The Optimal Population Size under Pollution and Migration Externalities: a Spatial Control Approach*

Davide La Torre[†] Danilo Liuzzi[‡] Simone Marsiglio[§]

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Abstract

We analyze the implications of pollution and migration externalities on the optimal population dynamics in a spatial setting. We focus on a framework in which pollution affects the mortality rate and decreases utility, while migration occurs within the spatial economy. Agents optimally determine their fertility rate which, along with pollution-induced mortality and spatial migration, determines the net population growth rate. This setting implies that human population follows an endogenous logistic-type dynamics where fertility choices determine what the optimal limit of human population will be. We compare the decentralized and the centralized outcomes showing that such fertility decisions generally differ, quantifying the extent to which pollution and migration induced externalities matter in determining the difference between the two outcomes. We show that, due to the effects of pollution on utility and mortality, both the optimal fertility rate and the population size are smallest in the centralized economy but migration effects change not only the size of these differences but also their direction, suggesting that the spatial channel is an important mechanism to account for in the process of policymaking.

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1 Introduction

Economists and demographers have debated for centuries about how human population, the economy and the environment are mutually related. Malthus (1798) was the first to conjecture the existence of feedback effects between population growth, economic performance and natural resources depletion. Following Malthus, several works have analyzed from different points of view the diverse nature of such feedback effects, by focusing on either those relating population and economic growth (see Bloom et al., 2003, for a survey) or those relating population and the environment (see Panayotou, 2000, for a survey). These issues are the main object of investigation of the optimal population size literature, which wonders which is the optimal number of lives in a population under given circumstances. Such a research question is generally addressed in an endogenous fertility framework aiming to understand how optimal fertility decisions are affected by specific (economic and environmental) factors. Several works analyze either the relation between endogenous fertility and economic growth (Palivos and Yip, 1993; Razin and Yuen, 1995; Boucekkine and Fabbri, 2013; Marsiglio, 2014) or the relation between endogenous fertility and natural resources (Nerlove, 1991; Schou, 2002; Jöst and Quaas, 2010; Marsiglio, 2011; Marsiglio, 2017). To the best of our knowledge, none of the

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[†]University of Dubai, Dubai Business School, Dubai, UAE; and University of Milan, Department of Economics, Business and Statistics, Milan, Italy. Contact: dlatorre@ud.ac.ae, and davide.latorre@unimi.it.

[‡]University of Milan, Department of Economics, Business and Statistics, Milan, Italy. Contact: danilo.liuzzi@unimi.it

[§]University of Pisa, Department of Economics and Management, via Cosimo Ridolfi 10, Pisa 56124, Italy. Contact: simone.marsiglio@unipi.it.

existing works takes into account migration flows which, by interacting with economic and environmental factors, are likely to play a major role in determining individuals' fertility decisions and population dynamics. This is exactly the goal of our paper which wishes to develop a tractable framework to understand how the determination of the optimal population size depends upon migration and pollution-induced mortality.

Specifically, we focus on a spatial endogenous fertility framework in which reproductive decisions affect both economic performance and the environment. On the one hand, fertility decisions directly impact on economic performance since they determine how much time is allocated to working and as such how much output the economy can produce. On the other hand, fertility decisions indirectly impact on the environment since by determining the population size they affect polluting emissions. These two effects are also interrelated since pollution affects the mortality rate, which by contributing to determine population growth and thus fertility and the population size, critically impacts on productive activities. The introduction of a spatial dimension in such a setting allows us to account for migration, which by being another important determinant of population growth, plays a critical role in determining fertility decisions. Since in our framework there are two externality sources (pollution and migration) the decentralized solution will likely be different from the centralized one. This allows us to study whether this is effectively the case and how these two different types of externality contribute to determine the eventual gap between the centralized and decentralized solutions. This is an important and novel contribution of our work, since, to the best of our knowledge, the comparison between the centralized and decentralized solutions in a spatial setup has not been discussed yet.

Our analysis allows us to derive some interesting conclusions. First, pollution by affecting the mortality rate puts a limit to the otherwise infinitely growing exponential population, and endogenous fertility choices critically determine what such an optimal limit will be. Since human population follows a logistic-type dynamics we can identify some upper bound for the total population level within the entire spatial economy, and such a level critically depends upon the maximum of the birth rate across all locations. Second, there are important differences between the optimal solutions in the centralized and decentralized settings, and such differences critically depend on the two sources of externalities, that is pollution and migration. Through our numerical analysis we can observe that, in our specific model's parametrization, pollution induced externalities lead the optimal fertility rate and population size to be larger in the decentralized than in the centralized economy; in some locations the migration induced externality changes not only the size of the difference between the centralized and decentralized solutions but also its direction, since the social planner, by internalizing the trend of migration flows tends to compensate for these effects by promoting fertility in the central locations and limiting it in the lateral ones. This result clearly suggests that the spatial channel represents an important mechanism to account for in the design of optimal policies.

This paper proceeds as follows. Section 2 introduces our spatio-temporal optimal control model of endogenous fertility choices with mutual feedback between population and pollution. Households need to determine how to allocate their time endowment between working and raising children; such a choice determines the evolution of human population which drives the level of pollutant emissions, which in turn impact the mortality rate; the mortality rate, jointly with the fertility rate, contributes to determining the population dynamics; another determinant of population dynamics is represented by migration flows which occur within the spatial economy. Section 3 analyzes the long run population dynamics generated by the pollution-induced mortality and endogenous fertility choices, showing that human population follows a logistic-type dynamics which allows us to determine some upper bounds for total population within the entire economy. Section 4 focuses on the optimal solution of our optimal control problem by analyzing separately the centralized and decentralized frameworks, and comparing the two outcomes. In a centralized setup the social planner quantifies and accounts for the externalities arising from pollution and migration, while in a decentralized setup these effects are not considered by individual agents in their optimal fertility decisions. By comparing the two solutions we conclude that they generally differ unless the two sources of externalities are completely removed. In order to gain a better understanding of how the two solutions differ, Section 5 presents some numerical simulations to characterize the size of the difference implied by the centralized and decentralized solutions, stressing the extent to which such difference is affected by pollution and migration induced externalities. Section 6 as usual concludes and suggests directions for future research.

2 The Model

We analyze a stylized spatio-temporal framework of endogenous fertility choices in which human population evolves over time and diffuses across space. We assume a continuous space structure to represent that the spatial economy develops along a linear city, where the population is mobile across different locations. We denote with $b_{x,t}$ and $N_{x,t}$ the birth rate and the population size in the position x at date t, in a compact interval $[x_a, x_b] \subset \mathbb{R}$, and we assume that there are no diffusion flows through the borders of $[x_a, x_b]$, that is the directional derivative is null, $\frac{\partial N_{x,t}}{\partial x} = 0$, at $x = \{x_a, x_b\}$. Population is mobile across different locations and in each single location population growth is determined by the difference between birth and mortality rates and the net migration flows. Migration flows are captured by $M_{x,t} = d\frac{\partial^2 N_{x,t}}{\partial x^2}$ where $d > 0$ represents the diffusion coefficient which describes the speed at which population tends to migrate from one location to the next. The mortality rate $m_{x,t}$ depends on the level of pollution $P_{x,t}$ as follows: $m_{x,t} = m(1 + \mu P_{x,t})$ where $\mu > 0$ quantifies the impact on pollution on mortality; this implies that in absence of pollution the mortality is determined by the biological mortality rate $m > 0$, while higher levels of polluting emissions increase mortality above this biological rate by a factor μ. The birth rate is endogenously determined as the result of the social planner's decision about how to allocate time between rising children and working. In each location, the social planner wishes to maximize social welfare $\mathcal{W}$ which is the weighted sum of two terms: the discounted ($\rho > 0$ is the discount factor) sum of instantaneous utilities $u(c_{x,t}, P_{x,t})$ depending positively on per capita consumption $c_{x,t}$ and negatively on pollution, and the end-of-planning horizon discounted disutility from pollution $d(P_{x,T})$. Both the instantaneous utility function and the disutility function take a logarithmic form as follows: $u(c_{x,t}, P_{x,t}) = \ln c_{x,t} - \beta \ln P_{x,t}$ where $\beta > 0$ is a measure of green preferences relative to consumption, and $d(P_{x,T}) = \ln P_{x,T}$. The weights of the instantaneous utility function and the disutility function are given by $0 < \theta < 1$ and $1 - \theta$, respectively. The social planner needs to choose how to allocate the unitary time endowment between working and rising children, and this choice is captured by the control variable $0 < n_{x,t} < 1$ which quantifies the time devoted to productive activities such that the rest of the time determines the birth rate as follows: $b_{x,t} = 1 - n_{x,t}$. Output $Y_{x,t}$ is produced through a linear production function employing only labor (which coincides with the population size since we abstract from unemployment): $Y_{x,t} = n_{x,t} N_{x,t}$; this implies that per capita output $y_{x,t} \equiv \frac{Y_{x,t}}{N_{x,t}}$ is given by $y_{x,t} = n_{x,t}$. For the sake of simplicity we assume that households entirely consume their income: $c_{x,t} = y_{x,t} = n_{x,t}$. Pollution is a flow variable and the level of emissions generated is proportional to the population size: $P_{x,t} = \eta N_{x,t}$, where $\eta > 0$ measures the degree of environmental inefficiencies in human activities. Therefore, population dynamics is described by the following partial differential equation (PDE): $\frac{\partial N_{x,t}}{\partial t} = d\frac{\partial^2 N_{x,t}}{\partial x^2} + [1 - n_{x,t} - m(1 + \eta N_{x,t})] N_{x,t}$. The social planner needs thus to determine the optimal time allocation $n_{x,t}$ in order to maximize social welfare, as summarized by the following spatial control problem:

$$\max_{n_t} \quad \mathcal{W} = \theta \int_0^T \int_{x_a}^{x_b} [\ln n_{x,t} - \beta \ln(\eta N_{x,t})] e^{-\rho t} dx dt - (1-\theta) \int_{x_a}^{x_b} \ln(\eta N_{x,T}) e^{-\rho T} dx \tag{1}$$

$$s.t. \quad \frac{\partial N_{x,t}}{\partial t} = d\frac{\partial^2 N_{x,t}}{\partial x^2} + [1 - n_{x,t} - m(1 + \eta\mu N_{x,t})] N_{x,t} \tag{2}$$

$$\frac{\partial N_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\} \tag{3}$$

$$N_{x,0} > 0 \text{ given} \tag{4}$$

Some comments on our modeling approach and assumptions are needed. (i) We model space by assuming that the economy develops along a line (Hotelling, 1929), and the presence of such a spatial structure implies that the dynamic constraint in our optimal control problem is given by a PDE and that social welfare within the entire economy is given by the sum of the welfare level in each single location. A recent and growing branch of the economics literature has adopted a similar approach to describe spatial spillovers in the context of capital accumulation (Brito, 2004; Camacho and Zou, 2004; Camacho et al., 2008; Boucekkine et al., 2009; 2013a, 2013b; Capasso et al., 2010) and environmental problems (Brock and Xepapadeas, 2008, 2010; Camacho and Pérez–Barahona, 2015; Anita et al., 2013, 2015; La Torre et al., 2015, 2018), but to the best of our knowledge none of existing works has analyzed endogenous fertility choices subject to spatial spillovers, representing migration in our setting. (ii) We assume that the time horizon is finite and the social planner cares for the pollution level remaining at the end of the planning horizon. This is consistent with a number of studies addressing optimal environmental policies and the sustainability issues related to intertemporal equity (Chichilnisky et al., 1995; Chichilnisky, 1997; Colapinto et al., 2017; La Torre et al., 2017), but none of these works accounts for how population and pollution are related and for the role of endogenous fertility choices. (iii) We also assume that pollution is entirely generated by human population's activities and not by productive activities. This is clearly a simplification of reality and consistent with the IPAT equation and its later refinements, stating that population is a driver of pollution per se and as such completely independent of technology and affluence, where the latter represents either production or consumption (Ehrlich and Holdren, 1971; Kaya, 1990, Rosa and Dietz, 1998). This assumption allows us to focus on one single driver of pollution characterizing mainly how households' lifestyle and daily life activities by producing waste contribute to pollution (Marsiglio, 2017). (iv) We also assume that pollution increases the mortality rate and thus critically impacts on population dynamics. This is in line with recent studies showing that high levels of pollution concentrations negatively affect health via increased morbidity and mortality (IPCC, 2007; Huang et al., 2011), and to the best of our knowledge such pollution-related mortality effects have never been accounted for in an endogenous fertility setting before.

Finally, note that in the above problem there are two different sources of externalities: pollution and migration. Pollution determines both a utility externality and a mortality externality, while migration determines a spatial externality. As we shall see later, the presence of such externalities imply that the optimal solution achieved by a social planner is different from the solution achieved in a decentralized economy. Understanding to what extent the two solutions differ and how both externality sources (i.e., pollution and migration) contribute to determine such a gap is important to derive eventual policy recommendations. In order to look at this we will proceed by analyzing the two problems separately and comparing their solutions. Since from our above problem the optimal time allocation determines the birth rate $b_{x,t} = 1 - n_{x,t}$, as a matter of expositional simplicity in the following we shall refer to $1 - n_{x,t}$ as either the birth rate or the fertility rate, and we wish to understand how such an optimal fertility rate is affected by the presence of externalities.

3 Long Run Population Behavior

Before proceeding to the derivation of the optimal fertility rate in the centralized and decentralized setups, it may be useful to analyze the long run behavior of the human population implied by (2), which suggests that because of the pollution-induced mortality human population follows a logistic-type dynamics. In order to do so, let us define the total population as follows:

$$N_t^{tot} = \int_{x_a}^{x_b} N_{x,t} dx \tag{5}$$

where:

$$N_0^{tot} = \int_{x_a}^{x_b} N_0(x) dx \tag{6}$$

The following classical result (Jensen's inequality) will be useful in the sequel.

Lemma 1. *If $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function and $f : [a, b] \to \mathbb{R}$ is an integrable function then:*

$$\phi\left(\frac{1}{x_b - x_a}\int_a^b f dx\right) \leq \frac{1}{x_b - x_a}\int_a^b \phi(f)dx$$

Lemma 1 allows us to derive the following result.

Proposition 1. *Define $n_{min} = \min_{\{x,t\}} n_{x,t}$. Then the following inequality holds:*

$$\dot{N}_t^{tot} \leq (1 - n_{min} - m)N_t^{tot}\left[1 - \frac{m\eta\mu}{(x_b - x_a)(1 - n_{min} - m)}N_t^{tot}\right].$$

Proof. If N and n solve the above equation (2) then the following chain of inequalities holds:

$$\begin{aligned}
\dot{N}_t^{tot} &= \frac{d}{dt}\int_{x_a}^{x_b} N_{x,t}dx \\
&= \int_{x_a}^{x_b} \frac{\partial N_{x,t}}{\partial t} \\
&= \int_{x_a}^{x_b} d\frac{\partial^2 N_{x,t}}{\partial x^2} + \left[1 - n_{x,t} - m(1 + \eta\mu N_{x,t})\right] N_{x,t} \\
&= \int_{x_a}^{x_b} \left[1 - n_{x,t} - m(1 + \eta\mu N_{x,t})\right] N_{x,t}dx \\
&\leq (1 - n_{min} - m)\int_{x_a}^{x_b} N_{x,t}dx - m\eta\mu\int_{x_a}^{x_b} N_{x,t}^2 dx \\
&\leq (1 - n_{min} - m)N_t^{tot} - \left[\frac{m\eta\mu}{x_b - x_a}\right](N_t^{tot})^2
\end{aligned}$$

where $n_{min} = \min_{\{x,t\}} n_{x,t}$ and the last step follows by noticing that:

$$\int_{x_a}^{x_b} d\frac{\partial^2 N_{x,t}}{\partial x^2}dx = d\left[\frac{\partial N_{x_b,t}}{\partial x} - \frac{\partial N_{x_a,t}}{\partial x}\right] = 0 \tag{7}$$

and using Jensen's inequality with $\phi(x) = x^2$

$$\frac{1}{x_b - x_a}(N_t^{tot})^2 \leq \int_{x_a}^{x_b} N_{x,t}^2 dx.$$

The above calculations show that:

$$\dot{N}_t^{tot} \leq (1 - n_{min} - m)N_t^{tot}\left[1 - \frac{m\eta\mu}{(x_b - x_a)(1 - n_{min} - m)}N_t^{tot}\right].$$

■

Proposition 1 identifies an upper bound for the total population size at any point in time and this is determined by the population size implied by a logistic dynamics of the following form: $\dot{N}_t^{tot} = g_N N_t^{tot}(1 - \frac{N_t^{tot}}{N^c})$, in which $g_N = 1 - n_{min} - m$ represents the intrinsic growth rate and $N^c = \frac{(x_b - x_a)(1 - n_{min} - m)}{m\eta\mu}$ the carrying capacity. In this view the above proposition states that human population is bounded from above from a logistic population in which the maximum of the birth rate across all locations $b_{max} = 1 - n_{min}$ determines both its intrinsic growth rate and its carrying capacity. The eventual knowledge of such a maximum birth rate (which is the result of the optimal decisions of welfare maximizing agents) will enable us to understand which level will not be exceeded by the optimal solution size. Further reasoning on this logistic-type population dynamics allows us to state also the following result.

Proposition 2. *Suppose that* $0 < N_0^{tot} < \frac{(x_b-x_a)(1-n_{min}-m)}{m\eta\mu}$ *and* $1-n_{min} \geq m$*. Then it follows that for any* $t \geq 0$*:*

$$N_t^{tot} \leq \frac{N_0^{tot}\frac{(x_b-x_a)(1-n_{min}-m)}{m\eta\mu}}{N_0^{tot} + \left(\frac{(x_b-x_a)(1-n_{min}-m)}{m\eta\mu} - N_0^{tot}\right)e^{-(1-n_{min}-m)t}} \leq \frac{(x_b-x_a)(1-n_{min}-m)}{m\eta\mu} \tag{8}$$

Proposition 2 can be easily proved by using a classical comparison theorem (Lakshmikantham and Leela, 1969). It states that provided that the intrinsic growth rate is positive, $g_N \geq 0$ (i.e., the maximum of the birth rate across all locations is larger than the mortality rate), and the initial population level is lower than its carrying capacity, $N_0^{tot} < N^c$, then the maximum level of the population clearly will not exceed such a carrying capacity level within the time horizon, and actually will not exceed an even lower level determined in the first inequality in (8). As mentioned earlier, the carrying capacity depends on the maximum birth rate which results from the optimal endogenous choices of welfare maximizing agents. Therefore, our framework with endogenous fertility choices and pollution-induced mortality characterizes an endogenous logistic-type population behavior: the pollution-induced mortality externality imposes a limit to the size that the otherwise infinitely growing exponential population will achieve, and agents' behavior with their fertility decisions determine what exactly this optimal limit will be. Understanding the extent to which agents' behavior might differ in the centralized and decentralized settings along with what this might imply for the optimal population size is our main goal in the rest of the paper.

4 Optimality

In this section we analyze the optimal solution in the centralized and decentralized settings, eventually comparing them. We focus first on the centralized framework in which the social planner quantifies and accounts for the externalities arising from pollution and migration. We then move to the decentralized framework in which the pollution and migration induced externalities are not considered by individual agents in the optimal fertility decisions. Lastly we compare the two solutions, even if the lack of analytical results limits the type of conclusions that we can effectively derive.

4.1 The Centralized Economy

The centralized problem, that is the problem faced by a benevolent social planner who internalizes all externalities, is given by the following equations:

$$\max_{n_t} \quad \mathcal{W} = \theta \int_0^T \int_{x_a}^{x_b} \left[\ln n_{x,t} - \beta \ln(\eta N_{x,t})\right] e^{-\rho t} dx dt - (1-\theta) \int_{x_a}^{x_b} \ln(\eta N_{x,T}) e^{-\rho T} dx \tag{9}$$

$$s.t. \quad \frac{\partial N_{x,t}}{\partial t} = d\frac{\partial^2 N_{x,t}}{\partial x^2} + \left[1 - n_{x,t} - m(1+\eta\mu N_{x,t})\right] N_{x,t} \tag{10}$$

$$\frac{\partial N_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\}, \tag{11}$$

$$N_{x,0} > 0 \text{ given} \tag{12}$$

We analyze the above spatial optimal control problem (9) – (12) by following a variational method (Troltzsch, 2010; Boucekkine et al., 2013a). The generalized current value Hamiltonian function, $\mathcal{H}(N_{x,t}, n_{x,t}, \lambda_{x,t})$, reads as follows:

$$\mathcal{H} \quad = \quad \theta\left[\ln n_{x,t} - \beta \ln(\eta N_{x,t})\right] + \lambda_{x,t}\left\{d\frac{\partial^2 N_{x,t}}{\partial x^2} + \left[1 - n_{x,t} - m(1+\eta\mu N_{x,t})\right] N_{x,t}\right\}, \tag{13}$$

where $\lambda_{x,t}$ is the costate variable. The FOCs for a maximum are given by the following expressions:

$$\begin{aligned}\frac{\partial \lambda_{x,t}}{\partial t} &= \rho\lambda_{x,t} - d\frac{\partial^2 \lambda_{x,t}}{\partial x^2} + \frac{\theta\beta}{N_{x,t}} + 2m\eta\mu N_{x,t}\lambda_{x,t} - (1 - n_{x,t} - m)\lambda_{x,t} && (14)\\ n_{x,t} &= \frac{\theta}{\lambda_{x,t}N_{x,t}} && (15)\end{aligned}$$

Substituting this last expression into (10) and (14) we obtain the following system of PDEs:

$$\begin{aligned}\frac{\partial N_{x,t}}{\partial t} &= d\frac{\partial^2 N_{x,t}}{\partial x^2} + [1 - m(1 + \eta\mu N_{x,t})]\, N_{x,t} - \frac{\theta}{\lambda_{x,t}} && (16)\\ \frac{\partial \lambda_{x,t}}{\partial t} &= \rho\lambda_{x,t} - d\frac{\partial^2 \lambda_{x,t}}{\partial x^2} + \frac{\theta(\beta+1)}{N_{x,t}} + 2m\eta\mu N_{x,t}\lambda_{x,t} - (1 - m)\lambda_{x,t} && (17)\end{aligned}$$

which, jointly with the following boundary conditions:

$$\begin{aligned}& N_{x,0} = N_0(x) && (18)\\ & \lambda_{x,T} = \frac{1-\theta}{N_{x,T}} && (19)\\ & \frac{\partial N_{x_a,t}}{\partial x} = \frac{\partial N_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0,T] && (20)\\ & \frac{\partial \lambda_{x_a,t}}{\partial x} = \frac{\partial \lambda_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0,T], && (21)\end{aligned}$$

characterize the optimal solution of our spatial control problem. Unfortunately deriving a closed-form solution for the population size and fertility rate from the above equations is not possible, thus we will need to rely upon numerical simulations in order to illustrate this solution and effectively compare it with the decentralized one.

4.2 The Decentralized Economy

The decentralized problem, that is the problem faced by a representative household who does not internalize the externalities, is given by the following equations:

$$\begin{aligned}\max_{n_t} \quad & \mathcal{W} = \theta \int_0^T \int_{x_a}^{x_b} (\ln n_{x,t} - \beta \ln P_{x,t})\, e^{-\rho t} dx dt - (1-\theta) \int_{x_a}^{x_b} \ln P_{x,T} e^{-\rho T} dx && (22)\\ s.t. \quad & \frac{\partial N_{x,t}}{\partial t} = M_{x,t} + [1 - n_{x,t} - m(1 + \mu P_{x,t})]\, N_{x,t} && (23)\\ & \frac{\partial N_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\}, && (24)\\ & N_{x,0} > 0 \text{ given} && (25)\end{aligned}$$

Single households do not understand that their individual actions contribute to determine the level of pollution in the economy and thus they do not account for how an increase in the population size impacts on pollution. They also do not understand how migration occurs and thus they do not consider how migration flows impact population growth. This means that, differently from the social planner, they simply take the level of pollution $P_{x,t}$ and migration $M_{x,t}$ as given. These differences lie at the basis of the gap between the centralized and decentralized solutions.

We use exactly the same approach which we followed earlier in the centralized framework to determine the optimal solution of the spatial optimal control problem (22) – (25). The generalized current value Hamiltonian function, $\mathcal{H}(N_{x,t}, n_{x,t}, \lambda_{x,t})$, in this case reads as follows:

$$\mathcal{H} = \theta (\ln n_{x,t} - \beta \ln P_{x,t}) + \lambda_{x,t} \{M_{x,t} + [1 - n_{x,t} - m(1 + \mu P_{x,t})]\, N_{x,t}\}, \quad (26)$$

where $\lambda_{x,t}$ is the costate variable. The FOCs for a maximum are given by the following expressions:

$$\frac{\partial \lambda_{x,t}}{\partial t} = \rho\lambda_{x,t} - [1 - n_{x,t} - m(1 + \mu P_{x,t})]\lambda_{x,t} \tag{27}$$

$$n_{x,t} = \frac{\theta}{\lambda_{x,t} N_{x,t}} \tag{28}$$

Substituting this last expression into (23) and (27) we obtain the following system of PDEs:

$$\frac{\partial N_{x,t}}{\partial t} = M_{x,t} + [1 - m(1 + \mu P_{x,t})]\, N_{x,t} - \frac{\theta}{\lambda_{x,t}} \tag{29}$$

$$\frac{\partial \lambda_{x,t}}{\partial t} = \frac{\theta}{N_{x,t}} - [-\rho + 1 - m(1 + \mu P_{x,t})]\lambda_{x,t} \tag{30}$$

which, jointly with the following boundary conditions:

$$N_{x,0} = N_0(x) \tag{31}$$

$$\lambda_{x,T} = \frac{1-\theta}{N_{x,T}} \tag{32}$$

$$\frac{\partial N_{x_a,t}}{\partial x} = \frac{\partial N_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{33}$$

$$\frac{\partial \lambda_{x_a,t}}{\partial x} = \frac{\partial \lambda_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T], \tag{34}$$

characterize the optimal solution of our problem. Recall that in a decentralized setting each individual household takes pollution and migration as given when determining its optimal fertility rate, thus in the above equations we need to replace $P_{x,t} = \eta N_{x,t}$ and $M_{x,t} = d\frac{\partial^2 N_{x,t}}{\partial x^2}$, leading to the following PDEs:

$$\frac{\partial N_{x,t}}{\partial t} = d\frac{\partial^2 N_{x,t}}{\partial x^2} + [1 - m(1 + \eta\mu N_{x,t})]\, N_{x,t} - \frac{\theta}{\lambda_{x,t}} \tag{35}$$

$$\frac{\partial \lambda_{x,t}}{\partial t} = \rho\lambda_{x,t} + \frac{\theta}{N_{x,t}} + m\eta\mu N_{x,t}\lambda_{x,t} - (1-m)\lambda_{x,t} \tag{36}$$

As for the centralized case, deriving a closed-form solution for the population size and fertility rate from the above equations is not possible, thus we will need to rely upon numerical simulations in order to illustrate this solution and effectively compare it with the centralized one.

4.3 Comparison

We have thus far focused on the centralized and decentralized solutions separately, while we now try to compare them. From the equations derived earlier, it is straightforward to prove that if a pair $(N_{x,t}, \lambda_{x,t})$, with $N_{x,t} > 0$ for any $x \in [x_a, x_b]$, solves the centralized problem, then it will also solve the decentralized system whenever the following condition is satisfied:

$$d\frac{\partial^2 \lambda_{x,t}}{\partial x^2} = \frac{\theta\beta}{N_{x,t}} + m\eta\mu N_{x,t}\lambda_{x,t} \tag{37}$$

Note that the above equation is automatically verified in absence of the pollution and migration induced externalities, that is whenever the utility externality is null ($\beta = 0$), the mortality externality is null ($\mu = 0$) and the spatial externality is null ($d = 0$). However, in presence of such externality (β, μ, and d taking strictly positive values) the above condition is never verified. Indeed, by integrating from x_a to x_b we obtain:

$$\int_{x_a}^{x_b} \left(\frac{\theta\beta}{N_{x,t}} + m\eta\mu N_{x,t}\lambda_{x,t} \right) dx = 0, \tag{38}$$

which is never satisfied due to the fact that the product $N_{x,t}\lambda_{x,t}$ is positive and $N_{x,t}$ is strictly positive. This allows us to conclude the following.

Proposition 3. *The centralized and decentralized solutions coincide whenever all externalities are null (i.e., $\beta = \mu = d = 0$) while they differ whenever the externalities are strictly positive (i.e., $\beta, \mu, d > 0$).*

Proposition 3 states that, as we would expect, in general the centralized and decentralized solutions differ, unless all sources of externalities are removed. However, the absence of a closed-form solution for the centralized and decentralized problems does not allow us to state how the two solutions differ and the extent to which they do. In order to look at this with more depth we will need to perform some numerical simulations to illustrate the two solutions and quantify the size of these differences, which is the goal of the next section.

5 Numerical Simulations

As mentioned earlier, since it is not possible to derive in closed-form the optimal solution of the centralized and decentralized problems, we will now proceed with some numerical simulations in order to quantify the size of the difference between the centralized and decentralized solutions. We focus on a framework where Proposition 2 is met, that is in which $N_0^{tot} < \frac{1-n_{min}-m}{m\eta\mu}$ and $1 - n_{min} > m$, meaning that the mortality rate is small enough in order to be strictly lower that the maximal birth rate. To the best of our knowledge, there is no availability of empirically relevant results to meaningfully calibrate the values of our parameters, thus in the following we will simply set the parameter values in order to make the figures as clear as possible, but it is possible to show that the results are robust since they are qualitatively identical whenever the above parameter restrictions are met. The specific parameter values that we employ in our simulations are the following: $\eta = 0.01$, $\theta = 0.01$, $\rho = 0.04$, $m = 0.11$, $\mu = 1$, $\beta = 0.9$, $N_0(x) = 0.75 + 0.25e^{-20x^2}$ and $T = 10$.

Recall that in our setting there are two sources of externality, pollution and migration, thus in order to understand the extent to which they contribute to the difference between the centralized and decentralized solutions we will analyze them separately. We first focus on pollution induced externalities (via utility and via mortality) and thus we set $d = 0$, meaning that the economy is spatially structured but the outcome in each location is completely independent from the outcome in other locations (La Torre et al., 2015). In order to disentangle the effects of the two pollution induced externalities, we first focus only on the utility externality by setting $\beta > 0$ and $\mu = 0$. The results of our simulations in this case are illustrated in Figure 1, which shows the spatio-temporal evolution of the population size (left panels) and the fertility rate (right panels) in the centralized (top panels) and decentralized (middle panels) cases. We can see that there are some important differences. The optimal fertility rate both in the centralized and decentralized settings is spatially heterogeneous (even if we cannot observe the spatial heterogeneity in the fertility rate as a matter of scale) in early times taking on larger values in the central than in lateral locations; over time it increases everywhere in the spatial domain becoming spatially homogeneous at the end of the planning horizon, where its value equal to $\frac{\theta}{1-\theta}$ is determined by the boundary conditions. Moreover, the fertility rate in the centralized framework is lower than in decentralized one and such differences, which are more pronounced in the central locations, tend to fade away as soon as time goes by. These differences are reflected and more noticeable in the population size which tends to grow faster in the decentralized than in centralized context, and the difference in such population sizes is largest in the central locations. In order to get a sense of magnitude it may be useful to look at the difference between the two solutions (bottom panels) which clearly shows the extent to which the fertility rate and the population size in the centralized and decentralized settings differ. The result that fertility is largest in the decentralized framework is intuitive and due to the effects of pollution which affects only utility since the mortality externality has been ruled out ($\mu = 0$): since pollution, which depends on the population size, negatively affects utility and such an effect is not accounted for in the decentralized economy, the social planner finds it optimal to limit fertility in order to reduce population growth and thus the negative effects of a larger population size on utility and social welfare.

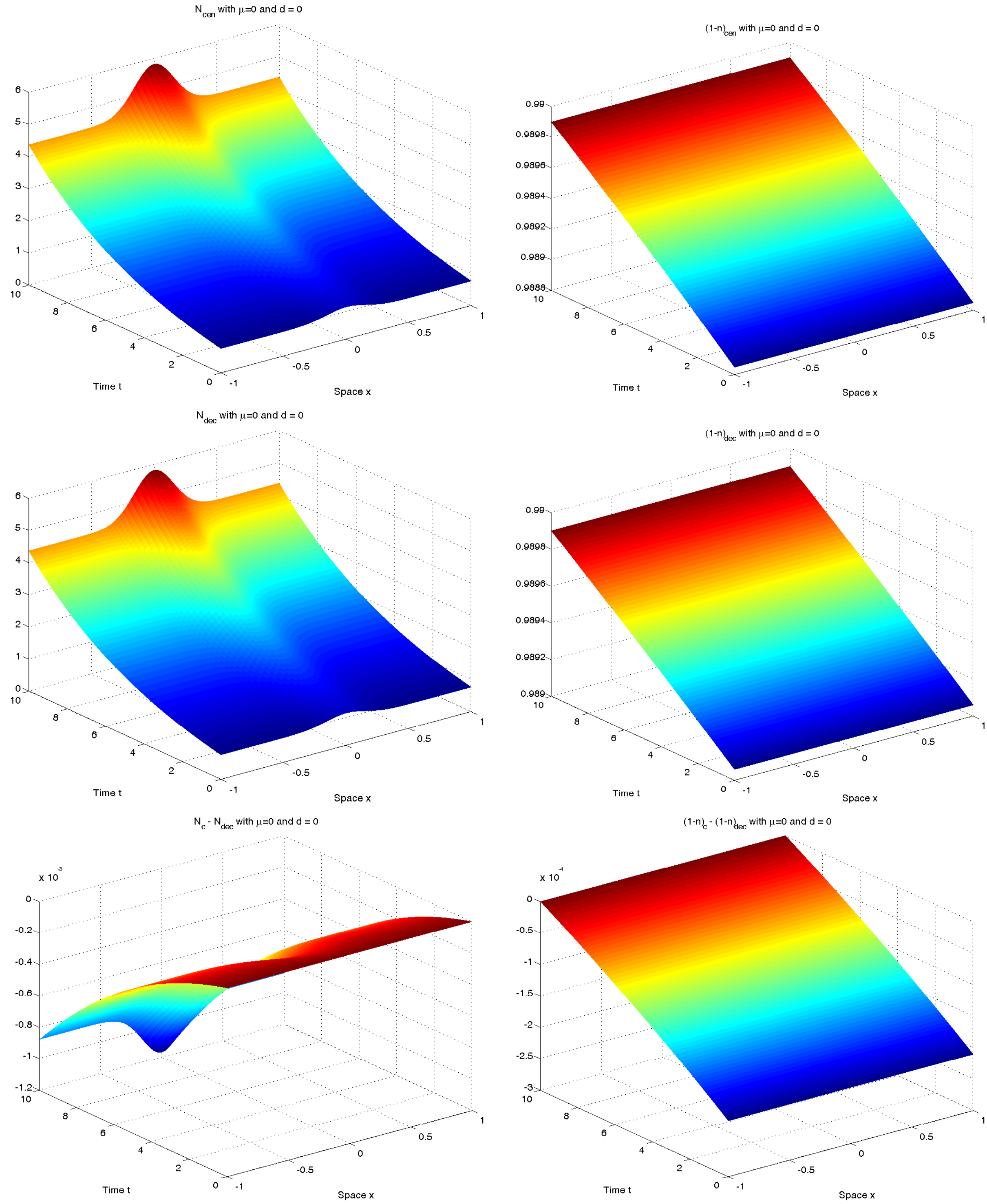

Figure 1: Evolution of the population size (left) and fertility rate (right) in the spatial economy with no spatial externality ($d = 0$), no mortality externality ($\mu = 0$) and with utility externality ($\beta > 0$), in the centralized setting (top), decentralized setting (middle) and difference between the two solutions (bottom).

We now introduce also the mortality externality by setting $\beta > 0$ and $\mu > 0$ in order to consider the implications of both the pollution induced externalities simultaneously. The results of our simulations in this case are illustrated in Figure 2, which from a qualitative point of view perfectly coincide with those discussed earlier, since both the optimal fertility rate and the population size in the centralized framework are lower than in decentralized one. The only noticeable difference is quantitative: the presence of the mortality

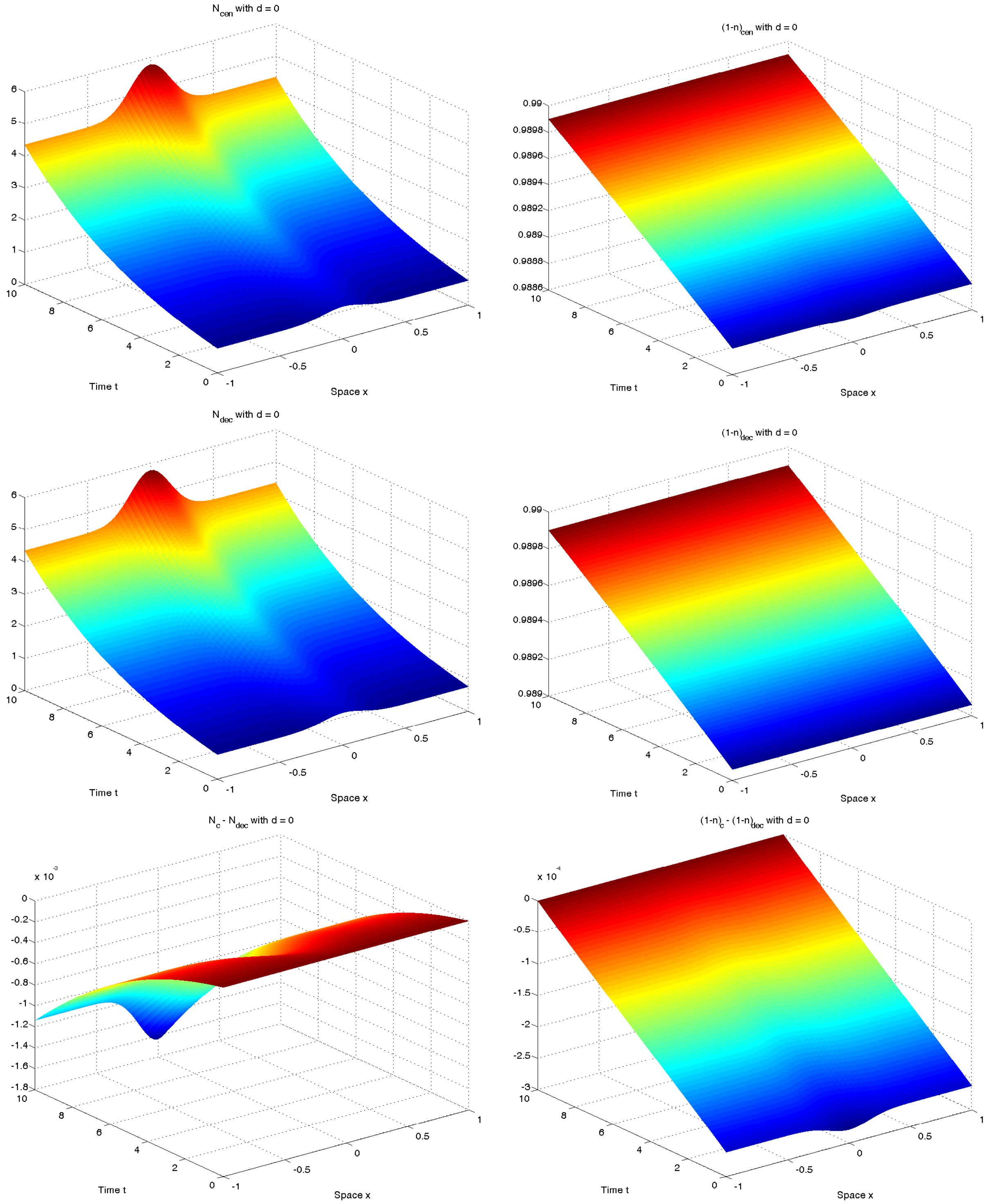

Figure 2: Evolution of the population size (left) and fertility rate (right) in the spatial economy with no spatial externality ($d = 0$), with mortality externality ($\mu > 0$) and with utility externality ($\beta > 0$), in the centralized setting (top), decentralized setting (middle) and difference between the two solutions (bottom).

externality increases the gap in the optimal fertility rate and population size between the centralized and decentralized settings. Also in this case the result that fertility is largest in the decentralized framework is intuitive and due to the effects of pollution which now affects not only utility but also mortality: since pollution increases mortality, which tends to decreases the population size and thus output, the mortality externality affects the fertility rate in the same direction as the utility externality, and so the social planner finds it optimal to limit fertility further in order to reduce pollution and its effects on mortality.

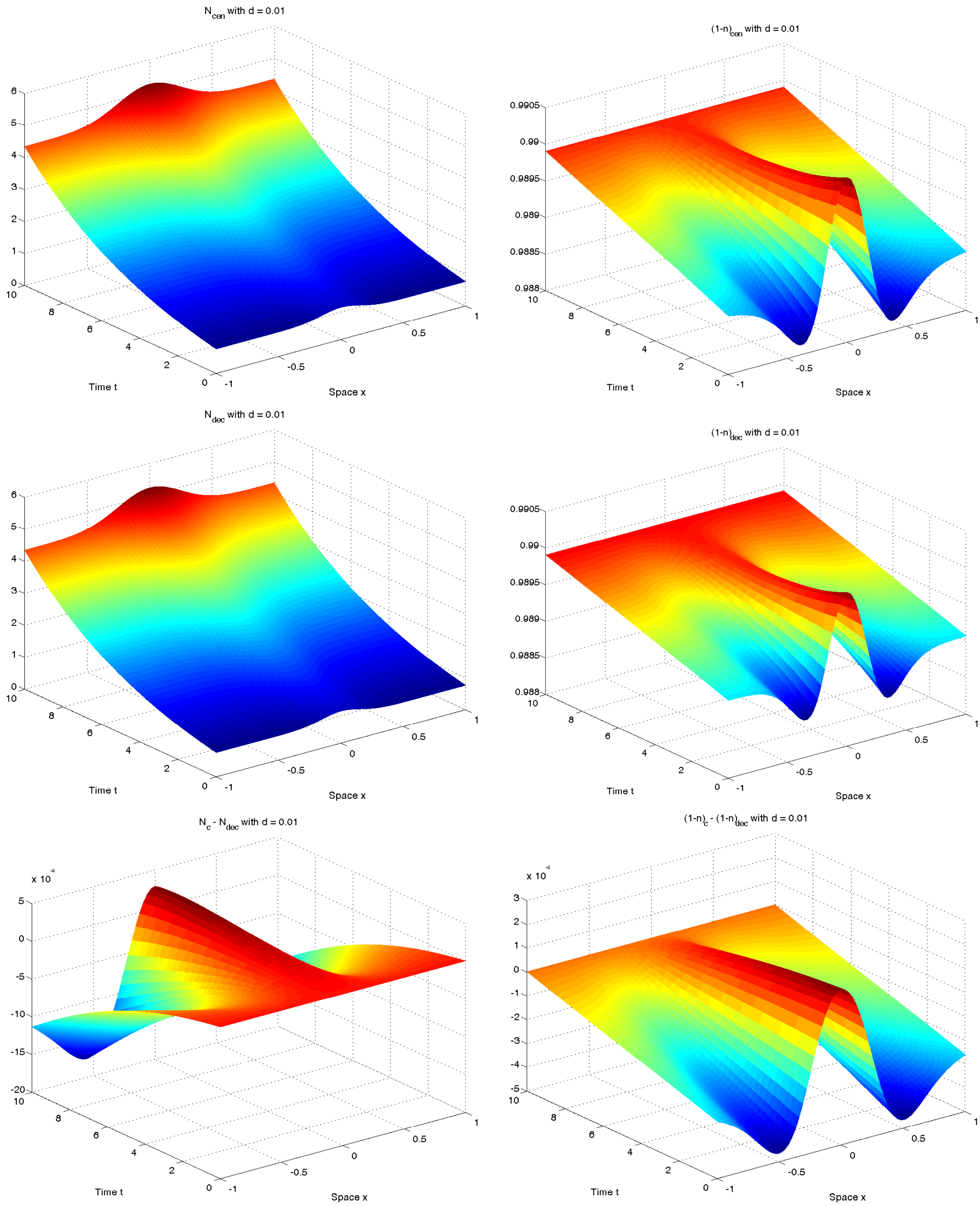

Figure 3: Evolution of the population size (left) and fertility rate (right) in the spatial economy with spatial externality ($d > 0$), with mortality externality ($\mu > 0$) and with utility externality ($\beta > 0$), in the centralized setting (top), decentralized setting (middle) and difference between the two solutions (bottom).

Apart from the above discussed effects of pollution, we now wish to understand how the migration induced externality affect the centralized and decentralized solutions. Therefore, we set $d > 0$ with $\beta > 0$ and $\mu > 0$, meaning that now the economy is not only spatially structured but the outcome in each location does depend on the outcome in other locations as well (La Torre et al., 2015). The results of our simulations in this case are illustrated in Figure 3. Due to the short time horizon we cannot clearly observe yet the effects of diffusion, which by acting as a convergence mechanism tends to smooth differences out (Boucekkine et al., 2009; La Torre et al., 2015). Exactly as before, the optimal fertility rate both in the centralized and decentralized economies is spatially heterogenous taking on larger values in the central than in lateral locations. This difference is clearly reflected also in the evolution of the population size. By looking at the size of the differences between the two solutions, we can note that in the centralized economy the fertility rate is higher than in the decentralized economy in the central locations while the opposite is true in lateral locations; moreover, the size of the difference between the two solutions is much more pronounced than in Figure 2, and this additional wedge between the two solutions is due to the role of the spatial externality which by affecting the net population growth rate provides the social planner with further incentives to differentiate its optimal fertility rate from what would be determined by single households. Since the planner internalizes the fact that, via diffusion, migration will partly move population from the most populated central locations to less populated lateral ones, it results most convenient to promote fertility in these central locations and limit it in lateral locations in order to compensate for such migration effects. The result that thanks to diffusion the direction of the difference between the centralized and decentralized solutions changes along with the fact that the gap between such solutions increases suggest that the spatial channel clearly matters and thus it is an important mechanism to account for in the policymaking process.

6 Conclusion

Since Maltus (1798) it is well known that human population, the economy and the environment are mutually related, thus it is important to take into account both economic and environmental factors in the determination of the optimal population size. This paper analyzes the optimal population size problem in a spatial framework with endogenous fertility with pollution and migration induced externalities: pollution, proportional to the population size, determines the mortality rate and decreases utility; net migration, determined by spatial diffusion, further contributes to population growth and thus to pollution. We show that in such a context human population follows an endogenous logistic-type dynamics where fertility choices determine what the optimal limit of human population will be. We also show that, because of pollution and migration induced externalities, the centralized and decentralized solutions differ, and through numerical simulations we compare the size of such a difference. This allows us to show that pollution induced externalities lead the optimal fertility rate to be larger in the decentralized than in the centralized economy and so is the optimal population size; in some locations the migration induced externality changes not only the size of the difference between the centralized and decentralized solutions but also its direction, since the social planner, by internalizing the trend of migration flows tends to compensate for these effects by promoting fertility in the central locations and limiting it in the lateral ones. This result clearly suggests that the spatial channel represents an important mechanism to account for in the design of optimal policies.

To the best of our knowledge, this is the first paper analyzing endogenous fertility choices in a spatial framework and also the first attempt to characterize the difference between the centralized and decentralized solutions in a spatial setting. In order to analyze these issues in the most intuitive way, we need to introduce some simplifying assumptions which partly abstract from reality. Specifically, polluting emissions are assumed to depend only on the population size while production activities are another important determinant of emissions (Marsiglio, 2017); output is assumed to be entirely consumed while partly is saved to allow for capital accumulation (Ramsey, 1928). Extending the analysis to account for these further issues is left for future research.

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