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DYNAMICAL ISSUES IN RENDEZVOUS OPERATIONS WITH THIRD BODY PERTURBATION

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The paper presents the complete 6-DOF set of equations of relative motion that describes the dynamics and the kinematic of two spacecraft in non inertial reference frames under the restricted three body problem hypotheses. The work was motivated by the increasing interest in missions that require the modelling of the third body perturbation to lead to an accurate synthesis if Guidance Navigation and Control systems, for this reason also the lienearized models of the complete coupled traslational-rotational dynamics are provided.

INTRODUCTION

The paper proposes a complete 6-DOF relative motion model of two vehicles, typically called *Target* and *Chaser*, that are orbiting in an environment in which the gravity field of two main bodies is not negligible. The equations are derived to improve the model the system and to allow the design of GNC components, they are also written in non-inertial frames.

The topic was inspired by the recent interest by the worldwide space agencies in assembling a new orbiting space station in NRHO around the Moon to be used as base for future exploration, for instance Mars (Reference 1). Several secondary vehicles will need to perform rendezvous and berthing/docking with the orbiting space station, then accurate relative dynamics models are necessary to guaratee the success of the operations. The first works dealing with the rendezvous problem in the restricted three-body problem appeared in the 60s and 70s, the golden era of the lunar exploration and of the Apollo missions, however most of the early literature discusses the rendezvous with a Lagrangian point, rather than with a vehicle that may orbit in its vicinity. Relative motion in the two-body problem and, in particular, in Near-Earth orbits, instead, has been studied extensively. The first and most remarkable models were proposed during the 1960s: the Clohessy-Wiltshire equations (Reference 2) and the Tschauner-Hempel equations (Reference 3). For years, these sets of equations have been the model of reference for the analysis and the design of relative guidance, navigation, and control systems (Reference 4), because they provide linear models for the study of relative motion in circular and elliptic orbits. However, the two sets are based on the following main assumptions: the spacecraft relative separation is significantly small compared with the distance from the primary body center of mass, and no orbital perturbations act on them. A wide variety of models have been proposed in the literature to overcome limitations introduced by these models (Reference 5) . The attention received by the study of relative motion in three-body scenarios is however not comparable to the two-body case. Several missions under study are targeting celestial bodies where the third-body influence must be explicitly taken into account in the relative

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dynamics, e.g. LOP-G. The two-body-based equation sets cannot be used in this context, because non-Keplerian orbits typical of three-body scenarios are considered.

In general, rigid-body dynamics can be represented as translation of the Centre of Mass (CoM) and rotation about the Centre of Mass (CoM). A spacecraft, in 1^{st} approximation can be considered as a rigid body, its attitude can be modelled by the second cardinal equation of motion (References 6 and 7). A relative attitude modelling approach is proposed by Reference 8 and 9, it is based on the difference of angular velocities and the consequent derivation of the relative attitude. Another possible approach, based on dual quaternion formulation is presented in References 10 and 11. *Chaser-Target* complete relative motion is then composed by the relative translational and rotational dynamics with respect to arbitrary points on the vehicles. Whenever one of these points does not coincide with the spacecraft CoM (e.g. the berthing port), a kinematic coupling between the rotational and translational dynamics of these points occurs (Reference 9).

In the case of rendezvous around Earth the coupling due to the rotational dynamics of features points can be treated as a kinematic perturbation (References 9 and 12), however when the relative distance becomes too small the results are not valid anymore, consequently it is not a good model for rendezvous manoeuvres, then the solution proposed by F. Ankersen in Reference 13 is usually considered.

This paper presents the complete non-linear sets of equations of motion and two different set of linear equation of motion are reviewed (Reference 14): the equations of motion based on the Elliptical Restricted Three Body Problem (ER3BP) hypothesis and those based on the Circular Restricted Three Body Problem (CR3BP) hypothesis.

The outline of the paper is as follow: first all the useful reference systems are introduced, then the non-linear set of equation of motion is discussed and linearized, comments are given in the conclusions.

FRAMES DEFINITION

The main coordinates systems used throughout the paper are now defined, and the relative notation is introduced.

Inertial Frame $\mathcal{I}$: The generic *inertial frame* centred in O and with unit vectors $\hat{\boldsymbol{I}}$, $\hat{\boldsymbol{J}}$, and $\hat{\boldsymbol{K}}$, will be denoted as follows,

$$\mathcal{I}\colon\ \left\{O; \hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}\right\}$$

In this paper we do not refer to any particular inertial or quasi-inertial frame.

Three-Body Rotating Reference Frames: Consider two primary bodies, with masses M_1 and M_2, orbiting around their composite center of mass C in a collinear formation. A convenient frame for describing the motion of spacecraft in such a system is the *synodic reference frame*. It can also be centered in one of the primaries center of mass, as in Reference 15, and the unit vectors are defined as follows:

- $\hat{\boldsymbol{i}}_s = -\boldsymbol{r}_{12}/\|\boldsymbol{r}_{12}\|$ where $\boldsymbol{r}_{12}$ is the position of M_2 with respect to M_1;
- $\hat{\boldsymbol{k}}_s$ is perpendicular to the plane where the primaries revolve, and is positive in the direction of the system angular velocity vector;
- $\hat{\boldsymbol{j}}_s = \hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{i}}_s$ completes the right-handed coordinate systems.

This choice may be convenient when spacecraft measurements are taken with respect to the nearest primary. In the following, we will refer to such a coordinate system as *primary-centred rotating frame*, and it will be denoted as

$$\mathcal{M}_i\colon\ \left\{O_i; \hat{\boldsymbol{\imath}}_m, \hat{\boldsymbol{\jmath}}_m, \hat{\boldsymbol{k}}_m\right\}$$

Where i is the index of the primary chosen for placing the coordinate frame. Fig. 6 shows the primary-centered rotating frame, centered on M_2 and the orbital frame (or Local-Vertical Local Horizon Frame).

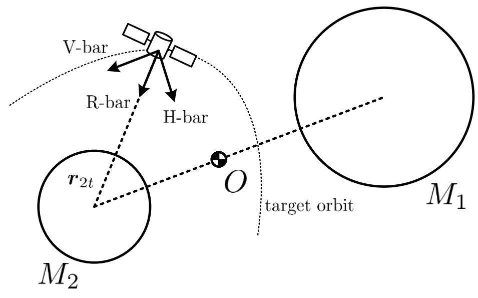
The Local-Vertical Local-Horizon Frame: Rendezvous trajectories are generally described in a frame local to the target. This eases the analysis and the trajectory monitoring of incoming vehicles, as well as the definition of keep-out zones and admissible approaching corridors. The local-vertical local-horizon (LVLH) frame is usually employed for rendezvous scenario analysis,

$$\mathcal{L}_i\colon\ \left\{\boldsymbol{r}_{it}; \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$$

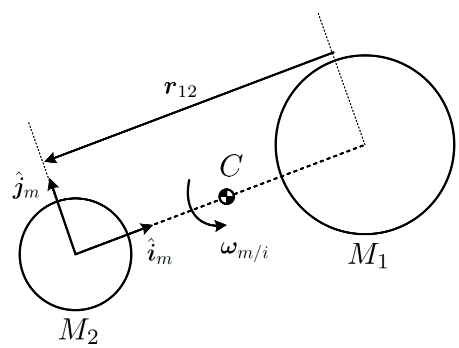
The LVLH* frame is defined with respect to the primary body around which the target is orbiting. Denoting with $\boldsymbol{r}_{it}$ the target position with respect to the primary i, with $\left[\,\dot{\boldsymbol{r}}_{it}\,\right]_{\mathcal{M}_i}$ the target velocity as seen from the primary, and with $\boldsymbol{h}_{it} = \boldsymbol{r}_{it} \times \left[\,\dot{\boldsymbol{r}}_{it}\,\right]_{\mathcal{M}_i}$ the target *specific angular momentum* with respect to the primary, the LVLH frame unit vectors are defined and named as follows,

- $\hat{\boldsymbol{k}} = -\boldsymbol{r}_{it}/\|\boldsymbol{r}_{it}\|$ points to the primary and is called *R-bar*;
- $\hat{\boldsymbol{\jmath}} = -\boldsymbol{h}_{it}/\|\boldsymbol{h}_{it}\|$, is perpendicular to the target instantaneous orbital plane and is called *H-bar*;
- $\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}$ completes the right-handed reference frame, and is called *V-bar*.

The above definition of the LVLH frame is consistent with the one given by Fehse in his classical reference book for spacecraft rendezvous and docking (Reference 4). The LVLH frame for a target orbiting around M_2 is shown in Fig. 6.



((a)) Local-vertical local horizon frame $\mathcal{L}_2$.



((b)) Primary-centered rotating reference frame $\mathcal{M}_2$

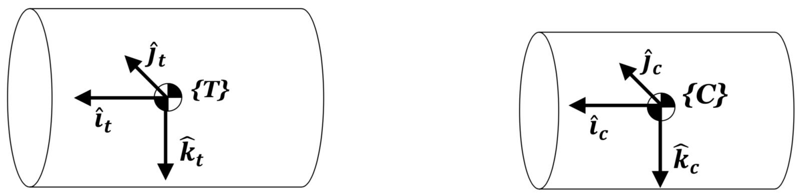
Figure 1: LVLH frame and Synodic frame.

*In this paper it is also called orbital frame.

Body Frames: The *body frame* is centered in center of mass and with the unit vectors parallel to principal axes of inertia. In this context the main body frames are the *Target* and *Chaser*'s one and they are defined as follow:

$$\mathcal{T} : \left\{ O_{t_{com}}; \hat{\boldsymbol{i}}_t, \hat{\boldsymbol{j}}_t, \hat{\boldsymbol{k}}_t \right\}, \quad \mathcal{C} : \left\{ O_{c_{com}}; \hat{\boldsymbol{i}}_c, \hat{\boldsymbol{j}}_c, \hat{\boldsymbol{k}}_c \right\}$$

Which are respectively for target and chaser. The frames above are shown in Fig. 2



((a)) Target body frame.

((b)) Chaser body frame.

Figure 2: Body frames.

Docking Reference Frame: The *Docking frame* has the axes in same direction of the body frame axes. It is attached to the docking port structure and its origin is located at $\mathbf{r}_{p_{t/c}}$, the port position vector with respect to body frame. The same frame is used if berthing is considered

$$\mathcal{D} : \left\{ \boldsymbol{r}_{p_{t/c}}; \hat{\boldsymbol{i}}_d, \hat{\boldsymbol{j}}_d, \hat{\boldsymbol{k}}_d \right\}$$

In Fig. 3 we can see a general docking frame

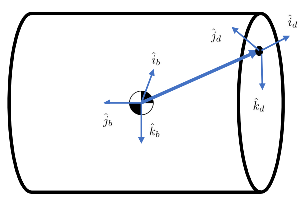


Figure 3: Docking frame.

RELATIVE MOTION EQUATIONS IN THE RESTRICTED THREE-BODY PROBLEM

In this section the complete non linear equation of motion are derived. First the traslational relative equation of motion, presented by G. Franzini in Reference 16 and 14, are recalled, then the relative attitude equations are shown and the relative non-linear attitude dynamics is derived.

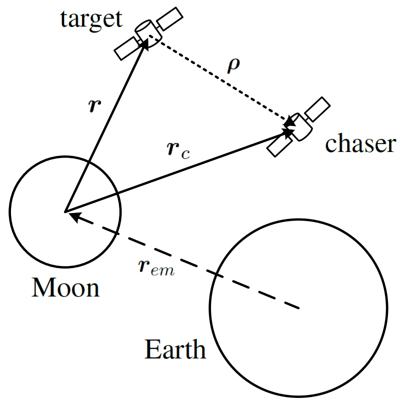


Figure 4: Target and chaser spacecraft in the three-body system.

Relative Traslational Dynamics

Consider a *Target* and a *Chaser* spacecraft, orbiting around the Moon, and subject to both Earth and Moon gravitational influence. Their equations of motions with respect to the Moon are:

$$\ddot{\boldsymbol{r}}_{\mathcal{I}} = -\mu \frac{\boldsymbol{r}}{r^3} - (1-\mu)\left(\frac{\boldsymbol{r}+\boldsymbol{r}_{em}}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^3} - \frac{\boldsymbol{r}_{em}}{r_{em}^3}\right) \tag{1}$$

$$\ddot{\boldsymbol{r}}_{c\mathcal{I}} = -\mu \frac{\boldsymbol{r}_c}{r_c^3} - (1-\mu)\left(\frac{\boldsymbol{r}_c+\boldsymbol{r}_{em}}{\|\boldsymbol{r}_c+\boldsymbol{r}_{em}\|^3} - \frac{\boldsymbol{r}_{em}}{r_{em}^3}\right) \tag{2}$$

where:

- μ is the Earth-Moon mass ratio;
- $\boldsymbol{r}_{em}$ is the position of the Moon with respect to the Earth, and $r_{em} = \|\boldsymbol{r}_{em}\|$ its norm;
- $\boldsymbol{r}$ and $\boldsymbol{r}_c$ are the target and chaser positions with respect to the Moon, with norms $r = \|\boldsymbol{r}\|$, and $r_c = \|\boldsymbol{r}_c\|$, respectively.

All the variables and parameters are normalized as follows:

- distances in units of the Moon orbit semi-major axis $a = 384\,400\,\mathrm{km}$;
- time in units of the inverse of the Moon mean angular motion $n = 2.661\,699 \times 10^{-6}\,\mathrm{rad\,s^{-1}}$.

For the sake of notation compactness, time-dependency will be omitted for the time-varying variables except when needed. The aim of this section is to give an outline of the tranlational equation computation described in Reference 16, in the LVLH frame defined with respect to the Moon. The frame in the following will be denoted with $\mathcal{L}$ for the sake of simplicity.

With reference to Fig. 4, chaser position with respect to the Moon is given by:

$$\boldsymbol{r}_c = \boldsymbol{r} + \boldsymbol{\rho} \tag{3}$$

where $\boldsymbol{\rho}$ is the relative position of the chaser with respect to the target. Starting from Eq. (1) (2) and (3) the non-linear relative translational dynamic is derived. However the result is highly non linear time variant set of equations as reported in Eq. (4), this set of equations is called NERM - *Non-linear Equation of Relative Motion*.

$$[\ddot{\boldsymbol{\rho}}]_{\mathcal{L}} + 2\boldsymbol{\omega}_{l/i} \times [\dot{\boldsymbol{\rho}}]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} \times \boldsymbol{\rho} + \boldsymbol{\omega}_{l/i} \times \left(\boldsymbol{\omega}_{l/i} \times \boldsymbol{\rho}\right) = \mu\left(\frac{\boldsymbol{r}}{r^3} - \frac{\boldsymbol{r}+\boldsymbol{\rho}}{\|\boldsymbol{r}+\boldsymbol{\rho}\|^3}\right) + (1-\mu)\left(\frac{\boldsymbol{r}+\boldsymbol{r}_{em}}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^3} - \frac{\boldsymbol{r}+\boldsymbol{\rho}+\boldsymbol{r}_{em}}{\|\boldsymbol{r}+\boldsymbol{\rho}+\boldsymbol{r}_{em}\|^3}\right) \quad (4)$$

where $\boldsymbol{\omega}_{l/i}$ is the angular velocity of the LVLH frame w.r.t. the inertial frame.

Relative attitude dynamics

To complete the relative dynamics equations the relative attitude equation are described herein. The authors remind the reader that it is convenient to refer all the quantities to the orbital reference frame since we are addressing relative equation of motion.

Nonlinear Chaser Attitude Dynamics and Kinematics: The chaser spacecraft is modelled as a rigid body, therefore the 2^{nd} Cardinal equation is used to model the vehicle attitude dynamics under external torques to describe attitude and attitude rates under externally applied torques:

$$I\dot{\boldsymbol{\omega}}_{c/i} + \boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i} = \mathbf{N} \quad (5)$$

Where $\mathbf{N}$ is the vector sum of all external torques, I is the inertia tensor matrix and $\boldsymbol{\omega}_{c/l}$ is the angular velocity vector of $\mathcal{C}$ w.r.t. $\mathcal{I}$. The equations in frame $\mathcal{C}$, becomes:

$$\begin{aligned} I^x\dot{\omega}^x_{c/i} + (I^z - I^y)\omega^y_{c/i}\omega^z_{c/i} &= N^x \\ I^y\dot{\omega}^y_{c/i} + (I^x - I^z)\omega^x_{c/i}\omega^z_{c/i} &= N^y \\ I^z\dot{\omega}^z_{c/i} + (I^y - I^x)\omega^x_{c/i}\omega^y_{c/i} &= N^z \end{aligned}$$

where the angular velocity of the frame $\mathcal{C}$ w.r.t. $\mathcal{I}$ can be thought as:

$$\boldsymbol{\omega}_{c/i} = \boldsymbol{\omega}_{c/l} + \boldsymbol{\omega}_{l/i}$$

Where $\boldsymbol{\omega}_{c/l}$ is the angular rate of $\mathcal{C}$ w.r.t. $\mathcal{L}$ and $\boldsymbol{\omega}_{l/i}$ is the angular rate of $\mathcal{L}$ w.r.t. $\mathcal{I}$. The latter can be computed also as:

$$\boldsymbol{\omega}_{l/i} = \boldsymbol{\omega}_{l/m} + \boldsymbol{\omega}_{m/i}$$

where $\boldsymbol{\omega}_{l/m}$ and $\boldsymbol{\omega}_{m/i}$ are the angular velocities of $\mathcal{L}$ whit respect to $\mathcal{M}$, and of $\mathcal{M}$ with respect to $\mathcal{I}$.

The kinematic motion is obtained by deriving the differential equations of motion of the body frame with respect to the reference frame LVLH, relating the Euler (3,2,1) angles $\boldsymbol{\theta}_c = (\theta_z, \theta_y, \theta_x)$ with the angular velocity vector $\boldsymbol{\omega}_{c/l}$ (Reference 6,13). Similarly it can also be described by means of quaternions (Reference 7), with the advantage that there are no singularities in this formalism of describing a rotation of one coordinate system into another one. In this paper we use the following definition:

$$\mathbf{q} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ e_1\sin(\frac{\theta}{2}) \\ e_2\sin(\frac{\theta}{2}) \\ e_3\sin(\frac{\theta}{2}) \end{bmatrix} = \begin{bmatrix} q_4 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where $\mathbf{e} = \left[e_1, \, e_2, \, e_3\right]$ is the Euler rotational eigen axis, and θ is the angle rotated around the eigenvector $\mathbf{e}$. The differential relationship between attitude quaternion and angular velocity is given by:

$$\dot{\mathbf{q}}_{c/l} = \frac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}_{c/l}^{\top} \\ \boldsymbol{\omega}_{c/l} & -[\boldsymbol{\omega}_{c/l}]^{\times} \end{bmatrix} \mathbf{q}_{c/l} = \frac{1}{2} \mathbf{Q}(\boldsymbol{\omega}_{c/l}) \mathbf{q}_{c/l} \tag{6}$$

with $[\cdot]^{\times}$ denote the skew matrix operator.

Target Attitude Motion The attitude motion of the target is typically described by a saw tooth type behaviour along the three axes of the LVLH reference frame (Reference 13).

Relative Dynamics and Kinematics The relative attitude between two spacecraft is based on the angular rate vectors, since quaternions are not cumulative. Thus:

$$\boldsymbol{\omega}_{ra} = \boldsymbol{\omega}_{c/l} - \mathbf{R}_{cl}(\mathbf{q}_{c/l}) \boldsymbol{\omega}_{t/l} \tag{7}$$

where $\boldsymbol{\omega}_{ra}$ is the relative angular rate in chaser body frame, $\boldsymbol{\omega}_{c/l}$ is the chaser angular rate expressed in chaser frame, $\boldsymbol{\omega}_{t/l}$ is the target angular rate expressed in orbital frame and $\mathbf{R}_{cl}(\mathbf{q}_{c/l})$ is the rotation matrix that transforms a vector from orbital frame components to body chaser frame components

$$\mathbf{R}_{cl}(\mathbf{q}_{c/l}) = \begin{bmatrix} 1 - 2(q_3^2 + q_4^2) & 2(q_2 q_3 + q_1 q_4) & 2(q_2 q_4 - q_1 q_3) \\ 2(q_2 q_3 - q_1 q_4) & 1 - 2(q_2^2 + q_4^2) & 2(q_3 q_4 + q_1 q_2) \\ 2(q_2 q_4 + q_1 q_3) & 2(q_3 q_4 - q_1 q_2) & 1 - 2(q_2^2 + q_3^2) \end{bmatrix}$$

Using Eq.(6) we can express the derivative of relative quaternion $\mathbf{q}_{ra}$ as follows:

$$\mathbf{q}_{ra} = \mathbf{q^*}_{t/l} \otimes \mathbf{q}_{c/l} \tag{8}$$

where the product indicates the difference of relative quaternions (Reference 17) and it is computed as follow:

$$\mathbf{q}_a \otimes \mathbf{q}_b = \left[\eta_a \eta_b - \boldsymbol{\xi}_a^T \boldsymbol{\xi}_b, (\eta_a \boldsymbol{\xi}_b + \eta_b \boldsymbol{\xi}_a + \boldsymbol{\xi}_a \times \boldsymbol{\xi}_b^T)\right]^T$$

where η is the real part and $\boldsymbol{\xi}$ is the vector of the imaginary elements of the quaternions.

Port to Port Motion For close proximity manoeuvres it is fundamental to model the port-to-port motion since rendezvous manoeuvre requires high accuracy. The port to port distance can be expressed as:

$$\boldsymbol{\rho}_{pp} = \boldsymbol{\rho} + \mathbf{r}_{pc} - \mathbf{r}_{pt}$$

where $\boldsymbol{\rho}$ is the relative distance between chaser and target Centre of Mass, $\mathbf{r}_{pc}$ and $\mathbf{r}_{dt}$ are the docking port position of chaser and target w.r.t their centre of mass (see Figure 5). The target (or chaser) docking port in the orbital reference frame LVLH can be represented by:

$$[\mathbf{r}_{pj}]_{\mathcal{L}} = \mathbf{R}_{jl}(\mathbf{q}_{j/l})^{\top} [\mathbf{r}_{pj}]_{\mathcal{J}} \tag{9}$$

with the rotation matrix defined as follow:

$$\mathbf{R}_{jl}(\mathbf{q}_{j/l}) = \begin{bmatrix} 1 - 2(q_3^2 + q_4^2) & 2(q_2 q_3 + q_1 q_4) & 2(q_2 q_4 - q_1 q_3) \\ 2(q_2 q_3 - q_1 q_4) & 1 - 2(q_2^2 + q_4^2) & 2(q_3 q_4 + q_1 q_2) \\ 2(q_2 q_4 + q_1 q_3) & 2(q_3 q_4 - q_1 q_2) & 1 - 2(q_2^2 + q_3^2) \end{bmatrix}$$

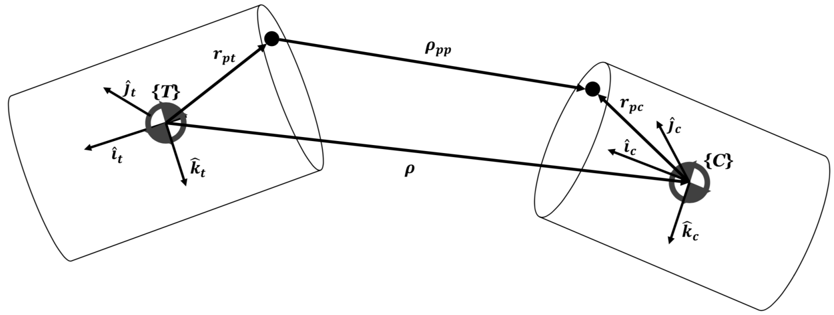


Figure 5: Relative position vectors

The port-to-port velocity can be derived by direct differentiation of the port-to-port position, taking into account that the differentiation is not performed in an inertial reference frame. The obtained port to port velocity then needs to be transformed in the orbital reference frame as for the positions and can be done as follows:

$$\begin{aligned} [\dot{\mathbf{r}}_{pj}]_{\mathcal{L}} &= \mathbf{R}_{jl}(\mathbf{q}_{j/l})^{\top} \left(\left[\boldsymbol{\omega}_{j/l}\right]_{\mathcal{J}} \times [\mathbf{r}_{pj}]_{\mathcal{J}} \right) \\ &= -\mathbf{R}_{jl}(\mathbf{q}_{j/l})^{\top} [\hat{\mathbf{r}_{pj}}]^{\times}_{\mathcal{J}} \left[\boldsymbol{\omega}_{j/l}\right]_{\mathcal{J}} \end{aligned} \tag{10}$$

Note that the subscripts $j = c, t$ refer to chaser or target respectively and $\hat{[]}^{\times}$ is the skew matrix operator.

LINEARIZED EQUATIONS OF MOTION

The linear complete 6-DOF model is here described since the classical of the control algorithms are designed based on the linear model, e.g. H-infinity control, H-2 or LQR, then two linear sets of equations of motions are proposed.

Traslational Linearized Motion

Due to the nonlinearity of the gravitational acceleration and the presence of several time-varying parameters, the equations of relative motion derived in Reference 14, may be difficult to use for the development of guidance and navigation systems. Two possible simplifications are briefly introduced, aimed at linearizing the equation set and at reducing the number of time-varying parameters: gravity linearization and Circular Restricted Three Body hypothesis. The equation sets presented in the following were firstly in the Reference 18.

Relative Motion Equation Sets The possible simplifications discussed in the previous section can be used to derive four equation sets, that describe the relative dynamics with different levels of accuracy.

In the following, the *skew symmetric matrices* associated with angular velocity and acceleration vectors $\boldsymbol{\omega}_{l/i}$ and $\left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}}$, i.e.:

$$\boldsymbol{\Omega}_{l/i} = \begin{bmatrix} 0 & -\omega^z_{l/i} & \omega^y_{l/i} \\ \omega^z_{l/i} & 0 & -\omega^x_{l/i} \\ -\omega^y_{l/i} & \omega^x_{l/i} & 0 \end{bmatrix} \quad \left[\dot{\boldsymbol{\Omega}}_{l/i}\right]_{\mathcal{L}} = \begin{bmatrix} 0 & -\dot{\omega}^z_{l/i} & \dot{\omega}^y_{l/i} \\ \dot{\omega}^z_{l/i} & 0 & -\dot{\omega}^x_{l/i} \\ -\dot{\omega}^y_{l/i} & \dot{\omega}^x_{l/i} & 0 \end{bmatrix}$$

where:

$$\boldsymbol{\omega}_{l/i} = \omega^x_{l/i}\hat{\boldsymbol{i}} + \omega^y_{l/i}\hat{\boldsymbol{j}} + \omega^z_{l/i}\hat{\boldsymbol{k}}, \quad \left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} = \dot{\omega}^x_{l/i}\hat{\boldsymbol{i}} + \dot{\omega}^y_{l/i}\hat{\boldsymbol{j}} + \dot{\omega}^z_{l/i}\hat{\boldsymbol{k}}$$

are introduced to express the equations of motion in a more compact form.

ER3BP Based Equations: ENERM - Elliptic Non-linear Equation of Relative Motion- and ELERM - Elliptical Linear Equation of Relative Motion If the elliptic restricted three-body problem (ER3BP) problem is considered, then the Moon motion is governed by the classical two-body problem equations, with the Earth as primary body. The quantities $\boldsymbol{r}_{em}$, $\boldsymbol{\omega}_{m/i}$, and $\left[\boldsymbol{\omega}_{m/i}\right]_{\mathcal{M}}$ can then be obtained accordingly.

With the name of ENERM we will refer to the following equation set:

$$\begin{aligned}[\ddot{\boldsymbol{\rho}}]_{\mathcal{L}} = -2\boldsymbol{\Omega}_{l/i}\,[\dot{\boldsymbol{\rho}}]_{\mathcal{L}} - \left(\left[\dot{\boldsymbol{\Omega}}_{l/i}\right]_{\mathcal{L}} + \boldsymbol{\Omega}^2_{l/i}\right)\boldsymbol{\rho} + \mu\left(\frac{\boldsymbol{r}}{r^3} - \frac{\boldsymbol{r}+\boldsymbol{\rho}}{\|\boldsymbol{r}+\boldsymbol{\rho}\|^3}\right) \\ + (1-\mu)\left(\frac{\boldsymbol{r}+\boldsymbol{r}_{em}}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^3} - \frac{\boldsymbol{r}+\boldsymbol{\rho}+\boldsymbol{r}_{em}}{\|\boldsymbol{r}+\boldsymbol{\rho}+\boldsymbol{r}_{em}\|^3}\right)\end{aligned} \quad (11)$$

characterised by time-varying parameters that depend on target and on Moon orbital motion, that is:

- $\boldsymbol{\omega}_{l/i} = \boldsymbol{\omega}_{l/m} + \boldsymbol{\omega}_{m/i}$;
- $\left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} = \left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}} + \left[\dot{\boldsymbol{\omega}}_{m/i}\right]_{\mathcal{M}} - \boldsymbol{\omega}_{l/m} \times \boldsymbol{\omega}_{m/i}$;
- $\boldsymbol{\omega}_{l/m}$ is the angular velocity of $\mathcal{L}$ w.r.t. $\mathcal{M}$
- $\left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}}$ it is the angular acceleration of $\mathcal{L}$ w.r.t. $\mathcal{M}$ in $\mathcal{L}$.

If the chaser acceleration is controllable by means of the control vector $\boldsymbol{u}$, then Eq. (11) can be written as a nonlinear system affine in the control:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x}) + \boldsymbol{B}_{CoM}\boldsymbol{u}, \quad \boldsymbol{x} = \begin{bmatrix}\boldsymbol{\rho} \\ [\dot{\boldsymbol{\rho}}]_{\mathcal{L}}\end{bmatrix}, \quad \boldsymbol{B}_{CoM} = \begin{bmatrix}\boldsymbol{0}_{3\times3} \\ \boldsymbol{I}_3\end{bmatrix}$$

where $\boldsymbol{x} \in R^6$, $\boldsymbol{f}\colon [0,+\infty) \times R^6 \to R^6$, $\boldsymbol{B} \in R^{6\times3}$, and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, and $\boldsymbol{0}_{n\times m}$ is the $n \times m$ zero matrix.

After the linearization of the gravity force, we obtain the ELERM:

$$\begin{aligned}[\ddot{\boldsymbol{\rho}}]_{\mathcal{L}} = -2\boldsymbol{\Omega}_{l/i}\,[\dot{\boldsymbol{\rho}}]_{\mathcal{L}} - \Bigg(\left[\dot{\boldsymbol{\Omega}}_{l/i}\right]_{\mathcal{L}} + \boldsymbol{\Omega}^2_{l/i} + \frac{\mu}{r^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}\boldsymbol{r}^T}{r^2}\right) \\ + \frac{1-\mu}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^3}\left(\boldsymbol{I} - 3\frac{(\boldsymbol{r}+\boldsymbol{r}_{em})(\boldsymbol{r}+\boldsymbol{r}_{em})^T}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^2}\right)\Bigg)\boldsymbol{\rho}\end{aligned} \quad (12)$$

with angular velocities and accelerations computed as for the ENERM. Assuming the chaser controllable in acceleration, Eq. (12) can be written in state-space form as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_{CoM}(t)\boldsymbol{x} + \boldsymbol{B}_{CoM}\boldsymbol{u} \quad (13)$$

with $\boldsymbol{A} \in^{6\times6}$ defined as:

$$\boldsymbol{A}_{CoM}(t) = \begin{bmatrix}\boldsymbol{0}_{3\times3} & \boldsymbol{I}_3 \\ \boldsymbol{A}_{\dot{\boldsymbol{\rho}}\boldsymbol{\rho}}(t) & -2\boldsymbol{\Omega}_{l/i}(t)\end{bmatrix}$$

$$A_{\dot{\rho}\rho} = -\dot{\boldsymbol{\Omega}}_{l/i\mathcal{L}} - \boldsymbol{\Omega}^2_{l/i} - \frac{\mu}{r^3}\left(\boldsymbol{I} - 3\frac{\boldsymbol{r}\boldsymbol{r}^T}{r^2}\right) - \frac{1-\mu}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^3}\left(\boldsymbol{I} - 3\frac{(\boldsymbol{r}+\boldsymbol{r}_{em})(\boldsymbol{r}+\boldsymbol{r}_{em})^T}{\|\boldsymbol{r}+\boldsymbol{r}_{em}\|^2}\right)$$

where in the last equation dependence on time was omitted for notation compactness.

CR3BP Based Equations: CNERM - Circular Non-linear Equation of Relative Motion and CLERM - Circular Linear Equation of Relative Motion If the circular restricted three-body problem (CR3BP) is considered, then the computation of the angular velocities simplifies as follows:

- $\boldsymbol{\omega}_{l/i} = \boldsymbol{\omega}_{l/m} + \boldsymbol{\omega}_{m/i}$ (as for the elliptic case);
- $\left[\dot{\boldsymbol{\omega}}_{l/i}\right]_{\mathcal{L}} = \left[\dot{\boldsymbol{\omega}}_{l/m}\right]_{\mathcal{L}} - \boldsymbol{\omega}_{l/m} \times \hat{\boldsymbol{k}}_m$;

The set in Eq. (11) with the CR3BP simplifications will be referred to as CNERM, whereas the set in Eq. (12) under the same simplifications as circular linear equation of relative motion (CLERM).

Linearized Attitude Motion Linearization is performed using a standard Taylor series expansion of Eq. (5) around the operating point $op = \left(\boldsymbol{\omega}_{l/i_0}, \mathbf{N}_0\right)$.

Since the attitude dynamics equation is a function of the torque vector $\mathbf{N}$ and the chaser velocity vector with respect to the inertial frame $\boldsymbol{\omega}_{c/i}$, we will linearize the equation around $\boldsymbol{\omega}_{l/i_0}$, the angular velocity of orbital frame LVLH at each point of the orbit, and $\mathbf{N}_0 = \mathbf{0}$, then no external torques are considered at the linenarization point.
The Jacobian with respect to $\boldsymbol{\omega}$ is then given by:

$$\begin{aligned}\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial\boldsymbol{\omega}_{c/i}} &= \frac{\partial\mathbf{N}}{\partial\boldsymbol{\omega}_{c/i}} - \frac{\left(\partial\boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i}\right)}{\partial\boldsymbol{\omega}_{c/i}}\\ &= \mathbf{0} - \frac{\left(\partial\boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i}\right)}{\partial\boldsymbol{\omega}_{c/i}}\\ &= -\frac{\left(\partial\boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i}\right)}{\partial\boldsymbol{\omega}_{c/i}}\end{aligned}$$

Evaluating the Jacobian at the operating point *op* we obtain:

$$\left.\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial\boldsymbol{\omega}_{c/i}}\right|_{op} = -\left.\frac{\partial\left(\boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i}\right)}{\partial\boldsymbol{\omega}_{c/i}}\right|_{op} = \mathbf{P}_c = \begin{bmatrix} 0 & (I^y - I^z)\omega^z_{l/i} & (I^y - I^z)\omega^y_{l/i} \\ (I^z - I^z)\omega^z_{l/i} & 0 & (I^z - I^x)\omega^x_{l/i} \\ (I^x - I^y)\omega^y_{l/i} & (I^x - I^y)\omega^x_{l/i} & 0 \end{bmatrix}$$

Where we assume a diagonal inertia matrix.
The Jacobian with respect to $\mathbf{N}$ is:

$$\begin{aligned}\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial\mathbf{N}} &= \frac{\partial\mathbf{N}}{\partial\mathbf{N}} - \frac{\left(\partial\boldsymbol{\omega}_{c/i} \times I\boldsymbol{\omega}_{c/i}\right)}{\partial\mathbf{N}}\\ &= \mathbf{I} - \mathbf{0}\\ &= \mathbf{I}\end{aligned}$$

The Taylor series expansion up to the first order is:

$$I\dot{\boldsymbol{\omega}}_{c/i} = \mathbf{N}_0 - \boldsymbol{\omega}_{l/i_0} \times I\boldsymbol{\omega}_{l/i_0} + \left.\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial \mathbf{N}}\right|_{op} (\mathbf{N} - \mathbf{N}_0) + \left.\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial \boldsymbol{\omega}}\right|_{op} \left(\boldsymbol{\omega}_{c/i} - \boldsymbol{\omega}_{l/i_0}\right)$$

where:

- $\left.\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial \mathbf{N}}\right|_{op} = \mathbf{I}_{3\times3}$
- $\left.\frac{\partial I\dot{\boldsymbol{\omega}}_{c/i}}{\partial \boldsymbol{\omega}}{}_{c/i}\right|_{op} = -\left.\frac{\partial\left(\boldsymbol{\omega}_{c/i}\times I\boldsymbol{\omega}_{c/i}\right)}{\partial \boldsymbol{\omega}_{c/i}}\right|_{\boldsymbol{\omega}_{l/i_0},\mathbf{N}_0} = \mathbf{P}_c$

Recalling that $\mathbf{N}_0 = \mathbf{0}$ (we assume negligible external torques in final phase of rendezvous/berthing) and $\mathbf{N}_0 - \boldsymbol{\omega}_{l/i_0} \times I\boldsymbol{\omega}_{l/i_0} = I\dot{\boldsymbol{\omega}}_{l/i_0}$. The linear vector equation for the attitude dynamics of chaser is then given by:

$$\dot{\boldsymbol{\omega}}_{c/l} = I^{-1}\mathbf{P}_c\boldsymbol{\omega}_{c/l} + I^{-1}\mathbf{N} \tag{14}$$

Chaser's kinematics: the Eq. (6) can be seen as a linear time-varying equation, so no more reduction are required, since the linearization of the kinematics, with an approach similar to the one above, produces a result that is valid only for small angular variations:

$$\dot{\mathbf{q}}_{c/l} = \frac{1}{2}\mathbf{Q}(\boldsymbol{\omega}_{c/l})\,\mathbf{q}_{c/l} + \mathbf{0}_{4\times3}\,\boldsymbol{\omega}_{c/l} \tag{15}$$

The linear attitude model is described by dynamics Eq. (14) and kinematics Eq. (15). Defining the state vector as $\mathbf{x}_c = \left[\mathbf{q}_{c/l}^\top, \boldsymbol{\omega}_{c/l}^\top\right]^\top$, the input vector as $\mathbf{u} = \mathbf{N}$, the state space form of linear attitude equations becomes:

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{u}$$

where the state matrix and input-state matrix are:

$$\mathbf{A}_c = \begin{bmatrix} \frac{1}{2}\mathbf{Q}(\boldsymbol{\omega}_{c/l}) & \mathbf{0}_{4\times3} \\ \mathbf{0}_{3\times3} & I^{-1}\mathbf{P}_c \end{bmatrix} \qquad \mathbf{B}_c = \begin{bmatrix} \mathbf{0}_{4\times3} \\ I^{-1} \end{bmatrix}$$

Linearized Target Motion We assume that for small angles $\mathbf{q}_{t/l} \approx \left[1, \frac{\theta_x}{2}, \frac{\theta_y}{2}, \frac{\theta_z}{2}\right]^\top$ the model of the dynamics of target is given by:

$$\dot{\boldsymbol{\omega}}_{t/l} = -\mathbf{K}_{qt}\mathbf{q}_{t/l}$$

where $\mathbf{K}_{qt}$ is the matrix of eigen frequencies:

$$\mathbf{K}_{qt} = \begin{bmatrix} 0 & k_x^2 & 0 & 0 \\ 0 & 0 & k_y^2 & 0 \\ 0 & 0 & 0 & k_z^2 \end{bmatrix}$$

The kinematic equation using quaternion formulation

$$\dot{\mathbf{q}}_{t/l} = \frac{1}{2} \begin{bmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & -q_4 & q_3 \\ q_4 & q_1 & -q_2 \\ -q_3 & q_2 & q_1 \end{bmatrix} \boldsymbol{\omega}_{t/l} = \frac{1}{2}\boldsymbol{\Xi}(\mathbf{q}_{t/l})\boldsymbol{\omega}_{t/l}$$

In state space form the model becomes:

$$\dot{\mathbf{x}}_t = \begin{bmatrix} \mathbf{0}_{3\times3} & \frac{1}{2}\boldsymbol{\Xi}(\mathbf{q}_{t/l}) \\ -\mathbf{K}_{qt} & \mathbf{0}_{3\times3} \end{bmatrix} \mathbf{x}_t = \mathbf{A}_t \mathbf{x}_t \tag{16}$$

with $\mathbf{x}_t = \left[\mathbf{q}_{t/l}^\top, \boldsymbol{\omega}_{t/l}^\top\right]^\top$. Note that the model describes the target attitude motion in target body frame with respect to the orbital frame (LVLH).

Linearized Relative Motion The best way to model the relative attitude dynamics is to treat it as an output equation (Reference 9 and 13).since it is a kinematic relation:

$$\mathbf{y}_{ra} = \begin{bmatrix} \mathbf{I}_{4\times4} & 0 & 0 \\ 0 & \mathbf{I}_{3\times3} & -\mathbf{R}_{cl}(\mathbf{q}_{c/l}) \end{bmatrix} \begin{bmatrix} \mathbf{q}_{ra} \\ \boldsymbol{\omega}_{c/l} \\ \boldsymbol{\omega}_{t/l} \end{bmatrix}$$

where $\mathbf{y}_{ra} = [\mathbf{q}_{ra}^T \; \boldsymbol{\omega}_{ra}^T]^T$.

Linear Port-to-Port motion As we can see, the Eq. (9) and (10) are kinematic relationships, so in a state space form they yield the output equation only:

$$\begin{aligned} \mathbf{y}_{pj} = \begin{bmatrix} [\mathbf{r}_{pj}]_{\mathcal{L}} \\ [\dot{\mathbf{r}}_{pj}]_{\mathcal{L}} \end{bmatrix} &= \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & -\mathbf{R}_{jl}(\mathbf{q}_{j/l})^\top [\hat{\mathbf{r}_{pj}}]_{\mathcal{L}}^\times \end{bmatrix} \begin{bmatrix} \mathbf{q}_{j/l} \\ \boldsymbol{\omega}_{j/l} \end{bmatrix} + \begin{bmatrix} \mathbf{R}_{jl}(\mathbf{q}_{j/l})^\top \\ \mathbf{0} \end{bmatrix} [\mathbf{r}_{pj}]_{\mathcal{J}} \\ &= \mathbf{C}_{pj} \begin{bmatrix} \mathbf{q}_{j/l} \\ \boldsymbol{\omega}_{j/l} \end{bmatrix} + \mathbf{D}_{pj} [\mathbf{r}_{pj}]_{\mathcal{J}} \end{aligned}$$

Coupled Relative Dynamics

In the coupled linear model (or small signal model) the state vector consists of relative CoM position, the chaser attitude, the target attitude, their rates and the relative attitude. So the state vector - $\mathbf{x}$ - and the input vector - $\mathbf{u}$ - are:

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\rho}^\top & \dot{\boldsymbol{\rho}}^\top & \mathbf{q}_{c/l}^\top & \boldsymbol{\omega}_{c/l}^\top & \mathbf{q}_{t/l}^\top & \boldsymbol{\omega}_{t/l}^\top & \mathbf{q}_{ra}^\top \end{bmatrix}^\top$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{F}^\top & \mathbf{N}^\top & [\mathbf{r}_{pc}]_{\mathcal{L}}^\top & [\mathbf{r}_{pt}]_{\mathcal{L}}^\top \end{bmatrix}^\top$$

for convenience the CoM dynamics are expressed in orbital frame (LVLH) and the rest in the body frames. Note that in the input vector only $\mathbf{F}$ and $\mathbf{N}$ are the control components, and $[\mathbf{r}_{pc}]_{\mathcal{L}}^\top$ $[\mathbf{r}_{pt}]_{\mathcal{L}}^\top$

are considered constant qualities since the two vehicles are considered rigid bodies.
Finally the output vector contains relative COM position, chaser and target attitude, the port to port position, relative attitude and their respective derivatives as

$$\mathbf{y} = \begin{bmatrix} \boldsymbol{\rho}^\top & \dot{\boldsymbol{\rho}}^\top & \mathbf{q}_{c/l}^\top & \boldsymbol{\omega}_{c/l}^\top & \mathbf{q}_{t/l}^\top & \boldsymbol{\omega}_{t/l}^\top & \boldsymbol{\rho}_{pp}^\top & \dot{\boldsymbol{\rho}}_{pp}^\top & \mathbf{q}_{ra}^\top & \boldsymbol{\omega}_{ra}^\top \end{bmatrix}^\top \tag{17}$$

The state space model is in standard form is the written as:

$$\begin{cases} \dot{\mathbf{x}} & = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} & = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where the state matrix is given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{CoM} & \mathbf{0}_{6\times7} & \mathbf{0}_{6\times7} & \mathbf{0}_{6\times4} \\ \mathbf{0}_{7\times6} & \mathbf{A}_c & \mathbf{0}_{7\times7} & \mathbf{0}_{7\times4} \\ \mathbf{0}_{7\times6} & \mathbf{0}_{7\times7} & \mathbf{A}_t & \mathbf{0}_{7\times4} \\ \mathbf{0}_{4\times6} & \begin{bmatrix}\mathbf{0}_{4\times4} & \frac{1}{2}\boldsymbol{\Xi}(\mathbf{q}_{ra})\end{bmatrix} & \begin{bmatrix}\mathbf{0}_{4\times4} & -\frac{1}{2}\boldsymbol{\Xi}(\mathbf{q}_{ra})\mathbf{R}_{c/l}(\mathbf{q}_{cl})\end{bmatrix} & \mathbf{0}_{4\times4} \end{bmatrix}$$

the input matrix is:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{CoM} & \mathbf{0}_{6\times3} & \mathbf{0}_{6\times6} \\ \mathbf{0}_{7\times3} & \mathbf{B}_c & \mathbf{0}_{7\times6} \\ \mathbf{0}_{7\times3} & \mathbf{0}_{7\times3} & \mathbf{0}_{7\times6} \\ \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times6} \end{bmatrix}$$

$\mathbf{A}_{CoM}$ and $\mathbf{B}_{CoM}$ come from linearized equations of relative motion in the three body problem (either ELERM or CLERM). The output matrix is:

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{6\times6} & \mathbf{0}_{6\times7} & \mathbf{0}_{6\times7} & \mathbf{0}_{6\times4} \\ \mathbf{0}_{7\times6} & \mathbf{I}_{7\times7} & \mathbf{0}_{7\times7} & \mathbf{0}_{7\times4} \\ \mathbf{0}_{7\times6} & \mathbf{0}_{7\times7} & \mathbf{I}_{7\times7} & \mathbf{0}_{7\times4} \\ \mathbf{I}_{6\times6} & \mathbf{C}_{p_c} & -\mathbf{C}_{p_t} & \mathbf{0}_{6\times4} \\ \mathbf{0}_{4\times6} & \mathbf{0}_{4\times7} & \mathbf{0}_{4\times7} & \mathbf{I}_{4\times4} \\ \mathbf{0}_{3\times6} & \begin{bmatrix}\mathbf{0}_{4\times4} & \mathbf{I}_{3\times3}\end{bmatrix} & \begin{bmatrix}\mathbf{0}_{4\times4} & -\mathbf{R}_{cl}(\mathbf{q}_{c/l})\end{bmatrix} & \mathbf{0}_{3\times4} \end{bmatrix}$$

and the output-input matrix is given by

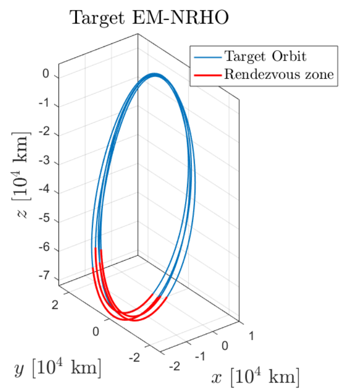
$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{6\times6} & \mathbf{0}_{6\times3} & \mathbf{0}_{6\times3} \\ \mathbf{0}_{7\times6} & \mathbf{0}_{7\times3} & \mathbf{0}_{7\times3} \\ \mathbf{0}_{7\times6} & \mathbf{0}_{7\times3} & \mathbf{0}_{7\times3} \\ \mathbf{0}_{6\times6} & \mathbf{D}_{pc} & -\mathbf{D}_{pt} \\ \mathbf{0}_{7\times6} & \mathbf{0}_{7\times3} & \mathbf{0}_{7\times3} \end{bmatrix}$$

OPERATIVE EXAMPLE

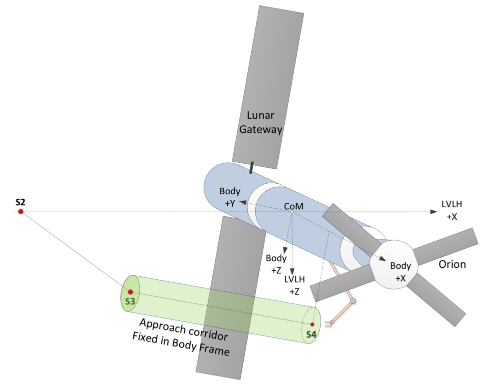
In this section an example of the use of the proposed non linear equation set of relative motion is given.
The scenario is part of the Heracles mission framework, one of the goals of the mission is to autonomously sample the Moon and return the sampling to the Lunar Orbiting Space Gateway (LOP-G), which is orbiting on NRHO around the Lagrangian point L2. The sample of Lunar terrain is

carried to the space station by a small vehicle called Lunar Ascend Module (LAE), and it has to perform rendezvous/ berthing manoeuvre with the LOP-G. In this context the LOP-G is consider the passive target vehicle and the LAE is the active chaser vehicle.

The proposed non linear equations of motion were used to model the relative motion between the



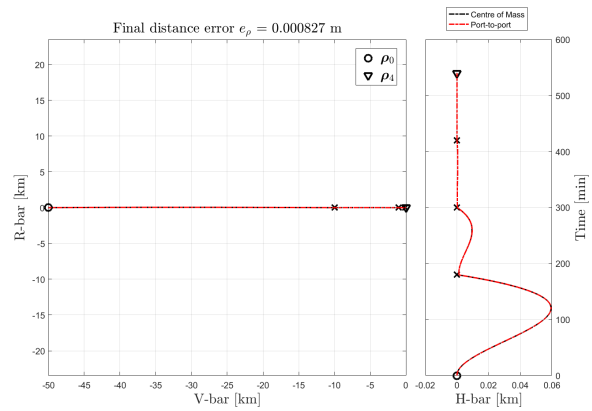
((a)) Target orbit around the Moon



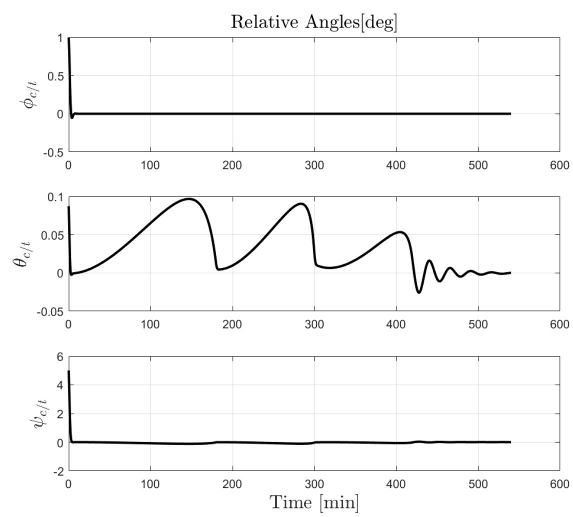
((b)) The target spacecraft and the Hold-Point sequence

Figure 6: Mission description

LOP-G and the LAE and design a guidance strategy that allows a successful rendezvous manoeuvre. The manoeuvre is composed by 4 hold-points located at 50km, 10km, 1km and 100m far from the *Target* along the V-bar, in each hold point the *Chaser* is required to have zero relative velocity and angular rate w.r.t. the *Target* and the relative attitude varies to allow the chaser x body axis to point towards the target for all the duration of the manoeuvre. The relative port-to-port motion in the LVLH frame and the relative attitude of the *Chaser* w.r.t. the *Target* are shown in Fig. 7, with the initial relative attitude is assumed non zero.



((a)) Port to port trajectory



((b)) Relative attitude

Figure 7: Mission example

CONCLUSIONS

The paper presents a complete relative dynamic and kinematic model of two spacecraft in relative motion in an environment that can be described only taking into account the third body perturbation and non inertial reference frames.

Starting from the traslational equations of relative motion in LVLH frame, the author added the relative attitude and the port-to-port equations, then they provided a linearized model of the 6-DOF dynamics to facilitate the design of GNC algorithms, with the most common state space control techniques.

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