PAYG pensions and economic cycles: exogenous versus endogenous fertility

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Abstract In this article we compare the dynamics and long-run outcomes of an overlapping generations closed economy under exogenous and endogenous fertility. Individuals have myopic foresight, and pay-as-you-go (PAYG) public pensions exist. Although large PAYG transfers may cause endogenous fluctuations in both contexts, cyclical instability and deterministic chaos more likely occur when fertility is an economic decision variable. We find that the existence of endogenous fertility and PAYG pensions can explain the occurrence of demographic oscillations that mimic the baby boom and baby bust, in contrast with the unrealistic case of constant population, and also economic cycles. We also show that an increase in the social security contribution rate under endogenous fertility, often advocated as a remedy against the crisis of public pension budgets, may prolong both the phases and size of demographic (economic) cycles around a lower (higher) long-run level of population growth (income per worker) than under exogenous fertility.

Keywords Myopic foresight; PAYG pension; Stability; OLG model

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1. Introduction

Unfunded pay-as-you-go (PAYG) social security is a controversial issue in the economic literature for several reasons. Some authors highlight the effects of pensions in reducing either saving (e.g., Feldstein 1974) or welfare if the capital stock is not too much larger (e.g., Diamond 1965), while others (e.g., Hansson and Stuart 1989) find that the crowding out effect of pensions on voluntary saving do not occur if individuals take care of their future infinite progeny, while leaving private inter-generational transfers.

Though the existence of the PAYG system has been justified on equity grounds in a political equilibrium model because of an inter-generational redistribution effect (e.g., Tabellini 1990), it has also been criticised due to the perils of unviable budgets, especially in countries facing with demographic changes because of population ageing (e.g., Italy, Japan and Spain). This has raised debates on the role PAYG pensions can play in overlapping generations (OLG) growth models when fertility changes (see, amongst many others, van Groezen et al. 2003). While in this class of models the issue of PAYG pensions has usually been studied under exogenous fertility (see, e.g., Samuelson 1975a), a recent body of theoretical literature, pioneered by the seminal paper by Becker (1960), argue that fertility should actually be considered as a decision variable influenced by economic incentives and constraints (Becker and Barro 1988; Barro and Becker 1989; Becker et al. 1990) rather than being leaved out the economic sphere.

However, to the best of our knowledge, most studies have not yet explored the dynamical features of an economy with endogenous fertility in comparison with those of the standard case of exogenous fertility when PAYG pensions are in existence. In particular, less attention has been paid to the role of PAYG pension and endogenous demographic changes on: (*i*) long-run demo-economic outcomes, and (*ii*) economic and demographic stability. The aim of this paper, therefore, is to fill this gap by reconsidering the issue of unfunded social security in the neoclassical OLG growth model à la Diamond (1965) under exogenous and endogenous fertility.

In particular, as in Eckstein and Wolpin (1985) and Galor and Weil (1996), the choice of fertility is assumed to be motivated by the so-called weak form of altruism of parents (see Zhang and Zhang 1998), that is individuals draw utility from the number of descendants they have and choose fertility by comparing benefits and costs of children. Moreover, as child bearing can reasonably be considered either a time-based or income-based activity irrespective of whether fertility is endogenous or exogenous (due, for instance, to religion belief, customs, unchecked sexuality and so on), in this paper we assume that raising children is costly even under exogenous fertility.

Given the difficulty of studying the dynamics of overlapping-generations models with endogenous fertility,[1] we postulate a lognormal utility function and a Cobb-Douglas production function, which permit closed form analytical solutions as well as economic interpretation of the dynamical outcomes, otherwise prevented by the use of other more general functional forms.[2] This assumption is usual in literature and, beyond its analytical simplification, has a large empirical support.[3]

Our results reveal that PAYG pensions strongly matter for both the steady-state and dynamical events depending on whether fertility is endogenous or exogenous. In particular, in the absence of public pensions fertility still remains constant even when it is an economic decision variable, and thus the model is not suited to explain the observed population dynamics in such a case. Indeed, the working of pensions generates an endogenous population dynamics which may mimic the observed evolution of fertility, especially in Western countries. Moreover, PAYG pensions are crucial to determine the long-run level of population growth as well as the appearance of permanent and chaotic phases of baby booms and baby busts.

Therefore, the interaction between PAYG pensions and endogenous fertility significantly matters for stability, and the destabilising effect of public pensions is higher than when fertility is exogenous. The economic reasons for this result are the following.

(*i*) Regardless of whether fertility is exogenous or endogenous, the existence of a PAYG transfer crowds out voluntary savings because the need to sustain consumption when old becomes lower than in the absence of it. As a consequence, when individuals are short sighted, raising pensions beyond a certain critical level may determine a sharp reduction in savings and capital accumulation as well as the destabilisation of the equilibrium point.

(*ii*) The negative crowding out effect of public pensions on private savings is lower the higher the subjective discount factor (because, *ceteris paribus*, the relative weight of old-age consumption increases as the inter-generational discount factor becomes larger).

(*iii*) In the case of endogenous fertility, the individual discount factor also reduces the weight of the positive effect of pensions on the demand for children (note that the positive relationship between fertility and PAYG pension exists because individuals save less than when inter-generational transfers are absent, and thus the disposable income for child bearing increases in such a case). Moreover, the weight of the subjective discount factor on saving and fertility through public pensions is exactly the same (see Eqs. 19 and 20 in Section 3).

(*iv*) Since capital accumulation per worker is determined as the ratio between savings and fertility, then the subjective discount factor does not affect the public pension component on capital accumulation when fertility is endogenous, while affecting in a negative way saving and the public pension component on capital accumulation when fertility is exogenous.

The present paper, therefore, contributes to three strands of literature. (*1*) The neoclassical OLG growth literature with public pensions. In this regard, the paper studies an issue so far not explored, such as the dynamical and steady-state demo-economic outcomes with PAYG pensions, by contrasting models with exogenous and endogenous fertility, while also focusing on the effects of increasing the contribution rate, which is currently highly debated by economists and politicians (see, e.g., Cigno 2007; Liikanen 2007). (*2*) The theory of endogenous population dynamics (see, e.g., Manfredi and Fanti 2006 and the literature cited therein), providing an explanation of the occurrence of long-run demographic cycles. (*3*) The endogenous economic cycles theory framed in the OLG context (Grandmont 1985). Though it is well known that OLG economies with myopic expectations may typically show cyclical (and even chaotic) dynamics when the elasticity of substitution in production (Farmer 1986; Reichlin 1986) and utility functions (Michel and de la Croix 2000; Fanti and Spataro 2008) are fairly low and high, respectively, it is remarkable that cyclical instability and deterministic chaos can occur in the standard double Cobb-Douglas economy because of the financing of PAYG pensions.

Our findings can have interesting policy implications. In particular, to the extent that fertility is an economic decision variable, as argued by the new home economics literature (Becker 1960), a rise in the contribution rate, often invoked as a remedy[4] against the crisis of public pension budgets, may reduce the desired number of children and trigger baby booms and baby busts around the long-run level of fertility.

The rest of the paper is organised as follows. Section 2 (Section 3) analyses the steady-state and local stability properties of an OLG economy with exogenous (endogenous) fertility. Section 4 compares the dynamical and steady-state outcomes under exogenous and endogenous fertility. Section 5 concludes.

2. Exogenous Fertility

Firms are identical and markets are competitive. Aggregate production at time t (Y_t) takes place by combining capital (K_t) and labour ($L_t = N_t$ in equilibrium, where N_t is the number of young individuals at t) through the Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. Assuming that output is sold at unit price and capital fully depreciates at the end of every period, profit maximisation implies that production factors are paid their marginal products, that is:

$$r_t = \alpha A k_t^{\alpha-1} - 1, \tag{1}$$

$$w_t = (1-\alpha) A k_t^{\alpha}, \tag{2}$$

where $k_t := K_t / N_t$ is capital per worker, r_t is the interest rate and w_t the wage rate.

At time t young population of measure N_t grows at the constant rate $\frac{N_t}{N_{t-1}} - 1 = \overline{n} - 1$, where $\overline{n}$ is the exogenously given number of children per each individual (by assuming, as usual, the existence of a single parent only). The government redistributes from the young to the old with unfunded public pensions. Therefore, the pension expenditure at t ($P_t = p_t N_{t-1}$) is constrained by the amount of tax receipts $\theta w_t N_t$, where $0 < \theta < 1$ is the (wage) contribution rate to the PAYG system. The (per pensioner) budget constraint of the government can then be written as:

$$p_t = \theta w_t \overline{n} . \tag{3}$$

Individuals are identical and live two periods in an OLG closed economy (Diamond, 1965): youth (working period) and old-age (retirement period). Agents of generation t draw utility (U_t) from young- and old-age consumptions, $c_{1,t}$ and $c_{2,t+1}$, respectively. Young individuals are endowed with one unit of labour inelastically supplied to firms and receive wage income w_t. Starting from the seminal papers by Diamond (1965), Samuelson (1975b) and Deardorff (1976), the economic literature did not consider children costs in models with exogenous fertility. However, whatever the reasons why a positive fertility rate exists, it is natural to conjecture that the bearing of children is costly even when fertility is not an economic decision variable (see Fanti and Gori 2011). In particular, in this paper we assume that the amount of resources needed to care for a child is $q w_t$, where $0 < q < 1$ is the percentage of child rearing cost on working income (see, e.g., Boldrin and Jones 2002). Therefore, the budget constraint of the young at t reads as

$$c_{1,t} + s_t + q w_t \overline{n} = w_t (1 - \theta), \tag{4}$$

i.e., the disposable income is used to consume, save (s_t) and take care of $\overline{n}$ descendants. Old individuals retire and live with the amount of resources saved when young plus the expected interest accrued from t to $t+1$ at the rate $r^e{}_{t+1}$ and the expected pension benefit, $p^e{}_{t+1}$. The budget constraint of the old both at $p^e{}_{t+1}$ therefore is:

$$c_{2,t+1} = \left(1 + r^{e}_{t+1}\right)s_t + p^{e}_{t+1}\,. \tag{5}$$

The typical agent at t chooses how much to save out of his/her disposable income to maximise the utility function

$$U_t = \ln\left(c_{1,t}\right) + \beta \ln\left(c_{2,t+1}\right), \tag{6}$$

subject to Eqs. (4) and (5), where $0 < \beta < 1$ is the subjective discount factor.[5] The saving function therefore is:

$$s_t = \frac{\beta w_t \left(1 - \theta - q\bar{n}\right)}{1+\beta} - \frac{p^{e}_{t+1}}{\left(1+\beta\right)\left(1 + r^{e}_{t+1}\right)}. \tag{7}$$

where $q < \left(1-\theta\right)/\bar{n}$. Now, inserting the one-period-forward pension accounting rule Eq. (3) into Eq. (7) to substitute out for p^{e}_{t+1} yields:

$$s_t = \frac{\beta w_t \left(1 - \theta - q\bar{n}\right)}{1+\beta} - \frac{\bar{n}\,\theta}{1+\beta} \cdot \frac{w^{e}_{t+1}}{1 + r^{e}_{t+1}}\,. \tag{8}$$

From Eq. (7) we see that saving is the result of two components: (*i*) a positive *private component* (the first term on the right-hand side of Eq. 7), which depends on the propensity to save out of wage income (net of contributes and children costs),[6] and (*ii*) a negative *public pension component* (the second term on the right-hand side of Eq. 7), because of the existence of a pension transfer to sustain consumption when old that induces the young to save less than in the absence of it.

Note that the negative effect of future pensions on current savings depends on the relative weight between the subjective discount rate (ρ), i.e. the individual's evaluation of future consumption, and the market interest rate (r), i.e. the market's evaluation of future consumption. The higher the relative importance of the former, the higher the negative effect of pensions on savings.

Given the government budget Eq. (3) and knowing that $N_{t+1} = \bar{n}\, N_t$, the equilibrium condition on the capital market can be expressed as:

$$\bar{n}\, k_{t+1} = s_t\,. \tag{9}$$

From Eqs. (8) and (9) we get:

$$k_{t+1} = \frac{\beta w_t(1-\theta-q\bar{n})}{(1+\beta)\bar{n}} - \frac{\theta}{1+\beta} \cdot \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}. \tag{10}$$

In an OLG economy the dynamic evolution of capital is different depending on whether expectations about factor prices are either rational or myopic (see, e.g., Michel and de la Croix, 2000; de la Croix and Michel, 2002; Chen et al., 2008).

While our economy does not exhibit any interesting dynamical events (i.e., the steady state is locally stable and the dynamics is always monotonic) if expectations are rational, non-monotonic dynamics and cyclical instability may occur if expectations are myopic. Therefore, below we concentrate on the case of short sighted agents to study the complex dynamical properties of this stylised economy.

2.1. Exogenous Fertility: Steady-State and Local Stability Analyses with Myopic Foresight

With myopic expectations both the expected interest and wage rates depend on the time-t stock of capital per worker, that is

$$\begin{cases} 1+r^e{}_{t+1} = \alpha A k_t^{\alpha-1} \\ w^e{}_{t+1} = (1-\alpha) A k_t^{\alpha} \end{cases}. \tag{11}$$

Therefore, exploiting Eqs. (1), (2), (10) and (11) the dynamic path of capital accumulation is:

$$k_{t+1} = \frac{1}{1+\beta}\left[\frac{\beta(1-\theta-q\bar{n})(1-\alpha)A}{\bar{n}} k_t^{\alpha} - \theta\frac{1-\alpha}{\alpha} k_t\right]. \tag{12}$$

Steady states of the phase map Eq. (12) are defined as $k_{t+1} = k_t = k^*$. Therefore, the unique steady state is:

$$k^* = \left\{\frac{\beta\alpha(1-\alpha)A(1-\theta-q\bar{n})}{\bar{n}[\alpha(1+\beta)+\theta(1-\alpha)]}\right\}^{\frac{1}{1-\alpha}}. \tag{13}$$

As expected, the equilibrium stock of capital negatively depends on the pension contribution rate because saving reduces for a twofold reason when θ raises: (*i*) the disposable income becomes lower because a larger fraction of wages is used for the payment of pensions; (*ii*) the substitution of private savings for public transfers to sustain consumption when old becomes more attractive.

From Eqs. (12) and (13), therefore, the following proposition therefore holds:

Proposition 1. *[Dynamics under exogenous fertility]. (1) Let* $0 < \alpha < \alpha_4$ *hold. Then* $\theta_1 < \theta_2 < 1$*, and (1.1) if* $0 < \theta < \theta_1$*, the dynamics of capital is monotonic and convergent to* k^**; (1.2) if* $\theta_1 < \theta < \theta_2$*, the dynamics of capital is oscillatory and convergent to* k^**; (1.3) if* $\theta = \theta_2$*, a flip bifurcation emerges;*[7] *(1.4) if* $\theta_2 < \theta < 1$*, the dynamics of capital is oscillatory and divergent to* k^**. (2) Let* $\alpha_4 < \alpha < \alpha_2$ *hold. Then* $\theta_1 < 1$*,* $\theta_2 > 1$*, and (2.1) if* $0 < \theta < \theta_1$*, the dynamics of capital is monotonic and convergent to* k^**; (2.2) if* $\theta_1 < \theta < 1$*, the dynamics of capital is oscillatory and convergent to* k^**. (3) Let* $\alpha_2 < \alpha < 1$ *hold. Then* $\theta_2 > \theta_1 > 1$ *and the dynamics of capital is monotonic and convergent to* k^* *for any* $0 < \theta < 1$*, where*

$$\theta_1 = \theta_1(\alpha, \beta) := \frac{\alpha^2(1+\beta)}{(1-\alpha)^2}, \tag{14}$$

$$\theta_2 = \theta_2(\alpha, \beta) := \theta_1 \frac{1+\alpha}{\alpha}, \tag{15}$$

$$\alpha_2 = \alpha_2(\beta) := \frac{-1+\sqrt{1+\beta}}{\beta} > 0, \quad 0 < \alpha_2 < 1/2, \tag{16}$$

$$\alpha_4 = \alpha_4(\beta) := \frac{-(3+\beta)+\sqrt{\beta^2+10\beta+9}}{2\beta} > 0, \quad 0 < \alpha_4 < 1/3. \tag{17}$$

Proof. See Appendix 1.

Indeed, a rise in the capital stock causes a twofold effect. It increases saving because the current wage raises, while also decreasing it because of the rise in the present value of next period transfers. Moreover, the lower the capital share in production and the subjective discount factor (or, alternatively, the higher the relative weight of both the labour income in production and public pension component in capital accumulation), the likely a rise in θ causes cyclical instability because the relative weight of the public pension component increases.

3. Endogenous Fertility

In this section we assume that individuals enjoy to have children and choose the number of descendants (the so-called weak form of altruism, see Zhang and Zhang 1998) by comparing between benefits and costs of upbringing. The model is outlined below.

The public pension budget and the individual budget constraints when young and old are still determined by Eqs. (3), (4) and (5), respectively, with the only difference that the number of children is now an economic decision variable, and, as can easily be ascertained below (see Eq. 19), it depends on both the wage and interest rates, which in turn depend on the dynamic variable k_t. As a consequence, the number of children is no longer constant.

Individuals have preferences towards material consumption over the life cycle and the number of children, as in Eckstein and Wolpin (1985) and Galor and Weil (1996). The typical agent that enters the working period at t chooses fertility and saving to maximise the utility function

$$U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}) + \phi \ln(n_t), \tag{18}$$

subject to Eqs. (4) and (5), where $\phi > 0$ captures the parents' taste for children. The constrained maximisation of Eq. (18) gives:[8]

$$n_t = \frac{1}{1+\beta+\phi}\left[\frac{\phi(1-\theta)}{q} + \frac{\phi p^e{}_{t+1}}{q w_t (1+r^e{}_{t+1})}\right], \tag{19}$$

$$s_t = \frac{1}{1+\beta+\phi}\left[\beta w_t (1-\theta) - \frac{(1+\phi) p^e{}_{t+1}}{1+r^e{}_{t+1}}\right]. \tag{20}$$

Then, upon substitution of the one-period forward pension accounting rule Eq. (3) for $p^e{}_{t+1}$ into Eqs. (19) and (20), fertility and the saving function are definitely given, respectively, by:

$$n_t = \frac{\phi w_t (1-\theta)}{(1+\beta+\phi) q w_t - \phi\theta \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}}, \tag{19'}$$

$$s_t = \frac{w_t(1-\theta)}{(1+\beta+\phi) q w_t - \phi\theta \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}} \left(\beta q w_t - \phi\theta \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}} \right). \tag{20'}$$

It is important to note that the young choose fertility by taking the future level of pensions as given, that is they consider their own number of children to be too small to affect the size of the benefit they will receive when old. In other words, each individual does not internalise the government pension budget when chooses fertility and saving. In contrast, if the pension system were of the so-called "fertility-related" type (see, amongst others, Kolmar, 1997; Fenge and Meier, 2005), i.e. pension transfers are contingent on the individual number of children, then each young parent would take into account the influence of the government budget on saving and fertility decisions by internalising it.

From Eq. (19') we note that fertility is constant at $n_t = n = \frac{\phi}{q(1+\beta+\phi)}$ if $\theta = 0$, while becoming a time-dependent variable, which positively depends on capital accumulation, if $0 < \theta < 1$.

Using Eqs. (9), (19') and (20'), the equilibrium in the capital market is:

$$k_{t+1} = \frac{\beta}{\phi} q w_t - \theta \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}. \tag{21}$$

3.1. Endogenous Fertility: Steady-State and Local Stability Analyses with Myopic Foresight

Exploiting Eqs. (1), (2), (11) and (21), the dynamics of capital becomes:

$$k_{t+1} = \frac{\beta}{\phi} q (1-\alpha) A k_t^{\alpha} - \theta \frac{1-\alpha}{\alpha} k_t. \tag{22}$$

Steady-state now implies

$$k^* = \left\{ \frac{q\beta\alpha(1-\alpha)A}{\phi[\alpha + \theta(1-\alpha)]} \right\}^{\frac{1}{1-\alpha}}. \tag{23}$$

Though Eq. (23) reveal that a rise in θ negatively affects capital accumulation also in the case of endogenous fertility, the intensity of such a negative effects is different depending on whether fertility is exogenous or endogenous. Comparison of Eqs. (13) and (23), therefore, gives the following remark.

Remark 1. *Ceteris paribus, the negative effect of a rise in the pension contribution rate on the long-run economic growth is smaller under endogenous fertility.*

Indeed, a rise in the contribution rate causes a negative income effect that reduces the disposable income irrespective of whether fertility is exogenous or endogenous. However, while in the former case such a negative effect only reduces savings, in the latter case it causes the same identical reduction in both savings and fertility, and then the negative income effect on capital accumulation is sterilised in such a case. An illustration of this result is postponed to Section 4.2.

We now proceed with the steady-state analysis of the relationship between fertility and the pension contribution rate. Using Eqs. (1), (2), (19'), (20') and (23), the long-run fertility rate can be written, after some straightforward algebra, as:

$$n^*(\theta) = \frac{\phi(1-\theta)[\alpha + \theta(1-\alpha)]}{q\{\alpha\beta + (1+\phi)[\alpha + \theta(1-\alpha)]\}}. \tag{24}$$

We now analyse how the long-run fertility rate reacts to a change in θ.[9] First, we define the following threshold value of the output elasticity of capital:

$$\hat{\alpha} = \frac{\beta}{1 + 2\beta + \phi} < \frac{1}{4}, \tag{25}$$

for any $0 < \phi < 1$. Then the following proposition holds.

Proposition 2. *(1) Let* $0<\alpha<\hat{\alpha}$ *hold. Then* $n^{*}(\theta)$ *is inverted U-shaped with* $\theta=\hat{\hat{\theta}}$ *being the fertility-maximising contribution rate. (2) Let* $\hat{\alpha}<\alpha<1$ *hold. Then a rise in* θ *always causes the fall of fertility in the long run.*

Proof. The proof uses the following derivative:

$$\frac{\partial n^{*}(\theta)}{\partial\theta}=\frac{\phi\left\{-(1-\alpha)^{2}(1+\phi)\theta^{2}-2\alpha(1-\alpha)(1+\beta+\phi)\theta-\alpha[\alpha(1+2\beta+\phi)-\beta]\right\}}{q\{\alpha\beta+(1+\phi)[\alpha+\theta(1-\alpha)]\}^{2}}. \tag{26}$$

Solving the numerator of Eq. (26) with respect to θ gives:

$$\theta=\hat{\theta}:=-\frac{\alpha(1+\beta+\phi)+\sqrt{\alpha\beta(1+\alpha\beta+\phi)}}{(1-\alpha)(1+\phi)}<0, \tag{27}$$

$$\theta=\hat{\hat{\theta}}:=\frac{-\alpha(1+\beta+\phi)+\sqrt{\alpha\beta(1+\alpha\beta+\phi)}}{(1-\alpha)(1+\phi)}, \tag{28}$$

with $\hat{\hat{\theta}}>\hat{\theta}$. Then, by applying the Descartes' rule of sign we find that: (*i*) if $0<\alpha<\hat{\alpha}$, then $\hat{\theta}<0$ and $0<\hat{\hat{\theta}}<1$, so that Eq. (27) can be ruled out. Therefore, $\frac{\partial n^{*}(\theta)}{\partial\theta}\gtrless 0$ if and only if $\theta\lessgtr\hat{\hat{\theta}}$ with $0<\hat{\hat{\theta}}<1$ being an interior global maximum. This proves Point (1); (*ii*) if $\hat{\alpha}<\alpha<1$, then $\hat{\theta}<0$ and $\hat{\hat{\theta}}<0$. Therefore, $\frac{\partial n^{*}(\theta)}{\partial\theta}<0$ for any $0<\theta<1$. This proves Point 2 and Proposition 2 follows. **Q.E.D.**

Note that the output elasticity of capital $\hat{\alpha}$ is found to be fairly below 0.2 for a wide range of values of both the subjective discount factor β and taste for children ϕ, and thus the restriction given by Eq. (25) is not satisfied for some realistic values of the output elasticity of capital recently observed in several developed countries, which is generally not lower than 0.25 (see, e.g., Gollin, 2002). Therefore, we may reasonably conclude raising PAYG transfers in actual economies reduces fertility.[10]

As regards local stability, the analysis of Eqs. (22) and (23) gives the following proposition:

Proposition 3. *[Dynamics under endogenous fertility]. (1) Let* $0 < \alpha < 1/3$ *hold. Then* $\theta_3 < \theta_4 < 1$, *and (1.1) if* $0 < \theta < \theta_3$, *the dynamics of capital is monotonic and convergent to* k^*; *(1.2) if* $\theta_3 < \theta < \theta_4$, *the dynamics of capital is oscillatory and convergent to* k^*; *(1.3) if* $\theta = \theta_4$, *a flip bifurcation emerges; (1.4) if* $\theta_4 < \theta < 1$, *the dynamics of capital is oscillatory and divergent to* k^*. *(2) Let* $1/3 < \alpha < 1/2$ *hold. Then* $\theta_3 < 1$, $\theta_4 > 1$, *and (2.1) if* $0 < \theta < \theta_3$, *the dynamics of capital is monotonic and convergent to* k^*; *(2.2) if* $\theta_3 < \theta < 1$, *the dynamics of capital is oscillatory and convergent to* k^*. *(3) Let* $1/2 < \alpha < 1$ *hold. Then* $\theta_4 > \theta_3 > 1$, *and the dynamics of capital is monotonic and convergent to* k^* *for any* $0 < \theta < 1$, *where*

$$\theta_3 = \theta_3(\alpha) := \frac{\alpha^2}{(1-\alpha)^2}, \tag{29}$$

$$\theta_4 = \theta_4(\alpha) := \theta_3 \frac{1+\alpha}{\alpha}. \tag{30}$$

Proof. See Appendix 2.

Similar to the case of exogenous fertility, large PAYG transfers may destabilise the economy when the output elasticity of capital is fairly low. However, we note the subjective discount rate now plays no role on stability. Indeed, comparison of Propositions 1 and 3 reveals that the stability properties dramatically change depending on whether fertility is endogenous or exogenous, as shown in the next section.

4. Exogenous Versus Endogenous Fertility: Some Analytical and Numerical Results

In order to compare exogenous and endogenous fertility economies focusing specifically on dynamical features, we now assume that the same steady-state number of children is raised in both contexts. This amounts to say that the same unique steady-state capital stock is achieved.[11] First of all, it is important to remark that:

Remark 2. *When PAYG pensions are absent ($\theta = 0$) and the number of children is the same under exogenous and endogenous fertility, the dynamic paths of capital accumulation and the steady states coincide.*

In contrast, when PAYG pensions exist ($0 < \theta < 1$), we find that the same positive steady-state is achieved but with two different dynamic adjustment processes. Moreover, the following proposition holds:

Proposition 4. *The risk of cyclical instability caused by PAYG transfers is higher under endogenous fertility. Moreover, under endogenous (exogenous) fertility the subjective discount factor is stability-neutral (acts as an economic stabiliser).*

Proof. Under exogenous (endogenous) fertility cyclical instability emerges only when $0 < \alpha < \alpha_4$ and $\theta_2 < \theta < 1$ ($0 < \alpha < 1/3$ and $\theta_4 < \theta < 1$). As $\alpha_4 < 1/3$, then $\theta_4 < \theta_2 := \theta_4(1+\beta)$. Moreover, $\frac{\partial \theta_2}{\partial \beta} = \theta_4 > 0$ and $\frac{\partial \theta_4}{\partial \beta} = 0$. Then Proposition 4 follows. **Q.E.D.**

Indeed, when fertility is exogenous, a rise in β reduces the relative weight of the public pension component on savings and this, in turn, tends to mitigate the negative effect of a higher contribution rate on capital accumulation (see Eq. 12). In contrast, when fertility is endogenous, the subjective discount factor does not alter capital accumulation through the public pension component (see Eq.

22), because it negatively affects in the same way both savings and the demand for children. Therefore, the cyclical unstable region in the parameter space (α,θ) is larger when fertility is endogenous.

[Figure 1 about here]

Figure 1 illustrates the content of Proposition 4 and displays the different size of the cyclically unstable regions under exogenous and endogenous fertility.

In the next sections (4.1 and 4.2) we compare the dynamic evolution of demo-economic variables and long-run outcomes in both economies when θ changes.

4.1. Change in PAYG Pensions When the Number of Children is the Same Under Exogenous and Endogenous Fertility

While the previous section has presented some analytical results, the quantitative implications will be better seen through numerical simulations, which can help to reveal other effects of social security whereby ambiguity exists in the analysis (e.g., the effects of the pension contribution rate on the population growth).

To illustrate the different dynamic adjustment processes under exogenous and endogenous fertility, we take the following parameter set: $A=10$, $\alpha=0.2$, $\beta=0.4$ for both economies. Moreover, without loss of generality we assume $\overline{n}=1$, i.e. stationary population. Then we choose $\phi=0.3$ and calibrate q (for different values of θ) to obtain $n^{*}=1$ under endogenous fertility. In particular, we assume $\theta=0.05$ (0.35) $[0.46]$ to obtain $q=0.1745$ (0.133) $[0.1124]$. The simulations results are summarised in Result 1 and Figures 2-4.

Result 1. *Given the same initial condition,* k_0*, economic and demographic variables under exogenous and endogenous fertility may show the following three distinct dynamical behaviours, depending on the relative size of the PAYG system.*

(1) Monotonic and convergent dynamics (the pension contribution rate is fairly low). Figure 2.1 illustrates the phase map under exogenous and endogenous fertility when $\theta = 0.05$ *and* $q = 0.1745$*. Figure 2.2 represents the corresponding time plot for* n_t *(*$k_0 = 0.01$ *and* $n_0 = 0.7$*) showing that (under endogenous fertility) population increases in the first stages of development and then approaches the stationary state.*

(2) Non-monotonic dynamics, which may result in: (2.i) non-monotonic convergence towards the stationary state in capital and fertility (intermediate values of the contribution rate). Figure 3.1 displays the phase map of capital accumulation, which is uni-modal in both cases, when $\theta = 0.35$ *and* $q = 0.133$*. The steady-state is approached with dampened oscillations (of course, given the different slopes of the phase maps at the equilibrium point, fluctuations in demo-economic variables under endogenous fertility are larger than under exogenous fertility, as shown in the corresponding time plot for both* n_t *and* k_t*, see Figures 3.2, 3.3 and 3.4, where* $k_0 = 0.01$ *and* $n_0 = 0.7$*); (2.ii) irregular fluctuations in capital and fertility (the pension contribution rate is fairly high). Figure 4.1 displays the phase map when* $\theta = 0.46$ *and* $q = 0.1124$*. The corresponding time plot for* n_t *and* k_t *reveal that oscillations are permanent also in the very long run when fertility is endogenous, while converging towards the stationary state when fertility is exogenous (see Figures 4.2, 4.3 and 4.4, where* $k_0 = 0.01$ *and* $n_0 = 0.7$*).*

[Figures 2.1 and 2.2 about here]

[Figures 3.1, 3.2, 3.3 and 3.4 about here]

[Figures 4.1, 4.2, 4.3 and 4.4 about here]

The economic intuition is the following. In the first stages of development saving grows faster than fertility, the latter being univocally of the Malthusian type (endogenous fertility), i.e. the higher income, the higher the number of children. When economies develop, however, the reduction in saving is, on average, larger than that of fertility and fluctuations are larger under endogenous fertility. It is important to note that when θ becomes larger and families rationally compares benefits and costs of having children, we can observe stages of development in which fertility subsequently increases and decreases. Moreover, the intensity of theses fluctuations either shrinks as the steady state is approached to or indefinitely persist. In the latter case (the contribution rate is fairly high), our model predicts economic cycles as well as persistent and marked baby booms and baby busts, which may be of irregular (bounded) amplitude and length (see Figures 4.2 and 4.4). This behaviour accords with (*i*) the observed income fluctuations over time, (*ii*) the baby boom and baby bust observed in the last century, especially in Europe and the United States (see Greenwood et al., 2005), and (*iii*) the persistent and steadily reduction in fertility experienced in the Western world (see, e.g., Murphy et al, 2008 as regards the U.S.). Notice that if fertility were exogenous, then *ceteris paribus* not only baby booms and baby busts, but also economic cycles would be prevented (see Figure 4.3).

Therefore the existence of large PAYG transfers is sufficient to generate realistic predictions about demo-economic outcomes in an OLG model with endogenous fertility, such as for instance the observed phase of fertility drop (which should obviously be seen as a temporary, though intense and possibly long lasting, event), which has otherwise been explained, for instance, by the quantity-quality theory of fertility (e.g., Becker and Lewis, 1973; Becker and Tomes, 1976; Barro and Becker, 1989) or by the theory of female labour participation (e.g., Mincer, 1962).

4.2. Change in PAYG Pensions: Policy Experiment

In the current political debate in several developed countries there is wide consensus to increase pension contributions with the aim to balance public pension budgets which are currently under strain. Which are the effects of an increase in pension contributions to the time evolution of capital accumulation and fertility?

Taking seriously into account some forecasts on the social security contribution rate, such as Liikanen (2007) (see Footnote 4), we now conduct a policy experiment (by using the same values of technological and preference parameters as in Section 4.1) to compare the short- and long-run outcomes in fertility and capital corresponding to a change from $\theta = 0.1$ to $\theta = 0.3$ at a certain point in time ($t = 2$) in models with exogenous and endogenous fertility. We start by assuming $\overline{n} = n^* = 1$ and $q = 0.1702$ in both economics. Note that before the change in the social security contribution rate, the steady-state stock of capital under exogenous and endogenous fertility is the same (see Figures 5.2 and 5.3).

Fertility. Figure 5.1 represents the time plot for n_t portrayed from $t = 2$ (the time of the change in θ) onwards. Note that the upper curve ($\theta = 0.1$) started out at $t = 0$ (not shown in the figure) with $k_0 = 0.01$ and $n_0 = 0.7$ as initial conditions. Then we assume $k_2 = 1.42$ (see Figure 5.3) as the initial condition for k at the time of the change in θ. Case $\theta = 0.1$: after a few oscillations, up to almost the third generation, fertility approaches the stationary state ($n = 1$). Case $\theta = 0.3$: the rise in θ causes an 18 percent reduction in fertility (i.e., almost 2 children per couple when $\theta = 0.1$ against 1.65 children per couple when $\theta = 0.3$) followed by phases of baby booms and baby busts, dampened over dozen generations, until the new and lower long-run fertility rate is approached to.

Capital. Figures 5.2 and 5.3 show that the capital stock installed at $t = 2$ when $\theta = 0.1$ is $k_2 = 1.35$ ($k_2 = 1.42$) under exogenous (endogenous) fertility. The rise in θ causes: (*i*) in the short run, $t = 2$, a 48 (24) percent reduction in the capital stock under exogenous (endogenous) fertility. This because the reduction in saving is mitigated by the reduced number of children when fertility is endogenous; (*ii*) in the long run, phases of small and fast (wide and slow) oscillations under exogenous (endogenous) fertility until the new low (high) steady state is achieved around the seventh (thirtieth) generations.

[Figures 5.1, 5.2 and 5.3 about here]

5. Conclusions

This paper dealt with the study of the dynamical and steady-state outcomes of a double Cobb-Douglas OLG economy with public PAYG pensions and short-sighted individuals under exogenous and endogenous fertility. We showed, *ceteris paribus*, that an economy is likely to be exposed to endogenous fluctuations when fertility is an economic decision variable. Indeed, in such a case our model generates a population dynamics that mimic: (*i*) the historical evolution of population with an increasing trend with increasing income (i.e., the Malthusian-type fertility observed in the first stages of development),[12] and (*ii*) phases of baby booms and baby busts before either approaching the steady state level of fertility or permanently oscillating around it. Indeed, such oscillations may be consistent, in a broad sense, with the baby boom and the subsequent fertility drop observed in some actual developed economies in the last sixty years (see, e.g., Greenwood et al., 2005). Moreover, the existence of an endogenous population dynamics allows for a novel realistic explanation of regular and even chaotic demo-economic cycles. Therefore, to the extent that fertility choices are based on a comparison between benefits and costs of children, the effects of PAYG pensions may be markedly different from those expected in the case in which fertility is determined out of the economic sphere.

As the rational choice of fertility may be considered going hand to hand with the stages of economic development,[13] and social security reforms are currently high on the political agenda in several developed economies, our findings constitute a policy advice about the economic and demographic effects of raising pension transfers when fertility is endogenous.

Appendix 1. Proof of Proposition 1

Differentiating Eq. (12) with respect to k_t and using Eq. (13) we get:

$$\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}=\alpha-\frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha}. \tag{A.1}$$

Monotonic and non-monotonic dynamics with exogenous fertility. From Eq. (A.1), $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}\underset{<}{\overset{>}{-}}0$ implies

$$\alpha-\frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha}\underset{<}{\overset{>}{-}}0\Leftrightarrow\theta\underset{>}{\overset{<}{-}}\theta_1, \tag{A.2}$$

so that $\theta_1<1$ ($\theta_1>1$) for any $0<\alpha<\alpha_2$ ($\alpha_2<\alpha<1$). Moreover, $\theta_1<1$ if and only if $\alpha_1<\alpha<\alpha_2$, where $\alpha_1=\alpha_1(\beta):=\frac{-1-\sqrt{1+\beta}}{\beta}<0$ and $0<\alpha_2<1/2$ is given by Eq. (16). Since $\alpha_1<0$ it is ruled out. Now, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ gives

$$\alpha-\frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha}<1\Rightarrow\theta>-\frac{\alpha(1+\beta)}{1-\alpha}. \tag{A.3}$$

Therefore, in the case of monotonic behaviour, $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ and thus the dynamics of capital converges towards the stationary state.

Non-monotonic dynamics: stability analysis with exogenous fertility. From Eq. (A.1), $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}\underset{<}{\overset{>}{-}}-1$ implies

$$\alpha-\frac{\theta}{1+\beta}\cdot\frac{(1-\alpha)^2}{\alpha}\underset{<}{\overset{>}{-}}-1\Rightarrow\theta\underset{>}{\overset{<}{-}}\theta_2, \tag{A.4}$$

where $\theta_2>\theta_1$, so that $\theta_2<1$ ($\theta_2>1$) for any $0<\alpha<\alpha_4$ ($\alpha_4<\alpha<1$), with $\alpha_4<\alpha_2$. Moreover, $\theta_2<1$ if and only if $\alpha_3<\alpha<\alpha_4$, where $\alpha_3=\alpha_3(\beta):=\frac{-(3+\beta)-\sqrt{\beta^2+10\beta+9}}{2\beta}<0$ and $0<\alpha_4<1/3$ is given by Eq. (17). Since $\alpha_3<0$ it is ruled out. Therefore, (*i*) if $0<\alpha<\alpha_4$ then

$\theta_1 < \theta_2 < 1$ and (1.1) $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < \theta_1$, (1.2) $-1 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 0$ for any $\theta_1 < \theta < \theta_2$, (1.3) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} = -1$ if and only if $\theta = \theta_2$, and (1.4) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < -1$ for any $\theta_2 < \theta < 1$. This proves point (1); (*ii*) if $\alpha_4 < \alpha < \alpha_2$ then $\theta_1 < 1$, $\theta_2 > 1$ and (2.1) $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < \theta_1$ and (2.2) $-1 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 0$ for any $\theta_1 < \theta < 1$. This proves point (2); (*iii*) if $\alpha_2 < \alpha < 1$ then $\theta_2 > \theta_1 > 1$ and $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < 1$. This proves point (3) and Proposition 1 follows.

Appendix 2. Proof of Proposition 3

Differentiating Eq. (22) with respect to k_t and using Eq. (23) in the main text gives:

$$\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} = \alpha - \theta \frac{(1-\alpha)^2}{\alpha}. \tag{B.1}$$

Monotonic and non-monotonic dynamics with endogenous fertility. From Eq. (B.1), $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} 0$ implies

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} 0 \Leftrightarrow \theta \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \theta_3, \tag{B.2}$$

so that $\theta_3 < 1$ ($\theta_3 > 1$) for any $0 < \alpha < 1/2$ ($1/2 < \alpha < 1$). Now, $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ yields

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} < 1 \Rightarrow \theta > -\frac{\alpha}{1-\alpha}. \tag{B.3}$$

Therefore, in the case of monotonic behaviour, $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ and the dynamics of capital converges to the steady state.

Non-monotonic dynamics: stability analysis with endogenous fertility. From Eq. (A.5), $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} \gtrless -1$ implies

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \gtrless -1 \Rightarrow \theta \lesseqgtr \theta_4, \tag{B.4}$$

where $\theta_4 > \theta_3$, so that $\theta_4 < 1$ ($\theta_4 > 1$) for any $0 < \alpha < 1/3$ ($1/3 < \alpha < 1$). Therefore, (*i*) if $0 < \alpha < 1/3$ then $\theta_3 < \theta_4 < 1$ and (1.1) $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < \theta_3$, (1.2) $-1 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 0$ for any $\theta_3 < \theta < \theta_4$, (1.3) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} = -1$ if and only if $\theta = \theta_4$, and (1.4) $\frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < -1$ for any $\theta_4 < \theta < 1$. This proves point (1); (*ii*) if $1/3 < \alpha < 1/2$ then $\theta_3 < 1$, $\theta_4 > 1$ and (2.1) $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < \theta_3$ and (2.2) $-1 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 0$ for any $\theta_3 < \theta < 1$. This proves point (2); (*iii*) if $1/2 < \alpha < 1$ then $\theta_4 > \theta_3 > 1$ and $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t=k^*} < 1$ for any $0 < \theta < 1$. This proves point (3) and Proposition 3 follows.

Appendix 3. Deterministic Chaos

The parameter set used in Section 4.1 generates the following flip bifurcation values of θ: $\theta_2 = 0.525$ (exogenous fertility) and $\theta_4 = 0.375$ (endogenous fertility). These critical values of the pension contribution rate are those beyond which the stable equilibrium becomes unstable through oscillations, and then a branch of additional stable or unstable equilibria of order 2. If the equilibrium is stable (unstable), the flip bifurcation is said to be super-critical or sub-critical. If the flip bifurcation is super-critical, the equilibrium point of order 2 repeats itself every two periods and a cycle of period 2 observed. This is the reason why a flip bifurcation is often called "period-doubling" bifurcation. Increasing further θ beyond either θ_2 (exogenous fertility) or θ_4

(endogenous fertility), implies that the initially stable two-period cycle looses stability at a certain new critical value of the contribution rate, where a new stable four-period cycle is created through flip bifurcation. This sequence of events can then be repeated leading to an infinite sequence of flip bifurcations and period-doubling cascades. Hence, the period-doubling bifurcation is also called “route to chaos”.

As the analytical proof of the existence of subsequent flip bifurcations and period-doubling cascades is beyond the scope of the present paper (given the economical rather than mathematical motivation of it), we resort to numerical experiments to show that deterministic chaos can occur. From Figures C.1, C.2, 4.2 and 4.4 random-like fluctuations in capital and fertility are quite evident. However, from a technical point of view we do not know whether this behaviour is either chaotic or only of high periodicity. Therefore, in order to check for the existence of deterministic chaos we use a theorem originally proposed by Li and Yorke (1975): if a cycle of period three is detected then chaos in the sense of Li and Yorke exists (*topological chaos*). For an example of an application of the Li and Yorke’s theorem in a one-dimensional OLG economy see, amongst many others, Zhang (1999).

Given the flip bifurcation values $\theta_2 = 0.525$ and $\theta_4 = 0.375$, we assume $\theta = 0.46$, that is the dynamics under exogenous (endogenous) fertility is cyclically stable (unstable). Though the pension contribution rate ($\theta = 0.46$), the initial ($k_0 = 0.568$) and steady state ($k^* = 0.3404$) stocks of capital are the same in both economies, Figures C.1 and C.2 reveal that the dynamic adjustment processes are dramatically different, and, *ceteris paribus*, showing the cyclical stable (unstable) non-monotonic trajectory under exogenous (endogenous) fertility. The difference is explained by the stabilising effect exerted by the subjective discount factor in reducing the relative weight of the public pension component when fertility is exogenous, which is instead absent under endogenous fertility: because the negative weight on savings and the positive weight on fertility of the subjective discount factor through public pensions are the same, capital accumulation, defined as the ratio between saving and fertility, is unaffected by β. Indeed, while the relative weight of the positive component in capital accumulation is higher under endogenous fertility, the negative effect of θ on the public pension component is not mitigated by the coefficient β. Then the weight of the destabilising component is higher (lower) under endogenous (exogenous) fertility, see Eq. (12) versus Eq. (22).

Figure C.3 is an application of the standard version of the Li and Yorke's (1975) theorem and shows that the dynamics under endogenous fertility (see Figure C.2) is chaotic. First, we define the right-hand side of Eq. (22) as $f(k_t)$. As known, Li and Yorke showed that $f(3) > k_0 > f(1) > f(2)$ is a sufficient condition for the occurrence of topological chaos, where $f(.)$ indicates the number of iterates of the function f. Figure C.3, therefore, displays three iterations of $f(k_t)$ that satisfy the Li and Yorke's theorem. Assume that the initial condition is $k_0 = 0.568$. Then, $f(1) = 0.529$, $f(2) = 0.081$, $f(3) = 0.576$. Therefore, the following result holds:

Result C.1. *In an economy with endogenous fertility and pay-as-you-go public pensions, deterministic chaos in income and fertility can occur when PAYG transfers are fairly high.*

To the extent that demo-economic variables show irregular fluctuations, the proof of the existence of chaotic dynamics represents a relevant result that can also have interesting policy implications.

[Figure C.1, C.2 and C.3 about here]

Notes

1. This is actually known for several extensions of the basic neoclassical OLG growth model. For instance, Michel and Pestieau (1999, 2) emphasise that the OLG model with endogenous labour supply "has not been used to study the

interaction between social security and retirement, the reason for the omission being in the analytical difficulty of studying the dynamics of overlapping-generations models with endogenous labor supply.", and then they use a double Cobb-Douglas framework.

2. It is important to note, however, the use of a Constant Elasticity of Substitution (CES) production function does not alter any of the conclusions of the present paper.

3. For instance, Epstein and Zin (1991) find values of the relative risk aversion around unity and then consistent with the logarithmic utility function. As regards technology, a wide consensus exists that argues that the use of the Cobb-Douglas function represents a good approximation for the real data (see, e.g., Mankiw et al. 1992; Jones, 2005).

4. For instance, Liikanen (2007, 4) argues that "if pension systems are not radically reformed, there will be a dramatic rise in the scale of the pension contributions levied to finance pay-as-you-go schemes. The Bank of Finland's calculations indicate that pension contributions in Europe would rise from their present level of around 16% of aggregate wages to around 28% by the year 2040."

5. Defining $\rho > 0$ as the subjective discount rate, the discount factor can then be written as $\beta = 1/(1+\rho)$.

6. The propensity to save is constant and defined as $\beta/(1+\beta)$. It positively depends on β which also measures the individual degree of "parsimony" or "thriftiness".

7. Note that the existence of flip bifurcation causes the emergence of a (stable or unstable) cycle of period 2. This phenomenon (common to several maps) represents the first step towards the so-called period-doubling cascade, namely the appearance of more complex and possibly chaotic phenomena.

8. From Eq. (19) we note that when $\theta = 0$ the fertility rate is constant. In such a case, therefore, the demo-economic outcomes of an economy with endogenous fertility are similar as those in the case with exogenous fertility.

9. A simple inspection of Eq. (19) reveals that this relationship may be ambiguous. A rise in θ, in fact, reduces the disposable income and increases the pension transfer. The former (latter) effect works as a fertility-reducing (enhancing) device.

10. This result is in accord with the view that large PAYG pensions may crowd out fertility. Indeed, Cigno (2007, 38) claim that "… it would thus seem that public pensions displace fertility… rather than voluntary saving. This makes public pension systems, all essentially pay-as-you-go, intrinsically unstable."

11. This can easily be ascertained by assuming $\overline{n} = n^* = \dfrac{\phi(1-\alpha)A(k^*)^\alpha(1-\theta)}{(1+\beta+\phi)q(1-\alpha)A(k^*)^\alpha - \phi\theta\dfrac{1-\alpha}{\alpha}k^*}$.

Substituting it into Eq. (13) to eliminate n and solving for k^* we get Eq. (23).

12. We note that the assumption of exogenous fertility (usual in OLG growth models) is at odds not only with the historical evolution of population but also with the economic sense, because a positive population growth rate is postulated without the corresponding resources for child bearing.

13. In fact, as argued by van Groezen et al. (2003, 237) "… at least in western countries, people are to a very large extent able to freely choose the number of children they desire. The rate of fertility should therefore be treated as an endogenous variable, that is, as the result of a rational choice which is influenced by economic constraints and incentives."

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