0\}$ and $\partial D_\varepsilon := \bar D_\varepsilon \setminus D_\varepsilon$. Denote by $H_0 : \bar D_\varepsilon \to \mathbb{H}$ (resp. H_r, for $r > 0$) the restriction of $\mathrm{d}_{\mathsf{h}} F(0)$ (resp. F_r) to $\bar D_\varepsilon$. The map $H : [0, \infty) \times \bar D_\varepsilon \to \mathbb{H}$, $(r, x) \mapsto H_r(x)$, is continuous in both variables, by uniform convergence of F_r towards $\mathrm{d}_{\mathsf{h}} F(0)$ (Lemma 8.3).

For any compact $C \subseteq D_\varepsilon$, by injectivity of H_0 and continuity of H, there is an $\bar r = \bar r(C) > 0$ such that, for $r \in [0, \bar r]$, $H_r(C) \cap H_r(\partial D_\varepsilon) = \emptyset$: otherwise, one could find sequences $r_k \to 0^+$, $x_k \in C$, $y_k \in \partial D_\varepsilon$ with $H_{r_k}(x_k) = H_{r_k}(y_k)$ and, by compactness and continuity, limit points $x \in C$, $y \in \partial D_\varepsilon$ with $H_0(x) = H_0(y)$.

Given a compact $K \subseteq \mathrm{d}_{\mathsf{h}} F(0)(\mathrm{B}_\varepsilon(0))$, we let $C = H_0^{-1}(K)$. From $H_0(C) = \mathrm{d}_{\mathsf{h}} F(0)(C) = K$, we deduce that $z \in K$ implies $z \notin H_r(\partial D_\varepsilon)$, for $r \leq \bar r$. By homotopy invariance of the degree of a continuous map, it follows that $\deg(H_r, \bar D_\varepsilon, z) = \deg(H_0, \bar D_\varepsilon, z) = \operatorname{sign} \det \nabla_{\mathsf{h}} F(0) \neq 0$, hence there is an $x \in D_\varepsilon$ with $z = H_r(x) = F_r(x)$. □

Proof of Theorem 8.2. We split the proof into several steps.

Reduction to a density computation. In view of the measure-theoretic coarea formula [19, Theorem 2.2], it is sufficient to show that the density with respect to $\mathcal{L}^3$ of the measure

$$\nu_F(U) := \int_{\mathbb{R}^2} \mathcal{S}^2_{\mathsf{d}} \left(U \cap F^{-1}(z) \right) \mathrm{d}\mathcal{L}^2(z)$$

coincides with $J_{\mathsf{h}} F(p)$, at $\mathcal{L}^3$-a.e. any point $p \in \mathbb{H}$. The coarea inequality [19, Theorem 4.2] (i.e. inequality $\leq$ in (81), up to some multiplicative factor) implies that the set

$$\{p \in \mathbb{H} : J_{\mathsf{h}} F(p) = 0\}$$

is ν_F-negligible, hence, without loss of generality, we let $p \in \mathbb{H}$ be nondegenerate for F. To simplify notation, we assume $p = 0$, (the general case follows by considering $q \mapsto F(pq) - F(0)$).

Functional density. Instead of proving the usual differentiation

$$\lim_{r \to 0^+} \frac{\nu_F(\bar{\mathrm{B}}_r(0))}{r^4} = \mathcal{L}^3(\bar{\mathrm{B}}_1(0)) J_{\mathsf{h}} F(0), \tag{83}$$

it is technically easier, but equivalent, to introduce $\varepsilon > 0$, to be specified below, and prove

$$\lim_{r \to 0^+} \frac{1}{r^4} \int_{\mathbb{H}} u \circ \delta_{\frac{1}{r}} \, \mathrm{d}\nu_F = J_{\mathsf{h}} F(0), \tag{84}$$

for all continuous functions $u : \mathbb{H} \to [0, \infty)$ with $\int_{\bar{\mathrm{B}}_\varepsilon(0)} u \, \mathrm{d}\mathcal{L}^3 = 1$ and $\mathrm{supp}(u) \subseteq \bar{\mathrm{B}}_\varepsilon(0)$. Indeed, if (84) holds, we let $u(x) := \mathrm{c}(\varepsilon - \mathsf{d}(0, x))^+ = \mathrm{c} \int_0^\varepsilon \mathbb{1}_{\bar{\mathrm{B}}_\varrho(0)}(x) d\varrho$, where $\mathrm{c} = \mathrm{c}(\varepsilon)$ is such that $\int_{\bar{\mathrm{B}}_\varepsilon(0)} u \, \mathrm{d}\mathcal{L}^3 = 1$. If 0 is a differentiability point for ν_F, i.e. the limit in the left hand side of (83) exists, which we denote by ℓ, then (84) yields

$$\begin{aligned} J_h F(0) &= \lim_{r\to 0^+} \frac{1}{r^4}\int_{\mathbb{H}} u\circ\delta_{\frac{1}{r}}\,\mathrm{d}\nu_F = \lim_{r\to 0^+}\frac{1}{r^4}\int_0^{\mathrm{c}\varepsilon} \nu_F(\mathrm{B}_{r\varepsilon-\frac{rt}{\mathrm{c}}}(0))\,\mathrm{d}t \\ &= \int_0^{\mathrm{c}\varepsilon}\left(\lim_{r\to 0^+}\frac{\nu_F(\mathrm{B}_{r\varepsilon-\frac{rt}{\mathrm{c}}}(0))}{(r\varepsilon-\frac{rt}{\mathrm{c}})^4}\right)\left(\varepsilon-\frac{t}{\mathrm{c}}\right)^4\mathrm{d}t = \ell\int_0^{\mathrm{c}\varepsilon}\left(\varepsilon-\frac{t}{\mathrm{c}}\right)^4\mathrm{d}t \\ &= \frac{\mathrm{c}\ell}{\mathcal{L}^3(\mathrm{B}_1(0))}\int_{\mathrm{B}_\varepsilon(0)}(\varepsilon-\mathsf{d}(0,x))\,\mathrm{d}\mathcal{L}^3 = \frac{\ell}{\mathcal{L}^3(\mathrm{B}_1(0))}\int_{\mathrm{B}_\varepsilon(0)} u\,\mathrm{d}\mathcal{L}^3 = \frac{\ell}{\mathcal{L}^3(\bar{\mathrm{B}}_1(0))}. \end{aligned}$$

Change of variables. To show (84), we write, for $r>0$,

$$\begin{aligned} \frac{1}{r^4}\int_{\mathbb{H}} u\circ\delta_{\frac{1}{r}}\,\mathrm{d}\nu_F &= \frac{1}{r^4}\int_{\mathbb{R}^2}\int_{F^{-1}(z)} u\circ\delta_{\frac{1}{r}}\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z) \\ &= \frac{1}{r^2}\int_{\mathbb{R}^2}\int_{F_r^{-1}((z-F(0))/r)} u(x)\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}(x)\,\mathrm{d}\mathcal{L}^2(z) \quad \text{by substitution } x\mapsto\delta_r x \\ &= \int_{\mathbb{R}^2}\int_{F_r^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z) \quad \text{by substitution } z\mapsto zr+F(0). \end{aligned}$$

Since $\mathrm{supp}(u)\subseteq\bar{\mathrm{B}}_\varepsilon(0)$, we can restrict the integration over $\mathbb{R}^2$ to any Borel $U\supseteq F_r(\bar{\mathrm{B}}_\varepsilon(0))$,

$$\frac{1}{r^4}\int_{\mathbb{H}} u\circ\delta_r\,\mathrm{d}\nu_F = \int_U\int_{F_r^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z), \tag{85}$$

and similarly, by (82) with $\mathrm{d}_{\mathsf{h}}F(0)$ in place of F (Lemma 8.3), if $U\supseteq \mathrm{d}_{\mathsf{h}}F(0)(\bar{\mathrm{B}}_\varepsilon(0))$, then

$$J_{\mathsf{h}}F(0) = J_{\mathsf{h}}(\mathrm{d}_{\mathsf{h}}F(0)) = \int_{\mathbb{H}} u\,J_{\mathsf{h}}\left(\mathrm{d}_{\mathsf{h}}F(0)\right)\mathrm{d}\mathcal{L}^3 = \int_U\int_{(\mathrm{d}_{\mathsf{h}}F(0))^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z). \tag{86}$$

By uniform convergence of F_r to $\mathrm{d}_{\mathsf{h}}F(0)$, there is an $\bar r=\bar r(\varepsilon)>0$ such that $F_r(\bar{\mathrm{B}}_\varepsilon(0))\subseteq \mathrm{d}_{\mathsf{h}}F(0)(\bar{\mathrm{B}}_{2\varepsilon}(0))$, hence both (85) and (86) hold with $\mathrm{d}_{\mathsf{h}}F(0)(\bar{\mathrm{B}}_{2\varepsilon}(0))$ in place of U. On the other hand, Lemma 8.4 applied with $p=0$, 3ε in place of ε and $K:=\mathrm{d}_{\mathsf{h}}F(0)(\bar{\mathrm{B}}_{2\varepsilon}(0))$ gives some $\bar r=\bar r(\varepsilon)>0$ such that

$$\mathrm{d}_{\mathsf{h}}F(0)(\bar{\mathrm{B}}_{2\varepsilon}(0))\subseteq F_r\left(\bar{\mathrm{B}}_{3\varepsilon}(0)\right), \quad \text{for } r\in(0,\bar r],$$

hence (85) and (86) hold, for $r\le\bar r$, with $U:=\bigcap_{r\le\bar r}F_r\left(\bar{\mathrm{B}}_{3\varepsilon}(0)\right)$, a choice that we fix in what follows.

Convergence of level sets. We are in a position to apply Corollary 5.7 to (some subsequence of) $\{F_r\}_{r>0}$, which converge to $\mathrm{d}_{\mathsf{h}}F(0)$, in $C^{1,\alpha}_{\mathsf{h}}(\mathbb{H},\mathbb{R}^2)$, as $r\to 0^+$. To simplify, we retain the notation $\{F_r\}_{r>0}$ instead of writing e.g. $\{F_{r_n}\}_{n\ge 1}$. We obtain some positive $\bar r$, δ_4, ε_4 and ϱ_4 such that, letting $I=[-\delta_4,\delta_4]$, for any $r\in(0,\bar r]$ and $q^r\in\bar{\mathrm{B}}_{\varepsilon_4}(0)$, there is an injective solution to the LSDE $\gamma^r: I\to\mathbb{H}$ associated to F_r, with $\gamma^r_0=q^r$, (42) and

$$\gamma^r(I)\cap\bar{\mathrm{B}}_{\varepsilon_4}(0) = (F_r)^{-1}(F_r(q^r))\cap\bar{\mathrm{B}}_{\varepsilon_4}(0).$$

Therefore, we choose ε such that $3\varepsilon\le\varepsilon_4$. Then, for any $z\in U$, $r\in(0,\bar r]$, there is a $q^r\in\bar{\mathrm{B}}_{3\varepsilon}(0)$ such that $F_r(q^r)=z$, hence we obtain from the area formula

$$\int_{F_r^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}} = \int_I u(\gamma^r_\tau)\,\mathrm{d}\tau \quad \text{for } r\in(0,\bar r], \tag{87}$$

due to Corollary 7.2 and Remark 7.3, since (42) holds. As $r \to 0^+$, along any (uniformly) converging subsequence γ^{r_k}, we obtain in the limit some injective solution γ to the LSDE associated to $\mathrm{d}_{\mathsf{h}} F(0)$, such that (23) holds and

$$\gamma(I) \cap \bar{\mathrm{B}}_{\varepsilon_4}(0) = \left(\mathrm{d}_{\mathsf{h}} F(0)\right)^{-1}(z) \cap \bar{\mathrm{B}}_{\varepsilon_4}(0)$$

using also $z = F_{r_k}(\gamma_0^{r_k}) \to \mathrm{d}_{\mathsf{h}} F(0)(\gamma_0)$, by uniform convergence of $\{F_{r_k}\}_k$. Since u is continuous, we have that

$$\lim_{r_k \to 0^+} \int_I u(\gamma_\tau^{r_k})\,\mathrm{d}\tau = \int_I u(\gamma_\tau)\,\mathrm{d}\tau = \int_{(\mathrm{d}_{\mathsf{h}} F(0))^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}},$$

the latter equality being an application of the area formula for γ. In particular, the limit value does not depend on the subsequence $\{r_k\}_k$, and recalling (87), we conclude that

$$\lim_{r \to 0^+} \int_{F_r^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}} = \int_{(\mathrm{d}_{\mathsf{h}} F(0))^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}},$$

for any $z \in U$. From (85) and (86), by dominated convergence, we have

$$\begin{aligned}\lim_{r \to 0^+} \frac{1}{r^4} \int_{\mathbb{H}} u \circ \delta_r\,\mathrm{d}\nu_F &= \lim_{r \to 0^+} \int_U \int_{F_r^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z) \\ &= \int_U \int_{(\mathrm{d}_{\mathsf{h}} F(0))^{-1}(z)} u\,\mathrm{d}\mathcal{S}^2_{\mathsf{d}}\,\mathrm{d}\mathcal{L}^2(z) = J_{\mathsf{h}} F(0),\end{aligned}$$

i.e. (84) is proven, hence the thesis. □

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