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SURVEY, EXPERIMENTAL TESTS AND MECHANICAL MODELING OF THE DOME OF PISA CATHEDRAL: A MULTIDISCIPLINARY STUDY

Keywords: Dome, Structural analysis, Limit analysis, FEM, Laser scanning, Photogrammetry

Abstract. *The present contribution illustrates the results obtained to date on an ongoing research study of the mechanical response and load capacity of the dome of Pisa Cathedral. A well-known feature of structural modelling is that it requires reliable data on the actual shape and material properties of the structure in question. Moreover, a comprehensive account of relevant historical and architectural aspects is needed as well. Hence, the starting point of our research work consisted of high-density, precision surveys. Both range-based (laser scanning) and image-based (3D photogrammetry) survey methodologies have been used to obtain different structural models. Furthermore, a set of experimental tests has been performed to evaluate the constituent masonry's properties. From the mechanical point of view, the research has focused mainly on structural analysis of the dome subject to vertical dead loads. The dome's mechanical behaviour is described by means of both analytical and numerical analyses. The results obtained via these different methods are discussed.*

1 INTRODUCTION

Pisa Cathedral is a world-renowned masonry monument, currently part of the UNESCO World Heritage *"Piazza del Duomo"*, which includes the baptistery, the bell tower (the famous 'Leaning Tower') and the cemetery. The Cathedral is an enormously important building from both the historical and architectural points of view, founded in 1064.

The Cathedral dome itself represents a structure of considerable interest as regards the construction techniques used and possible interpretation of its mechanical response. Since an adequate documentation on the construction process is lacking, the historical and architectural aspects related to the dome have been a matter of discussion [1]. As an example, some scholars have doubted that the dome could have been built as part of the original cathedral fabric, since its elliptical plan and pointed profile are unique in the coeval Roman architecture [2]; furthermore, close examination of its technical details and masonry properties [3] is very difficult. According to Sanpaolesi [4], the dome can be reasonably dated back to as early as the 12th century. This statement is supported by the detailed examination of the materials making up the dome (brick, tuff, stone of *S. Giuliano*) and the comparison with some contemporary domes (church of *S. Paolo a Ripa d'Arno*). If this dating is accepted, the dome of Pisa Cathedral can be considered the first dome in Europe that is visible from outside the building (a feature termed *estradossato*). Also its elliptical plan and pointed profile are unique in coeval Romanesque architecture. Sanpaolesi observes that the dome has undergone the most profound and ancient transformations on the outside. As for the external loggia with the eight underlying arches, performed starting from 1383 by Puccio di Landuccio, there is no clear evidence on the reasons that led the *Opera* to promote this project: they could be either completion or restoration works. According to the first hypothesis, this addition would have been necessary in order to access roofs and attics, as well as to favour water conveyance. In the second case it would instead be a hoop ring, perhaps due to the onset of some cracking in the dome.

The present research project, focused on the dome of Pisa Cathedral, took advantage of the opportunity provided by the scaffolding (Fig. 1) recently erected for the restoration works being carried out on the Cathedral in preparation for the 900th anniversary of its dedication (1118). This privileged circumstance, also thanks to the cooperation of the *Opera della Primaziale Pisana*, has allowed for thorough examination of the dome, which involved its building details, material properties and state of preservation. The final objective of the current research is to investigate the structural response of the dome, through a profound knowledge of its geometry and constituent materials, and by taking into account historical and architectural aspects as well.

The research activities performed so far and described in the following can be subdivided into two main stages. The first stage was to collect a substantial dataset in order to describe the dome's shape and the building materials' properties in some detail. To this end, an accurate geometrical survey of the intrados and extrados, as well as an extensive campaign of experimental tests have been performed. It should be observed that, to the purposes of the present paper, a detailed survey has been necessary because of the absence of dome surveys characterized by high-density and precision. Without being exhaustive, the *status quo* on the surveys of the Cathedral dome, from the most ancient ones up today is briefly reviewed in [5].

In the subsequent stage, the information collected by means of the geometrical surveys and experimental tests were then used as input data to the different models developed for the study of the dome's structural response. Although this study is still ongoing, a first set of results has been obtained via analytical as well as numerical models by limiting the analysis to vertical dead loads only. The choice to perform a preliminary study via analytical models working within the framework of limit analysis has brought, despite their simplicity, to some significant results concerning preliminary estimations of the dome of Pisa Cathedral safety factor. Further steps of the ongoing research will be devoted to the refinement and extension of the analysis, addressing the global stability conditions of the system formed by the dome and drum.

2 THE 3D GEOMATIC SURVEY OF THE DOME

The management, preservation, improvement and restoration of the world's historic architectural heritage commonly has to deal with the lack of high-scale, refined geometric surveys, which would allow proper planning of any required interventions and would provide the needed support for structural checks and modeling as well.

From this perspective, the monumental masonry construction, which is the object of the present research, is no exception. Although extensively studied in the past, the dome of Pisa Cathedral has to date never been subjected to a comprehensive survey of its extrados and intrados surfaces, performed by fully exploiting the potentialities of modern surveying technologies.

Taking advantage of the undisputed potentialities provided by 3D methodologies, the survey was aimed at creating a detailed 3D model of the structure, including both its interior and its exterior (Fig. 2, top). This has enabled achieving a number of important results, namely, precise measurements of the structure (at 1:50 or greater scale), detailed views of both structural and ornamental components, an idealized breakdown of the dome into its building units and determinations of the building materials used.

The presence of scaffolding inside and outside the Cathedral (Fig. 1) made it possible to access the dome and closely examine its intrados and extrados. However, the visual obstruction consequent to the internal scaffolding during its setup in stage, made it necessary to proceed with the geometry survey by steps, as illustrated in [5].

Fig. 1. The recently erected scaffolding for the restoration works,
in preparation for the 900th anniversary of the Cathedral dedication.

In order to achieve a 3-D model featuring both high geometric precision and high-quality photorealistic texture, two surveying methodologies have been combined, *i.e.* Terrestrial Laser Scanning (TLS) and photogrammetry [6]. TLS usually allows for geometric precision, independent of the surveyed object's surface texture, whereas photogrammetry-based models, which provide high-resolution textures (both geometric and chromatic), yield variable, texture-dependent precision levels [7].

Although the surveying methodologies for conventional buildings are well established, no such standard workflow exists for complex masonry constructions. This is true even more so in the case at hand, where working construction and restoration sites prevented laying out an ideal surveying network. Major factors contributing to the overall complexity of the survey included the unsteadiness of some of the structures on which total stations and laser scanners were set and the linkup path between the interior and exterior.

The first abovementioned issue affecting the short-term stability of the survey reference system was dealt with by setting up and redundantly measuring a large number of steady points. As regards the second issue, reciprocal visibility between the interior and exterior was achieved solely through two small doors at the bottom of the tambour; this required collimating six interior points from the outside, though at distances of less than 1 m and in a nearly straight spatial layout. These also provided the survey with the necessary rigidity by acting as end points for different survey paths, which included eight ground-level points on the Cathedral's exterior (Fig. 2, bottom).

Advances in digital photogrammetry, in particular regarding Structure from Motion (SfM) and Multi-View Stereo (MVS), ensure good results even under extreme image taking conditions, which, on the contrary, would make analogue or analytical photogrammetry unusable [8].

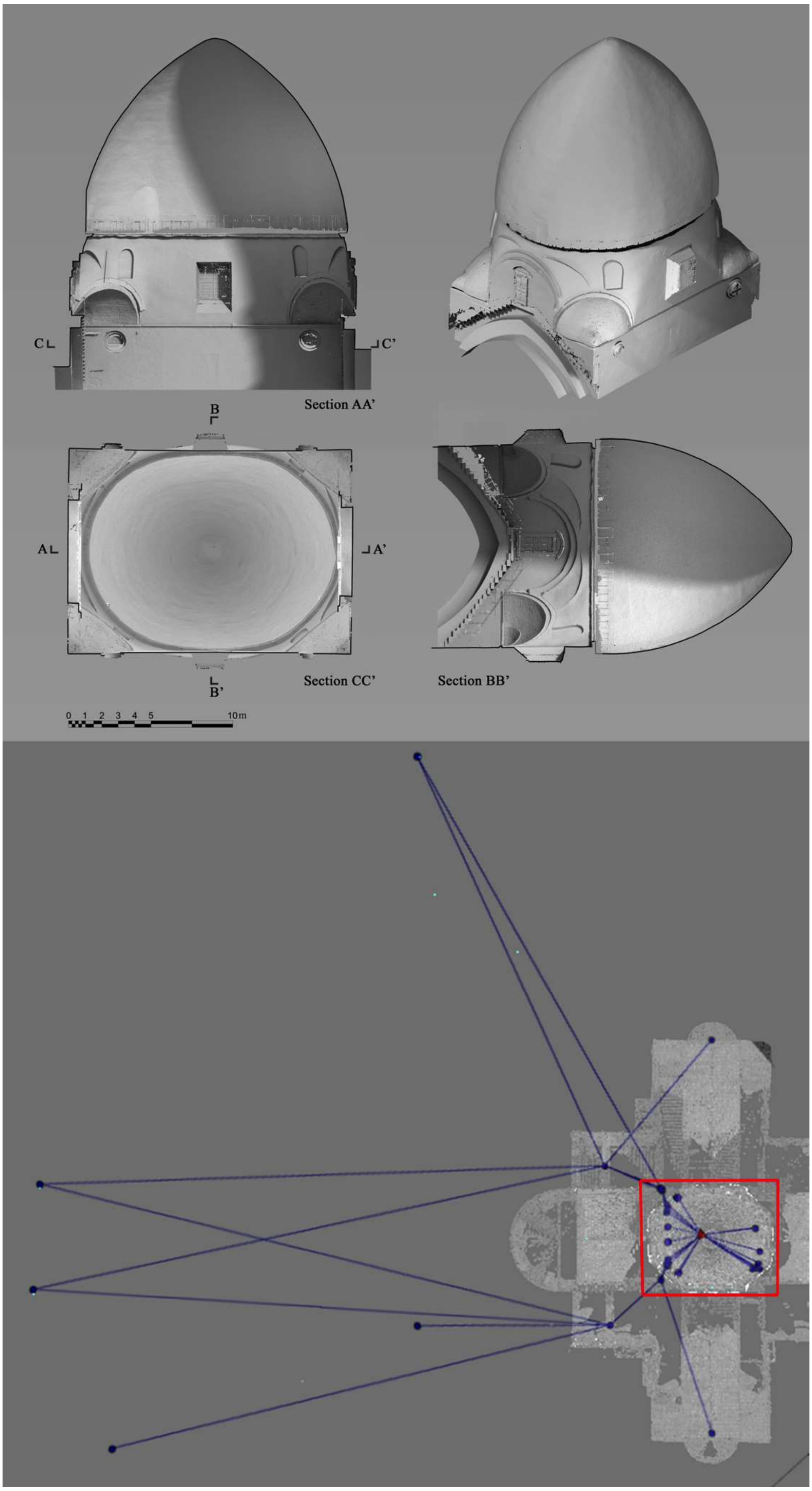


Fig. 2. Sections and axonometric projection (top);
Survey network for reference system definition, 2D view (bottom).

3 IDEAL SURFACES APPROXIMATING THE DOME INTRADOS AND EXTRADOS

The dome's actual intrados and extrados surfaces, which are obviously irregular, have been approximated by smooth surfaces having relatively simple analytical expressions. The approximating surfaces' parametric representation, obtained as described in the following, has been used in all the structural analyses reported in the next section.

The already completed first stage of the research has focused on studying the part of the dome above the colonnade of the external *loggia*, corresponding to the level where the lead covering begins. The dome's top, in the neighborhood of the central hole, has also been excluded from the analysis because of the sharp variation in the curvature, which can be clearly observed from inside the cathedral. The interaction between the dome and the external colonnade, connected to the dome through stone beams and vaults, will be investigated in a continuation of the research.

The actual shapes of the dome intrados and extrados have been carefully reconstructed by means of the laser scanner survey, whose result consists of a cloud of millions of points. Such point clouds are however not suited for use in finite element codes for structural analysis. In order to implement a finite element code, some post processing is generally needed to extract a coarser representation of the surfaces from the point cloud and to build, for example, a mesh with polygonal surfaces. For this study, we have used a post processing procedure that yields the dome's intrados and extrados horizontal sections in CAD format, from which a representative subset of points has been selected. The coordinates of these points have been used as input data to find the analytically determined best approximating surfaces (in the following denoted as *ideal* surfaces).

The analytical expressions of the approximating surfaces have been determined (as suggested by the laser-scanner survey output) by assuming that:

1. the ideal surface level curves are ellipses, approximating the actual oval level curves; all ellipses share the same projection onto the base plane of their center and principal axes;
2. the end points of all ellipses' semidiameters, for both principal directions, lie on a circumference arc.

It is worth stressing that while the assumed ideal surface shape is just one of many possible alternatives, it is described by relatively simple expressions in terms of a few parameters.

The search for the parameters' optimal values has been performed by means of a simplified automatic procedure, which makes use of a minimization routine in Mathematica®. The procedure consists of a geometric fitting operation in which the object function to be minimized is set equal to the sum of the squares of the distances between the points selected from the laser scanner survey and the ideal surface. In order to minimize the error, the Mathematica® "NMinimize" function was used.

The average value of the oriented distances between the survey points and the corresponding points on the surface, positive if the point is outside the surface and negative contrariwise, is null for both the intrados and the extrados. The average value of the absolute distances is equal to 7 centimeters for the intrados and 6 centimeters for the extrados. By way of example, the survey points and ideal surfaces are shown on two representative sections in Fig. 3.

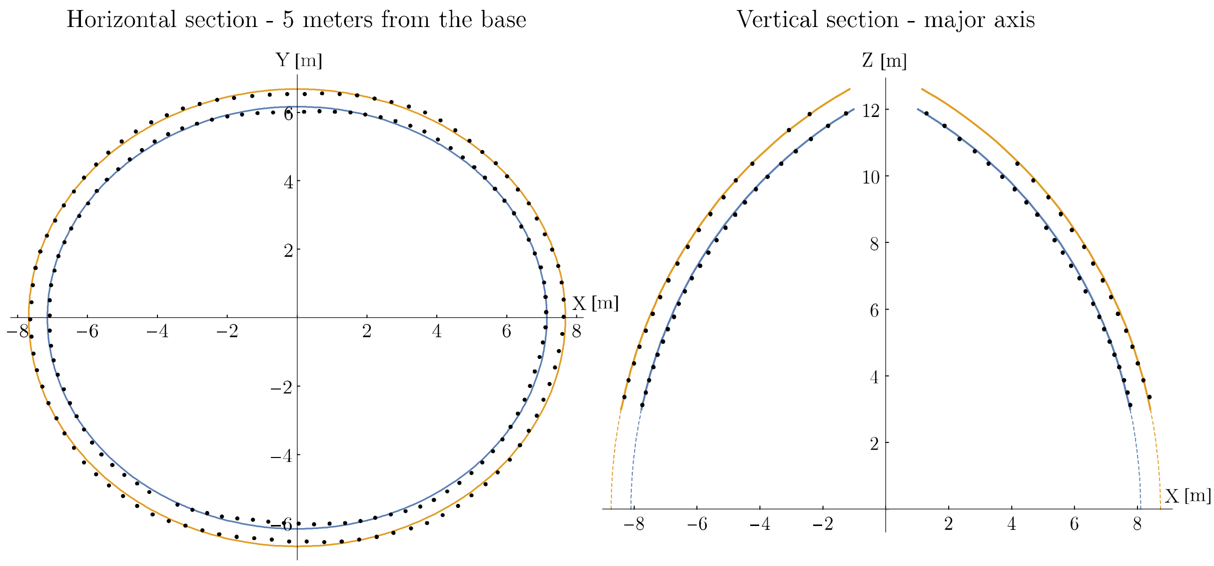


Fig. 3. Comparison between the survey points and the "ideal" intrados and extrados surfaces.

4 MASONRY PROPERTIES AND EXPERIMENTAL TESTS

The experimental campaign on the dome of Pisa Cathedral included execution of endoscopic tests, flat-jack tests, mechanical analysis on masonry samples obtained by core drilling, continuous and periodical monitoring of open cracks (Figure 4).

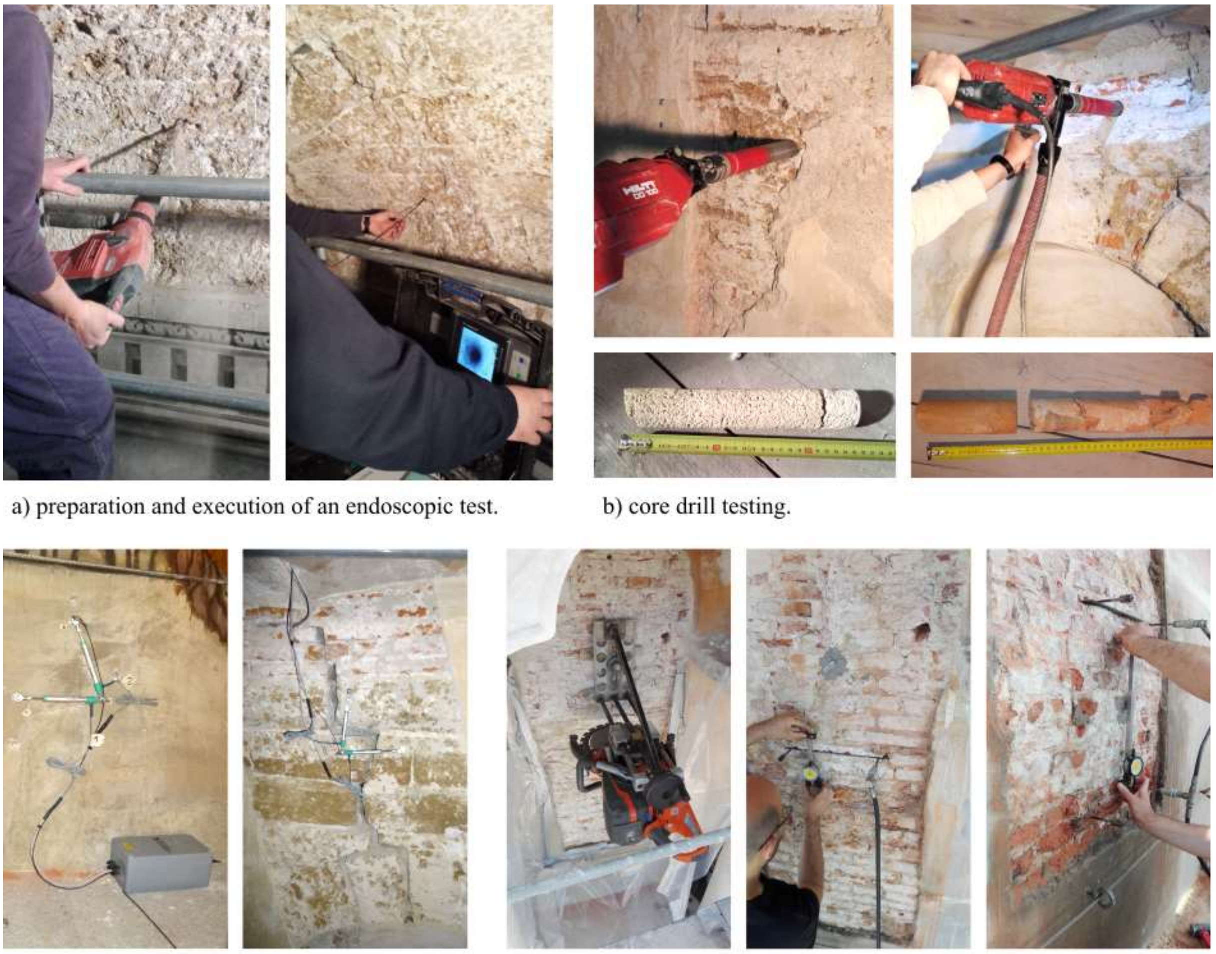
Endoscopic tests were performed to investigate the internal stratigraphy of the masonry dome. A Boviar endoscope was introduced into suitably prepared and cleaned 16 mm diameter holes. The different types of masonry identified were sampled through dry core drilling. Cylindrical specimens were obtained from the samples and subsequently subject to chemical or mechanical analyses.

The endoscopic tests and core-drilling samples enable verifying the dome's actual thickness, as well as the kind of masonry contained within. In agreement with historical documentation, a constant thickness of some 60 cm has been found. Uniform brick masonry seems to have been used in constructing the dome; no internal core has been detected.

Flat-jack tests were executed on four unplastered areas of the dome extrados, located at the level of the external *loggia* in correspondence to the four cardinal points. Single flat-jack tests were executed at all four points to evaluate the masonry's internal compressive stress, while double flat-jack tests were performed at the western and southern points to estimate the main mechanical characteristics of the constituent masonry (Figure 5).

An average compressive stress of about 0.05 MPa has been estimated from the results of the single flat-jack tests. The mean compressive stress level obtained from the single flat-jack tests turned out to be of the same order as that deduced from the numerical finite element analyses, which will be illustrated in the following.

The two stress-strain diagrams that have been obtained from the double flat-jack tests show some not negligible differences (Figure 6). More specifically, stiffness of the southern-side test MD4 turned out to be consistently lower than that of the other test MD2. The lack of stiffness clearly detectable in the diagram obtained from the test MD4 can reasonably be attributed to a local weakening of the constraint exerted from the rest of the masonry on the panel being tested. The absence of an effective constraint on which to contrast made it necessary also to limit the working pressure of the jacks. By considering the results yielded by the double flat-jack test MD2, the masonry Young's modulus, E, and compressive strength, f_c, have been set respectively equal to E = 3 GPa and f_c = 3 MPa.



a) preparation and execution of an endoscopic test.

b) core drill testing.

c) installation of the monitoring system.

d) preparation and execution of single and double flat jack tests.

Fig. 4. Experimental tests performed on the dome.

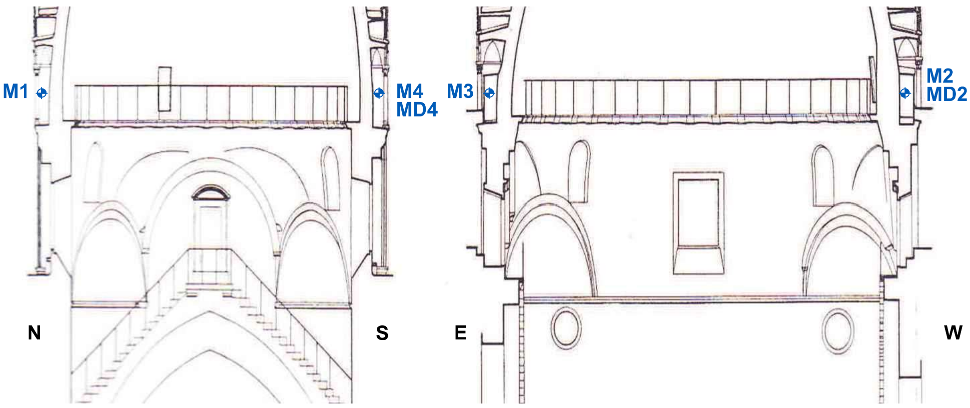


Fig. 5. Positions of single and double flat-jack tests.

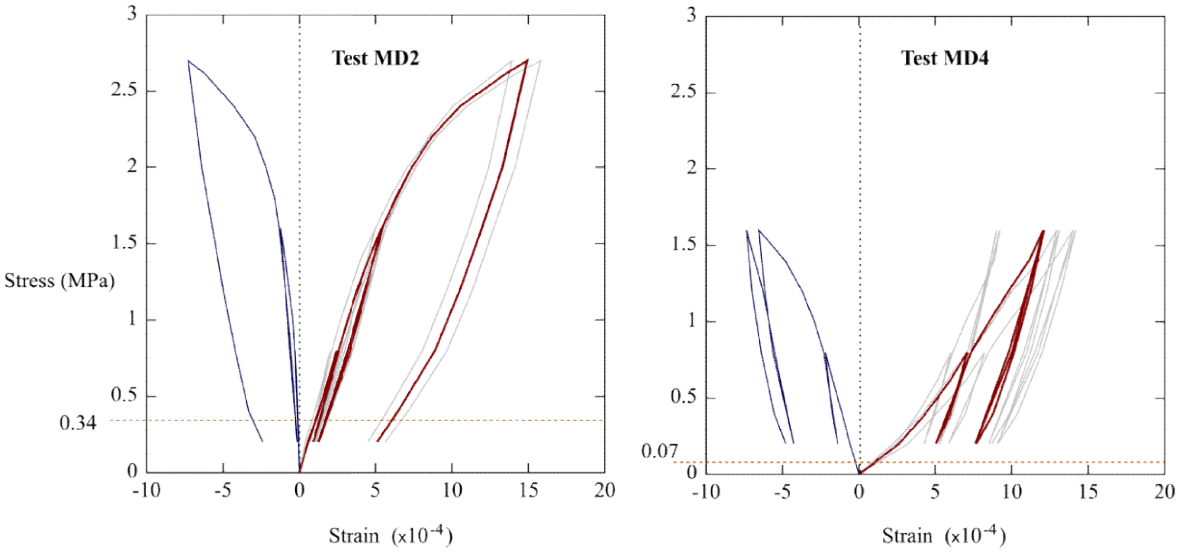


Fig. 6. Conventional stress-strain diagrams obtained from the double flat-jack tests: (blue) horizontal strain; (red) vertical strain; dashed orange lines represent compressive stress levels estimated within the masonry (compressive stresses are positive).

The experimental survey included monitoring of the cracking pattern clearly observable on the dome intrados. Any further opening of each of the four large vertical cracks spreading from the base in an approximately vertical direction was monitored by means of both a continuously operating automatic system of transducers and a set of manual point measurements. In this way it has been possible to check for potential crack enlargements over an observation period of about 6 months. From an initial analysis of the data collected, no significant evolution of the cracking pattern has emerged. By way of example, Figure 7 shows the crack opening values recorded by the continuously operating automatic system of transducers during 2017. The recorded relative displacements vary accordingly to the temperature variations over the seasons. No long-term trends have been observed.

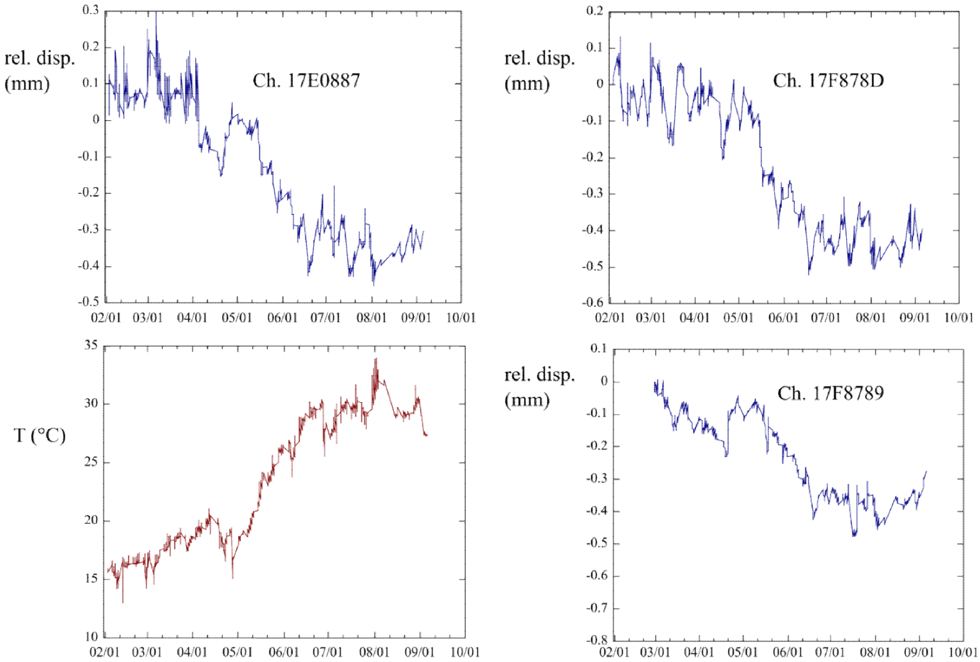


Fig. 7. Evolution of crack opening (mm) recorded by the continuously operating automatic system of transducers during 2017 (bottom-left: temperature recorded within the same period).

5 STRUCTURAL MODELS OF THE DOME OF PISA CATHEDRAL

The information collected by means of the geometrical surveys and experimental tests described in the preceding section were used as input data for the two different analytical models developed to study the dome's structural response. The same idealized dome shape, obtained as described in the previous section, has been used as input for the expressly developed structural models in order to enable comparison of the results yielded by the different methods.

The present paper presents some preliminary results of an ongoing study aimed at deeply understanding the structural response of the dome. The analysis does not pretend to be complete nor exhaustive, and the following illustrates the results for vertical dead loads alone. The effects of horizontal loads, as well as those produced by settlement of the underlying support structures will be considered in a subsequent stage of the research, which is still ongoing. The dome structural safety factor against vertical loads is investigated by considering the dome portion standing above the external loggia in its current conditions. In this regard, the complex arrangement of transition elements on which the dome rests, as well as the cracking pattern, which spreads from the underlying octagonal drum and extend to the bottom part of the dome, are not comprised within the present analysis. Moreover, monitoring of the clearly detectable cracks, already reported by Sanpaolesi during his repairs of 1957 [4], enables considering them as a stable cracking pattern.

Two analytical models have been formulated within the framework of limit analysis. The first searches for statically admissible stress fields through determination of suitable thrust surfaces contained within the dome thickness; the second model is instead a modern reinterpretation of Durand-Claye's method for domes. We recall that, since the present analysis is aimed at determining statically admissible stress fields within the dome, accordingly to the static theorem of limit analysis the results obtained would not be affected by possible settlements of the support or stable cracking patterns.

As an additional term of comparison, a first set of linear elastic numerical analyses has been carried out by means of FE models. The adopted structural models can reasonably be considered simple compared to more burdensome numerical models. Nonetheless, some significant results are obtained, concerning preliminary estimations of the dome of Pisa Cathedral safety factor. As will be illustrated in the following, the results yielded by the linear FE models show that the dome is compressed almost everywhere; moreover, FE models turned out to be in agreement with the structural analysis based on the static theorem of limit analysis.

As already stated in the Introduction of the present paper, we remark that an adequate documentation on the dome construction process is missing. In particular, for our aims, it is worth observing that there is no clear evidence about the reasons of the construction of the external loggia, executed starting from 1383 by Puccio di Landuccio [4]. It could be either completion or restoration works. In this first step of structural analysis, the reinforcing masonry ring provided by the external loggia has been supposed to contribute to fully sustain the thrust of the dome.

5.1 Thrust surfaces for determination of statically admissible stress fields

The well-known static theorem of limit analysis enables evaluating a structure's safety level by searching for statically admissible stress fields. Herein, the safe theorem is assumed to hold true for the dome, and the existence of at least one statically admissible stress field is checked, thus ensuring the structure's stability. In the analysis, Heyman's hypotheses [9] are adopted for the material, *i.e.*, the masonry is assumed to have no tensile strength. Moreover, no sliding is allowed between masonry units.

The search for statically admissible stress fields is performed by using the concept of *"thrust surface"* [10], which for our purposes is defined as the average surface of a thin shell in equilibrium with the external loads, wholly contained within the dome thickness and in which only membrane forces are present (from a mechanical point of view, such a thin shell can be considered representative of the resisting part of an ideal dome). It can be shown that if it is possible to find a thrust surface wholly contained within the masonry thickness, and if the principal stresses of the corresponding membrane stress field are compressive (negative) throughout, then according to Heyman's hypotheses the structure can be considered able to withstand the action of the external loads, and is therefore safe.

As is well known, the membrane equilibrium problem can be conveniently expressed by means of the Pucher formulation [11], which makes use of a suitable potential function, the Airy's stress function. In this way, it becomes possible to express the problem in terms of a single second-order partial differential equation to be completed by proper boundary conditions on the free edges of the thrust surface. The stress state at any given point on the thrust surface complies with Heyman's hypotheses provided that two conditions are fulfilled regarding the sign of the stress tensor invariants. As illustrated in [12], it can be proved that such conditions are equivalent to requiring that the stress function be concave. Solution of the equilibrium problem of a thin shell in a membrane stress state is therefore reformulated as solution of a single second-order partial differential equation for both the stress function and the function describing the surface (both scalar functions), under the condition that the stress function be concave.

In the problem at hand, it can be shown that, in the case only vertical loads are present, it is always possible to find a statically admissible stress field defined on an elliptical paraboloid shaped thrust surface. Thus, checking the dome's safety is converted into a purely geometrical problem.

A first assessment of the dome's safety level is performed by determining the *geometric safety factor*, ν_g, which is defined as the ratio between the actual dome thickness and the minimum thickness that would yield at least one statically admissible thrust surface. If the complete set of all possible statically admissible thrust surfaces were considered, the true minimum thickness and the corresponding actual geometric safety factor could be determined. In the current analysis only a subset of statically admissible thrust surfaces has been taken into account, namely elliptical paraboloid-shaped thrust surfaces. Therefore, only a rounded down approximation of the geometric safety factor has been determined. The results obtained by a trial-and-error graphical procedure allow us to conclude that the dome's geometric safety factor is at least 1.50. Hence, ideally, a certain amount of material could be removed from the structure in its current configuration before it collapses. More precisely, the result obtained means that the dome would be safe even if its thickness were only 2/3 of its actual value, provided that the masonry compressive stress is unbounded, so as to allow the thrust surface to reach the dome intrados or extrados. The dashed line in Fig. 8 shows the minimum thickness extrados obtained via this procedure.

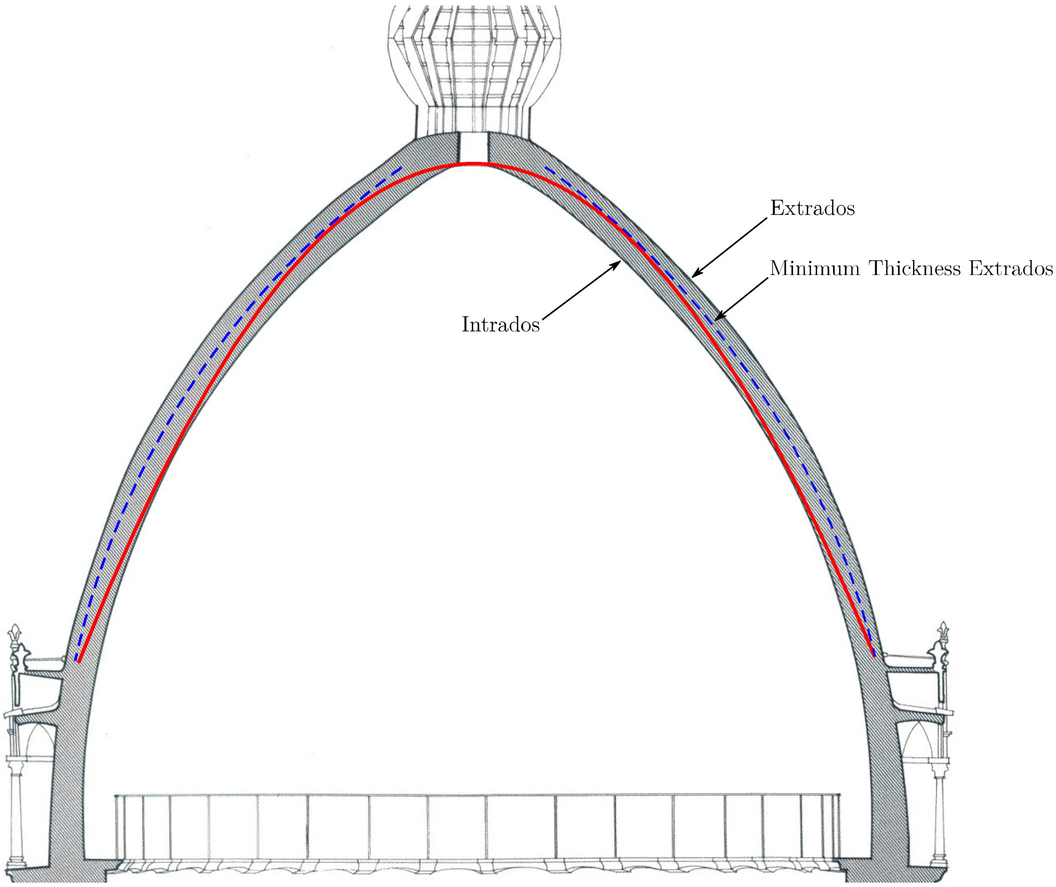


Fig. 8. The dome of minimum thickness used for assessing the geometric safety factor ν_g.

In the analysis illustrated so far, the masonry's compressive strength has been assumed unbounded. Consequently, the magnitude of the compressive stresses within the masonry can be disregarded. In order to propose an alternative assessment of the dome safety level able to take into account limited masonry compressive strength, a second analysis has been performed by taking into consideration a suitable collection of thrust surfaces as far as possible from the dome intrados and extrados.

Since the thrust surface has been defined as the mean surface of the resisting part of the dome, it is straightforward to conclude that half the thickness of the resisting part, say t_r, cannot exceed the distance of the thrust surface from the extrados or intrados, whichever is smaller. By hypothesis, the resisting part is assumed to behave as a thin shell in a membrane stress state. Hence, the maximum stress in the resisting part, σ_{max}, can be computed as $\sigma_{max} = N_{max}/t_r$, where N_{max} is the maximum (compressive) axial force. Accordingly, the mechanical safety factor, ν_m, is defined as the ratio between the masonry compressive strength, f_c, and the maximum compressive stress:

$$\nu_m = \frac{f_c}{\sigma_{max}}. \tag{1}$$

In order to provide an estimate of the mechanical safety factor, the idea is to identify the resisting part whose thickness is as large as possible and which is wholly contained within the masonry thickness, so as to reduce the internal stresses as much as possible (in absolute terms).
The mechanical safety factor has been estimated by determining the maximum compressive stress for several thrust surfaces, all lying in the neighborhood of the dome's mean surface. The best estimate has been obtained by considering the stress distribution shown in Fig. 9. The results obtained from the analysis enable concluding that $\nu_m \geq 4.60$.

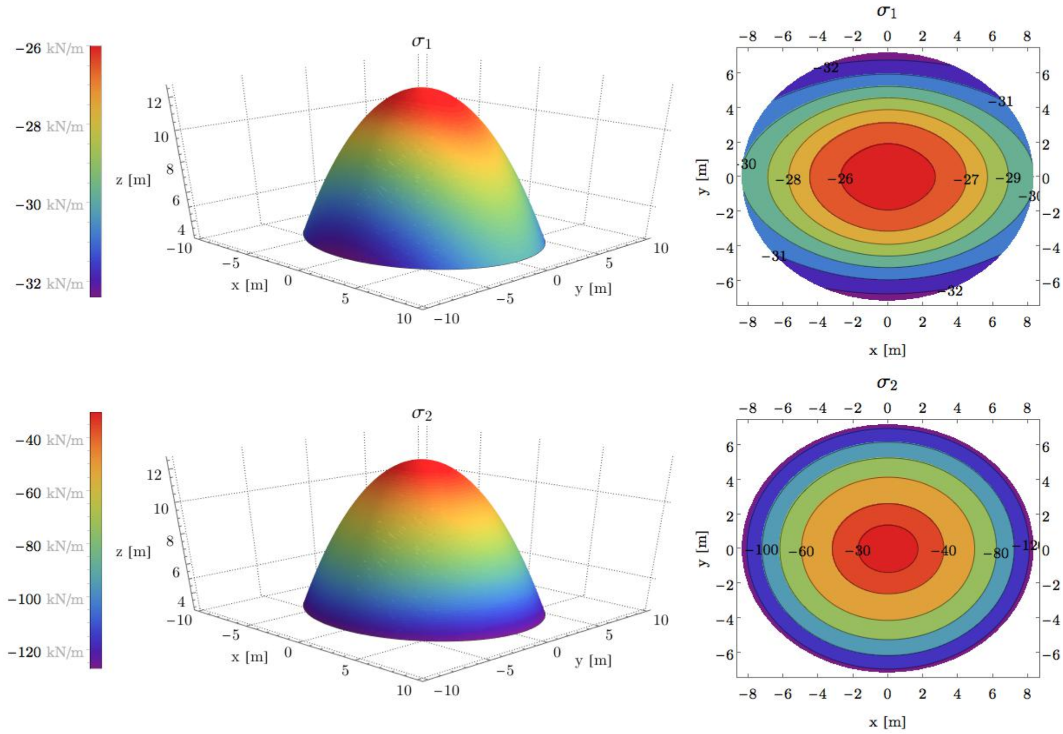


Fig. 9. Plot of the principal stresses σ_1 and σ_2 corresponding to the thrust surface used for assessing the mechanical safety factor.

It is worth noting that there is a very large difference between the mechanical and geometrical safety factor ($\nu_g = 1.50$, $\nu_m = 4.60$), both under the vertical loading condition representative of the dome self weight. This finding is fully consistent with the very different meaning of these two safety factors. While the geometrical safety factor measures the amount of material that could ideally be removed from the dome before a collapse condition is attained (assuming unbounded masonry compressive strength), the mechanical safety factor measures how much the dome weight could be increased before collapse occurs (this instead under the assumption of limited compressive strength). Hence, it can be concluded that the dome self-weight could be increased more than four times before the compressive stresses reach the masonry's compressive strength.

This finding confirms that, as already argued by Heyman, the collapse of a masonry structure is seldom due to compression failure of the material, and that it is the construction's shape that governs its collapse conditions.

5.2 A modern reworking of Durand-Claye's method for the equilibrium analysis of the dome

Like the first model described in the previous section, the second analytical model used to study the Pisa dome's mechanical response also falls within the framework of the static theorem of limit analysis. More precisely, this section reports on some results of a preliminary study performed by means of a modern reworking of Durand-Claye's method.

The starting point of such analysis is the so-called *"stability area method"*, *i.e.* the 19th-century graphical procedure originally introduced by Durand-Claye [13] for symmetrical masonry arches, which he later extended in order to assess the equilibrium of *voûtes sphériques*, *i.e.* domes of revolution [14].

With reference to the stability of masonry arches, Durand-Claye's contribution aims at determining the set of statically admissible solutions by means of a graphical method able to define the so-called *area of stability* at the crown section of a symmetric arch.

At each arch joint *a-a* (Fig. 10, left), the cross-sectional bending moment and shear capacities are determined as functions of the masonry compressive and tensile strengths and friction coefficient along the joints. By scanning all joints along the arch, a double set of limitations can be obtained and represented graphically in the (P, e) plane, where P and e denote the crown thrust and its eccentricity with respect to the cross section's center of gravity, respectively. The so-called *area of stability* is obtained by considering the region formed by all the points of coordinates (P, e) that fulfil all of the aforementioned limitations (Fig. 10, right). Thus, starting with an analysis of the *stability area*, the set of statically admissible *lines of thrusts* can be identified. When such an area shrinks to a point (or a segment) the arch attains a limit equilibrium condition, *i.e.*, there is only one admissible value left for the crown thrust.

This method has been extensively described in some works by Aita *et al.* with reference to symmetrical masonry arches [15-18] and applied to both symmetrical masonry arches and domes of revolution [19, 20]. In particular, in [20] some critical remarks on the original method have been advanced, with reference to its application to domes of revolution. Here, it is recalled that when Durand-Claye applied his method to the domes of revolution, he imposed equilibrium on each of the dome's lunes obtained by conducting meridian planes. Each lune is assumed to be independent of the others, under the assumption that tensile hoop stresses are not admissible in masonry structures. Thus, in its original form, the Durand-Claye approach does not adequately account for the influence of hoop forces on dome stability. Furthermore, it ignores some aspects regarding kinematic compatibility issues of the corresponding collapse mechanisms.

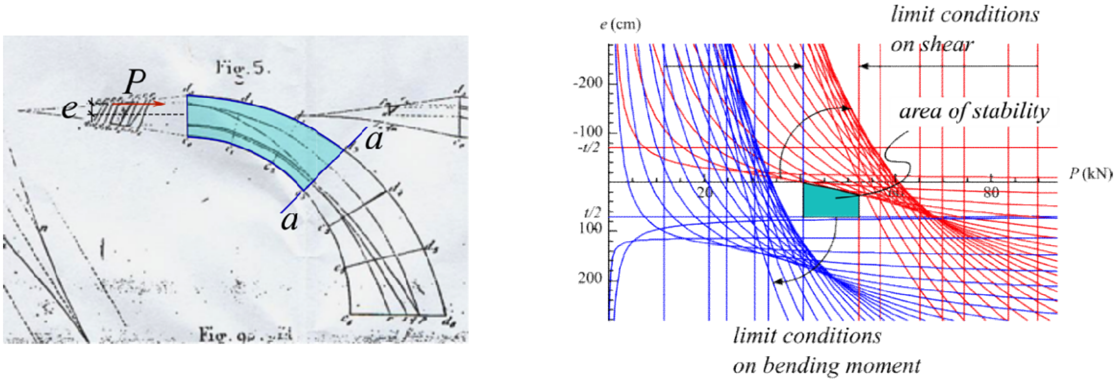


Fig. 10. Durand-Claye's method for masonry arches. Left: the generic joint *a-a* (reworked from [13]); right: the *area of stability* obtained by scanning all the joints *a-a*.

Despite the above-mentioned critical remarks, the method proposed by the French scholar is nevertheless deserving of attention, as it provides a comparatively easy way to obtain all the statically admissible solutions. Aita *et al.* have thus extended its application to assessing the stability of the domes of revolution in order enable suitably taking into account the influence of compressive hoop forces [19, 20]. The *area of stability* has been determined by means of an in-house, expressly developed algorithm implemented in Mathematica. Despite its simplicity, the method enables finding a conventional limit thickness for the dome and adequately defining a safety factor.

With reference to the structural response of the dome of Pisa Cathedral, it is necessary to duly consider its peculiar geometry, which is elliptical in plan. As a starting point, in place of the actual elliptical dome, we have considered two theoretical domes of revolution, loaded by their own weight alone and with meridian sections corresponding to those of the actual dome along the major and minor axes of its elliptical plan, respectively (Fig. 11). The masonry making up the two ideal domes has been assumed to have infinite compressive strength and nil tensile strength; moreover an infinite friction coefficient has been assumed along the joints.

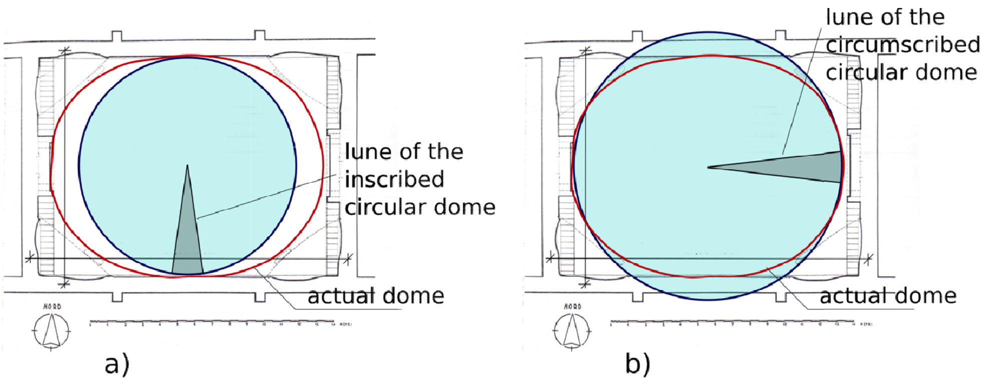


Fig. 11. The two theoretical domes of revolution: (*a*) inscribed dome; (*b*) circumscribed dome.

It is worth noting in passing that since the load condition is reduced to the dome self weight, very similar results would be obtained by supposing limited compressive strength.

A first set of results has been obtained by applying the Durand-Claye method in its original formulation to the two theoretical domes of revolution described above. A single lune (of amplitude ζ), ideally extracted from both the two theoretical domes has been considered as an independent arch, subjected to its own weight and a horizontal thrust, P, of eccentricity e, acting at the ‘*crown*’ joint. The profile plotted in Fig. 12a, corresponding to the theoretical dome of revolution circumscribed on the actual dome, is obtained by considering the vertical meridian section along the major axis of the actual dome. The curves in the (P, e) plane plotted in Fig. 12b show that the stability area is the empty set; thus, this simplified approach, which neglects any interaction between adjacent lunes, would not be compatible with a thrust surface wholly contained within the dome thickness. For example, in Fig. 12a, the thrust line *a*, corresponding to the green point plotted in Fig. 12b, falls outside the lune’s thickness above joint θ_1.

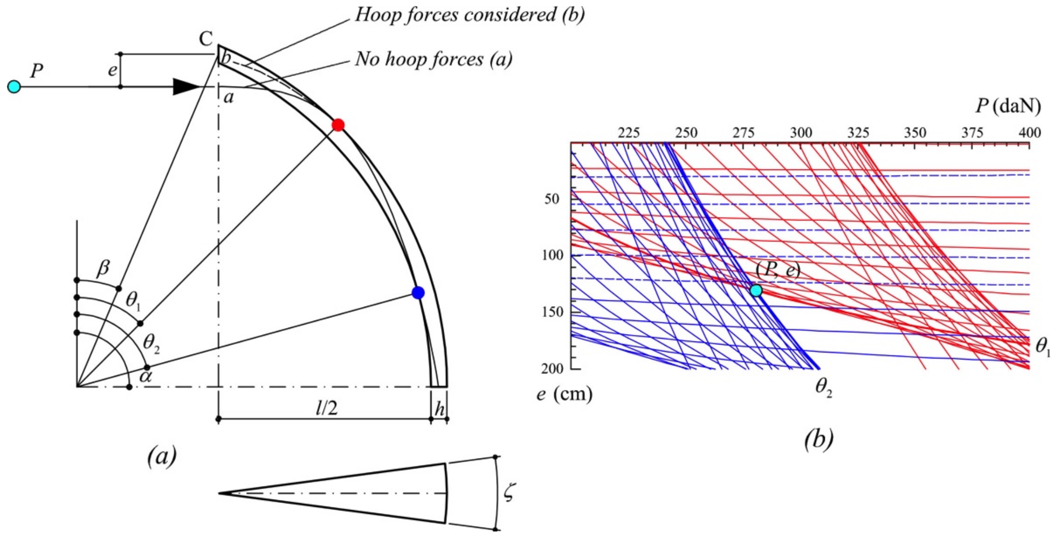


Fig. 12. The dome meridian profile along the major axis of its elliptical plan: thrust surfaces *a* and *b* (*a*); the *“stability area”* referring to the corresponding dome lune of amplitude $\zeta = 1°$ (*b*).

This preliminary and simplified analysis has been extended by also considering the presence of some hoop forces acting on the lateral surfaces of the *voussoirs*. A more detailed description of the expressly developed procedure is available in [20]. The presence of hoop forces allows for finding statically admissible solutions. By way of example, the thrust line *b* plotted in Fig. 12a is obtained by considering the presence of hoop forces above joint θ_1. It is a simple matter to verify that it falls within the lune’s thickness. Analogous results have been obtained by examining the second theoretical dome of revolution, *i.e.* that obtained by the meridian profile along the minor axis of the actual dome.

The equilibrium analysis performed here allows for easily determining the geometric safety factors, ν_g, for the two theoretical domes. As recalled in Section 5.1, ν_g is defined as the ratio between the dome thickness and the minimum thickness that would enable finding at least one thrust surface fully contained within the masonry. Since the geometric safety factors are, respectively, $\nu_g = 1.99$ and $\nu_g = 2.41$, the stability of the two theoretical domes of revolution is guaranteed.

This last finding suggests that the actual dome of Pisa Cathedral can also be considered to be a safe masonry structure when the loading condition is limited to its own weight. In this regard, Rankine’s theorem of the transformation of structures [21, 22] can be exploited. Since a geometric correspondence can be easily established between the two domes of revolution and the actual one, elliptical in plan, the equilibrium analysis of the two theoretical domes can be reinterpreted as a tool to assess the stability of the actual dome.

We would like to remark that, unlike what was done in Section 5.1, the tracing of the thrust line in the dome can be ideally extended up to the dome's springing (corresponding to angle α = 90° in Fig. 12a), by considering also the ideal dashed portion represented in Fig. 3.

5.3 Finite element models of the dome: some first results

The information collected on the dome shape and material properties were used as input data to the finite element (FE) models specifically developed for the dome structural analysis. The FE model meshes were built using the analytical expressions for the dome intrados and extrados ideal surfaces described in section 3. The ideal surfaces were used instead of the cloud of points directly provided by the laser scanner survey so as to enable comparing the results yielded by the analytical methods with those obtained by the numerical model. The masonry has been modelled as a homogeneous, isotropic, linear elastic material. As per the results obtained from the *in situ* experimental survey, Young's modulus and Poisson's ratio have respectively been set to E = 3 GPa and ν = 0.24. Lastly, the dome has been assumed to be perfectly clamped at its base.

A well-known property of masonry constructions is that they usually exhibit a markedly nonlinear response. Due to their intrinsic inability to transmit appreciable tensile stresses, the onset and spread of cracking is very likely to ensue even at load levels quite low relative to the ultimate collapse values. Consequently, severe alterations of the stress field with respect to the linear elastic solution are expected in the general case. Hence, it is clear that the results obtained via linear elastic FE models described in the following cannot be considered a close approximation of the actual stress field that would affect the dome. Nonetheless, they can still be considered useful in an important aspect, that is, the overall qualitative stress distribution yielded by the linear elastic analysis can help identify the regions in which cracking due to the tensile stresses is more likely to be expected, as well as those in which large compressive stresses could appear.

The dome's structural behaviour was analysed by means of two different FE models, composed respectively of shell and solid elements, both developed by using the Ansys® software package. The models were formulated in parametric form, thereby making it easier to refine each mesh as needed. The models' minimum overall number of degrees of freedom (DOF) has been determined considering the variations in the natural frequency values corresponding to increasingly refined model meshes. In particular, the natural frequencies become almost constant from a minimum required DOF number onward. In this way the minimum DOF number has been estimated as 150,000 for the solid model, and 60,000 for the shell model, corresponding respectively to some 50,000 and 10,000 nodes. In the numerical structural analyses described in the following, the solid model consisted of 58,560 nodes, while shell model consisted of 11,712 nodes.

The dome's structural behaviour was analysed under the effect of wind and self-weight loads. The results on the wind loads clearly shown that the stresses and strains were very low compared to those due to the dome's self weight. Hence, the wind load was subsequently disregarded. Snow loads have not been considered either because the steep slope of the dome surface does not allow for snow accumulation.

The two different FE models used for the analysis of the vertical loading condition representative of the dome self weight yield very similar results in terms of stresses, displacements and natural frequencies. The dome is mainly compressed in the meridian direction, with compressions increasing from the top to the base (Fig. 13a). Tensile stresses are very low compared to compressions, and can be disregarded, as shown in Fig. 13b. It is worth noting that the tensile stresses are restricted to a very small region within the meridian. The large blue region (solid model) and white region (shell model) in Fig. 13b both correspond to the area covering almost the entire meridian, where the principal stresses are negative or, in other terms, where tensile stresses are absent.

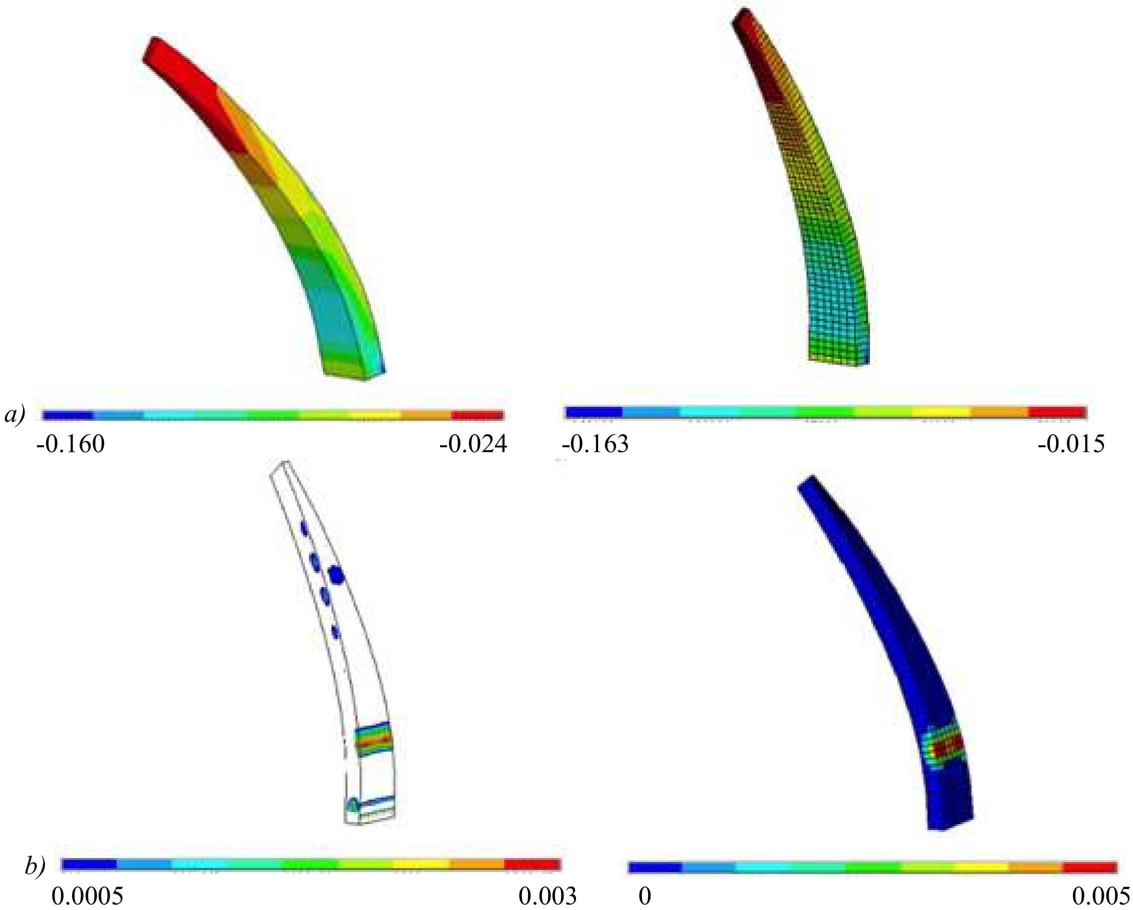


Fig. 13. Stress distribution along the minor axis meridian: *a*) principal compressive stresses (expressed in MPa) obtained from the shell (left) and solid (right) model; *b*) principal tensile stresses (expressed in MPa) obtained from the shell (left) and solid (right) model.

Fig. 14 shows the distribution of circumferential normal stress, where it is evident that the circumferential compressive stresses are larger at the top and decrease towards the base. The presence of the aforementioned, barely appreciable tensile stresses is confirmed, with a maximum value of 0.005 MPa at about two meters from the base.

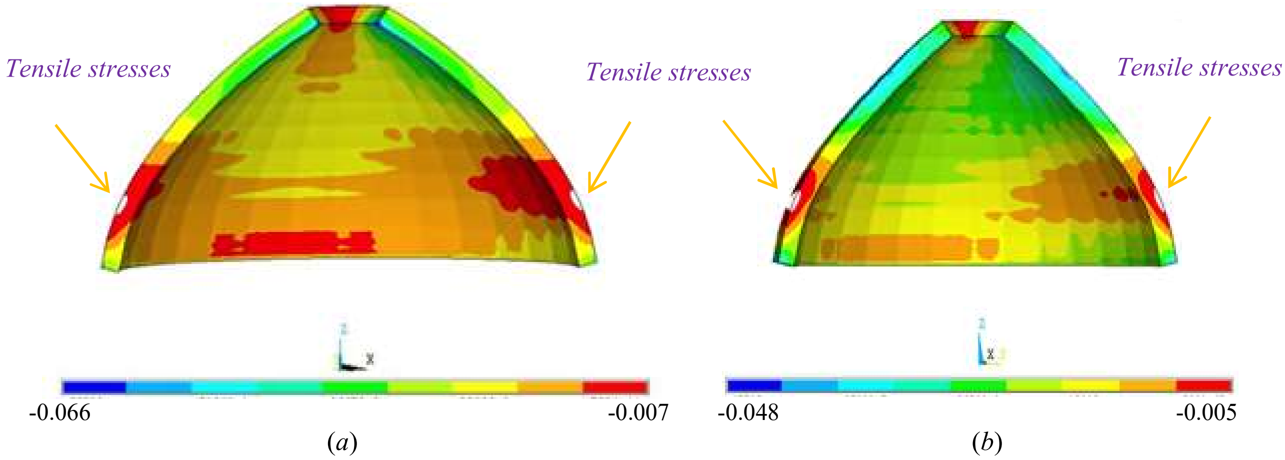
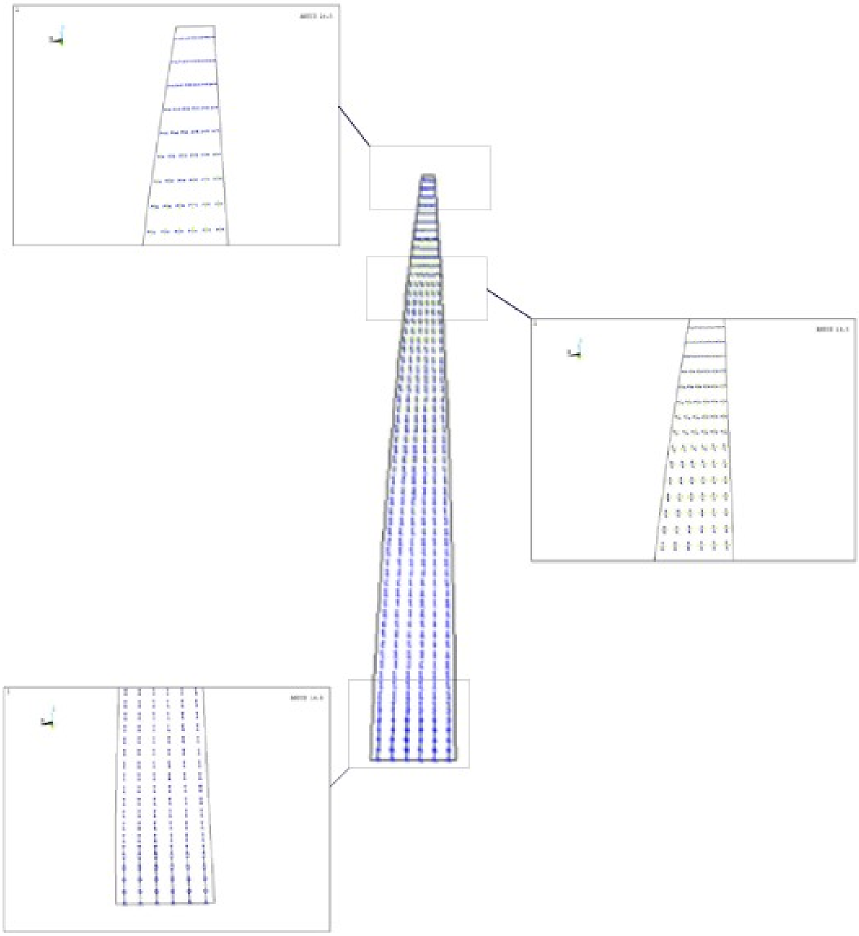


Fig. 14. Amplitude of circumferential compressive stresses in the dome, expressed in MPa:
a) major axis section; *b*) minor axis section.

The orientation of the direction of the principal maximum compressive stress is another point worth discussing. A change in the orientation of principal directions is clearly observed along a meridian. At the dome base, the maximum compressive stress is oriented along the meridian, approximately in the vertical direction (Fig. 15a). On the contrary, in the dome's upper part, the maximum compressive stress is oriented along the parallels (Fig. 15c). In the intermediate region, the maximum stresses gradually rotate from the meridian to the parallel direction (Fig. 15b).



(*a*)

(*b*)

(*c*)

Fig. 15. Principal maximum compressive stress directions at the dome's top (*a*), at about two meters from the top (*b*) and at the dome's base (*c*).

As already stated, in the finite element analysis a linear elastic behavior has been assumed for the masonry material. The linear elastic solution shows that the dome, subjected to its own weight, is compressed everywhere, except for a very narrow region of less than 3 % of the overall volume, at about two meters from the base, in which very low tensile stresses equal to some 0.005 MPa appear. As a consequence, in this case the linear elastic model – despite its simplicity – can be considered a reasonable approximation of the 'actual' response of the dome structural scheme.

As tensile stresses are negligible and restricted to a very small region within the dome, it is reasonable to assume that the dome could have been able to withstand the load due to its self-weight without developing any appreciable cracking, provided that settlement of the dome base would actually be limited. Therefore, although further studies are needed to determine the actual cause of the current cracking pattern, it could, as a tentative hypothesis, be assumed that the vertical cracks clearly visible on the intrados have been caused by a set of horizontal outward displacements of the dome's base.

As a last remark, it is worth noting that the analysis yields very small displacements. The maximum downward vertical displacement turned out to be equal to 0.29 mm (Fig.16a, b).

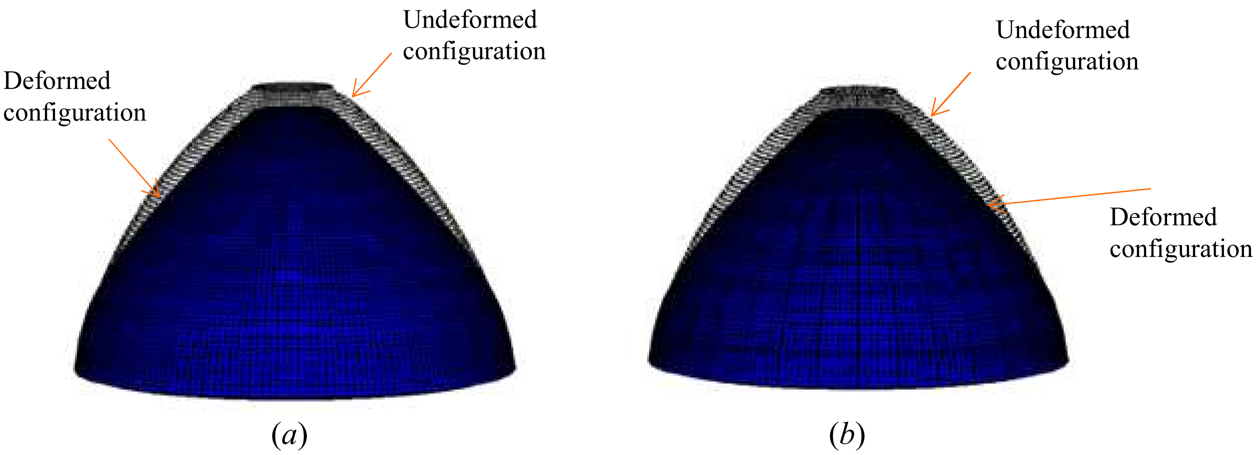


Fig. 16. Maximum vertical displacement in the dome: (a) solid model, 0.27 mm; (b) shell model, 0.29 mm.

More details on the non-linear response of the dome will be given in a forthcoming paper with reference to other load conditions and possible settlements of its base.

ACKNOWLEDGMENTS

The financial support of the *Opera della Primaziale Pisana* is gratefully acknowledged.

CONCLUSIONS

The present contribution is part of a currently ongoing research project whose main objective is to accurately investigate the structural response of the dome of Pisa Cathedral. This is a challenging task, given its oval shape, the complex arrangement of the building elements connecting the dome to the underlying drum, and, obviously, the complex mechanical properties of masonry and the large number of mechanical and geometrical parameters to be taken into account.

The dome's actual shape has been carefully reconstructed by means of high-density, suitably precision surveys using both range-based (laser scanning) and image-based (3D photogrammetry) methodologies. Experimental tests have been performed in order to evaluate the masonry's main mechanical properties and the evolution of the cracking pattern currently affecting the dome.

The dome's structural response has been examined via two different solution strategies. The first is based on analytical or semi-analytical models: the attention is focused on the dome's global stability by determining statically admissible solutions, as well as safety factors with respect to a collapse condition. The second approach consists of numerical analyses, aimed at studying the problem in terms of both displacements and stresses using 2D and 3D FE models.

The results obtained with the two semi-analytical methods working within the framework of limit analysis, *i.e.*, determination of admissible thrust surfaces and characterization of the stability area according to Durand Claye's method, suggest that a safety factor of approximately two can be estimated with respect to the dead vertical loads. The results yielded by FE linear analyses do not contradict those obtained via limit analysis. Moreover, as the dome is almost entirely compressed in FE analyses, it is reasonable to conclude that thanks to its shape and thickness in its actual configuration, the dome can withstand its self weight without developing any cracking, naturally provided that settlements of its base are absent or limited.

A detailed description of the analytical and numerical studies performed, as well as a more comprehensive discussion of the results obtained will be provided in forthcoming papers.

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