The dynamics of a differentiated duopoly with quantity competition

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Abstract We analyse the dynamics of a Cournot duopoly with heterogeneous players to investigate the effects of micro-founded differentiated products demand. The present study, which indeed modifies and extends Zhang et al. (2007) (Zhang, J., Da, Q., Wang, Y., 2007. Analysis of nonlinear duopoly game with heterogeneous players. Economic Modelling 24, 138–148) and Tramontana, F., (2010) (Tramontana, F., 2010. Heterogeneous duopoly with isoelastic demand function. Economic Modelling 27, 350–357), reveals that a higher degree of product differentiation may destabilise the market equilibrium. Moreover, we show that a cascade of flip bifurcations may lead to periodic cycles and ultimately chaotic behaviours. Since a higher degree of product differentiation implies weaker competition, then a theoretical implication of our findings, that also constitute a policy warning, is that a fiercer (weaker) competition tends to stabilise (destabilise) the unique positive Cournot-Nash equilibrium of the economy.

Keywords Bifurcation; Chaos; Cournot; Oligopoly; Product differentiation

JEL Classification C62; D43; L13

1. Introduction

In this paper we analyse the dynamics of a Cournot duopoly within the framework developed by the recent literature (see, Bischi et al. 2010) that studies the dynamics of oligopoly models

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based on expectations different from the simple naïve formation mechanism implicit in the original model by Cournot (1838). In particular, we consider differentiated products and focus on the dynamic role played by the degree of product differentiation (see the original contributions by Hotelling, 1929, and Chamberlin, 1933, for the notion of differentiated goods and services).

While Cournot (1838) considers a duopoly with a single homogenous product, more recently the economic literature offered duopoly models with differentiated products, see for instance, the works by Dixit (1979) and Singh and Vives (1984), which allow goods and services to be substitutes or complements, in models with a standard linear demand structure.

As is known, the forecasts as regards the behaviour of the competitor in a duopoly game are crucial in order to make the optimal (rational) output choice. The pioneering work by Cournot (1838) introduced the first formal theory of oligopoly in economics, and treated the case with naive expectations, so that in every step each player assumes the last values taken by competitors without any forecasts about their future reactions.

Recently, several works have considered more realistic mechanisms through which players form their expectations on the decisions of the competitors, and have shown that the Cournot model may lead to periodic cycles and deterministic chaos. While several articles (see, e.g., Kopel, 1996; Agiza, 1999); Bischi and Kopel, 2001; Agliari et al., 2005, 2006) assume that both duopolists adopt the same decision mechanism as regards expectation formation (i.e. the case of homogeneous players), another branch of literature exists where firms are assumed to have heterogeneous expectations (Leonard and Nishimura, 1999; Den Haan, 2001; Agiza et al., 2002; Agiza and Elsadany, 2003, 2004; Zhang et al., 2007; Tramontana, 2010). In particular, the present paper is strictly related to Zhang et al. (2007) and Tramontana (2010) and analyses a Cournot duopoly game with heterogeneous players. However, in contrast with these two works, which consider a market for single homogenous product, we introduce a micro-economic founded demand of differentiated goods and services, which may be substitutes or complements between them. Other differences that distinguish the present study with those of the existing literature are the following: (*i*) production costs are assumed, as in Tramontana (2010), to be linear to simplify the analysis, while Zhang et al. (2007) assume non-linear (quadratic) costs, and (*ii*) a system of linear demand, as in Zhang et al. (2007), exists, while Tramontana (2010) assumes, following Puu (1991), a non-linear (isoelastic) demand system.

The horizontal differentiated duopoly considered here introduces microeconomic foundations proposed, among many others, by Singh and Vives (1984). Note that while the investigation of the static Cournot differentiated duopoly has produced a huge amount of works (see Footnote 1), less attention has been paid to the study of the dynamics in such a model. We aim therefore to fill this gap within the literature on nonlinear dynamic oligopolies.

The main result of the present analysis that an increase in product differentiation may destabilise the unique Cournot-Nash equilibrium: despite the rise in profits that an increase in the extent of product differentiation can lead to, it may also cause undesirable and unpredictable fluctuations, while contributing to reduce the market size. Moreover, from a mathematical point of view, we show that the destabilisation of the fixed point can occur through a flip bifurcation and also that a cascade of flip bifurcations may lead to periodic cycles and deterministic chaos.

The paper is organised as follows. Section 2 develops the model with micro-foundations of the differentiated products demand and presents the two-dimensional dynamic system of a duopoly game with heterogeneous expectations (bounded rational and naïve). Section 3 studies both the steady state and dynamics of the Cournot differentiated duopoly, showing explicit parametric conditions of the existence, local stability and bifurcation of the market equilibrium. Section 4 presents numerical simulations of the analytical findings, while also showing that complex behaviours through standard numerical tools (i.e., bifurcation diagrams, Lyapunov exponents, shape of the strange attractors and basins of attraction, sensitive dependence on initial conditions and fractal dimension of the chaotic attractor). Section 5 concludes.

2. The model

Since in the present study we concentrate on the effects on stability of horizontal product differentiation in a Cournot duopoly, it is of importance to set up the microeconomic foundations of the differentiated commodity setting and clarify the economic reasons why we assume specific demand and cost functions.

We assume the existence of an economy with two types of agents: firms and consumers. There exists a duopolistic sector with two firms, firm 1 and firm 2, and every firm i produces differentiated goods and services, whose price and quantity are given by p_i and q_i, respectively, with $i=\{1,2\}$. In addition to the duopolistic sector, a competitive sector that produces the numeraire good y exists.

We also assume the existence of a continuum of identical consumers which have preferences towards q and y represented by a separable utility function $V(q;y)$, which is linear in the numeraire good. The representative consumer maximises $V(q;y)=U(q)+y$ with respect to quantities subject to the budget constraint $p_1q_1+p_2q_2+y=M$, where $q=(q_1,q_2)$, q_1 and q_2 are non-negative and M denotes the consumer's exogenously given income. The utility function $U(q)$ is assumed to be continuously differentiable and satisfies the standard properties required in consumer theory (see, e.g., Singh and Vives, 1984, pp. 551–552). Since $V(q;y)$ is separable and linear in y, there are no income effects on the duopolistic sector. This implies that for a large enough level of income, the representative consumer's optimisation problem can be reduced to choose q_i to maximise $U(q)-p_1q_1-p_2q_2+M$. Utility maximization, therefore, yields the inverse demand functions (i.e., the price of good i as a function of quantities): $p_i=\frac{\partial U}{\partial q_i}=P_i(q)$, for $q_i>0$ and $i=\{1,2\}$. Inverting the inverse demand system above gives the direct demand functions (i.e., the quantity of good i as a function of prices): $q_i=Q_i(p)$, where $p=(p_1,p_2)$ and p_1 and p_2 are non-negative.

In order to have explicit demand functions for the goods and services of variety 1 and 2, a specific utility function should be assumed. We consider a simplified version of the model proposed by Singh and Vives (1984), which is usually adopted to represent a micro-founded demand system of differentiated products. On the demand side of the market, the representative consumer's utility is a quadratic function of two differentiated products, q_1 and q_2, and a linear function of a numeraire good, y.[1]

Therefore, we assume that preferences of the representative consumer over q are given by:

$$U(q_i,q_j)=a_iq_i+a_jq_j-\frac{1}{2}\left(\beta_iq_i^{\,2}+\beta_jq_j^{\,2}+2d\,q_iq_j\right), \tag{1}$$

[1] The quadratic utility function is the usual specification of preferences proposed by Dixit (1979) and subsequently used, amongst many others, by Singh and Vives (1984), Qiu (1997), Häckner (2000), Correa-López and Naylor (2004), Gosh and Mitra (2010), Fanti and Meccheri (2011). The important feature of such a utility function is that it generates a system of linear demand functions.

where $-1 < d < 1$ represents the degree of horizontal product differentiation. More in detail, if $d = 0$, then goods and services of variety 1 and 2 are independent. This implies that each firm behaves as if it were a monopolist in its own market; if $d = 1$, then products 1 and 2 are perfect substitutes or, alternatively, homogeneous; $0 < d < 1$ describes the case of imperfect substitutability between goods. The degree of substitutability increases, or equivalently, the extent of product differentiation decreases as the parameter d raises; a negative value of d instead implies that goods 1 and 2 are complements, while $d = -1$ reflects the case of perfect complementarity.

If $a_i \neq a_j$, then a demand asymmetry between firms i and j exists, which can be interpreted as a quality difference between products supplied by the two firms, as in Häckner (2000). This asymmetry implies a vertical (quality) differentiation between the two products. Since we are interested to exclusively analyse the dynamic role played by the degree of horizontal differentiation (i.e., the parameter d) we assume that $a_i = a_j = a$. Furthermore, we normalise the coefficients of the squared terms in the utility function (i.e., the slopes of the inverse demand functions) to unity, that is $\beta_i = \beta_j = 1$. Therefore, the present utility specification slightly differs from that adopted by Singh and Vives (1984), because the notation has been simplified without loss of generality.[2]

The inverse demand functions of products of variety 1 and 2 (as a function of quantities) that come from the maximisation by the representative consumer of Eq. (1) subject to the budget constraint $p_1 q_1 + p_2 q_2 + y = M$, are given by:

$$p_1(q_1, q_2) = a - q_1 - d q_2, \tag{2.1}$$

$$p_2(q_1, q_2) = a - q_2 - d q_1. \tag{2.2}$$

Following Correa-López and Naylor (2004) and Fanti and Meccheri (2011), we assume that firm i produces output of variety i through the following production function with constant (marginal) returns to labour: $q_i = L_i$, where L_i represents the labour force employed by the ith firm. Firms face the same (constant) average and marginal wage cost $w \geq 0$ for every unit of output produced. Therefore, the firm i's cost function is linear and described by:

$$C_i(q_i) = w L_i = w q_i. \tag{4}$$

Profits of firm i in every period can be written as follows:

$$\pi_i(q_i, q_j) = p_i(q_i, q_j) q_i - w q_i = \left[p_i(q_i, q_j) - w\right] q_i. \tag{5}$$

From the profit maximisation by firm $i = \{1,2\}$, marginal profits are obtained as:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = a - 2q_1 - d q_2 - w, \tag{6.1}$$

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = a - 2q_2 - d q_1 - w. \tag{6.2}$$

The reaction or best reply functions of firms 1 and 2 are computed as the unique solution of Eqs. (6.1) and (6.2) for q_1 and q_2, respectively, and they are given by:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 0 \Leftrightarrow q_1(q_2) = \frac{1}{2}\left[a - w - d q_2\right], \tag{7.1}$$

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = 0 \Leftrightarrow q_2(q_1) = \frac{1}{2}\left[a - w - d q_1\right] \tag{7.2}$$

Following Zhang et al. (2007) and Tramontana (2010) (to which we refer for details), we assume heterogeneous expectations: i.e., firm 1 (2) has bounded rational (naïve)

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[2] En passant, we note that this simplification is usual, e.g. Correa-López and Naylor (2004), Gosh and Mitra (2010), Fanti and Meccheri (2011).

expectations about the quantity to be produced in the future. Therefore, given these types of expectations formation mechanisms, the two-dimensional system that characterises the dynamics of the economy is the following:

$$\begin{cases} q_{1,t+1} = q_{1,t} + \alpha\, q_{1,t} \dfrac{\partial \pi_{1,t}}{\partial q_{1,t}} \ , \\ q_{2,t+1} = q_{2,t} \end{cases} \tag{8.1}$$

where $\alpha > 0$ is a coefficient that captures the speed of adjustment of firm 1's quantity at time $t+1$ with respect to a marginal change in profits when q_1 varies at time t. Using Eqs. (7.1), (7.2), the two-dimensional system Eq. (8.1) that characterises the dynamics a differentiated Cournot duopoly can alternatively be written as follows:

$$\begin{cases} q_{1,t+1} = q_{1,t} + \alpha\, q_{1,t}\left[a - 2q_{1,t} - d\, q_{2,t} - w\right] \\ q_{2,t+1} = q_{2,t} = \dfrac{a - w - d\, q_{1,t}}{2} \end{cases} . \tag{8.2}$$

From Eq. (8.1) it can be seen that the degree of horizontal product differentiation, d, plays a twofold role on marginal profits of firm 1 and then on the quantity it will produce in the future. Indeed, for any $0 < d < 1$ ($-1 < d < 0$), a rise in the absolute value of d, i.e. the degree of substitutability (complementarity) increases: (1) directly reduces (increases) the weight of the reply of firm 1 because marginal profits reduces (increases) since the degree of competition becomes lower (higher), (2) indirectly tends to reduce (increase) the reaction of firm 1 through a negative (positive) effect on the production of firm 2. Definitely, a rise in the (absolute value) of d at time t has a potentially uncertain effect on the quantity produced by the bounded rational firm at time $t+1$.

3. Local stability analysis of the unique positive Cournot-Nash equilibrium

From an economic point of view we are only interested to the study of the local stability properties of the unique positive output equilibrium, which is determined by setting $q_{1,t+1} = q_{1,t} = q_1$ and $q_{2,t+1} = q_{2,t} = q_2$ in (8.2) and solving for (non-negative solutions of) q_1 and q_2, that is:

$$q^*_1 = q^*_2 = q^* = \frac{a - w}{2 + d}, \tag{9}$$

where $w < a$ should hold to ensure $q^* > 0$.

The Jacobian matrix evaluated at the equilibrium point (9) is the following:

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{2 + d - 2\alpha(a - w)}{2 + d} & \dfrac{-d\alpha(a - w)}{2 + d} \\ -\dfrac{d}{2} & 0 \end{pmatrix}. \tag{10}$$

The trace and determinant of the Jacobian matrix (10) are respectively given by:

$$T := Tr(J) = J_{11} + J_{22} = \frac{2 + d - 2\alpha(a - w)}{2 + d}. \tag{11}$$

$$D := Det(J) = J_{11}J_{22} - J_{12}J_{21} = \frac{-\alpha d^2 (a - w)}{2(2 + d)}, \tag{12}$$

so that the characteristic polynomial of (10) is:

$$G(\lambda) = \lambda^2 - tr(J)\lambda + \det(J), \tag{13}$$

whose discriminant is $Q := [Tr(J)]^2 - 4Det(J)$.

We now study the local stability properties of the Cournot-Nash equilibrium Eq. (9) by means of well-known stability conditions for a system in two dimensions with discrete time (see, e.g., Medio, 1992; Gandolfo, 2010), which are generically given by:

$$\begin{cases} (i) \quad F := 1 + T + D > 0 \\ (ii) \quad TC := 1 - T + D > 0 \\ (iii) \quad H := 1 - D > 0 \end{cases}. \tag{14}$$

The violation of any single inequality in (15), with the other two being simultaneously fulfilled leads to: (*i*) a flip bifurcation (a real eigenvalue that passes through -1) when $F = 0$; (*ii*) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC = 0$; (*iii*) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H = 0$, namely $Det(J) = 1$ and $|Tr(J)| < 2$. For the particular case of the Jacobian matrix (10), the stability conditions stated in (14) can be written as follows:

$$\begin{cases} (i) \quad F = \dfrac{-\alpha(a-w)(4+d^2)+4(2+d)}{2(2+d)} > 0 \\ (ii) \quad TC = \dfrac{\alpha(2-d)(a-w)}{2} > 0 \\ (iii) \quad H = \dfrac{2(2+d)+\alpha(a-w)d^2}{2(2+d)} > 0 \end{cases}. \tag{15}$$

From (15) it can easily be seen that while conditions (*ii*) and (*iii*) are always fulfilled, condition (*i*) can be violated. Therefore, the Cournot-Nash equilibrium $q^*_1 = q^*_2 = q^*$ can loose stability through neither a transcritical nor Neimark-Sacker bifurcation. The stability condition (*i*) in (15) represents a region F in the (α, d) plane, i.e., the speed of adjustment and the degree of horizontal product differentiation, bounded by the economic model assumption $\alpha > 0$ and $-1 < d < 1$. Therefore, the following equation $\mathrm{B}(d)$, i.e. the numerator of F in (15), represents a bifurcation curve at which the positive equilibrium point $q^*_1 = q^*_2 = q^*$ looses stability through a flip (or period-doubling) bifurcation, that is:

$$\mathrm{B}(d) := -\alpha(a-w)(4+d^2)+4(2+d) = 0. \tag{16}$$

A simple inspection of Eq. (16) leads to the following remarks.

Remark 1. *The bifurcation curve $\mathrm{B}(d)$ is hump-shaped*[3] *and intersects the horizontal axis at $d = d^F_1 := C - K$ and $d = d^F_2 := C + K$, where*

$$C := \frac{2}{\alpha(a-w)}, \quad K := \frac{2\sqrt{-\alpha^2(a-w)^2 + 2\alpha(a-w)+1}}{\alpha(a-w)}. \tag{17}$$

The fixed point q^* is locally asymptotically stable ($\mathrm{B}(d) > 0$) when $d^F_1 < d < d^F_2$ (see Figure 1). Moreover, there are no real solutions of $\mathrm{B}(d)$ for d when $\alpha(a-w) > 2.41$ (see the Appendix for details).

Therefore, when the combination of the speed of adjustment and the market size[4] is fairly high, i.e. $\alpha(a-w) > 2.41$ (resp., low, i.e. $\alpha(a-w) < 0.8$), the Cournot-Nash equilibrium (9) of the dynamic system (8.2) is locally unstable (locally asymptotically stable) irrespective of the degree of product differentiation d. While within the intermediate range $0.8 < \alpha(a-w) < 2.41$,

[3] This can be ascertained by looking at the (negative) sign of the coefficient of d^2 in Eq. (16).

[4] Broadly speaking, $a - w > 0$ captures the size of market demand.

the degree of product differentiation crucially matters for stability. However, we must investigate whether the real solutions (if any) for d are feasible from an economic point of view in such a case.

In particular, it is of importance for economics to establish whether the stability region is reduced when products of variety 1 and 2 tends to become either substitutes or complements (i.e., whether the loss of stability of the market equilibrium may occur only through a reduction in the degree of substitutability between products), because the preceding mathematical analysis has revealed that the Cournot-Nash equilibrium Eq. (9) might occur through either an increase or decrease in the value of the parameter d.

In other words, in order to have an interesting economic interpretation of the results, it is crucial to know whether and how the bifurcation values $d = d^{F}{}_{1}$ and $d = d^{F}{}_{2}$ are included between -1 and 1 or not.

By using the Budan-Fourier theorem we are able to establish that the introduction of a higher differentiation between products has always a clear-cut stability effect, as the following proposition claims.

Proposition 1. *Let $0.8 < \alpha(a - w) < 2.41$ hold. Then, starting from a stability situation, when the parameter d is reduced (i.e., the degree of product differentiation increases), the Cournot-Nash equilibrium looses stability through a flip bifurcation when $d = d^{F}{}_{1}$.*

Proof. See the Appendix.

From an economic point of view, Proposition 1 shows that when a firm attempts to increase profits by reducing the degree of competition through an increase in product differentiation, it also tends to destabilise the market equilibrium. Moreover, *ceteris paribus* as regards the size of market demand, $a - w$, the higher the speed of adjustment α is the closer d is to unity (perfect substitutability).

Therefore, depending on the relative size of both the market demand and speed of adjustment, we have the following three cases:

Case (1). $\alpha(a - w) < 0.8$. In this case there exists two real solutions of $\mathrm{B}(d)$ for d, namely $d^{F}{}_{1} < -1$ and $d^{F}{}_{2} > 1$. The Cournot-Nash equilibrium Eq. (9) is locally asymptotically stable irrespective of the degree of product market differentiation.

Case (2). $\alpha(a - w) > 2.41$. No real solutions exist of $\mathrm{B}(d)$ for d. The Cournot-Nash equilibrium Eq. (9) is locally unstable irrespective of the degree of product market differentiation.

Case (3.1). $0.8 < \alpha(a - w) < 2.41$ and $0 < \alpha < \dfrac{2}{a - w}$. Then $-1 < d^{F}{}_{1} < 0$. The Cournot-Nash equilibrium Eq. (9) is locally asymptotically stable for any $0 < d^{F}{}_{1} < 1$. It looses stability through a flip bifurcation when the degree of products of variety 1 and 2 become complements.

Case (3.2). $0.8 < \alpha(a - w) < 2.41$ and $\alpha = \dfrac{2}{a - w}$. Then $d^{F}{}_{1} = 0$. The Cournot-Nash equilibrium Eq. (9) is locally asymptotically stable for any $0 < d^{F}{}_{1} < 1$. It looses stability through a flip bifurcation when the degree of product differentiation increases up to the point in which the two firms act as two separate monopolists in their own market.

Case (3.3). $0.8 < \alpha(a-w) < 2.41$ and $\alpha > \dfrac{2}{a-w}$. Then $0 < d^{F}{}_1 < 1$. The Cournot-Nash equilibrium Eq. (9) looses stability through a flip bifurcation when products of variety 1 and 2 from perfect substitutes (homogeneous) tend to become less substitutable between them.

4. A numerical illustration

The main purpose of this section is to show that the qualitative behaviour of the solutions of the duopoly game with heterogeneous player described by the dynamic system (8.2) can generate, in addition to the local flip bifurcation and the resulting emergence of a two-period cycle, complex behaviours. To provide some numerical evidence for the chaotic behaviour of system (8.2), we present several numerical results, including bifurcations diagrams, strange attractors, Lyapunov exponents, sensitive dependence on initial conditions and fractal structure.

According with the aim of the paper, we take the degree of product differentiation d as the bifurcation parameter, and choose the following parameter set only for illustrative purposes: $\alpha = 2.2$, $a = 2$ and $w = 1$, which represents Case (3.3).

Figure 2 depicts the bifurcation diagram for d. The figure clearly shows that an increase in the extent of product differentiation (i.e., the parameter d moves from 1 to values smaller than 1), implies that the map (8.2) converges to a fixed point for $1 > d > 0.2287$. Starting from this interval, in which the positive fixed point (9) of system (8.2) is stable, Figure 2 shows that the equilibrium output undergoes a flip bifurcation at $d^{F}{}_1 = 0.2287$. Then, a further increase in product differentiation implies that a stable two-period cycle emerges for $0.2287 > d > -0.2$. As long as the parameter d reduces a four-period cycle, cycles of highly periodicity and a cascade of flip bifurcations that ultimately lead to unpredictable (chaotic) motions are observed when product are complements. As an example, the phase portrait of Figure 3 depicts the strange attractor and basin of attraction for $d = -0.46$.

Another numerical tool useful in order to determine the parameter sets for which the system (6) converges to periodic cycles, quasi-periodic and chaotic attractors, is the study of the largest Lyapunov exponent, as a function of the parameter of interest (which, in the present paper, is the degree of product differentiation). As is known, there exists evidence for quasi periodic behaviour (chaos) when the largest Lyapunov exponent is zero (positive). Let $Le1$ be the largest Lyapunov exponent of our system. Then, for the above parameter constellation and initial conditions, in Figure 3 we plot $Le1$ against the parameter d (see, e.g., Fanti and Manfredi, 2007). In order to better characterise the largest exponent from a quantitative point of view, and take account for the fact that since there may be very long (periodic or aperiodic) transients, the dynamical system is left to evolve for $t = 10^5$ time units and then the Lyapunov exponents are calculated during a time of order $t = 10^5$. This allows to unambiguously detect the existence of chaotic motions in the range of values of d with respect to which $Le1$ is steadily positive. Moreover, the Lyapunov dimension evaluated according to the well-known Kaplan-Yorke conjecture (see Kaplan and Yorke, 1979), corresponding to $d = -0.46$ is $DL = 1.175$.[5]

[5] The Lyapunov dimension is computed as $DL \leq s + \dfrac{\sum_{k=1}^{s} \lambda_k}{|\lambda_{s+1}|}$, where λ_k is the kth Lyapunov exponent, s is the largest number for which $\sum_{k=1}^{s} \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_{s+1} < 0$ (see Medio, 1992).

As known, the sensitivity to initial conditions is a characteristic of deterministic chaos. In order to show the sensitivity to initial conditions of system (8.2), we have computed two orbits of the variable q_1 whose coordinates of initial conditions differ by 0.00001. Figures 4 depicts the orbits of q_1 with initial conditions $q_{1,0} = 0.03$ and $q_{2,0} = 0.01$, and $q_{1,0} = 0.03001$ and $q_{2,0} = 0.01001$ at $d = -0.46$. As expected, the orbits rapidly separate each other, thus suggesting the existence of deterministic chaotic.

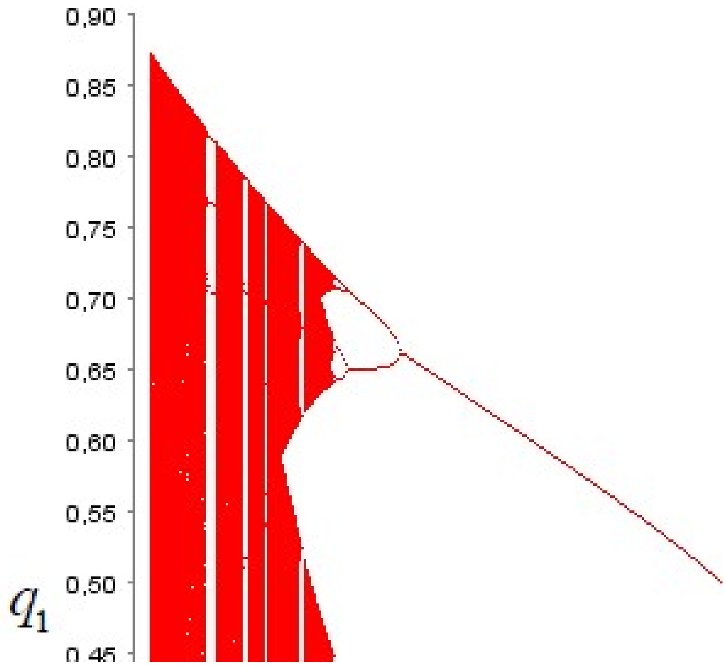

Figure 1. Bifurcation diagram for d. Initial conditions: $q_{1,0} = 0.03$ and $q_{1,0} = 0.01$.

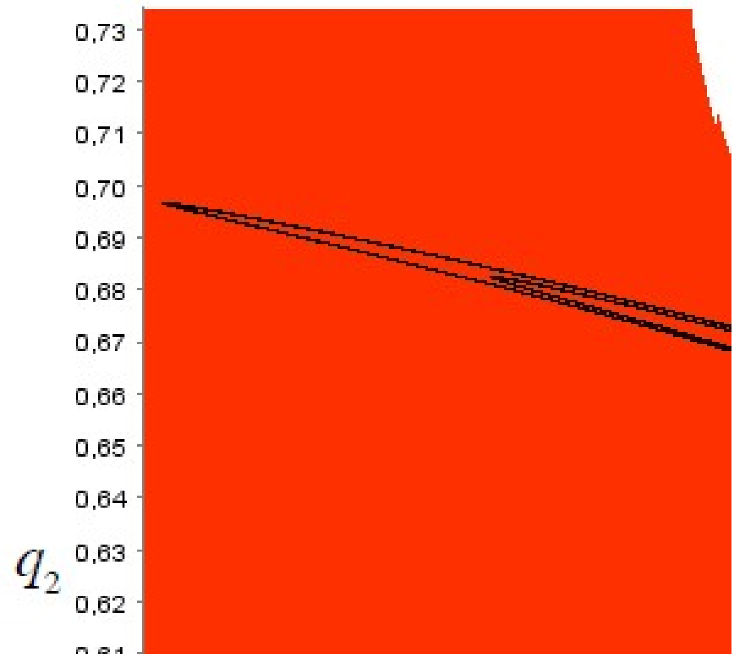

Figure 2. Phase portrait ($d = -0.46$).

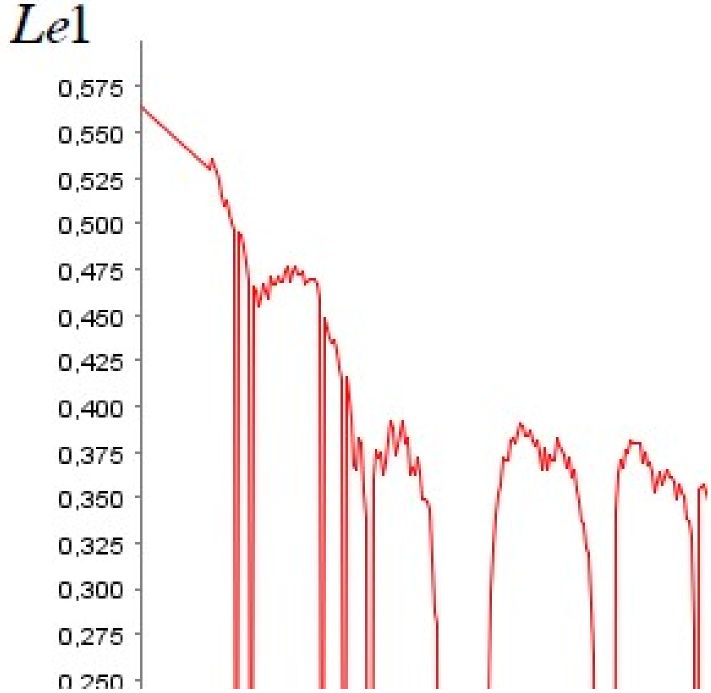

Figure 3. Largest Lyapunov exponent for $-0.5 < d < -0.15$ (one million iterations).

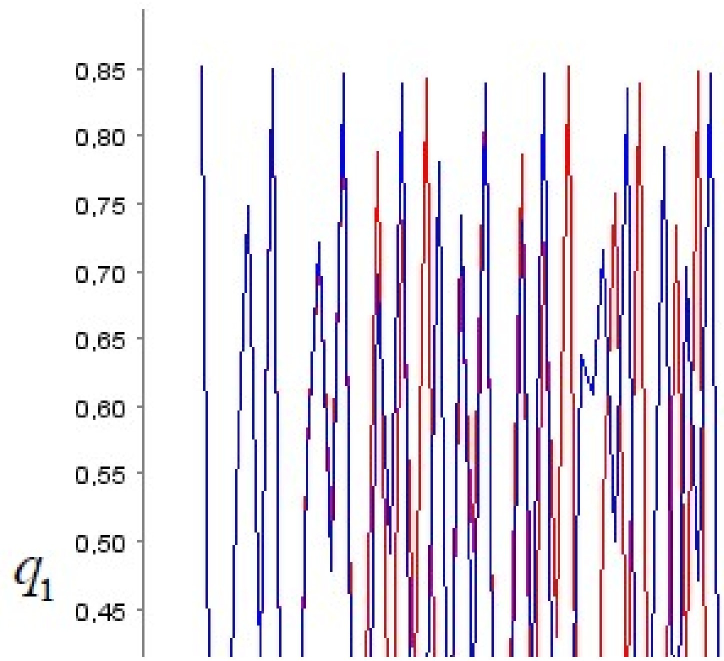

Figure 4. Sensitivity dependence to initial conditions (q_1 versus time). Initial conditions: $q_{1,0} = 0.03$ and $q_{2,0} = 0.01$ (red line), and $q_{1,0} = 0.03001$ and $q_{2,0} = 0.01001$ (blue line). ($d = -0.46$).

5. Conclusions

We analysed the dynamics of a differentiated Cournot duopoly with firms' heterogeneous expectations, and investigated the effects of a micro-founded differentiated products demand. The main result is that a higher degree of product market differentiation may destabilise the unique Cournot-Nash equilibrium, while also showing the existence deterministic chaos. This result suggests a twofold effect: while an increase in the extent of product differentiation tends to increase profits, it may also cause the loos of stability of the equilibrium through a flip bifurcation. In this sense, our findings constitute a policy warning for firms that want to differentiate their products in order to reduce competition.

The economic intuition behind the result is that the higher the degree of product differentiation, the lower the level of competition and the higher the output produced by each firm whatever the quantity produced by the rival. The larger amount of output produced by each single firm in comparison with the case of homogenous products is responsible for the loss of stability of the market equilibrium and the resulting complex dynamic events. An interesting theoretical implication is that a fiercer (weaker) competition tends to stabilise (destabilise) the economy.

However, we ask ourselves whether and how this result is robust to the underlying economic theoretical extensions (for instance, when returns to labour are decreasing (i.e. quadratic wage costs) or the labour market is unionised). The answers to these questions are left for future research.

Appendix

Proof of Proposition 1

The proof of Proposition 1 amounts to simply show that (*i*) at most only the root $d = d^F{}_1$ can be included in the interval $(-1,1)$, and (*ii*) $d = d^F{}_2 > 1$ always holds.

Let us begin by providing a standard version of the Budan-Fourier theorem.

Theorem 1. *[Budan-Fourier Theorem]. For any real number a and b such that $b > a$, let $F(a) \neq 0$ and $F(b) \neq 0$ be real polynomials of degree n, and $C(x)$ denote the number of sign changes in the sequence $\{F(x), F'(x), F''(x), \ldots, F^n(x)\}$. Then the number of zeros in the interval (a,b) (each zero is counted with proper multiplicity) equals $C(a) - C(b)$ minus an even non-negative integer.*

Armed with this theorem, the following proposition holds.

Proposition A.1. *Only one of the two roots for d ($d = d^F{}_1$) of the flip bifurcation boundary $\mathrm{B}(d) = 0$ is included between -1 and 1, while the root $d = d^F{}_2$ is always larger than 1.*

Proof. Let us rewrite the flip bifurcation boundary as:

$$\mathrm{B}(d) := -\alpha(a-w)d^2 + 4d + 8 - 4\alpha(a-w) = 0. \tag{A.1}$$

Then, by denoting $z = \alpha(a-w)$ we define the function

$$G(d) := -zd^2 - 4z + 4d + 8. \tag{A.2}$$

By a simple inspection of $G(d)$, it is easy to establish that the discriminant of $G(d)$ is negative for $z > \frac{12}{5} = 2.41$ and thus real solutions for d of $\mathrm{B}(d)$ do exist if, and only if, $z < \frac{12}{5}$. Then, we find that $G'(d) = -2zd + 4$ and $G''(d) = -2z$. Therefore, the following inequalities hold:

$$(i) \quad G(1) \underset{<}{\overset{\geq}{-}} 0 \Leftarrow z \underset{>}{\overset{\leq}{-}} \frac{12}{5}; \quad G'(1) \underset{<}{\overset{\geq}{-}} 0 \Leftarrow z \underset{>}{\overset{\leq}{-}} 2; \quad G''(1) < 0, \tag{A.3}$$

$$(ii) \quad G(-1) \underset{<}{\overset{\geq}{-}} 0 \Leftarrow z \underset{>}{\overset{<}{-}} \frac{4}{5} = 0.8; \quad G'(-1) > 0; \quad G''(-1) < 0. \tag{A.4}$$

Tables 1 and 2 resume the numerical results of the application of the Budan-Fourier theorem. As is shown: (1) in the last row of Table 1 only one root of d included between -1 and 1 does exist; 2) in the last row of Table 2 two sign changes when $d = -\infty$ does exist; (3) by comparing the number of sign changes when $d = -\infty$ and when $d = -1$, we observe that there exists no roots (one root) for d included between $-\infty$ and -1 when $\frac{12}{5} > z > \frac{4}{5}$ ($z < \frac{4}{5}$); therefore, since from Table 1 we observe that there is one root (no roots) for d included between -1 and 1 when $\frac{12}{5} > z > \frac{4}{5}$ ($z < \frac{4}{5}$), then we conclude that for any $\frac{12}{5} > z > 0$ one root $d > 1$ always exists.

Table 1. Threshold values and application of the Budan-Fourier theorem for the number of zeros in the interval $d \in (-1,1)$.

	$12/5 < z < 2$		$4/5 < z < 2$		$z < 4/5$	
	-1	1	-1	1	-1	1

$G(d)$	-	+	-	+	+	+
$G'(d)$	+	+	+	-	+	-
$G''(d)$	-	-	-	-	-	-
Number of sing changes (C)	2	1	2	1	1	1
Variation ($C(-1)-C(1)$)	1		1		0	

Table 2. Threshold values and application of the Budan-Fourier theorem for the number of zeros in the interval $d \in (-\infty, \infty)$.

	$12/5 < z < 0$	
	$-\infty$	$+\infty$
$G(d)$	-	-
$G'(d)$	+	-
$G''(d)$	-	-
Number of sing changes (C)	2	0
Variation ($C(-1)-C(1)$)	2	

It follows that since the Cournot-Nash equilibrium Eq. (9) of the two-dimensional system (8.2) is stable for any $d^F{}_1 < d < d^F{}_2$, and since $-\infty < d^F{}_1 < 1$ and $d^F{}_2 > 1$, then starting from a stability situation, the Cournot-Nash equilibrium Eq. (9) can loose stability only when d decreases beyond $d = d^F{}_1$. Moreover, it can easily be ascertained that $d^F{}_1 = 0$ if $\alpha = \dfrac{2}{a-w}$, $d^F{}_1 < 0$ for any $0 < \alpha < \dfrac{2}{a-w}$ and $d^F{}_1 > 0$ for any $\alpha > \dfrac{2}{a-w}$. **Q.E.D.**

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