Local and global bifurcations in an economic growth model with endogenous labour supply and multiplicative external habits

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Local and global bifurcations in an economic growth model with endogenous labour supply and multiplicative external habits

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This paper analyses the mathematical properties of an economic growth model with overlapping generations, endogenous labour supply, and multiplicative external habits. The dynamics of the economy is characterised by a two-dimensional map describing the time evolution of capital and labour supply. We show that if the relative importance of external habits in the utility function is sufficiently high, multiple (determinate or indeterminate) fixed points and poverty traps can exist. In addition, periodic or quasiperiodic behaviour and/or coexistence of attractors may occur. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4865787]

Endogenous preferences represent a relevant topic in economic theory. They influence individual choices about consumption and saving paths over time and may be responsible of endogenous fluctuations and nonlinear dynamics in macroeconomic variables (e.g., per capita output). This study examines an economic growth model with overlapping generations and endogenous labour supply where habit formation is a source of local and global bifurcations.

I. INTRODUCTION

The study of economic models with habit formation has received in depth attention in the economic literature. Several works have dealt with this topic in recent years. Among them are the papers by Abel (1990), Boldrin *et al.* (1997), and Lahiri and Puhakka (1998) that are concerned with the relationship between habit formation and the equity premium puzzle. Boldrin *et al.* (2001) introduces habit preferences in the standard real business cycle model to explain the joint behaviour of asset prices and consumption. Chen and Hsu (2007) analyse a continuous time one-sector neoclassical growth model with inelastic labour supply and show that consumption externalities can be a source of local indeterminacy when the degree of impatience is large enough, while Alonso-Carrera *et al.* (2008) generalises their model by introducing endogenous labour supply. More recently, Rozen (2010) provides an axiomatic theory of habit formation, where habits enter linearly the utility function of individuals, while Orrego (2014) finds indeterminacy in a pure exchange economy with overlapping generations (OLG), habits and three-period lived individuals. From an empirical point of view, there exists evidence of the role of habit formation on consumption and other macroeconomic variables (Ferson and Constantinides, 1991; de la Croix and Urbain, 1998; Fuhrer, 2000; Carrasco *et al.*, 2005; Smith and Zhang, 2007; Chen and Ludvigson, 2009).

It is useful to recall that in the economic literature different concepts of habit formation have been studied. An important distinction that clarifies the differences between internal and external habits can be found in Alonso-Carrera *et al.* (2007). Specifically, (internal) habits refer to the case in which preferences of an individual depend on his/her own consumption as well as on a benchmark level that weights the consumer's own past consumption experience. Aspirations (or external habits) refer to the case in which preferences of an individual depend on his/her own consumption as well as on a benchmark level that weights the consumption experience of others.

The present study aims at analysing the role of multiplicative external habits on the dynamics of a two-dimensional OLG economy (in particular, endogenous fluctuations and local and global indeterminacy) where an individual works when he/she is young and consumes when he/she is old (Woodford, 1984; Reichlin, 1986). The study of growth models that generate endogenous deterministic fluctuations dates back to Grandmont (1985), Farmer (1986), Reichlin (1986), and Azariadis (1993). Subsequently, several other authors have dealt with this topic in OLG models with either exogenous labour supply (Yokoo, 2000) or endogenous labour supply (Nourry, 2001; Nourry and Venditti, 2006) (Some applications of nonlinear dynamics in macroeconomic models can be found in Zhang (1999), Antoci *et al.* (2004), Chen and Li (2011, 2013), and Fanti *et al.* (2013)). The works of de la Croix (1996) and de la Croix and Michel (1999) essentially represent the OLG literature related to the present paper. Both papers deal, however, with an economy where the labour supply is inelastic and individuals consume in both the first period and second period of life (Diamond, 1965). In particular, de la Croix (1996) shows that when the intensity of aspirations in utility is large, individuals want to increase consumption because the standard of living of their parents is high and then savings becomes low. This can generate a Neimark-Sacker bifurcation because savings may

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experience too high a contraction. Subsequently, de la Croix and Michel (1999) show that the negative externality due to endogenous preferences can be corrected through investment subsidies and lump sum transfers.

The novelty of this study is the introduction of external habits in an OLG growth model with endogenous labour supply à la Reichlin (1986). First, we show that the relative importance of aspirations in utility is responsible for the existence of either one (normalised) fixed point (which can be determinate or indeterminate) or two interior fixed points. Second, some interesting local and global dynamic properties of the two-dimensional decentralised economy emerge: when the relative importance of aspirations in utility is strong enough, periodic or quasiperiodic behaviour and/or coexistence of attractors may occur. Since in the rest of the paper we sometimes refer to local and global (in)determinacy (by following a well-established economic literature), it could be useful to clarify the difference between these two concepts for potential readers that are not familiar with terms used in economic theory. A fixed point is locally indeterminate (resp. determinate) if for every arbitrarily small neighbourhood of it, and for a given value of the state variable close enough to its coordinate value at the stationary state, there exists a continuum of values (resp. a unique value) of the control variable for which an equilibrium trajectory converge towards the fixed point. From a mathematical point of view, a locally indeterminate (resp. determinate) fixed point is a sink (resp. a saddle). Differently, the system is globally indeterminate when there exist values of the state variable such that different choices on the control variable lead to different invariant sets. In this case, the initial condition of the stock of capital is not sufficient to define the long-term dynamics of the system.

The rest of the paper is organised as follows. Section II outlines the model. Section III studies the conditions for the existence of fixed points of the two-dimensional map and analyses local bifurcations and stability. Section IV describes the global properties of the model. Concluding remarks are in Sec. V.

II. THE MODEL

We consider an OLG closed economy populated by a continuum of perfectly rational and identical two-period lived individuals of measure one per generation (Diamond, 1965). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. A new generation is born in every period. Each generation overlaps for one period with the previous generation and then overlaps for one period with the next one. In the first period of life (youth), the individual of generation t is endowed with two units of labour. He/she supplies $\ell_t \in (0\,,\,2)$ units of labour to firms and receives the wage w_t per unit of labour. The remaining share $2 - \ell_t$ is used for leisure activities. Individuals consume only in the second period of life (Woodford, 1984; Reichlin, 1986; Galor and Weil, 1996; Grandmont *et al.*, 1998; Antoci and Sodini, 2009; Gardini *et al.*, 2009; Gori and Sodini, 2011, 2013).

The budget constraint of a young individual of generation t is $s_t = w_t\,\ell_t$, implying that labour income is entirely saved (s_t) to consume when old (C_{t+1}). Old individuals retire and consumption is constrained by the amount of resources saved when young plus expected interest accrued from time t to time $t+1$, so that $C_{t+1} = R^e_{t+1}\,s_t$, where R^e_{t+1} is the expected interest factor, which will become the realised interest factor at time $t+1$. Therefore, the lifetime budget constraint of an individual of generation t is

$$C_{t+1} = R^e_{t+1} w_t \ell_t. \tag{1}$$

Individuals have preferences towards leisure when young and consumption when old. In addition, we assume the existence of a reference level against which consumption of the current generation is compared with. This implies that effective consumption of individuals of generation t is negatively affected by the consumption experience of their parents (a_t), which gives rise to a form of external habits (de la Croix, 1996; Carroll *et al.*, 1997, 2000; de la Croix and Michel, 1999; Gori and Sodini, 2013). It is important to specify that we are considering external habit under the flow concept of it.

The lifetime utility index of generation t is described by a twice continuously differentiable function $U_t(\bullet)$. Since individuals consume only when they are old, consumption of generation $t-1$ (C_t) affects consumption of generation t, so that $a_t = C_t$ for every t. By assuming that external habits take the multiplicative form $C_{t+1}/{a_t}^{\rho}$ (Abel, 1990; Galí, 1994; Carroll, 2000; Bunzel, 2006; Hiraguchi, 2011), we specify the lifetime utility function of individuals of generation t by using the following Constant Inter-temporal Elasticity of Substitution (CIES) formulation (Christiano *et al.*, 2010)

$$U_t(\ell_t, a_t, C_{t+1}) = \frac{(2-\ell_t)^{1-\gamma}}{1-\gamma} + B\,\frac{(C_{t+1}/a_t^{\rho})^{1-\sigma}}{1-\sigma}, \tag{2}$$

where B is a scale parameter that allows us to define the normalised fixed point $(1\,,\,1)$ when the other parameters of the model are continuously changed, $\sigma > 0$ ($\sigma \neq 1$) and $\gamma > 0$ ($\gamma \neq 1$) represent a measure of the constant elasticity of utility with respect to consumption and leisure, respectively, and $\rho \geq 0$ is the aspiration intensity. If $\rho = 0$, external habits are irrelevant; if $\rho = 1$ current and past consumptions are equally weighted; if $\rho > 1$ external habits strongly matter.

By taking factor prices and external habits as given, the individual representative of generation t chooses ℓ_t to maximise utility function (2) subject to (1) and $\ell_t \in (0\,,\,2)$. Therefore, the first order conditions for an interior solution are given by

$$-(2-\ell_t)^{-\gamma} + \frac{B}{\ell_t}(R^e_{t+1}\,w_t\,\ell_t/a_t^{\rho})^{1-\sigma} = 0. \tag{3}$$

At time t, identical and competitive firms produce a homogeneous good, Y_t, by combining capital and labour, K_t and L_t, respectively, through the constant returns to scale Cobb-Douglas technology $Y_t = A\cdot F(K_t, L_t) = A{K_t}^{\alpha}{L_t}^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. Different from Grandmont *et al.* (1998) and Cazzavillan (2001), there are no externalities in the production sector. The temporary equilibrium condition in the labour market at time t is given by $L_t = \ell_t$.

By assuming full depreciation of capital and a unit price of output Y_t, profits maximisation implies

$$R_t = \alpha A K_t^{\alpha-1} \ell_t^{1-\alpha}, \tag{4}$$

$$w_t = (1-\alpha) A K_t^{\alpha} \ell_t^{-\alpha}, \tag{5}$$

i.e., the marginal productivities of capital and labour equal the interest factor and the wage rate, respectively. The market-clearing condition in the capital market is

$$K_{t+1} = s_t = w_t \ell_t. \tag{6}$$

By using (3)–(6) and knowing that: (i) individuals have perfect foresight, and (ii) $a_t = C_t = \alpha A K_t^{\alpha} \ell_t^{1-\alpha}$, equilibrium implies

$$-(2-\ell_t)^{-\gamma} + \frac{B}{\ell_t}\left[\frac{\alpha(1-\alpha)A^2 K_{t+1}^{\alpha-1}\ell_{t+1}^{1-\alpha} K_t^{\alpha}\ell_t^{1-\alpha}}{(\alpha A K_t^{\alpha}\ell_t^{1-\alpha})^{\rho}}\right]^{1-\sigma} = 0, \tag{7}$$

$$K_{t+1} = A(1-\alpha) K_t^{\alpha} \ell_t^{1-\alpha}. \tag{8}$$

The dynamic system described by (7) and (8) defines the variables K_{t+1} and ℓ_{t+1} as functions of K_t and ℓ_t.

III. EXISTENCE OF FIXED POINTS

The aim of this section is to study existence and stability properties of the fixed point of the system given by (7) and (8). To this purpose, we use the geometrical-graphical method developed by Grandmont *et al.* (1998).

A. Existence of fixed points

In economic models with overlapping generations, the existence of fixed points is not generally guaranteed. Thus, we now impose some restrictions on parameters such that the (normalised) fixed point

$$(K, \ell) = (1, 1), \tag{9}$$

always exists. This allows us to analyse the effects on stability due to changes in some parameter values by avoiding that the fixed point vanishes (Grandmont *et al.*, 1998; Cazzavillan, 2001). Therefore, through (7)–(9), we get

$$A = A^* := \frac{1}{1-\alpha}, \tag{10}$$

$$B = B^* := \left(\frac{\alpha}{1-\alpha}\right)^{-(1-\sigma)(1-\rho)}. \tag{11}$$

By substituting out (10) and (11) into (7) and (8), the two-dimensional map that characterises the dynamics of the economy is the following:

$$M : \begin{cases} K_{t+1} = V(K_t, \ell_t) := K_t^{\alpha}\ell_t^{1-\alpha} \\ \ell_{t+1} = Z(K_t, \ell_t) := K_t^{\frac{\alpha(\rho-\alpha)}{1-\alpha}} \ell_t^{\frac{1+(1-\alpha)(1-\sigma)(\rho-\alpha)}{(1-\alpha)(1-\sigma)}} (2-\ell_t)^{\frac{-\gamma}{(1-\alpha)(1-\sigma)}} \end{cases}. \tag{12}$$

Given the couple (K_t, ℓ_t) it is possible to compute its subsequent iterate if and only if we start from a point on $D := \{(K_t, \ell_t) \in R^2 \ : \ K_t > 0, \ 0 < \ell_t < 2\}$. However, feasible trajectories lie in a set smaller than D since by starting from an initial condition in D it is possible to have an iterate from which the existence of the subsequent one is not guaranteed. Then, the set of feasible trajectories is $G := \{(K_0, \ell_0) \in R^2 \ : \ K_t > 0, \ 0 < \ell_t < 2, \forall t > 0\}$. In addition, by the first equation of (12) and $\ell_t < 2$, it follows that diverging trajectories cannot exist.

Since $K = \ell$ always holds as a coordinate of a stationary state of map M, then stationary-state coordinate values of ℓ are determined as solutions of $\ell = Z(\ell, \ell)$ or they are equivalently obtained by solving the following equation:

$$g(\ell) := \ell^{\rho+\sigma(1-\rho)}(2-\ell)^{-\gamma} = 1. \tag{13}$$

Of course, $\ell = 1$ is a solution of (13) for every constellation of parameters. Then, the following proposition holds.

Proposition 1. [Existence of fixed points]. If either $\sigma < 1$ or $\sigma > 1$ and $\rho < \frac{\sigma}{\sigma-1}$, then $(1,1)$ is the unique fixed point of map M. If $\sigma > 1$ and $\frac{\sigma}{\sigma-1} < \rho < \frac{\sigma+\gamma}{\sigma-1}$ (resp. $\sigma > 1$ and $\rho > \frac{\sigma+\gamma}{\sigma-1}$), then another fixed point $(\hat{K}, \hat{\ell})$ exists with $\hat{\ell} < 1$ (resp. $\hat{\ell} > 1$).

Proof. From (13) we have that $\lim_{\ell\to 2} g(\ell) = +\infty$, while $\lim_{\ell\to 0} g(\ell) = 0$ (resp. $\lim_{\ell\to 0} g(\ell) = +\infty$) if and only if (*a*) $\sigma < 1$ or (*b*) $\sigma > 1$ and $\rho < \frac{\sigma}{\sigma-1}$ (resp. $\sigma > 1$ and $\rho > \frac{\sigma}{\sigma-1}$). In addition, with direct computation we have that

$$\mathrm{sgn}\{g'(\ell)\} = \mathrm{sgn}\{2[\sigma - \rho(\sigma-1)] - \ell[\sigma - \rho(\sigma-1) - \gamma]\}$$

and

$$\mathrm{sgn}\{g'(1)\} = \mathrm{sgn}\{\sigma + \gamma - \rho(\sigma-1)\}.$$

It follows that $g(\ell)$ is a monotone function or unimodal function. In the former case, no solution other than $\ell = 1$ of (13) does exist. In the latter case, there exists another solution of (13) on the left (resp. on the right) of $\ell = 1$ if $\rho < \frac{\sigma+\gamma}{\sigma-1}$ (resp. $\rho > \frac{\sigma+\gamma}{\sigma-1}$). □

Proposition 1 shows the crucial role played by σ (which is a measure of the elasticity of utility with respect to effective consumption) and ρ (the intensity of aspirations in utility) in determining the long-term behaviour of the economy. When σ and/or ρ are low, a unique (normalised) fixed point exists. A second fixed point appears (with leisure being lower or higher than one) when σ and ρ raise. In particular, when ρ is sufficiently high, the supply of labour becomes higher than 1 because the relative importance of past consumption is high and individuals want to increase the amount of time spent at work when they are young to increase consumption when they are old.

B. Local bifurcations and stability

This section starts by analysing the local dynamics around the normalised fixed point. In the present model, the stock of capital K_t is a state variable, so its initial value K_0 is given, while the supply of labour ℓ_t is a control variable. It follows that individuals of the first generation ($t = 0$) choose the initial value ℓ_0. If the normalised fixed point is a saddle and the initial condition of the stock variable K is close enough to 1, then, given the expectations on the interest rate, there exists a unique initial value of ℓ_t (ℓ_0) such that the orbit that passes through (K_0, ℓ_0) approaches the fixed point. When the fixed point is a sink, given the initial value K_0 and expectations on the interest factor, there exists a continuum of initial values ℓ_0 such that the orbit that passes through (K_0, ℓ_0) approaches the fixed point. As a consequence, the orbit that the economy will follow is "locally indeterminate" because it depends on the choice of ℓ_0. When the fixed point is a source, given the initial value K_0, the choice on ℓ_0 and expectations on the interest factor, the trajectory may be captured by another fixed point or it may be not feasible. The Jacobian matrix of map M evaluated at $(1\,,\,1)$ is

$$J = \begin{pmatrix} \alpha & 1-\alpha \\ \dfrac{\alpha(\rho-\alpha)}{1-\alpha} & \dfrac{(\rho-\alpha)(1-\sigma)(1-\alpha)+\gamma+1}{(1-\sigma)(1-\alpha)} \end{pmatrix}. \quad (14)$$

The trace and determinant of (14) are the following:

$$Tr(J) = \frac{1+\gamma}{(1-\alpha)(1-\sigma)} + \rho, \quad (15)$$

$$Det(J) = \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}. \quad (16)$$

Ceteris paribus, when ρ varies the point

$$(P_1, P_2) := \left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)} + \rho,\ \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right), \quad (17)$$

drawn in the $(Tr(J), Det(J))$ plane, describes a horizontal half-line T_1 that starts from

$$(\bar{P}_1,\, \bar{P}_2) := \left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)},\ \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right), \quad (18)$$

when $\rho = 0$. When α varies, the point $(\bar{P}_1,\, \bar{P}_2)$ drawn in $(Tr(J), Det(J))$ plane describes a half-line T_2 (with slope equal to 1) that starts from $\left(\frac{1+\gamma}{1-\sigma},\, 0\right)$ when $\alpha = 0$. If $\sigma \in (0\,,\,1)$ (resp. $\sigma > 1$), then $(\bar{P}_1,\, \bar{P}_2) \to (+\infty, +\infty)$ (resp. $(\bar{P}_1,\, \bar{P}_2) \to (-\infty, -\infty)$) for $\alpha \to 1$. Moreover, regardless of the value of σ, $(P_1,\, P_2) \to \left(+\infty, \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right)$ for $\rho \to +\infty$.

From the above geometrical findings and Proposition 1, we can state the following proposition with regard to local bifurcations.

Proposition 2. [Local bifurcation]. Let $\rho_{fl} := \frac{2+\sigma(\alpha-1)+\gamma(1+\alpha)}{(1-\alpha)(\sigma-1)}$ *and* $\rho_{tc} := \frac{\sigma+\gamma}{\sigma-1}$ *hold. Then, (1) if* $\sigma \in (0,1)$*, the normalised (unique) fixed point is a saddle (determinate); (2) if* $\sigma > 2+\gamma$ *and* $\frac{\sigma-2-\gamma}{\sigma+\gamma} < \alpha < \frac{\sigma-1}{\gamma+\sigma}$ *or* $1 < \sigma < 2+\gamma$ *and* $\alpha < \frac{\sigma-1}{\gamma+\sigma}$*, the normalised fixed point is a saddle (determinate) for* $\rho < \rho_{fl}$*, it undergoes a supercritical flip bifurcation for* $\rho = \rho_{fl}$*, it is a sink (indeterminate) for* $\rho \in (\rho_{fl}, \rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho = \rho_{tc}$*, and it is a saddle (determinate) for* $\rho > \rho_{tc}$*; (3) if* $\sigma > 2+\gamma$ *and* $\alpha < \frac{\sigma-2-\gamma}{\sigma+\gamma}$ *the normalised fixed point is a sink (indeterminate) for* $\rho \in (0, \rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho = \rho_{tc}$*, and it is a saddle (determinate) for* $\rho > \rho_{tc}$*; (4) if* $\alpha > \frac{\sigma-1}{\gamma+\sigma}$*, the normalised fixed point is a saddle (determinate) for* $\rho \in (0, \rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho = \rho_{tc}$*, it is a source (determinate) for* $\rho \in (\rho_{tc}, \rho_{fl})$*, it undergoes a flip bifurcation for* $\rho = \rho_{fl}$*, and it is a saddle (determinate) for* $\rho > \rho_{fl}$.

Proof. In order to find the bifurcation values of ρ, we impose the condition that (P_1, P_2) belongs to: (*i*) the straight line $1 - Tr(J) + Det(J) = 0$, to obtain the transcritical bifurcation value ρ_{tc}, and (*ii*) the straight line $1 + Tr(J) + Det(J) = 0$, to obtain the flip bifurcation value ρ_{fl}. Then, we identify cases 1–4 by considering the position of the starting points $(\bar{P}_1, \bar{P}_2)$ and (P_1, P_2) with respect to the stability triangle delimited by $1 \pm Tr(J) + Det(J) = 0$ and $Det(J) = 1$ (see Grandmont *et al.*, 1998 for details). ☐

Proposition 2 is represented in Fig. 1. In particular, the first quadrant is referred to point (1) of the proposition, the second and third quadrants to points (2), (3), and (4).

It is important to note that map M may not be defined at fixed point $(0, 0)$. In particular, this happens if and only if either (*i*) $\rho < \alpha$ or (*ii*) $\sigma > 1$ and $\rho < \frac{1+(\sigma-1)(1-\alpha)\alpha}{(\sigma-1)(1-\alpha)}$. In the remaining cases, however, the map results to be not differentiable at $(0, 0)$. Then, it is not possible to apply the linearization method with the purpose of studying the stability of the map at $(0, 0)$. However, we can classify the local properties of the map at $(0, 0)$ by considering the sign of $\Delta K \equiv K_{t+1} - K_t$ and $\Delta\ell \equiv \ell_{t+1} - \ell_t$ in the phase plane. By this study, we can deduce that $(0, 0)$ is an attracting fixed point when two interior stationary equilibria do exist. In this case, $(0, 0)$ can be

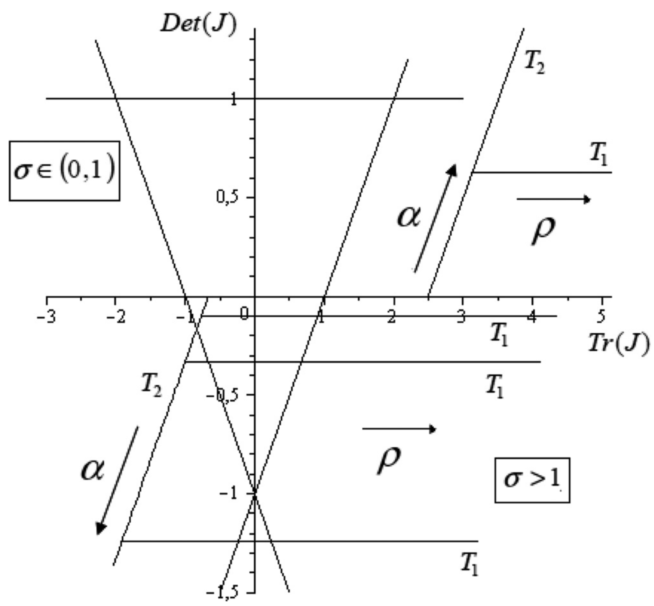

FIG. 1. Stability triangle and local indeterminacy. If $\sigma \in (0, 1)$, a unique fixed point (saddle) exists. If $\sigma > 1$, multiple fixed points and local indeterminacy may exist.

interpreted as a poverty trap (e.g., Azariadis, 1996; Chakraborty, 2004).

IV. GLOBAL ANALYSIS

The importance of the global analysis for economic models is recognised by the fact that studying just the local behaviour of a map does not give information of the structure of the basins of attraction and their qualitative changes when parameters vary. Since in economics it is also important to understand the long-term behaviour of variables given initial conditions, a characterisation of the basins of attraction is necessary if one wants to explain phenomena that occur by starting from initial conditions far away from a fixed point or an attracting set.

In this section, we show how the study of: (i) the dynamics around non-normalised fixed points and (ii) the global structure of map M, permit us to explain some interesting events that cannot be investigated with the local analysis (Pintus *et al.*, 2000). We start the global analysis by showing that map M is invertible. The invertibility of a map is an important result when the global properties of a dynamic system are studied. For instance, it implies that the basins of attraction of any attracting set of a map are connected sets. In addition, by making use of the inverse map, we can obtain the boundary of the attracting sets and, more generally, the stable manifolds of saddle points.

With regard to map M, the following lemma holds.

Lemma 1. Map M is invertible on set D.

Proof. First, we note that it is impossible to have a closed-form expression of the inverse map of M (M^{-1}). However, it is possible to find that M^{-1} is solution of the following system:

$$M^{-1}:\begin{cases}\dfrac{(2-\ell_t)^{\gamma}}{\ell_t}=\left(\dfrac{K_{t+1}^{\rho-\alpha}}{\ell_{t+1}^{1-\alpha}}\right)^{1-\sigma}\\ K_t=K_{t+1}^{\frac{1}{\alpha}}\ell_t^{\frac{\alpha-1}{\alpha}}\end{cases}. \tag{19}$$

The left-hand side of the first equation in (19) defines a bijection from $(0,2)$ to $(0,+\infty)$. It follows that the first equation of (19) admits a unique solution of ℓ_t. The second equation of the system implies the result. □

Before performing the global analysis of map M, we recall the definitions of both the stable manifold

$$W^s(p)=\{x: M^{zn}(x)\to p\ as\ n\to+\infty\}, \tag{20}$$

and unstable manifold

$$W^u(p)=\{x: M^{zn}(x)\to p\ as\ n\to-\infty\}, \tag{21}$$

of a periodic point p of period z. If the periodic point $p\in R^2$ is a saddle cycle (we recall that a hyperbolic cycle is called a saddle cycle if it has multipliers both inside and outside the unit circle (Kuznetsov, 2003)) then the stable (resp. unstable) manifold is a smooth curve through p, tangent at p to the eigenvector of the Jacobian matrix evaluated at p corresponding to the eigenvalue λ with $|\lambda|<1$ (resp. $|\lambda|>1$). Outside the neighbourhood of p, the stable and unstable manifolds may even intersect each other with dramatic consequences on the global dynamics of the model.

Non-trivial intersection points of stable and unstable manifolds of a unique saddle cycle are known as homoclinic points. However, when multiple saddle cycles exist, heteroclinic bifurcations may also occur. Given two saddle cycles h_1 and h_2, a heteroclinic bifurcation is defined as the birth

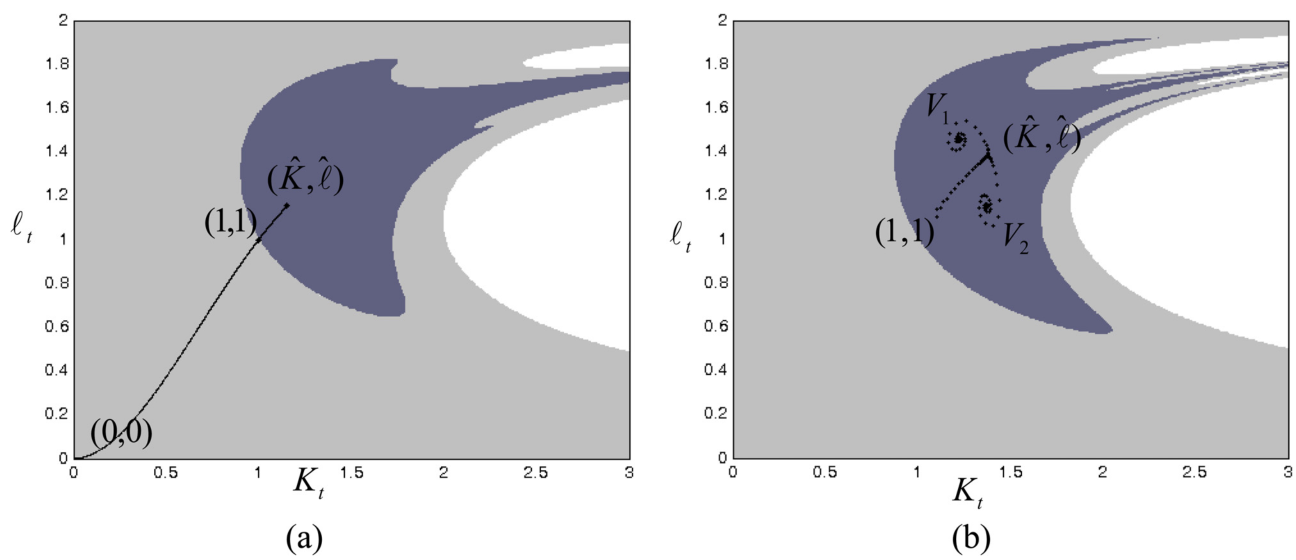

FIG. 2. (a) ($\rho=2.2$). The fixed point $(\hat{K},\hat{\ell})$ ($\hat{K}=\hat{\ell}\cong 1.1532$) is the unique attractor of the system (indeterminate equilibrium). The normalised fixed point $(1,1)$ belongs to the boundary of the attractor. The basin of the attraction of $(\hat{K},\hat{\ell})$ is dark-grey-coloured. Trajectories that start in the light-grey region converge to $(0,0)$. The white region describes the set of initial conditions that generate unfeasible trajectories. The curve that connects $(\hat{K},\hat{\ell})$ and $(1,1)$ is the simulated unstable manifold of $(1,1)$ (i.e., the heteroclinic connection), obtained by iterating a small segment in the direction of the unstable eigenvector. If we consider an economy that starts on the stable manifold that converges to $(1,1)$, a small change (shock) in expectations about the interest factor R^e_{t+1} may cause the convergence to $(\hat{K},\hat{\ell})$, where the capital stock is the same and the labour supply higher than the normalised fixed point. (b) ($\rho=2.4$). Map M has an attracting two-period cycle of points V_1 and V_2. The stable manifold of $(1,1)$ defines the boundary of the basin of attraction of the two-period cycle (dark-grey region).

of a non trivial point of intersection between the stable manifold of one cycle and the unstable manifold of the other cycle. From this new configuration, it is possible to find a path on the two manifolds that connects the cycles. This phenomenon is of importance in economic theory because it is related to global indeterminacy. We recall that global indeterminacy occurs when (starting from the same initial condition of the state variable) different fixed points or other ω-limit sets can be reached according to the initial value of the jumping variable chosen by individuals of the first generation (Agliari and Vachadze, 2011; Gori and Sodini, 2011). In addition, when saddle cycles exist heteroclinic connections can be another source (alongside the Neimark-Sacker bifurcation) of existence of repelling or attracting closed invariant curves (Agliari *et al.*, 2005). Let us remind that the dynamics of the restriction of a map to a closed invariant curve is either quasiperiodic or periodic of high periodicity so that numerically indistinguishable from a quasiperiodic one.

In the analysis performed below, we fix the following parameter values: $\alpha = 0.35$ (Gollin, 2002), $\gamma = 1.2$, $\sigma = 3$ and let ρ vary. The study starts with $\rho = 2.2$, corresponding to which the normalised fixed point $(1, 1)$ is a saddle, while the other points $(\hat{K}, \hat{\ell})$ and $(0, 0)$ are attracting and their basins of attraction are separated by the stable manifold of $(1, 1)$. Notice that a heteroclinic connection exists between $(1, 1)$ and $(\hat{K}, \hat{\ell})$ as well as between $(1, 1)$ and $(0, 0)$, each of which is given by one of the branches of the unstable manifold of the saddle (Fig. 2(a)). This implies that by considering a small neighbourhood of $(1, 1)$, there exist feasible trajectories that converge either to the interior equilibrium or $(0, 0)$. While this result is not surprising and generically occurs when the saddle point belongs to the border of the basin of attraction of an attracting fixed point, the economic literature on local (in)determinacy has not stressed the importance of the existence of a continuum of equilibria around the determinate fixed point (an exception is Agliari and Vachadze, 2011), even though this can be of interest

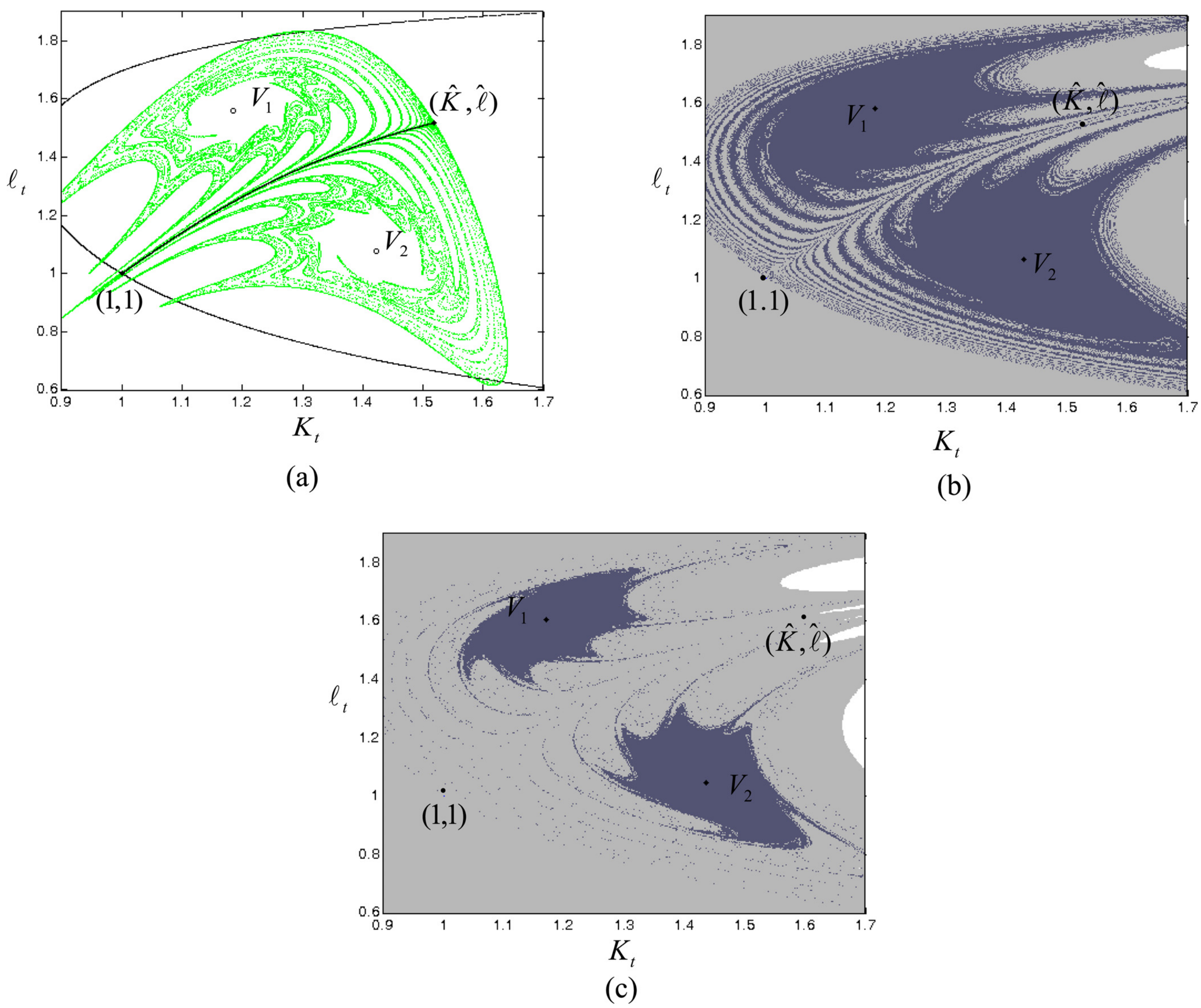

FIG. 3. (a) The stable and unstable manifolds of $(1, 1)$ when $\rho = 2.58$. The black lines describe the upper branch of the unstable manifold of $(1, 1)$ converging to $(\hat{K}, \hat{\ell})$ (that coincides with the lower branch of the stable manifold of the saddle $(\hat{K}, \hat{\ell})$) and the stable manifold of $(1, 1)$. The green line represents the unstable manifold of $(\hat{K}, \hat{\ell})$. (b) Basin of attraction of M when $\rho = 2.58$. (c) Evolution of the basin of attraction of M when $\rho = 2.628$.

especially from a policy perspective. This implies that given an initial condition K_0 close enough to 1, there exist several feasible trajectories converging to other attractors of the system.

If we let ρ increase, a flip bifurcation occurs at $\rho \cong 2.3078$ for $(\hat{K}, \hat{\ell})$, and a two-period cycle captures almost all feasible trajectories beyond the stable manifold of $(1, 1)$. Trajectories (as those drawn in Fig. 2(b)) that start in the dark-grey region just out the left or the right of the stable manifold of $(\hat{K}, \hat{\ell})$, follow the curve almost until the saddle $(\hat{K}, \hat{\ell})$ but converge to a two-period cycle of points V_1 and V_2. This implies that (after the transient) the economy oscillates between the two points. With regard to economic implications, by considering a historically given value of the capital stock (K_0) close to 1, the system may follow very different paths and long-term behaviours according to individual decisions.

If we let ρ increase, numerical simulations reveal that more and more convolutions of the unstable manifold of $(\hat{K}, \hat{\ell})$ evolve. This means that a new heteroclinic connection is close to be born. In particular, for $\rho \cong 2.5$, a branch of the unstable manifold of $(\hat{K}, \hat{\ell})$ has a tangential contact with a branch of the stable manifold of $(1, 1)$, while for larger values of ρ a transversal intersection between these branches of manifolds exists (Fig. 3(a)). When also this connection occurs, there are heteroclinic orbits from $(1, 1)$ to $(\hat{K}, \hat{\ell})$ and heteroclinic orbits from $(\hat{K}, \hat{\ell})$ to $(1, 1)$ leading to a heteroclinic connection, which has the same dynamic properties of a homoclinic orbit. That is, in any neighbourhood of a heteroclinic connection (the union of two heteroclinic orbits), there exists an invariant set on which the map is chaotic. This chaotic repellor is responsible for the complex structure of the basins of attraction of the two coexisting attractors (Figs. 3(b) and 3(c) show the evolution of the basins of attraction). In addition, if we let ρ increase further, we can also have interesting economic consequences because even if the 2-period cycle is stable for map M, starting from values of the capital stock around $\hat{K}$ only a few values of ℓ_0 (Fig. 3(c)) can generate trajectories converging to the attractor or interior fixed points, while the remaining values of ℓ_0 generate trajectories approaching to $(0, 0)$.

The last part of this section is devoted to the study of periodic and/or quasiperiodic orbits generated by the system when ρ is fixed at higher values. In order to better illustrate the dynamic properties of the map, we study through numerical simulations the second (forward) iterate of map M, namely, M^2. We now recall that fixed points of M hold as fixed points of M^2, while two-period cycles of M become fixed points of M^2. In other words, the flip bifurcation of M is a pitchfork bifurcation of M^2 and two attracting fixed points V_1 and V_2 exist. In what follows we concentrate on the evolution of the dynamics on the upper part of the domain D around V_1 (the dynamics on the lower part around V_2 being similar).

The bifurcation diagram depicted in Fig. 4, which is obtained by using $(K_0, \ell_0) = (1.12, 1.6)$ as the initial condition, shows some apparent discontinuities starting from $\rho \cong 2.6295$. They are caused by one of the two couples of

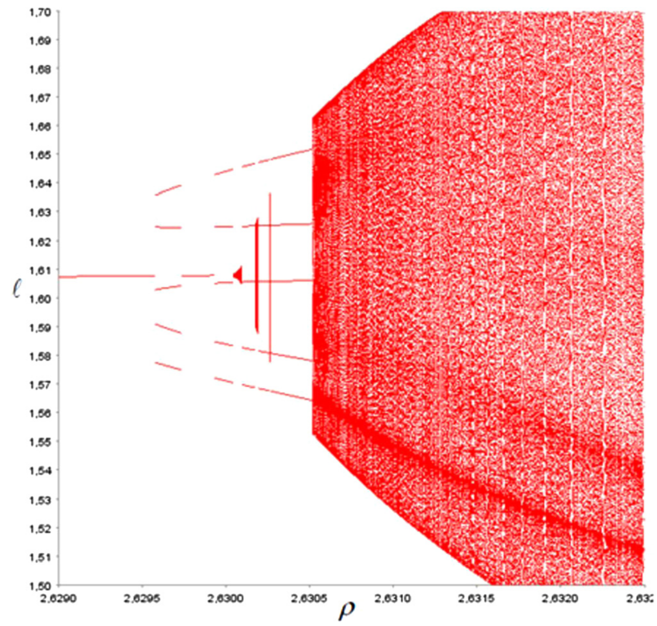

FIG. 4. Bifurcation diagram for ρ. We follow the long-term evolution of the starting point $(K_0, \ell_0) = (1.12, 1.6)$ when ρ increases. The discontinuities in the picture are due to the birth of another attractor. The apparent superposition of the curves just beyond $\rho = 2.63$ is due to the projection of the dynamics on the ℓ axis.

5-period cycles born through a saddle node bifurcation that captures the given initial condition for some ranges of ρ. At this stage, the dynamics of M^2 starting from the upper part of the phase plane may converge to: V_1, $(0, 0)$ or one of the two attracting 5-period cycles. The basins of attraction of the two attracting 5-period cycles are defined by the stable manifolds of the corresponding saddle cycles (Fig. 5(a)). The unstable branches of the saddle cycles tend either to V_1 or to the attracting 5-period cycle.

Following now the evolution of fixed point V_1 of M^2, it undergoes a supercritical Neimark-Sacker bifurcation at $\rho \cong 2.629732$ (Fig. 4), and an attracting invariant curve (Γ) may be observed around V_1 on which the dynamics may be periodic or quasiperiodic. Fig. 5(b) (plotted for $\rho = 2.6302$) shows coexistence of attractors together with two trajectories. The stable manifolds of the saddle cycle (S) bound the basin of attraction of the attracting 5-period cycle, and the unstable branches of the saddle cycles tend either to the invariant curve Γ or to the attracting 5-period cycle.

For the values of ρ used to depict Figs. 5(a) and 5(b), the dynamics of M^2 is characterised by four coexisting interior attractors. In this case, it is difficult to predict the long-term dynamics of the economy, and an exogenous change in some parameter values and in expectations of individuals may cause the switch to another attractor of the system. For instance, if a trajectory is converging to V_1 (Fig. 5(a)), the coordination on a large value of ℓ causes the switch to an attractor with larger oscillations.

If we let ρ increase, the invariant curve becomes larger and a collision between the stable manifold of the 5-period cycle and the curve itself occur at $\rho \cong 2.63044$. This causes the death of the invariant curve and a unique attracting 5-period cycle of the system does exist (Fig. 5(c)).

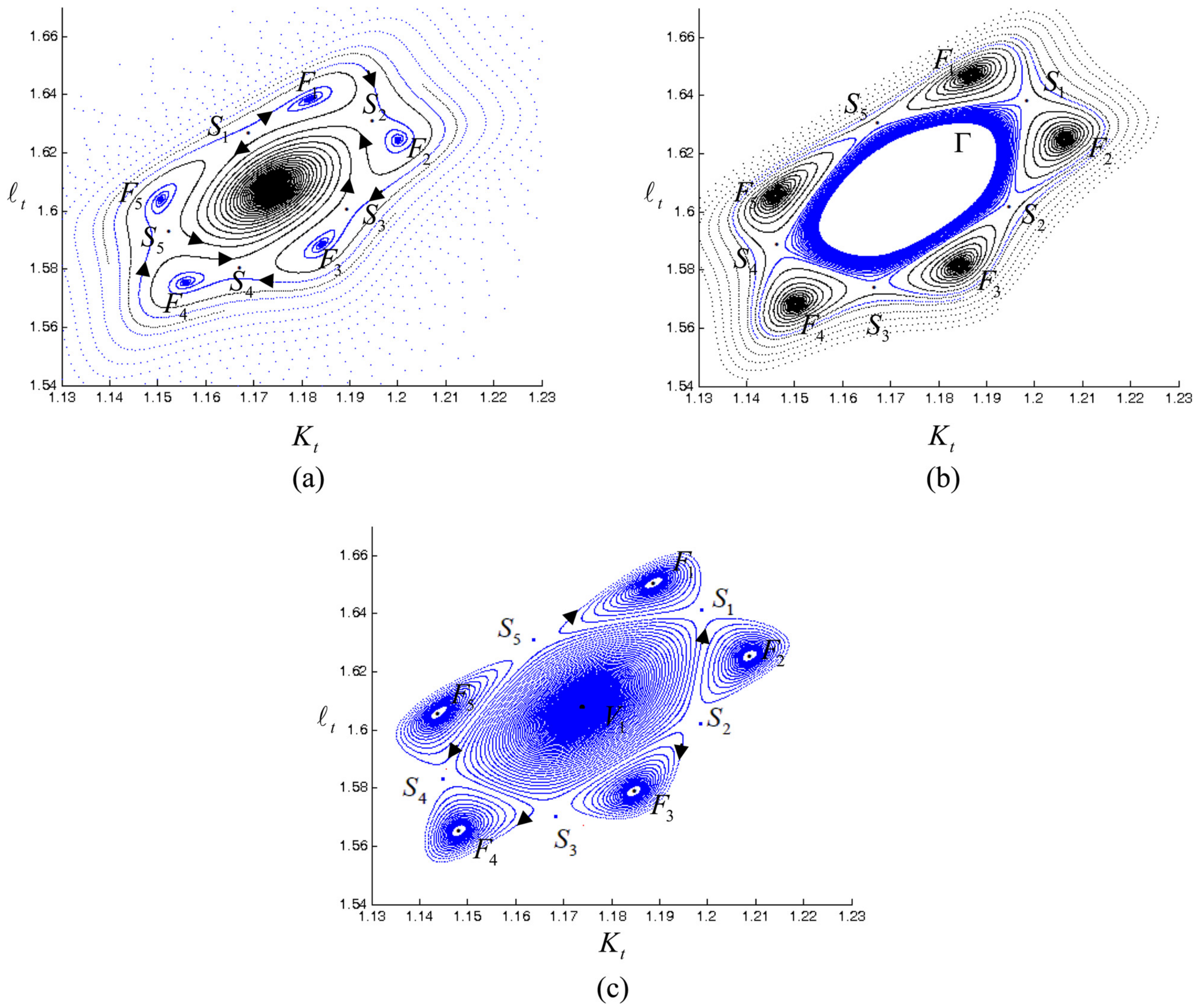

FIG. 5. (a) ($\rho = 2.6297$). A couple of 5-period cycles coexists with an attracting fixed point. Two converging trajectories are depicted (black-coloured and blue-coloured). F_i (S_i) indicates the ith point of the attracting (saddle) 5-period cycle. (b) ($\rho = 2.6302$). Coexistence of both an attracting 5-period cycle and closed invariant curve (Γ). (c) ($\rho = 2.63044$). A basin boundary bifurcation has destroyed the closed invariant curve and a unique attracting 5-period cycle survives.

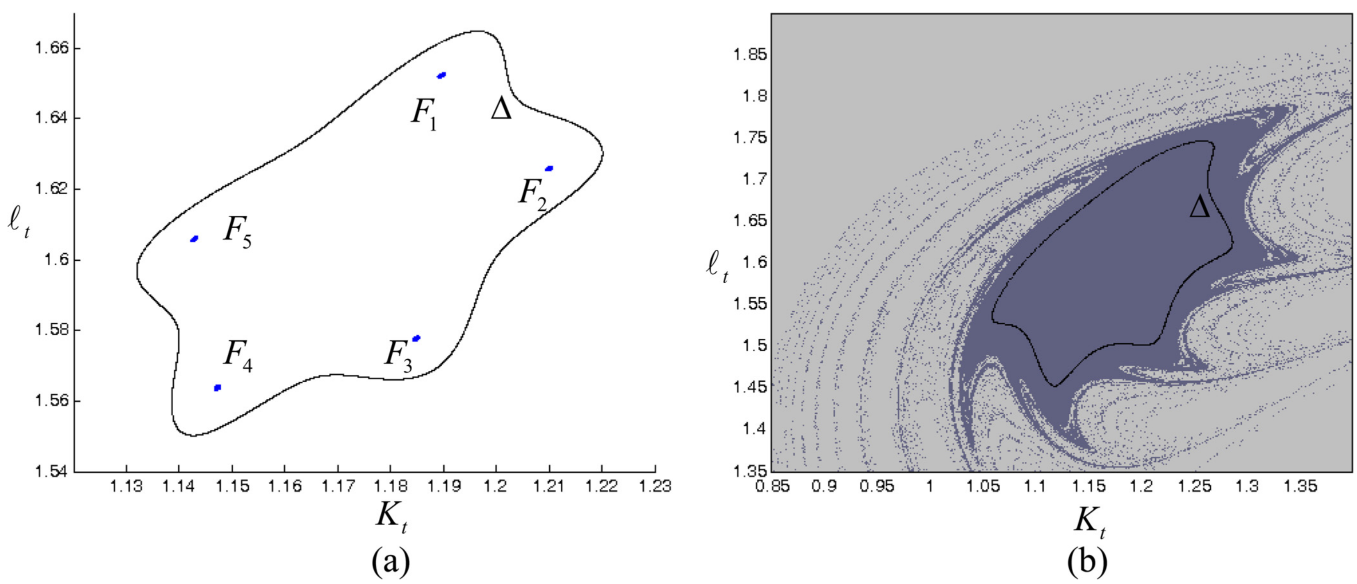

FIG. 6. (a) A large invariant curve (Δ) surrounds a 5-period cycle ($\rho = 2.63056$). (b) When ρ increases (in Fig. 3(b) $\rho = 2.633$), the closed invariant curve (Δ) remains the unique interior attractor of M^2 in this portion of the phase plane. Note that in Fig. 6(a) the basins of attraction of Δ and F are not reported because of the long transient. Nevertheless, we have that the basin of F lies in a portion of the phase plane bounded by Δ (this is due to the invertibility of the map). In Fig. 6(b), both the attractor and its basin of attraction are depicted.

From the bifurcation diagram (Fig. 4), we note a sudden enlargement of the attractor at $\rho \cong 2.63052$. In order to understand this finding, we refer to the theoretical results on invariant curves proposed by Agliari *et al.* (2005, 2006, 2007). We start from the situation described by Fig. 5(c). At this stage, we have that the stable set associated to the 5-period saddle cycle bounds the basin of attraction of the attracting 5-period cycle (F), and the two branches of the unstable set have different limit sets: F and another attractor. No intersection between the stable and unstable sets of S exists. If we let ρ increase, we have that a heteroclinic loop or a heteroclinic tangle is created by the branches of the unstable manifold of S non-converging to F and the stable manifold. After this global bifurcation, a new closed attracting curve exists (Δ) and the two 5-period cycles are both inside the closed invariant curve Δ (Fig. 6(a)). The stable set of S separates the basins of attraction of Δ and F (this phenomenon is shown in Fig. 6(a), plotted for $\rho = 2.63056$). If we let ρ increases, a new saddle node bifurcation causes the disappearance of the 5-period cycles and Δ remains the unique attractor of the system in this portion of the phase plane (this phenomenon is shown in Fig. 6(b), plotted for $\rho = 2.633$).

V. CONCLUDING REMARKS

This paper has studied the dynamic properties of a two-dimensional growth model with overlapping generations, endogenous labour supply (Reichlin, 1986), and multiplicative external habits (de la Croix, 1996). We have shown that the intensity of aspirations in utility matters for the existence of multiple attractors and business cycles dynamics.

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