Local and global dynamics in a duopoly with price competition and market share delegation

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Abstract

This paper aims at studying a nonlinear dynamic duopoly model with price competition and horizontal product differentiation augmented with managerial firms, where managers behave according to market share delegation contracts. Ownership and management are then separate and managers are paid through adequate incentives in order to achieve a competitive advantage in the market. In this context, we show that complexity arises, related both to the structure of the attractors of the system and the structure of their basins, as multistability occurs. The study is conducted by combining analytical and numerical techniques, and aims at showing that slight different initial conditions may cause very different long-term outcomes.

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1 Introduction

The issue of strategic delegation in oligopoly models has received in depth attention starting from the pioneering contributions of Baumol (1958), Fama and Jensen (1983), Vickers (1985), Fershtman (1985), Fershtman and Judd (1987) and Sklivas (1987). This literature has investigated several types of contracts amongst firms' owners and managers. These contracts are observable, and the performance of managerial firms also depends on whether they behave as price-setters or quantity setters. Specifically, when ownership and management are separate (this is the case of large companies, where governance is different from competitive firms) managers are likely to be driven by other motives than just maximizing profits, thus owners may try to motivate them through adequate incentives in order to achieve a competitive advantage in the market. To this purpose, managerial incentive schemes are essentially based on a weighted average of profits and output (Vickers, 1985), profits and revenues (Fershtman and Judd, 1985; Sklivas, 1987), and relative performance evaluation (Fumas, 1992; Miller and Pazgal, 2001, 2002; Kopel and Lambertini, 2012). The main interest of these studies lies in ranking outputs in oligopolies with managerial firms adopting these kind of contracts, and contrasting them with the case of profit maximisation.

More recently, there has been a burgeoning interest in strategic incentive schemes where compensations of managers are based on profits and market shares (Jansen et al., 2007, 2009; Ritz, 2008, and Wang and Wang, 2010; Kopel and Lambertini, 2013), called market share delegation contracts. In particular, Jansen et al. (2007) have studied a two-stage market share delegation game with two competing managerial firms, finding that a duopoly with market share delegation contracts performs better in terms of profits than a sales delegation game. Both cases of sales delegation and market share delegation lead to more aggressive managerial behaviors, causing lower profitability and higher social welfare than the standard Cournot and Bertrand duopolies. Market share delegation is found to be the dominant strategy in an asymmetric duopoly delegation game. Ritz (2008) has tackled the issue of market share contracts showing that this kind of incentives for managers dominate delegation output-based contracts as well as standard profit maximisation. They have also found that in an equilibrium with a market share contract firms turn out to be less competitive than under sales delegation contracts and players are captured in a prisoner's dilemma. Later, Kopel and Lambertini (2013) have revisited Jansen et al. (2007) showing for Bertrand competition that they have used a misspecified demand system for horizontally differentiated products. By deriving correct demand functions (i.e., prices collapse to marginal costs when products are homogeneous), they have concluded that under market share delegation firms result in less competition (higher profits) than under sales delegation *à* la Fershtman and Judd (1987). In addition, some empirical studies have evidenced that market share rather than sales may provide an important objective for managerial firms, thus market share delegation contracts may become of importance in actual economies. For instance, Peck (1988) has founded that increasing market shares are the second and third objectives for several Japanese and American top managers, while Gray (1995) has substantially confirmed such a result finding that the market share objective is ranked second for a sample of managers of US firms with subsidiaries in Japan. Also in countries such as Canada, Germany and UK the market share objective seems to be relevant in the managerial performance evaluation (Borkowski, 1999).

The present paper revisits the nonlinear duopoly model with price competition and horizontal differentiation developed by Fanti et al. (2013), thus obtaining a discrete time two dimensional dynamic system T which describes the price evolution in the economic setup. System T is then studied in order to explain how managerial incentive contracts based on market share affects local and global dynamics. To this purpose, we distinguish between the cases of: symmetric weight attached to the managers' bonus in their objective function (i.e., managers are of the same type and T is symmetric), and asymmetric weight (i.e., managers are of different types and the symmetry of system T is broken). We assume that products are substitutes and we show that there is an important relationship between the degree of horizontal product differentiation and the manager's bonus.

Specifically, we find different dynamic outcomes depending on whether the manager's bonus is equally weighted or weighted differently in the managers' objective function.

Symmetric case. In the case of symmetric delegation contracts, an interior Nash equilibrium[1] exists only if managers do not behave aggressively in the market. We find the existence of an upper bound of the weight of manager's bonus in the objective function such that the Nash equilibrium is locally and globally stable when it tends to such a threshold. With regard to stability outcomes, we find that when the bonus is close to its maximum threshold value ($\bar{b}$), the Nash equilibrium is locally (and globally) stable. The threshold $\bar{b}$ depends on the degree of product differentiation. In particular, when products tend to become homogeneous (substitutes), $\bar{b}$ decreases. This means that the higher product homogeneity is the less aggressive managers should be to guarantee the existence of a Nash equilibrium. In addition, when players start from the same initial condition coordination occurs in the long term. The attractor will become more complex if managers' behaviors are driven by contracts that assign an intermediate weight to the market share bonus in their objective function or the degree of substitutability between products is high or small. A chaotic attractor can be obtained if managers are driven less aggressively. In contrast, when players start from different initial conditions, if the attractor on the diagonal is transversely stable, the system synchronizes. In fact, given the manager's behavior, when products are substitutes (i.e., an increase in the market demand of product of variety 1 implies a decrease in the market demand of product of variety 2) managerial firms cannot coordinate themselves. We also find that the phenomenon of multistability of two or more attractors may occur depending on the relative weight of the manager's type and the attractors may be complex (the complexity in the structure of the attractor increases when the manager's bonus decreases sufficiently or products tend to be perfect substitutes). However, since products are imperfect substitutes not only the structure of the attractor but also the structure of the basin of attraction may be complex as well. This gives rise to problems of unpredictability (and then of policy rule) of the final outcome of the economy.

A*symmetric case.* If delegation contract are weighted differently, there exists a unique interior Nash equilibrium with different coordinates values. Specifically, the lower price is associated with the good produced by the firm where the manager behaves more aggressively. The local stability of the Nash equilibrium is obtained when managers behave not aggressively and similarly. By starting from the same kind of contract, we find that a slight perturbation on the size of the bonus (heterogeneity) causes the emergence of cycles. This because system T is no more symmetric and synchronised trajectories (coordination) are avoided. However, the multistability phenomenon continues to exist.

The present paper is connected with the work of Fanti et al. (2012), that has developed a nonlinear Cournot duopoly with quantity setting firms and managerial incentive contracts based on relative profit delegation. Despite the assumption of homogeneous players (symmetric system), they have shown that an increase in the degree of competition between managers may be a source

[1]For the notion of Nash equilibrium see the seminal contributions of Nash (1950, 1951).

of on-off intermittency, blow-out phenomena and multistability. Similar events are found by Bischi et al. (1998) under profit-maximizing quantity-setting firms but only when they are heterogeneous (asymmetric map). From a mathematical point of view, this holds because - unlike Bischi et al. (1998) - the two-dimensional map that characterises the Cournot duopoly with relative profit delegation of Fanti et al. (2012) contains a parameter (i.e., the relative managers' attitude in their objective functions) that weights the mixed term without affecting the pure quadratic term.

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 describes some preliminary properties of the two-dimensional dynamic system T, whose iteration defines the time evolution of price of each variety in the Bertrand duopoly model with market share delegation. In particular, it describes the feasible set and its structure and determines the fixed points and other invariant sets, while reaching some conditions for the local stability. Given the analytical form of system T and the high number of parameters, its dynamic properties are quite difficult to be analyzed. Hence, Section 4 considers the simpler case in which owners assign the same weight to the market share, thus obtaining the symmetric system T_b. First, we prove that T_b admits a unique Nash equilibrium for suitable parameter values. Second, we study the synchronized trajectories occurring when both players start from the same initial condition. Finally, we consider the question of synchronization and show that multistability may occur. Section 5 comes back to the non-symmetric model T and underlines the main differences with the symmetric case. Section 6 outlines the conclusions.

2 The model

Consider an economy that consists of two types of agents (firms and consumers) and two sectors. One sector is competitive and produces the numeraire good $k \geq 0$, whose unit price is normalised to 1 without loss of generality. There also exists a duopolistic sector where firm 1 and firm 2 produce (horizontally) differentiated products of variety 1 and variety 2, respectively. The price (per unit of good) and quantity of product of firm i ($i = 1, 2$) are respectively denoted by $p_i \geq 0$ and $q_i \geq 0$.

Consumers. Let the economy be comprised of a continuum of identical consumers with preferences towards goods q_1, q_2 and k captured by the separable utility function $V(q_1, q_2, k) : \mathbb{R}^3_+ \to \mathbb{R}_+$, and specified by the following quasi-linear formulation: $V(q_1, q_2, k) = U(q_1, q_2) + k$, where $U(q_1, q_2) : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a twice differentiable function. The representative consumer maximises $V(q_1, q_2, k)$ subject to budget constraint $p_1q_1 + p_2q_2 + k = M$, where $M > 0$ is the exogenous nominal income of the consumer. This income is high enough to avoid the existence of corner solutions. Since $V(q_1, q_2, k)$ is a separable function and it is linear in k, there are no income effects on the duopolistic sector. The consumer's optimization problem can then be written as: $\max_{\{q_1,q_2\}} U(q_1, q_2) - p_1q_1 - p_2q_2 + M$. By following Bowley (1924), Spence (1976), Dixit (1979), Singh and Vives (1984), Vives (1985), Qiu (1997), Häckner (2000) and Kopel and Lambertini (2013), we specify utility function $U(q_1, q_2)$ as follows:

$$U(q_1, q_2) = q_1 + q_2 - \frac{1}{2}(q_1^2 + q_2^2 + dq_1q_2), \tag{1}$$

where $d \in (0, 1)$ captures the degree of horizontal product differentiation in the case of substitutability. More precisely, if $d > 0$ then products are (imperfect) substitutes, while when $d \to 1$ they are perfect substitutes. On the other hand, if $d \to 0^+$ then products of variety 1 and variety 2 tend to be independent and each firm behaves as a monopolist in such a case.

By using (1), the consumer's maximization program above gives the following inverse demands of good 1 and good 2, respectively:

$$p_1 = 1 - q_1 - dq_2 \ \text{ and } \ p_2 = 1 - q_2 - dq_1. \tag{2}$$

From (2) the corresponding direct demands are then given by:

$$q_1 = \frac{1 - p_1 - d(1 - p_2)}{1 - d^2} \ \text{ and } \ q_2 = \frac{1 - p_2 - d(1 - p_1)}{1 - d^2}. \tag{3}$$

Observe that, since prices and quantities of products of both varieties must be nonnegative, then according to direct demands in (3) the following relations must hold,

$$1 - p_1 - d(1 - p_2) \geq 0 \ \text{ and } \ 1 - p_2 - d(1 - p_1) \geq 0, \ \ p_1, p_2 \geq 0.$$

As a consequence economic meaningful prices must belong to an opportune subset $Q \subset \mathbb{R}^2_+$, where Q is the convex polygon with vertices $(0, 0)$, $(0, 1 - d)$, $(1, 1)$ and $(1 - d, 0)$.

Duopolistic firms with managerial incentive contracts based on market share bonuses. We assume firms with the same average and marginal cost of producing an additional unit of output, represented by $0 \leq w < 1$. Therefore, firms i's cost function is $c_i = wq_i$ $(i = 1, 2)$, that is there exist constant marginal returns to labor (Correa-Lopez and Naylor, 2004), implying that $q_i = L_i$, where L_i is the labor force of firm i. We assume that owners of both firms hire a manager and delegate output decisions to him. By following Kopel and Lambertini (2013), each manager receives a fixed salary (which is set to zero without loss of generality) plus a bonus offered in a contract which is publicly observable[2] and based on market share $\frac{q_i}{q_i+q_j}$, with $i, j = 1, 2$, $i \neq j$, where $q_i + q_j$ is the total output. The bonus of manager hired by firm i is then given by:

$$W_i = \Pi_i + b_i \frac{q_i}{q_i + q_j}, \tag{4}$$

that represents the manager's utility function, where $\Pi_i = (p_i - w)q_i$ are profits of ith firm and $b_i > 0$ is the delegation variable of manager hired in firm i, which is usually considered as a control variable in the industrial economics literature chosen in the contract-stage of the game (Jansen et al., 2007; Kopel and Lambertini, 2013). However, since the aim of this paper is to analyse the dynamics of a duopoly with price competition when the degree of market share in the managers' objective varies, we assume b_i as an exogenously given parameter[3] and look at the dynamic properties of a two-dimensional map, as is shown below. Then, by using (3) and knowing that $q_i + q_j = \frac{2-p_i-p_j}{1+d}$ we may write (4) as follows:

$$W_i = \frac{1 - p_i - d(1 - p_j)}{1 - d^2} \left[p_i - w + b_i \frac{1 + d}{2 - p_i - p_j} \right], \ \ i, j = 1, 2, \ \ i \neq j. \tag{5}$$

From (5), the marginal bonus of manager $i, j = 1, 2$ is given by:

$$\frac{\partial W_i}{\partial p_i} = \frac{1 - 2p_i - d(1 - p_j) + w}{1 - d^2} - b_i \frac{(1 + d)(1 - p_j)}{(1 - d)(2 - p_i - p_j)^2}, \ \ i, j = 1, 2, \ \ i \neq j. \tag{6}$$

[2]A contract which is publicly available implies that firms know the kind of contract agreed upon the manager, and the type of manager behaviour as well. In contrast, in the case a contract was agreed upon private information there should be problems of signalling and uncertainty.

[3]Since the focus of the paper is on the mathematical properties of a two-dimensional map embodying market share delegation, we limit ourselves only to positive values of b for the sake of simplicity. A possible analysis of negative values of b is beyond the scope of this paper and it is left for future research.

Dynamic setting. Consider now a dynamic setting where time is indexed by $t \in \mathbb{Z}_+$. Following Bischi et al. (1998) and Fanti et al. (2012, 2013), we assume that both players have limited information regarding manager bonuses (no knowledge of the market). However, they follow an adjustment process based on local estimates of the marginal bonus to overcome this informational lacuna (for similar mechanisms see Bischi and Naimzada, 2000 or Bischi and Lamantia 2012 with regard to a discrete time model, and Dixit, 1986 with regard to a continuous time model). This behavioral rule is given by:

$$p_{i,t+1} = p_{i,t} + \alpha p_{i,t} \frac{\partial W_i(p_{i,t}, p_{j,t})}{\partial p_{i,t}}, \quad i = 1, 2, \quad t \in \mathbb{Z}_+, \tag{7}$$

where $\alpha > 0$ is a coefficient that captures the speed of adjustment of firm i's price with respect to a marginal change in the manager's bonus, $\alpha p_{i,t}$ is the intensity of the reaction of player i, and $\frac{\partial W_i}{\partial p_i}$ is determined by (6).

By taking into account equation (7), the two-dimensional system that characterizes the dynamics of a Bertrand duopoly with horizontal differentiation, linear demands, constant average and marginal cost and market share delegation is the following:

$$T : \begin{cases} x' = xF(x,y) = x\left[1 + \alpha\left(\frac{1-2x-d(1-y)+w}{1-d^2} - b_1\frac{(1+d)(1-y)}{(1-d)(2-x-y)^2}\right)\right] \\ y' = yG(x,y) = y\left[1 + \alpha\left(\frac{1-2y-d(1-x)+w}{1-d^2} - b_2\frac{(1+d)(1-x)}{(1-d)(2-x-y)^2}\right)\right] \end{cases}, \tag{8}$$

where $x' = p_{1,t+1}$, $x = p_{1,t}$, $y' = p_{2,t+1}$, and $y = p_{2,t}$. System T is a two-dimensional dynamic system, whose iteration defines the time evolution of price of each variety.

3 Some preliminary properties

3.1 The feasible set and its structure

First of all we observe that system (8) is economically meaningful only whether, at any time t, the two state variables x and y belong to Q, where Q is given by the convex polygon with vertices $(0,0)$, $(0,1-d)$, $(1-d,0)$ and $(1,1)$.

We now pursue the following definition.

Definition 1. *Let $T^t(x(0), y(0))$, $t \in \mathbb{N}$, denote the $t-th$ iterate of system T for a given initial condition (i.c.) $(x(0), y(0)) \in Q$. Then the sequence $\psi_t = \{(x(t), y(t))\}_{t=0}^{\infty}$ is called trajectory. A trajectory ψ_t is said to be feasible for system T if $(x(t), y(t)) \in Q$ for all $t \in \mathbb{N}$, otherwise it is said to be unfeasible. The set $D \subseteq Q$ whose points generate feasible trajectories is called feasible set.*

For system T considered in the convex polygon Q it is easy to verify that feasible set D is non-empty and such that $D \subset Q$. The proof of this statement is straightforward. In fact, $(0,0)$ produces a feasible trajectory while $(1-\epsilon, 1-\epsilon)$, with $\epsilon > 0$ small enough, generates an unfeasible trajectory.

In Figures 1 (a) and (b), the feasible set is depicted in white for two different parameter constellations while the grey points represent initial conditions belonging to Q producing unfeasible trajectories. It can be immediately noticed that D may have a simple structure (as in (a)) or a complex structure (as in (b)): the study of the structure of set D is of great interest both from an economic and a mathematical perspective since the long-term evolution becomes path-dependent, and a thorough knowledge of the properties of D becomes crucial in order to predict the feasibility of the economic system.

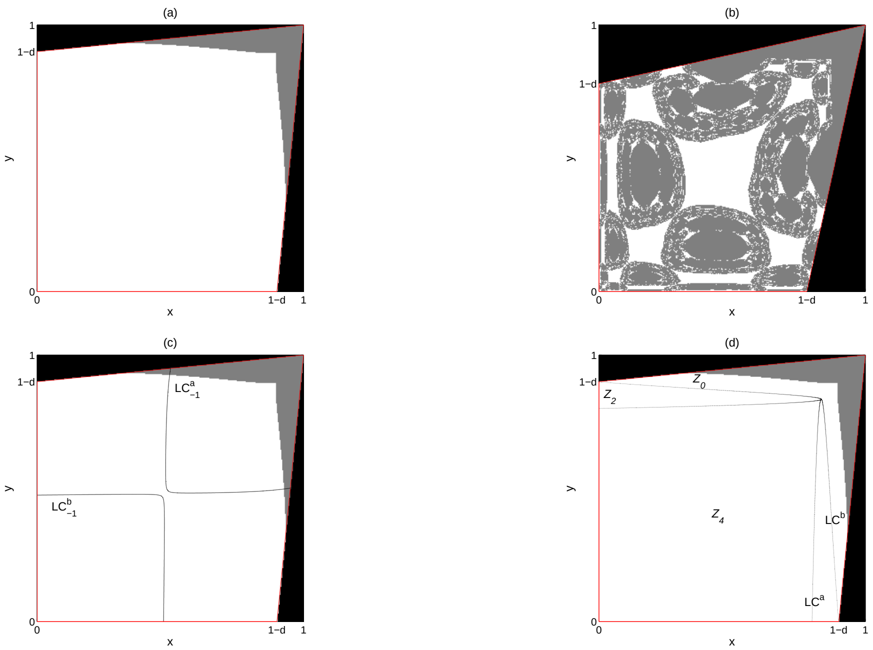

Figure 1: (a), (b). The feasible set $D \subset Q$ is depicted in white; the gray points are initial conditions producing unfeasible trajectories. $\alpha = 1.7$, $w = 0.5$, $b_1 = b_2 = 0.1$; in (a) $d = 0.1$ while in (b) $d = 0.22$. (c) Critical curves of rank-0, LC_{-1}, for system T and the parameter values as in (a). (d) Critical curves of rank-1, $LC = T(LC_{-1})$, for the same parameter values as in panel (a). These curves separate the plane into the regions Z_4, Z_2 and Z_0, whose points have different number of preimages.

The main purpose of this section is to analyze the properties of critical curves in order to determine the structure of the feasible set inside which feasible asymptotic trajectories are bounded. The procedure we will follow is mainly numerical and similar to the one used for the study of the basins structure in two-dimensional noninvertible maps (see, for instance, Bischi et al. 2000; Bischi and Lamantia 2002; Brianzoni et al. 2012).

Map (8) is noninvertible on set Q. In fact, given an initial condition $(x(0), y(0)) \in Q$, the forward iteration of T defines a unique trajectory, while its backward iteration is not uniquely defined since a point belonging to Q may have more than one preimage (a description of the main properties of noninvertible maps of the plane is in Mira et al., 1996). More precisely, the rank-1 preimages of a given point $(x', y') \in Q$ are solutions of system

$$(x', y') = T(x, y),$$

and the corresponding two fourth degree algebraic equations may have four or two real solutions or no real solution at all. As a consequence, T is of $Z_4 - Z_2 - Z_0$ type as Q can be subdivided into regions whose points have 4, 2 or 0 preimages and the boundaries of such regions are characterised by the existence of at least two coincident (merging) preimages. By following the notation used by Mira et al. (1996) and Abraham et al. (1997), we denote the critical curve of rank-1 by LC (it represents the locus of points with two or more coincident preimages) and the curve of merging preimages by LC_{-1}.

The locus LC_{-1} for a two dimensional continuous and differentiable map is given by the set of points such that the determinant of the Jacobian matrix is null. For system T the Jacobian matrix is given by

$$J(x,y) =$$
$$\begin{pmatrix} 1+\alpha\left(\frac{1-4x-d(1-y)+w}{1-d^2} - \frac{b_1(1+d)(1-y)(2+x-y)}{(1-d)(2-x-y)^3}\right) & \alpha x\left(\frac{d}{1-d^2} - b_1\frac{(1+d)(x-y)}{(1-d)(2-x-y)^3}\right) \\ \alpha y\left(\frac{d}{1-d^2} - b_2\frac{(1+d)(y-x)}{(1-d)(2-x-y)^3}\right) & 1+\alpha\left(\frac{1-4y-d(1-x)+w}{1-d^2} - \frac{b_2(1+d)(1-x)(2-x+y)}{(1-d)(2-x-y)^3}\right) \end{pmatrix} \quad (9)$$

and consequently its determinant vanishes in points $(x, y) \in Q$ such that $|J(x, y)| = 0$.

The set of points such that $|J(x, y)| = 0$ is depicted in Figure 1 (c) while curves $LC = T(LC_{-1})$ separate the polygon Q into regions having a different number of preimages (see Figure 1 (d)).

The noninvertibility property plays a crucial role in the investigation of the topological structure of the feasible set and of the global bifurcations that occurs as some parameters are changed.

Let $D \subset Q$ be the set of initial conditions that generate feasible trajectories (the white regions in Figures 1 (a) and (b)), and let $\bar{D} \subset Q$ be the set of points that generate unfeasible trajectories (the grey regions in Figures 1 (a) and (b)). Then $D = Int(Q/\bar{D})$, where $Int(M)$ are the interior points of set M. In order to determine the feasible set D, we observe that its boundary, namely ∂D, coincides with the boundary of $\bar{D}$, given by $\partial\bar{D}$.

We now fix the following parameter values $\alpha = 1.7$, $w = 0.5$, $b_2 = 0.1$, and present some numerical simulations to show the increasing complexity of the feasible set D when parameters d and b_1 are changed. Both d and b_1 are of importance from an economic point of view. In fact, they respectively represent the extent of product differentiation and the weight of market share bonus in manager 1's objective function if the weight of market share bonus of manager 2 is fixed. Under this assumption, managers' bonuses are evaluated differently, i.e. there exist heterogeneous market share delegation contracts.

Let $b_1 = 0.2$. Then if $d = 0.15$, set D has a simple structure (connected set), as it is shown in Figure 2 (a), while if $d = 0.25$ set D consists of infinitely many non-connected sets (see Figure

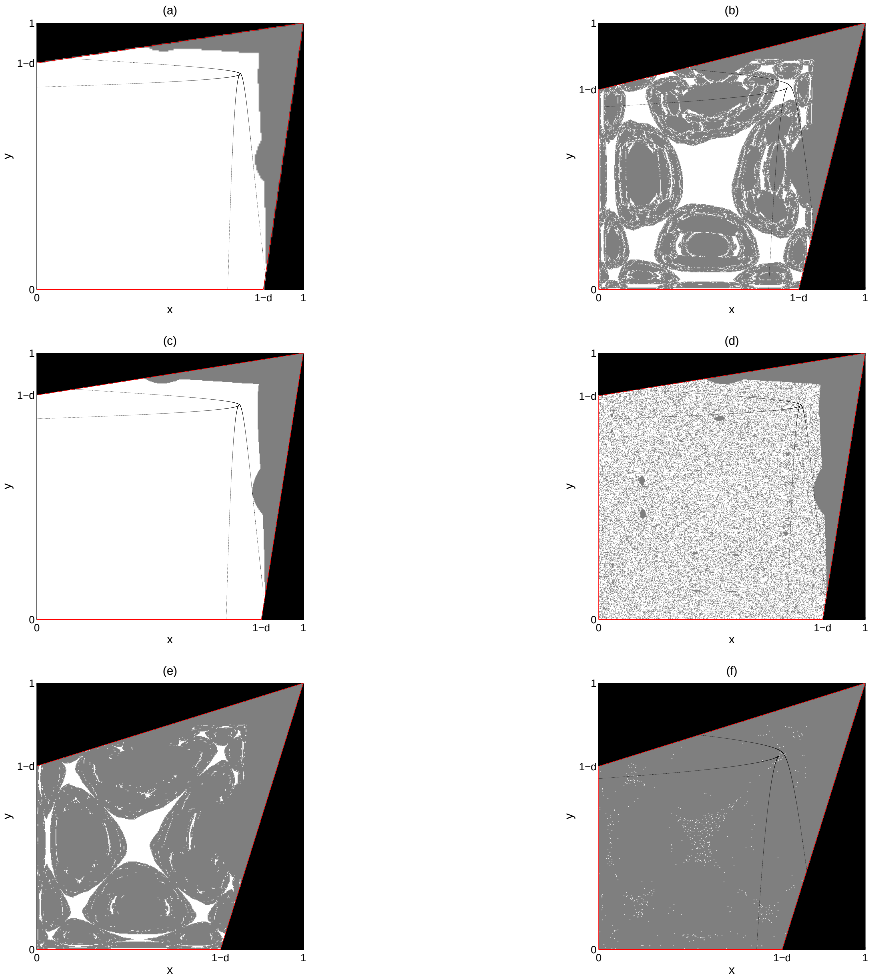

Figure 2: The feasible set $D \subset Q$ is depicted in white; the gray points are initial conditions producing unfeasible trajectories. (a) D has a simple structure ($d = 0.15$). (b) D has a complex structure ($d = 0.25$). (c) Immediately before the contact bifurcation ($d \simeq 0.1575$), the tangency between the critical curve and the boundary of the feasible set is shown. (d) Set D after the contact bifurcation ($d = 0.16$): gray holes are depicted. (e) For $d = 0.311$ the unfeasible set is increased in size. (f) Immediately before the final bifurcation ($d = 0.33185$) almost all trajectories are unfeasible.

2 (b)). By comparing the two panels, it can be observed that the LC^b curve moves upwards as parameter d decreases so that, given the other parameter values, a threshold value $\bar{d} \simeq 0.1575$ does exist such that a contact between a critical curve and the boundary of the feasible set occurs. At this parameter value, LC^b is tangent to the boundary (see Figure 2 (c)) and a bifurcation occurs causing the change of D from a connected set to non-connected set, i.e. it is given by an infinite sequence of non-connected regions (or holes) inside Q. This bifurcation occurs because a portion of $\bar{D}$ enters in a region characterised by a high number of preimages (and hence preimages of any rank of such a portion also belong to $\bar{D}$) and new components of the unfeasible set suddenly appear after the contact (in Figure 2 (d) the feasible set is depicted after the contact bifurcation creating gray holes inside the white region). Obviously, a subset $\bar{D}_0$ exists such that trajectories starting from $\bar{D}_0$ exit from Q at the first iteration. On the other hand, after the contact bifurcation, $\bar{D}_0$ admits new preimages given by $\bar{D}_{-1} = \{(x, y) \in Q : T(x, y) = \bar{D}_0\}$ and consequently initial conditions belonging to $\bar{D}_{-1}$ also generate unfeasible trajectories, as $\bar{D}_{-1}$ is mapped into set $\bar{D}_0$ after one iteration. The previous procedure can be repeated while considering the preimages of rank-2 of set $\bar{D}_0$, namely $\bar{D}_{-2}$. Again, initial conditions belonging to set $\bar{D}_{-2}$ generate trajectories converging to $\bar{D}_0$ after two iterations.

Let $\bar{D}_0 = \{(x, y) \in Q : T(x, y) \notin Q\}$ and $\bar{D}_{-i} = \{(x, y) \in Q : T(x, y) = \bar{D}_{-i+1}\}$, $i = 1, 2, ...$, then the set of points generating unfeasible trajectories is given by

$$\bar{D} = \bar{D}_0 \cup_{i \geq 1} \bar{D}_{-i}.$$

For the chosen parameter constellation, the complexity of the structure of the feasible set increases if d increases and the gray area increases as well (i.e. the set of initial conditions generating unfeasible trajectories), as is shown previously; however, it can also be observed that if d further decreases a final bifurcation occurs (see Figure 2 (e)), while as d crosses a value $d \simeq 0.33186$ almost all trajectories become unfeasible (see Figure 2 (f)).

The numerical results shown in Figure 2 describe the bifurcations occurring in the structure of set D for certain parameter values and $b_1 \neq b_2$. However, it can be observed that a similar behavior also holds if $b_1 = b_2$ (that is, managers' bonuses are equally weighted in both firms representing symmetric subgame-perfect equilibrium in a static game). In this particular case the contact bifurcation between the critical set and the boundary of the feasible set occurs into two points that are symmetric with respect to the diagonal $\Delta = \{(x, y) \in Q : x = y\}$, so that the resulting feasible set is symmetric as well (see Figure 1 (a) and (b)).

The previous arguments show that the bifurcation concerning the structure of the feasible set is strictly related to the values of the two key parameters d and b_1, hence it depends on both the degree of horizontal product differentiation and the level of market share bonuses.

Although the previous analysis has been carried out from a numerical point of view, the structure of D can be studied analytically in some limit cases as shown in the following Proposition.

Proposition 1. *Let T given by (8).*

(i) Assume $b_i > w$, $i = 1, 2$. If $d \to 1^-$ then $D = \{(0, 0)\}$.

(ii) If $b_i \to +\infty$, $i = 1, 2$, then $D = \{(0, 0)\}$.

Proof. (i) If $d \to 1^-$, then the convex polygon Q tends to the segment $I = \{(x, y) \in \Delta : x \in [0, 1]\}$. Hence, to prove the statement, we consider an initial condition $(x(0), y(0))$ such that $x(0) = y(0)$ and $x(0) \in (0, 1]$ and we show that it produces an unfeasible trajectory if $b_i > w$.

Trivially the initial condition $(1,1)$ is unfeasible as T is not defined in such a point. Let us consider $x(0) \in (0,1)$. By taking into account the first equation in system (8), if $d \to 1^-$ we obtain

$$x(1) \to \lim_{d \to 1^-} x(0)\left[1 + \frac{\alpha}{1-d}\left(\frac{w - x(0)}{2} - \frac{b_1}{2(1-x(0))}\right)\right] < \lim_{d \to 1^-}\left(1 + \frac{\alpha}{1-d}\frac{w - b_1}{2}\right)$$

and consequently if $b_1 > w$, $\lim_{d \to 1^-} x(1) < 0$, that is $(x(0), y(0))$ exits from Q at the first iteration. Similarly it can be proved that this condition must hold also for parameter b_2 once considering the second equation of system (8).

(ii) This statement can be proved simply considering the limits $\lim_{b_1 \to +\infty} x(1)$ and $\lim_{b_2 \to +\infty} y(1)$ for any given initial point $(x(0), y(0)) \in Q$, $x(0) + y(0) \neq 0$.

□

This proposition shows that if the bonus of manager i (b_i) is sufficiently high then $\forall d \in (0, d_+)$ the feasible set is composed only by the origin. In addition, $\forall d \in (0,1)$ both b_1 and b_2 should not be fixed at too high a level in order to have feasible trajectories produced by an interior initial condition. For these reasons, economically meaningful long-term dynamics can be produced only for small enough values of d and b_i ($i = 1,2$), thus confirming numerical experiments previously presented.

From an economic point of view, feasible trajectories do exist only when the extent of product differentiation is sufficiently high and the weight attached to market share in managers' bonuses is not too high. Therefore, policies aiming at increasing the degree of competition between the two firms by favoring production of homogeneous goods may cause the exit from the market of firms. The same result applies as long as firms' owners try to motivate their managers to behave aggressively in the market for achieving an adequate competitive advantage with respect to the rival. In this case, in fact, given the differentiation parameter d, firm i would react by fixing a price for the subsequent period at too high a level (because marginal profits are high) and then it may exit the market.

Given these results, in what follows we will focus on the study of the dynamics produced by T while assuming b and d relatively small.

3.2 Invariant sets, fixed points and local stability.

In order to describe the long-term dynamics produced by system T, it is very important to observe that each coordinate semiaxis is invariant for (8), that is $T(x,0) = (x',0)$ and $T(0,y) = (0,y')$, and consequently the dynamics of T on such lines are governed by the two one-dimensional maps $\phi_x = xF(x,0)$ and $\phi_y = yG(0,y)$, given by

$$x' = \phi_x(x) = x\left[1 + \alpha\left(\frac{1 - 2x - d + w}{1 - d^2} - b_1\frac{(1+d)}{(1-d)(2-x)^2}\right)\right], \quad x \in [0, 1-d] \tag{10}$$

and

$$y' = \phi_y(y) = y\left[1 + \alpha\left(\frac{1 - 2y - d + w}{1 - d^2} - b_2\frac{(1+d)}{(1-d)(2-y)^2}\right)\right], \quad y \in [0, 1-d] \tag{11}$$

respectively.

Given the analytical form of maps (10) and (11), and the high number of parameters, the dynamics generated by map T on the two invariant axes are difficult to be analytically investigated. However, for what it concerns the aim of our study, the following properties can be proved.

Proposition 2. *Consider maps ϕ_x and ϕ_y given by (10) and (11). Then:*

(a) $\phi_x(0) = \phi_y(0) = 0$.

(b) Let

$$\beta^* = \frac{(1-d)(1+d-\alpha)}{\alpha} + w.$$

If $b_1 = \beta^*$ *then* $\phi_x(1-d) = 0$*; if* $b_1 > \beta^*$ *then* $\phi_x(1-d) < 0$*; if* $b_1 \in (0, \beta^*)$ *then* $\phi_x(1-d) > 0$.

(c) Let

$$\tilde{\beta} = d + w - 1 < \beta^*.$$

If $b_1 > \tilde{\beta}$ *then* $\phi_x(1-d) < 1-d$*; if* $b_1 = \tilde{\beta}$ *then* $\phi_x(1-d) = 1-d$*; if* $b_1 < \tilde{\beta}$ *then* $\phi_x(1-d) > 1-d$.

(d) Let

$$\bar{\beta} = 4\frac{1-d+w}{(1+d)^2} \text{ and } \bar{\bar{\beta}} = 4\frac{(1-d)(1+d+\alpha)+\alpha w}{\alpha(1+d)^2}, \ 0 < \bar{\beta} < \bar{\bar{\beta}}.$$

If $b_1 \in (0, \bar{\beta})$ *then* $\lim_{x\to 0^+} \phi_x'(x) > 1$*; if* $\bar{\beta} \le b_1 < \bar{\bar{\beta}}$ *then* $\lim_{x\to 0^+} \phi_x'(x) \in (0,1]$*;* $b_1 \ge \bar{\bar{\beta}}$ *then* $\lim_{x\to 0^+} \phi_x'(x) \le 0$*. Furthermore* $\tilde{\beta} < \bar{\beta}$ *and* $\beta^* < \bar{\bar{\beta}}$.

(e) $\phi_x(x)$ *and* $\phi_y(y)$ *are strictly concave in* $[0, 1-d]$.

Statements (b)*,* (c) *and* (d) *also apply to* $\phi_y(y)$ *while replacing* b_1 *with* b_2.

Proof. Statement (a) is trivial. Part (b) can be easily proved by considering that

$$\phi_x(1-d) = (1-d)\left[1 + \alpha\frac{d-1+w-b_1}{1-d^2}\right]$$

and similarly for ϕ_y. The previous equation can also be used to prove part (c). To prove part (d) we observe that

$$\lim_{x\to 0^+} \phi_x'(x) = 1 + \alpha\left(\frac{4(1-d+w) - b_1(1+d)^2}{4(1-d^2)}\right)$$

and that similar computations hold for ϕ_y'. To prove (e), we note that

$$\phi_x''(x) = -\frac{4\alpha}{1-d^2} - \frac{\alpha b_1(1+d)}{(1-d)(2-x)^3}\left(\frac{14-4x}{2-x}\right) < 0, \forall x \in [0, 1-d],$$

and that this relation similarly holds for $\phi_y''(y)$. □

In order to discuss about the dynamics generated by map T on the two invariant axes we consider initial conditions belonging to the segments I_x and I_y given by

$$I_x = \{(x,y) : 0 \le x \le 1-d, y = 0\} \text{ and } I_y = \{(x,y) : 0 \le y \le 1-d, x = 0\},$$

and we describe the qualitative dynamics produced by $\phi_x(x)$ and $\phi_y(y)$ by taking into account the properties stated in Proposition 2. The following remark holds.

Remark 1. *Consider map ϕ_x given by (10).*

- *If* $b_1 \ge \bar{\bar{\beta}}$ *then* $x = 0$ *is the unique fixed point and all initial conditions* $x(0) \in (0, 1-d]$ *produce unfeasible trajectories;*
- *if* $\bar{\beta} \le b_1 < \bar{\bar{\beta}}$ *then* $x = 0$ *is the unique fixed point and it is locally asymptotically stable; at* $b_1 = \bar{\beta}$ *a fold bifurcation occurs;*

- *if $\tilde{\beta} \leq b_1 < \bar{\beta}$ then two fixed points are owned, $x = 0$, which is locally unstable, and $x = x_0 > 0$, which can be locally stable or unstable depending on the other parameter values; at $b_1 = \tilde{\beta}$ $x_0 = 1 - d$;*
- *if $b_1 < \tilde{\beta}$ then $x = 0$ is the unique fixed point and it is locally unstable.*

These results apply also to map ϕ_y while considering parameter b_2.

According to the previous remark we observe that the dynamics involved on the two segments I_x and I_y may be complex only if $\tilde{\beta} \leq b_i < \bar{\beta}$, $i = 1, 2$. Since $\tilde{\beta}$ can be positive or negative this means that the market share bonus must be not too high. We also observe that if $\tilde{\beta} \leq b_i < \bar{\beta}$ then ϕ_x is strictly concave and admits a unique maximum point $\bar{x} \in (0, 1-d)$. As a consequence the positive fixed point x_0 is locally stable along the invariant axis as long as $\phi_x'(x_0) \in (-1, 1)$ and it may become unstable via a period doubling bifurcation when $\phi_x'(x_0)$ crosses -1. Similarly to what happens with the logistic map, a period doubling bifurcation cascade is observed as b_1 decreases as long as $\phi_x(\bar{x}) \leq 1 - d$, i.e. the set

$$S_x = [\phi_x^2(\bar{x}), \phi_x(\bar{x})]$$

is trapping and positively invariant. It also contains the attractor of T along the x-axis. Such an attractor disappears when a final bifurcation occurs, i.e. the maximum value exits from the invariant interval S_x. A more detailed discussion on the dynamics generated by T on the two invariant semiaxis can be done with a further analysis of the two one-dimensional maps in (10) and (11). However, to preserve the standard economic properties of utility function (1) we are mostly interested in the study of the dynamics produced by T for a feasible initial condition belonging to the interior of Q.

Consider system T given by (8). Equilibria or fixed points of T are solutions of the following equation:

$$T(x, y) = (x, y),$$

i.e.

$$\begin{cases} x(1 - F(x, y)) = 0 \\ y(1 - G(x, y)) = 0 \end{cases}. \tag{12}$$

By taking into account Remark 1, the number of fixed points on the two invariant segments I_x and I_y can be obtained and the following Proposition can be proved.

Proposition 3. *Let system T be given by (8). The origin $E_0 = (0, 0)$ is a fixed point for all parameter values. Up to two more fixed points on I_y and I_x are owned, they are given by $E_1 = (0, y_0)$ and $E_2 = (x_0, 0)$.*

The local stability analysis of fixed points of T can be carried out by considering the Jacobian matrix associated with system T given by (16). We observe that $J(E_0)$ is a diagonal matrix while $J(E_1)$ and $J(E_2)$ are triangular matrix so that their eigenvalues are given by elements of the main diagonal. More precisely, the eigenvalues evaluated at fixed point E_0 are given by $\lambda_1(E_0) = \phi_x'(0)$ and $\lambda_2(E_0) = \phi_y'(0)$ so that, by taking into account Proposition 2 (d), the origin can be a stable node (if $\bar{\beta} < b_i < \bar{\bar{\beta}}$, $i = 1, 2$), an unstable node (if $b_i < \bar{\beta}$, $i = 1, 2$) or a saddle point. Similarly, taking into account the results previously described, E_1 and E_2 can be saddle points or unstable nodes.

In the general case, the existence of (at least) one interior fixed point is quite difficult to be proven given the analytical form of system T and the high number of parameters. However, the following Proposition shows that if the difference between the two market share bonuses is not too large, then any interior fixed point of T, if it exists, must be close to the diagonal Δ.

Proposition 4. *Let $b_1 = b$ and $b_2 = b+\Delta b$, $\Delta b \in (-b, +\infty)$. If system (12) has an interior solution $E^* = (x^*, y^*)$ then: (i) if $\Delta b > 0$ then $x^* > y^*$ while if $\Delta b < 0$ then $x^* < y^*$; furthermore (ii) if $\Delta b \to 0$ then $(x^* - y^*) \to 0$.*

Proof. Let $b_1 = b$ and $b_2 = b + \Delta b$ and let $E^* = (x^*, y^*)$ be an interior fixed point. From $1 - F(x^*, y^*) = 0$ and $1 - G(x^*, y^*) = 0$ it must be $G(x^*, y^*) - F(x^*, y^*) = 0$, that is

$$(x^* - y^*)\left[\frac{(2+d)(2-x^*-y^*)^2}{(1+d)^2} + b\right] = \Delta b(1 - x^*).$$

Hence $\Delta b > 0$ (resp. $\Delta b < 0$) implies $(x^* - y^*) > 0$ (resp. $(x^* - y^*) < 0$) while if $\Delta b \to 0$ also $(x^* - y^*) \to 0$. □

From an economic point of view, the result of the previous proposition is of importance as it implies that although there exists heterogeneity between the two managers' bonuses (because of the different type of delegation contract), the Nash equilibrium (if it exists) tends to have the same coordinate values if the difference between b_1 and b_2 is sufficiently small. As a consequence, small heterogeneities do not matter and the system tends to mimic the homogeneous behavior in such a case. On the other hand, when heterogeneity between b_1 and b_2 is relevant we observe that (at the Nash equilibrium) the price is greater for the variety associated to a lower market share bonus.

Given the analytic form of system T, the determination and the study of the stability properties of the interior equilibria is not an easy task since we cannot have their analytical expressions. For this reason, in what follows we consider the simpler problem obtained under the assumption that owners assign the same weight to the market share into the utility function of managers, i.e. $\Delta b = 0$. We will come back to the study of different market share bonuses later on, while perturbing Δb, mainly from a numerical point of view.

4 The symmetric case of identical market share bonus

In the case of identical market share bonuses, that is $\Delta b = 0$, the two players behave in the same manner and consequently system T in (8) assume the symmetric form T_b given by

$$T_b : \begin{cases} x' = xf(x,y) = x\left[1 + \alpha\left(\frac{1-2x-d(1-y)+w}{1-d^2} - b\frac{(1+d)(1-y)}{(1-d)(2-x-y)^2}\right)\right] \\ y' = yg(x,y) = y\left[1 + \alpha\left(\frac{1-2y-d(1-x)+w}{1-d^2} - b\frac{(1+d)(1-x)}{(1-d)(2-x-y)^2}\right)\right] \end{cases} . \tag{13}$$

More precisely, system T_b remains the same under the exchange of players, that is $T_b \circ S = S \circ T_b$ where $S : (x, y) \to (y, x)$ is the reflection through the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2_+ : x = y\}$. We call this *symmetry property.* Since map T_b is symmetric, we recall that either an invariant set (attractors, basins of attraction, etc.) of the map is symmetric with respect to Δ, or its symmetric set is invariant as well.

Equilibria or fixed points of T_b are solutions of the following equation: $T_b(x, y) = (x, y)$, hence, for what it concerns the existence of fixed points belonging to the boundary of set Q, Proposition 3 still holds, i.e. one (the origin) or three fixed points (two more fixed points on the invariant semiaxes) can be owned, depending on parameter values.

On the other hand, an interior fixed point of system (13) must satisfy

$$\frac{\partial W_i}{\partial x} = 0 \text{ and } \frac{\partial W_i}{\partial y} = 0$$

which yields maximum managers' utility. Thus, if prices coincide with the ones that maximise the marginal utility of both managers, then the system will remain in such a steady state forever. Obviously, the existence of a Nash equilibrium does not imply that a generic trajectory converges to it, since it may be locally stable, i.e. it attracts only trajectories starting close to the steady state, or locally unstable, i.e. another attractor is approached. In addition, since several coexisting attractors may exist, the structure of the basin of attraction is of importance as the final evolution of the economy may become path dependent.

By taking into account Proposition 4, the following result can easily be proved.

Proposition 5. *Let*

$$\frac{(1-w)+\sqrt{(1-w)^2+b(2-d)(1+d)^2}}{2(2-d)}<1, \tag{14}$$

then system T_b given by (13) admits a unique interior fixed point $E_b^* = (x_b^*, x_b^*)$, where

$$x_b^* = 1-\frac{(1-w)+\sqrt{(1-w)^2+b(2-d)(1+d)^2}}{2(2-d)} \in (0,1).$$

If condition (14) does not hold, then T_b has no interior fixed points.

By substituting out the equilibrium price x_b^* into the direct demand functions we obtain the equilibrium value of quantities of both firms related to the unique interior fixed point, that is,

$$q_b^* = \frac{(1-w)+\sqrt{(1-w)^2+b(2-d)(1+d)^2}}{2(2-d)(1+d)}, \tag{15}$$

while the correspondent level of profits is given by $\Pi_b^* = (x_b^* - w)q_b^*$. Notice that if w is small enough, profits are non negative (in the rest of the paper we will consider w-values that guarantee such a property).

According to Proposition 5 it can be observed that a sufficient condition for the existence of E_b^* is that b should be small enough. Hence, by taking also into account Proposition 1 (*ii*), in what follows we focus on the study of system T_b under the assumption $b \in (0, \bar{b})$, where

$$\bar{b} = \frac{4(1-d+w)}{(1+d)^2},$$

so that a unique interior fixed point E_b^* exists for all given values of d, α and w. We also observe that equilibrium prices at the Nash equilibrium are decreasing with respect to b, while the corresponding quantities are increasing with respect to b. This is in line with the literature on static oligopolies with market share delegation contracts (Kopel and Lambertini, 2013).

The local stability analysis of the interior fixed point of T_b can be carried out by considering the Jacobian matrix associated with system T_b given by

$$J_b(x,y) = \begin{pmatrix} 1+\alpha\left(\frac{1-4x-d(1-y)+w}{1-d^2} - \frac{b(1+d)(1-y)(2+x-y)}{(1-d)(2-x-y)^3}\right) & \alpha x\left(\frac{d}{1-d^2} - b\frac{(1+d)(x-y)}{(1-d)(2-x-y)^3}\right) \\ \alpha y\left(\frac{d}{1-d^2} - b\frac{(1+d)(y-x)}{(1-d)(2-x-y)^3}\right) & 1+\alpha\left(\frac{1-4y-d(1-x)+w}{1-d^2} - \frac{b(1+d)(1-x)(2-x+y)}{(1-d)(2-x-y)^3}\right) \end{pmatrix}. \tag{16}$$

Since T_b is symmetric with respect to Δ, then its Jacobian matrix evaluated at a point on the diagonal Δ is of the kind

$$J_b(x,x) = \begin{pmatrix} J_1(x) & J_2(x) \\ J_2(x) & J_1(x) \end{pmatrix},$$

where

$$J_1(x) = 1 + \frac{\alpha}{1-d^2}\frac{4(4-d)(1-x)^3 + 4(w-3)(1-x)^2 - b(1+d)^2}{4(1-x)^2} \tag{17}$$

and

$$J_2(x) = \frac{\alpha d}{1-d^2}x. \tag{18}$$

As a consequence, the eigenvalues of $J_b(x,x)$ are both real and they are given by:

$$\lambda_{b\|}(x) = J_1(x) + J_2(x) \;\text{ and }\; \lambda_{b\perp}(x) = J_1(x) - J_2(x), \tag{19}$$

while the corresponding eigenvectors are respectively given by $\underline{\mathrm{v}}_{b\|} = (1,1)$ and $\underline{\mathrm{v}}_{b\perp} = (1,-1)$.

The eigenvalues evaluated at the fixed point E_b^* are then given by

$$\lambda_{b\|}(E_b^*) = J_1(E_b^*) + J_2(E_b^*)$$

and

$$\lambda_{b\perp}(E_b^*) = J_1(E_b^*) - J_2(E_b^*)$$

so that the interior fixed point E_b^* can be attracting for suitable values of the parameters such that both $\lambda_{\|}(E_b^*)$ and $\lambda_{\perp}(E_b^*)$ belong to the set $(-1,1)$. Observe that, different from Fanti et al. (2013), the two eigenvalues evaluated at the interior fixed point may be less or greater than 1.

An interesting result can easily be obtained while considering the following Proposition.

Proposition 6. $\lim_{d\to 0^+} \lambda_{\|}(E_b^*) = \lim_{d\to 0^+} \lambda_{\perp}(E_b^*)$ *and* $\lambda_{b\|}(E_b^*) > \lambda_{b\perp}(E_b^*)$, $\forall d \in (0,1)$.

From the previous statement it follows that if E_b^* is a hyperbolic fixed point for $d = 0$ then, $\exists\ I_+(0,\delta)$ such that E_b^* is a stable or an unstable node $\forall d \in I_+$ (i.e. the Nash equilibrium cannot be a saddle point if d is positive but sufficiently close to zero). This is important also from an economic point of view because when the equilibrium is a stable node (rather than a saddle), there are several trajectories that may lead the economy towards the equilibrium. Furthermore, conditions $\lambda_{b\perp}(E_b^*) > 1$ or $\lambda_{b\|}(E_b^*) < -1$ are sufficient for E_b^* to be an unstable node.

Since we are mainly interested in the role played by parameters d and b in Figure 3 (a) and (b) we separate the plane (d,b) into regions depicted with different colors for which the interior fixed point is locally stable, locally unstable or it is a saddle point, respectively. More precisely, we distinguish between the following cases: $|\lambda_{b\|}(E_b^*)| < 1$ and $|\lambda_{b\perp}(E_b^*)| < 1$ (blue region), $|\lambda_{b\|}(E_b^*)| < 1$ and $|\lambda_{b\perp}(E_b^*)| > 1$ (light blue region), or $|\lambda_{b\|}(E_b^*)| > 1$ and $|\lambda_{b\perp}(E_b^*)| > 1$ (yellow region). In the red region condition (14) does not hold and T_b has no interior fixed point, while the white curve is $\bar{b}(d)$. Observe that α affects the stability properties of the Nash equilibrium. The following statement holds.

Proposition 7. *Let system T_b given by (13). Then a $\epsilon > 0$ does exist such that E_b^* is locally asymptotically stable $\forall b \in (\bar{b}-\epsilon, \bar{b})$, given the other parameter values.*

Proof. Since $\lim_{b\to\bar{b}^-} x_b^* = 0^+$, then it can be easily verified that,

$$\lim_{b\to\bar{b}^-} \lambda_{b\|}(E_b^*) = \lim_{b\to\bar{b}^-} \lambda_{b\perp}(E_b^*) = \bar{\lambda}_b = 1^-$$

so that $\exists\ \epsilon > 0$ such that $|\lambda_{b\|}(E_b^*)| < 1$ and $|\lambda_{b\perp}(E_b^*)| < 1$, if $b \in (\bar{b}-\epsilon, \bar{b})$. $\square$

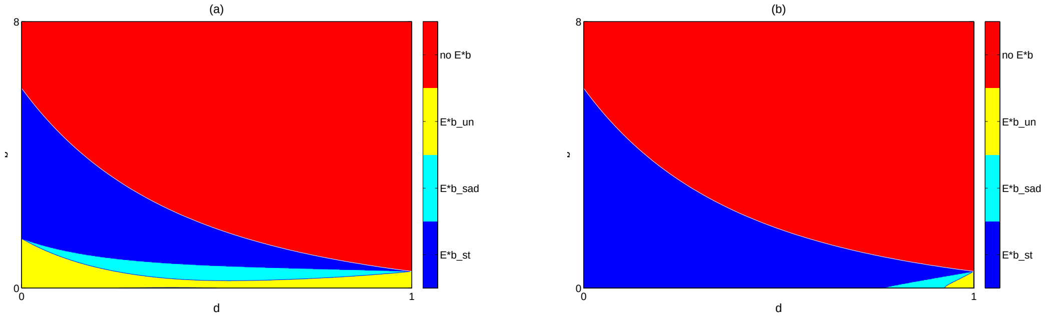

Figure 3: Subsets of the parameter plane (d, b) identifying regions at which different stability regimes occurs for $w = 0.5$. In panel (a) $\alpha = 1.5$ while in panel (b) $\alpha = 0.5$. *st* means locally stable, *un* means locally unstable while *sad* means saddle.

From an economic point of view, Proposition 7 gives us an important result. In fact, it shows that the size of the bonus in the manager's compensation scheme matters (in the symmetric case $b_1 = b_2 = b$) to distinguish between the cases of local stability and instability. The equilibrium tends to become locally unstable for high values of b. This because an increase in b increases the manager's marginal bonus and when b is sufficiently high the manager fixes the price at too high a level.

4.1 Synchronized trajectories

As it has been discussed, the symmetry property implies that the diagonal Δ is invariant for system T_b, i.e. equal initial conditions imply equal dynamic behavior $\forall t$. In order to consider the dynamics generated by T_b on the invariant submanifold

$$I = \{(x, x) : x \in [0, 1)\},$$

we can consider the following one-dimensional restriction:

$$\phi(x) = xf(x, x) = x\left[1 + \frac{\alpha}{1-d}\left(\frac{(1-d)(1-x) - x + w}{1+d} - b\frac{(1+d)}{4(1-x)}\right)\right], \quad x \in [0, 1), \tag{20}$$

where trajectories embedded into I, i.e. those characterized by $x = y$ for all t, are called *synchronized trajectories* (see Bischi and Gardini 2000 and Bischi et al. 1998).

In this section we want to describe synchronized trajectories of system T_b, i.e. the properties of the sequences generated when both firms start from the same initial feasible state, i.e. $x(0) = y(0)$.

Consider map $\phi(x)$ given by (20). Then the following properties can easily be verified: $\phi(0) = 0$, $\lim_{x \to 1^-} \phi(x) = -\infty$, $\phi'(0) > 1$ $\forall b \in (0, \bar{b})$, $\phi''(x) < 0$ $\forall x \in [0, 1)$. As a consequence there exists a unique $x_- \in (0, 1)$ such that $\phi(x_-) = 0$ and a unique maximum point $x_M \in (0, x_-)$ such that $\phi(x_M)$ is the maximum value of ϕ in $[0, 1)$. This fact implies that ϕ is unimodal in $[0, x_-]$.

Hence, the following cases may occur.

(i) If $\phi(x_M) \leq x_-$, then the set $S = [\phi^2(x_M), \phi(x_M)]$ is trapping and any trajectory generated from an i.c. $x(0) \in (0, x_-)$ enters in S after a finite number of iterations, i.e. S is absorbing. This means that ϕ admits an attractor A belonging to S. In this case synchronized trajectories starting from an interior feasible point converge to A, which may consist of a fixed point x_b^*, a

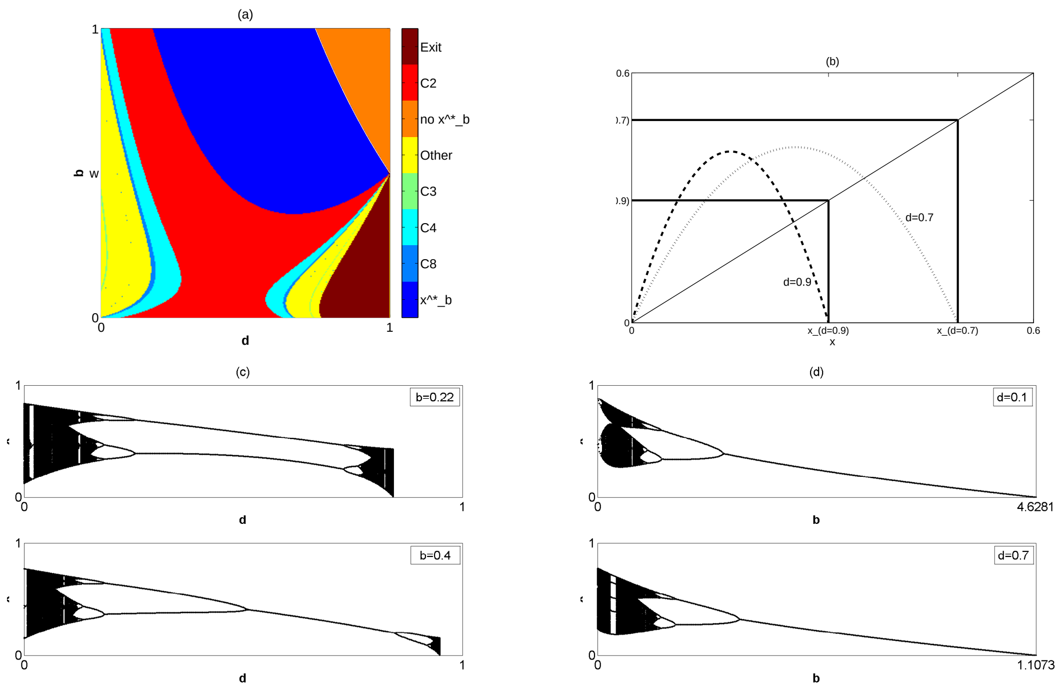

Figure 4: (a) Two dimensional bifurcation diagram of map ϕ in the plane (d, b) for $\alpha = 1.7$ and $w = 0.5$. (b) Map ϕ is plotted for different $d-$values and $b = 0.1$. (c) One dimensional bifurcation diagrams w.r.t. d for two fixed $b-$values. (d) One dimensional bifurcation diagrams w.r.t. b $(0 < b < \bar{b})$ for two fixed $d-$values.

cycle, an $m-$piece chaotic attractor etc.. Anyway, observe that the only way in which x_b^* may lose stability is via period-doubling bifurcation.

(ii) If $\phi(x_M) = x_-$ a final bifurcation occurs and the attractor A disappears.

(iii) If $\phi(x_M) > x_-$ then almost all synchronized trajectories are unfeasible.

The cases (i) and (iii) just described can be observed in Figure 4 (b) for two different values of d, once fixed the other parameters values. In order to know the structure of the attractor A, when it exists, we present the long-term synchronized dynamics qualitatively in the cycle cartogram depicted in Figure 4 (a). It shows a two-parameter bifurcation diagram where each color describes a long-run dynamic behavior for a given combination of d and b. Cycles of different order i (i.e. Ci) are exhibited and *other* more complex dynamics can be obtained. The brown region indicates parameter values such that an unfeasible trajectory is produced while the orange region indicates parameter values such that no interior fixed point exists (i.e. $b > \bar{b}$).

In order to obtain a necessary condition for the attractor A located on the main diagonal to exist for T_b, we recall the general result stated in Proposition 1 and condition (14) and observe that d or b must be not too high, as it can easily be checked while observing Figure 4 (a). This fact can be also confirmed while considering that $\phi(x_M)$ monotonically crosses x_- if $d \to 1$, providing that a $\bar{d}$ does exist such that $\phi(x_M) > x_-$ if $\bar{d} < d < 1$ (see Figure 4 (b) and the related bifurcation diagram in panel (c)). On the other hand, while considering the role of parameter b, a period doubling bifurcation cascade is observed as b decreases as it can be noticed while looking at Figure 4 (d).

To sum up, by taking into account the previous results and looking at the one dimensional bifurcation diagrams in Figure 4 (c) and (d), the following evidences can be pursued.

- Synchronized trajectories converge to the unique interior fixed point if b is close to $\bar{b}$. This means that there exist initial conditions such that trajectories converge to the fixed point only whether the manager bonus is large enough, but not fixed at too high a level (otherwise, the interior fixed point does not exist).
- Unfeasible trajectories are produced if d is close to its limit value $+1$ thus confirming the results in Fanti et al. 2013.
- Cycles can emerge due to period doubling and halving bifurcations of ϕ, similar to that occurring with the logistic map, if b decreases or d increases-decreases.

The synchronized behavior of managerial firms can tend to the Nash equilibrium only whether the bonus is close to (but less than) the threshold $\bar{b}$ (otherwise, feasible trajectories do not exist), i.e. owners try to motivate managers to behave aggressively in the market by increasing competition between them. Interestingly, the dynamic result of this kind of behavior is a sort of coordination along the diagonal to avoid an increasing competition that will lead to unfeasible trajectories, no profits and the exit from the market. A reduction in b may cause the emergence of cycles of higher periodicity through period doubling bifurcations. This implies that if the behavior of managers is driven by contracts with an intermediate value of the bonus, every manager will try to gain with respect to the rival's choice and then prices may experience a non-monotonic behavior.

In this subsection we have studied the dynamics of synchronized trajectories, showing conditions for which system T_b admits an attractor $A \subseteq I$ on the diagonal. Obviously, if firms start from the same initial condition and the attractor A exists, then the duopoly model converges to A in the long term. Thus the fate of any synchronized trajectory (i.e. any trajectory starting from identical initial conditions) is known.

An important question arising is whether an attractor A of ϕ is also an attractor of the two-dimensional map T_b on the invariant submanifold $I \cap D$. In fact, it is of importance to know whether a duopoly with identical players starting from different feasible initial conditions evolves toward synchronization.

4.2 Synchronization

In order to consider producers that start from different initial conditions, we recall that a feasible trajectory starting from $(x(0), y(0)) \in D - I$, that is with $x(0) \neq y(0)$, is said *to synchronize* if $|x(t) - y(t)| \to 0$ as $t \to +\infty$.

Assume that $A \subseteq I$ is an attracting set of ϕ, i.e. it is stable with respect to perturbations along Δ, then, in order to study the stability of A for T_b we have to consider the transverse stability (hence, the stability of A with respect to perturbations transverse to Δ).

Let $A = E_b^*$ for $d \to 0^+$. Then, by taking into account equation (19) and Proposition 6, and knowing that x_b^* and both eigenvalues are continuous with respect to d, the following statement trivially holds.

Proposition 8. *Let x_b^* be an attracting fixed point of ϕ for $d \to 0^+$. Then $\exists\, d_+ > 0$ such that E_b^* is an attracting fixed point of T_b $\forall d \in (0, d_+) \cap (0, 1)$, while at $d = d_+$ the fixed point E_b^* loses its transverse stability.*

According to Proposition 8, if synchronized trajectories converge to x_b^* for $d \to 0^+$, then trajectories starting from initial conditions close to it, with $x(0) \neq y(0)$, synchronize in the long term as long as $d \in I_+(0)$. When d increases the fixed point loses first its transverse stability, i.e. trajectories do not synchronize.

Consider now the case in which A is an m-cycle. For what it concerns the transverse stability of A, we recall that for an m-cycle $\{(x_1, x_1), ..., (x_m, x_m)\}$ of T_b embedded into the invariant set I where synchronized dynamics take place and corresponding to the cycle $\{x_1, ..., x_m\}$ of ϕ, multipliers are given by

$$\lambda_{b\|}^{(m)} = \prod_{i=1}^{m} \lambda_{b\|}(x_i) \text{ and } \lambda_{b\perp}^{(m)} = \prod_{i=1}^{m} \lambda_{b\perp}(x_i),$$

where $\lambda_{b\|}(x_i)$ and $\lambda_{b\perp}(x_i)$ are the eigenvalues of the Jacobian matrix evaluated at a point (x_i, x_i) associated with eigenvectors parallel to Δ and with eigenvectors normal to Δ, respectively. Then, several numerical computations show that, similarly to what happens for the fixed point, if A is an m-cycle, then as d increases the m-cycle loses its transverse stability and synchronization does not occur. This situation is depicted in Figures 5 (a) and (b) where we have used parameter values as in Figure 4 (c) and $b = 0.4$. It can be observed that for all d-values taken into the interval $(0.27, 0.315)$, A consists of a 2-period cycle. However, as d is increased, the cycle of period 2 becomes transversely unstable and trajectories starting outside the diagonal do not synchronize. In this last case the attractor existing out of the diagonal is complex and given by the black points in Figure 5 (b).

Consider now a more complex situation, i.e. A is a chaotic attractor on Δ. In order to study its transverse stability we follow the procedure proposed in Bischi et al. (1998), Bischi and Gardini (2000) and Bignami and Agliari (2010), based on the use of the *transverse Lyapunov exponent*

$$\Lambda_{b\perp} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \ln |\lambda_{b\perp}(x_n)|, \tag{21}$$

where $x_0 \in A$ and x_n is a generic trajectory generated by ϕ, and the use of the *natural transverse Lyapunov exponent* $\Lambda_{b\perp}^n$,

$$\Lambda_{b\perp}^{min} \leq ... \leq \Lambda_{b\perp}^{n} \leq ... \leq \Lambda_{b\perp}^{max}$$

where *natural* means that the exponent is computed for a typical trajectory taken in the chaotic attractor A (see Bischi and Gardini, 2000). By following Bischi et al. (1998) we recall that if $\Lambda_{b\perp}^{max} < 0$, then A is asymptotically stable in the Lyapunov sense, while if $\Lambda_{b\perp}^{n} < 0$ and $\Lambda_{b\perp}^{max} > 0$ then A is not stable in the Lyapunov sense but it is a stable attractor in the Milnor sense.

In order to investigate the existence of a Milnor attractor A, we estimate the natural transverse Lyapunov exponent $\Lambda_{b\perp}^{n}$ for a high number of parameter constellations, while also considering situations in which A is stable with respect to perturbations along the diagonal. The following cases may emerge: (i) A is also transversely stable, so that synchronization occurs (i.e. $\Lambda_{b\perp}^{max} < 0$); ($ii$) A is transversely unstable in the Lyapunov sense and also in the Milnor sense (i.e. $\Lambda_{b\perp}^{max} > 0$ and also 0), so that the existence of a Milnor attractor is avoided and synchronization does not take place.

In Figure 6 (a) the attractor A consisting of a 4-period cycle is transversely stable, anyway not all trajectories synchronize as T_b also admits a coexisting attractor, i.e. a 4-period cycle whose basin is depicted in yellow. This phenomenon will be clarified in the following Section.

4.3 Multistability

As is shown in the previous sections, the Nash equilibrium E_b^* is an attracting fixed point if b is great enough, i.e. if the weight of the market share bonus in the manager's utility is close to its

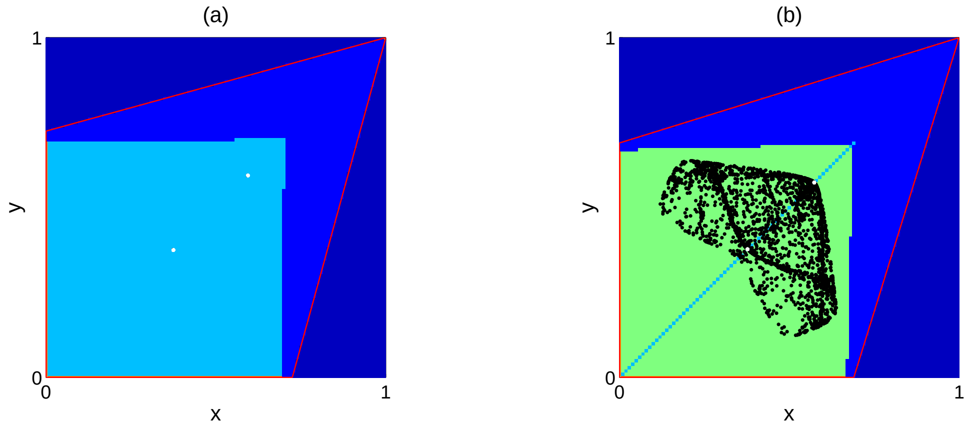

Figure 5: Parameter values $\alpha = 1.7$, $w = 0.5$, $b = 0.4$. (a) Attracting cycle-2 of system T_b belonging to the diagonal for $d = 0.275$. (b) The cycle-2 on the diagonal (white points) is transversally unstable and a complex attractor (black points) exists out of the diagonal for $d = 0.31$.

maximum admissible value $\bar{b}$ (given the other parameter values). Furthermore, if b decreases, the Nash equilibrium loses stability along the diagonal via a period doubling bifurcation which creates an attracting cycle of period 2, and this first flip bifurcation is followed by a sequence of flip bifurcations, similar to those occurring with the logistic map (see Figure 4 (d)). In addition, if the attractor A on the diagonal is also transversely stable, then it is also an attractor for system T_b, for suitable initial conditions taken in its own basin; otherwise, as it has been discussed in Section 4.2, synchronization cannot occur.

In this section we want to deal with the following open question: assume that T_b admits an attractor $A \subset I$ belonging to the diagonal, and that there exist feasible trajectories starting from interior points which do not synchronize. Which is the final outcome of the economy? An answer to this question must be given while considering that *multistability* may arise, i.e. several attractors may coexist, each of which with its own basin of attraction, so that the selected long-term state becomes path dependent as in the situation depicted in Figure 6 (a). In this case the structure of the basins of different attractors becomes crucial for predicting the long-term outcome of the economic system.

In order to better describe this phenomenon, we start by considering the case in which the unique Nash equilibrium is locally stable, that is the market share bonus is close to its upper bound, and in addition the Nash equilibrium E_b^* is also *globally stable*, in the sense that it attracts every initial condition taken on the interior of the feasible set D, representing economic meaningful initial conditions. This case is illustrated by the numerical simulation in Figure 6 (b), where the green region represents the basin of attraction of the Nash equilibrium E_b^* for $b \in (\bar{b} - \epsilon, \bar{b})$.

Observe that, since $\frac{\partial \bar{b}}{\partial d} < 0$, then $\bar{b}$ decreases as d increases and consequently $\bar{b}$ tends to its minimum value $b_- = w$ as $d \to 1^-$. As a consequence, since a level of the weight of market share bonus close to $\bar{b}$ is a sufficient condition to obtain the convergence to the unique Nash equilibrium, then b must be settled at lower levels as the degree of horizontal product differentiation d increases (that is, products tend to be substitutes). Furthermore, if b is close to $\bar{b}$, then the equilibrium price is at low level while equilibrium quantities are high. This implies that in order to converge towards the interior Nash equilibrium, a sufficient condition is to set a contract with respect to which the weight

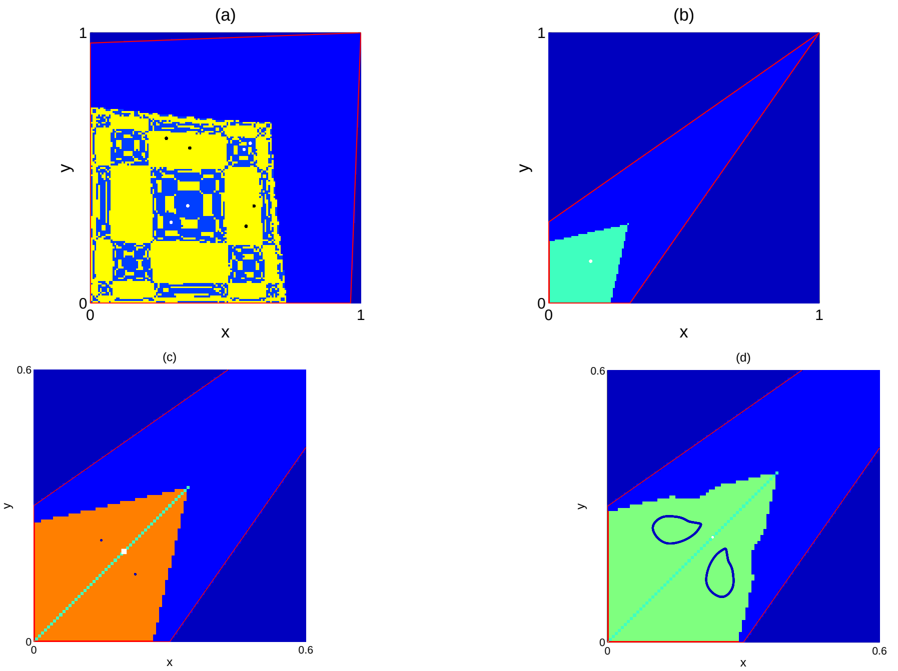

Figure 6: Parameter values $\alpha = 1.7$ and $w = 0.5$. (a) For $b = 0.9$ and $d = 0.038$ two coexisting cycle-4 are owned and their basins are depicted in yellow and blue. (b) The green region represents the basin of E_b^* for $b = 0.7$ and $d = 0.7$. (c) For $d = 0.7$ and $b = 0.6$ non synchronized trajectories converge to a 2-period cycle. (d) For $d = 0.7$ and $b = 0.53$ two cyclic attracting closed invariant curves have been created out of the diagonal.

of the bonus is close to the threshold $\bar{b}$ (for which the degree of competition between managers is relatively high). However, since $\bar{b}$ reduces when the degree of product differentiation increases, a rise in the degree of substitutability between products of different variety needs an increase in the weight of bonus in the manager's utility function to get synchronization (because products are more similar and then an increasing effort of managers is required to get a competitive advantage in the market, so that prices tend to reduce and demanded quantities tend to increase).

The situation depicted in Figure 6 (b) increases in complexity as b decreases. In fact, by considering the result shown in Proposition 6, as b decreases then $\lambda_{b\perp}$ crosses -1 and a flip bifurcation occurs transversely to the diagonal. When this bifurcation occurs, $\lambda_{b\|}$ is still less than one in modulus and consequently, the Nash equilibrium attracts all synchronized trajectories. In Figure 6 (c) an attracting 2-period cycle (black points) is created out of the diagonal attracting all interior feasible initial conditions having $x(0) \neq y(0)$ (orange region), while E^* (white point) is still an attractor along the diagonal (green line). As b further decreases two cyclical attracting closed invariant curves are created around the unstable 2-period cycle, due to a Neimark-Sacker bifurcation, and not synchronized trajectories starting from an interior point exhibits a quasi-periodic or a periodic behavior (see Figure 6 (d)).

Finally, a particular dynamic scenario occurs if $d \to 0^+$, i.e. products tends to be independent to each other and each manager tends to behave as a monopolist. Let $d = 0$, by taking into account Proposition 6, when a flip bifurcation along the diagonal creates a k-period cycle, then a k-period cycle is simultaneously created out of the diagonal as the eigenvalues of cycles embedded into the diagonal are identical. As a consequence, any period doubling bifurcation along Δ is associated with a period doubling bifurcation orthogonal to Δ. A similar phenomenon of multistability is presented in Bischi and Kopel (2003). In Figure 6 panel (a) this case is shown for d close to zero: blue points represent initial conditions that converge to a 4-period cycle on the diagonal while the yellow points represent initial conditions converging to a 4-period cycle out of the diagonal. This scenario occurs when the 2-period cycle along the diagonal undergoes the second period doubling bifurcation: two stable 4-period cycles are created, one along the diagonal (white points) and one with periodic points symmetric to it (black points). Observe that an economy starting far away from the diagonal may synchronize, as the basin of attraction of attractor A on the diagonal is composed by several non connected sets. This result is relevant from an economic point of view as it implies coordination although the manager hired in each firm behaves as a monopolist in his own market.

Since multistability may occur and the basins of attraction may have a complex structure, the previous study enables us to show that slightly different initial conditions may lead to distinct long-term evolution the economy may follow.

5 Breaking the symmetry

In the previous Section we have analysed the symmetric model in which market share bonuses were equally weighted. We now want to come back to the more general setting in which managers' bonuses are evaluated differently in order to describe its main dynamic properties and to compare them with those analyzed in the particular case of homogeneous weight. Specifically, we will set the value of $b_1 = b$ and assume $b_2 = b + \Delta b$. Then, we move the value of $\Delta b \in (-b, +\infty)$ in order to break the symmetry and proceed by making use of numerical techniques.

With regard to the existence of a Nash equilibrium, we recall that it is given by the feasible

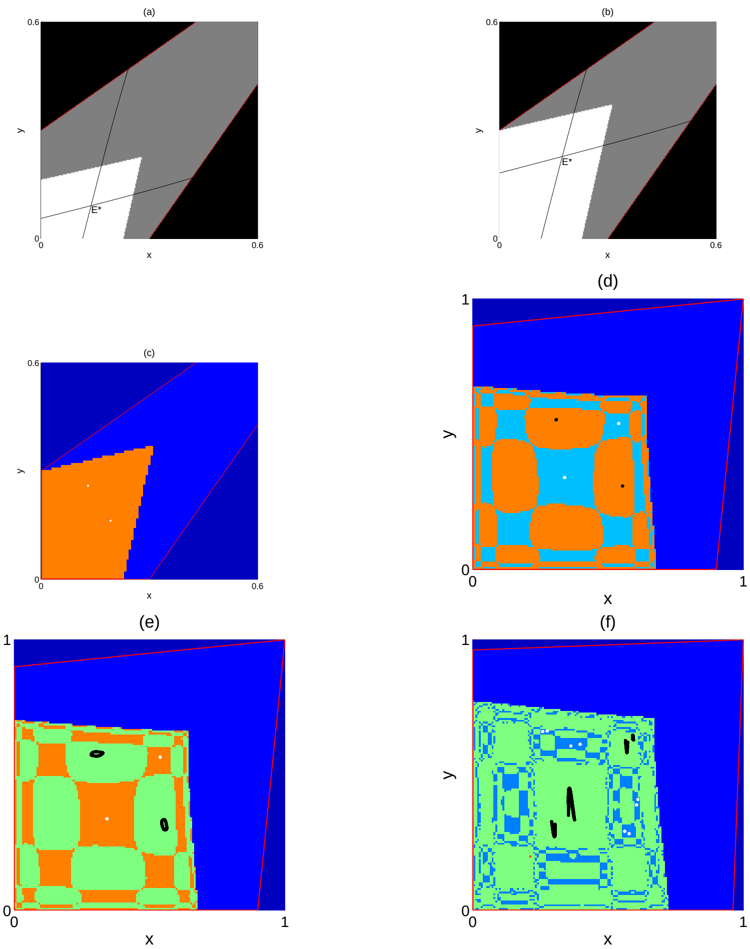

Figure 7: Parameter values $\alpha = 1.7$, $w = 0.5$, $b = 0.7$ and $d = 0.7$. The Nash equilibrium for (a) $\Delta b = 0.2$ and (b) $\Delta b = -0.2$. (c) If $\Delta b = -0.2$ the Nash equilibrium is unstable and a 2-period cycle is globally stable. (d) Coexisting attractors for $\alpha = 1.7$, $d = 0.1$, $w = 0.5$, $b_1 = b_2 = 0.9$. (e) Coexisting attractor for $\Delta b = -0.1$ and other parameters as in (d). (f) Coexisting attractor for $\Delta b = -0.1$ and $d = 0.038$ and other parameters as in (d).

solutions of system:

$$\left\{ \begin{array}{l} F(x,y) = \alpha \left(\frac{1-2x-d(1-y)+w}{1-d^2} - b \frac{(1+d)(1-y)}{(1-d)(2-x-y)^2} \right) = 0 \\ G(x,y) = \alpha \left(\frac{1-2y-d(1-x)+w}{1-d^2} - (b+\Delta b) \frac{(1+d)(1-x)}{(1-d)(2-x-y)^2} \right) = 0 \end{array} \right. , \tag{22}$$

i.e. by $(x^*, y^*) \in Q$ such that $F(x^*, y^*) = G(x^*, y^*) = 0$. An interior fixed point can be obtained by considering the intersection points of the two curves $F(x,y) = 0$ and $G(x,y) = 0$ in the phase plane. Obviously, if these curves intersect in a point $E^* = (x^*, y^*) \in Q$ then it is a Nash equilibrium for system T. By taking into account Proposition 1, a necessary condition for the existence of a feasible Nash equilibrium is d not too close to its extreme value $+1$ and, in addition, b and Δb have to be small enough.

In Figure 7 (a) and (b), we depict the two curves $F(x,y) = 0$ and $G(x,y) = 0$ in the phase plane (x,y) for the parameter values used in Figure 6 (b), and assuming $\Delta b \neq 0$. After several numerical simulations, and by considering also the analytical properties of F and G, the following evidences can be stated.

(i) With regard to the asymmetric case, if the Nash equilibrium exists then it is unique.

(ii) As is shown in Proposition 4, different from the symmetric case, the equilibrium prices differ between the two varieties; in particular, as long as E^* exists, if $\Delta b > 0$ (resp. $\Delta b < 0$) then $x^* > y^*$ (resp. $x^* < y^*$). This confirms that the equilibrium price is greater for the variety associated to a lower market share bonus; in the limit cases, if Δb is great enough y^* tends to zero while the corresponding quantity is high.

(iii) Equilibrium prices in the non symmetric Nash equilibrium do not depend on the speed of adjustment α, as for the symmetric case.

As expected, when the weight of market share bonus is different between firm 1 and firm 2 (and this difference is not negligible), the coordinate values of the Nash equilibrium are different and the higher price is associated with the good produced by the firm in which the manager behaves less aggressively. By choosing to be more aggressive, in fact, the rival's manager will set a lower price to capture the benefits of increasing market share and profits.

About the local stability of E^*, by taking into account Proposition 8, it can easily be observed that since E_b^* is locally stable for T_b if $b \in (\bar{b} - \epsilon, \bar{b})$, then E^* is locally stable for T if $b \in (\bar{b} - \epsilon, \bar{b})$ and $\Delta b \in I(0)$. Hence, if the Nash equilibrium is locally stable in the symmetric case, then it is also locally stable in the asymmetric case, iff the perturbation is small enough (i.e. Δb is close to zero). This case have been presented, for instance, in Figure 6 (b), as the Nash equilibrium is *globally stable*. With small heterogeneities, i.e. if Δb is low, the Nash equilibrium moves away from the diagonal but it remains globally stable until a standard flip bifurcation occurs at which it becomes a saddle point and a stable 2-period cycle appears near E^*. This situation is presented in Figure 7 (c).

Furthermore, as system T is not symmetric, then the diagonal Δ is no longer invariant, i.e. if firms start from the same initial feasible condition $(x(0), x(0)) \in D$, they will behave differently in the long term. For this reason, synchronized trajectories do not emerge and synchronization cannot occur. This fact can be clarified by contrasting Figure 7 (d) and Figure 7 (e). In Figure 7 (d) a situation in which synchronization may occur has been presented: the symmetric system T_b admits a cycle-2 on the diagonal that coexists with a 2-period cycle out of the diagonal. If we consider a difference between the weights attached to market share bonuses, i.e. $\Delta b \neq 0$, we obtain the situation depicted in Figure 7 (e): the attractor on the diagonal disappears while a 2-pieces chaotic attractor coexists with one of the 2-period cycles previously found.

Finally, observe that in the case in which products tend to be independent and homogeneous manager types are considered, the situation presented in Figure 6 (a) occurs, while the situation drastically change if $\Delta b = 0.01$. In fact, as shown in Figure 7 (f), a small perturbation on b causes the disappearance of the attracting 4-period cycle on the diagonal, while a 4-pieces chaotic attractor cycle close to the diagonal exists together with the 8-period cycle existing out of the diagonal. Obviously, due to the heterogeneity between the weights b_i, the shape of the boundaries of the coexisting attractors is no longer symmetric with respect to the diagonal.

6 Conclusions

We have examined how strategic delegation contracts (based on market share) affects local and global dynamics of a nonlinear duopoly with price competition and horizontal differentiation (Kopel and Lambertini, 2013). This kind of contracts is of increasing importance in firms where ownership and management are separate (as also stressed by empirical studies, e.g., Peck, 1988; Borkowski, 1999), and thus managers are likely to be driven by motives other than just maximizing profits. This paper aims at providing results about long-term outcomes from local and global perspectives in a context with managerial firms have limited information. These outcomes depend on the mutual relationship between the degree of horizontal product differentiation and the weight of market share bonus in the manager's objective function.

First of all, we have shown that there exist no feasible economic meaningful trajectories if: 1) the weight attached to the market share bonus in the manager's objective is too high, or 2) products are close to be perfect substitutes. Then, we have studied long-term dynamics in two distinct cases: in the former case we have concentrated on the symmetric case in which managers' bonuses are equally weighted in their objective functions; in the latter one we have taken into account the case of heterogeneous manager types (asymmetric case). From a policy perspective, our findings suggest that competition between this kind of firms require that products should not be too similar and managers should not behave too much aggressively in the market. A natural extension of the present paper is the analysis of a duopoly with quantity competition and managerial firms with market share delegation contracts, as well as a comparison with Bertrand's findings. In addition, our analysis may also be performed by using a different definition of market share delegation, i.e. the logarithmic difference of quantities produced by both firms (Wang et al., 2009) rather than the quantity of one of them divided by total production.

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