On the Reynolds time-averaged equations and the long-time behavior of Leray-Hopf weak solutions, with applications to ensemble averages

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Abstract

We consider the three dimensional incompressible Navier-Stokes equations with a non stationary source term $\mathbf{f}$, chosen in a suitable space. We prove the existence of global Leray-Hopf weak solutions and also that it is possible to characterize (up to sub-sequences) their long-time averages, which satisfy the Reynolds averaged equations, involving the additional Reynolds stress. Moreover, we show that the turbulent dissipation is bounded by the sum of the Reynolds stress work and the external turbulent fluxes, without any additional assumption, other than that of using Leray-Hopf weak solutions.

Finally, in the last section we consider ensemble averages of solutions, associated to a set of different forces and we prove that the fluctuations continue to have a dissipative effect on the mean flow.

Keywords: Navier-Stokes equations, time-averaging, Reynolds equations, Boussinesq hypothesis.

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1 Introduction

We consider the 3D homogeneous incompressible Navier-Stokes equations (NSE in the sequel),

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} & \text{in }]0,+\infty[\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in }]0,+\infty[\times\Omega, \\ \mathbf{v} = \mathbf{0} & \text{on }]0,+\infty[\times\Gamma, \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, $\Gamma = \partial\Omega$ its boundary, $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ denotes the fluid velocity, $p = p(t,\mathbf{x})$ the kinematic pressure, $\mathbf{f} = \mathbf{f}(t,\mathbf{x})$ the external source term, and $(t,\mathbf{x}) \in]0,+\infty[\times\Omega$. The main aim of this paper is to study the long-time average of weak solutions to the NSE (1.1), namely

$$\overline{\mathbf{v}}(\mathbf{x}) := \lim_{t\to+\infty} M_t(\mathbf{v}), \quad \text{where} \quad M_t(\mathbf{v}) := \frac{1}{t}\int_0^t \mathbf{v}(s,\mathbf{x})\,ds, \tag{1.2}$$

when the source term $\mathbf{f}$ is time dependent, and to link the results to the Reynolds averaged equations (see (4.6) below). We also will consider the problem of ensemble averages, which is closely related to long-time averages.

Long-time averages for "tumultuous" flows, today turbulent flows, seem to have been considered first by G. Stokes [29] and then by O. Reynolds [28]. One important idea is that *steady-state* turbulent flows are oscillating around a stationary flow, which can be expressed through long-time averages. L. Prandtl used them to introduce the legendary "Prandtl mixing length" (see [25] and [26, Sec. 3.4]) to model a turbulent boundary layer over a plate. However, although long-time averaging plays a central role in turbulence modeling it is not clear whether the limit in (1.2) is well defined.

The mathematical problem of properly defining long-time averages and investigating the connection with the Reynolds equations has been already studied, but only when the source term does not depend on time, namely $\mathbf{f} = \mathbf{f}(\mathbf{x})$.

To our knowledge, the first who considered this problem is C. Foias [11, Sec. 8], when $\mathbf{f} \in H$, where

$$H := \{\mathbf{u} \in L^2(\Omega)^3 \text{ s.t. } \nabla\cdot\mathbf{u} = 0, \text{ with } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}. \tag{1.3}$$

Foias' analysis is based on the notion of "statistical solution" introduced in [11, Sec. 3] and one important result is that for any given Leray-Hopf solution $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ to (1.1), the following hold:

- There exists $\overline{\mathbf{v}} \in H$, such that, up to a sub-sequence, $M_t(\mathbf{v}) \to \overline{\mathbf{v}}$ in H as $t \to +\infty$;
- There is a *stationary statistical solution* μ to the NSE, which is a probability measure on H, such that $\overline{\mathbf{v}} = \int_H \mathbf{w}\,d\mu(\mathbf{w})$;
- There exists a random variable on (H,μ) called $\mathbf{v}'$ (in the notation of Foias called $\delta\mathbf{v}$) such that

$$\overline{\mathbf{v}'} = \int_H \mathbf{v}'\,d\mu = 0 \qquad \text{and} \qquad \overline{\|\nabla\mathbf{v}'\|^2} = \int_H \|\nabla\mathbf{v}'\|^2\,d\mu < \infty,$$

such that $(\overline{\mathbf{v}}, \mathbf{v}')$ is a solution to the Reynolds Equations given by (4.6) below. If μ is a statistical solution, $\mathbf{v}'$ is the random variable expressed by its probability law,

$$\text{Prob}(\mathbf{v}' \in F) = \mu(\overline{\mathbf{v}} + F), \tag{1.4}$$

for any Borel set $F \in H$.

- According to our notations, the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ is given by

$$\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} := \int_H \nabla \cdot \big[(\mathbf{v} - \overline{\mathbf{v}}) \otimes (\mathbf{v} - \overline{\mathbf{v}})\big]\, d\mu,$$

and it should be proved that it is *dissipative on the mean flow.* Proving the dissipative effect will confirm the Boussinesq assumption/hypothesis (see [7, Ch.4, Sec. 4.4.3]) and this result is one of the main challenges associated with Reynolds equations. In fact the result proved is much better, since Foias in [11, Sec. 8-2-a, Prop. 1, p. 99] was able to prove that the *turbulent dissipation* ε is bounded by the work of the Reynolds stress on the mean flow,

$$\varepsilon := \nu \overline{\|\nabla \mathbf{v}'\|^2} \leq (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}), \tag{1.5}$$

where $(\cdot, \cdot)$ and $\| \,.\, \|$ denote the standard $L^2(\Omega)$ scalar product and norm, respectively. (Sometimes norm is normalized by $|\Omega|$, the measure of the domain, but this is not essential). This analysis is partially reported in [12, Ch. 3, Sec. 3], where the limit in (1.2) is replaced by the abstract Banach limit, but without any link to the Reynolds equations and dissipation inequality such as (1.5).

The original Foias' analysis is very deep and essential to the field. In this paper, we compare Foias' analysis to our results and make some connections between the two

i) The natural time filter used to determine the Reynolds stress (initially suggested by Prandtl)

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \boldsymbol{\sigma}^{(\mathrm{R})}(\mathbf{x}) := \lim_{t \to +\infty} \frac{1}{t} \int_0^t \big[(\mathbf{v} - \overline{\mathbf{v}}) \otimes (\mathbf{v} - \overline{\mathbf{v}})\big](s, \mathbf{x})\, ds,$$

is replaced by an abstract probability measure that it is not possible to calculate in practical simulations, although the ergodic assumption implies that they coincide (see for instance Frisch [14]); We note, however, that the results have been recently improved by Foias, Rosa, and Temam [13], where the convergence of the time averages has been proved almost everywhere with respect to any invariant measure, again only for the case of a stationary external force. Although in the context of time independent forcing, this work is certainly very relevant also in the context of convergence of time averages considered here. (Note that in [13] the convergence is not up to a sub-sequence).

ii) The fluctuation given by (1.4), when the force is time independent, is a time independent random variable, which may be questionable from the physical point of view. In fact, when $\overline{\mathbf{v}}$ is given by (1.2), one cannot conclude that this $\mathbf{v}'$ yields the Reynolds decomposition

$$\mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}),$$

in which the fluctuation is time dependent for a realistic non stationary flow.

iii) Concerning items i) and ii) above, the case of a time dependent source term also remains to be considered, to characterize in a sense similar to [13] the time-averages within the framework of statistical solutions (see the warning in [11, Part I, Sec. 5, p. 313]).

We also point out that, in the case of a stationary source term, the long-time averaged problem has been also recently considered in [22] and at the time the author was not aware of the connection between time-averaging and statistical solutions in Foias' work. In [22] the equation satisfied by $M_t(\mathbf{v})$ (see equations (5.1) below) are studied for a Leray-Hopf weak solution of the NSE, and the limit $t \to +\infty$ is taken in the case of a domain of class $C^{9/4,1}$ and $\mathbf{f} = \mathbf{f}(\mathbf{x}) \in L^{5/4}(\Omega)^3 \cap V'$, where

$$V := \{\mathbf{u} \in H_0^1(\Omega)^3 \text{ s.t. } \nabla \cdot \mathbf{u} = 0\}, \tag{1.6}$$

and V' denotes the dual space of V, with duality pairing $< \cdot, \cdot >$. The analysis in [22] is based on the energy inequality, which yields a uniform estimate in time of the L^2 norm of $\mathbf{v}$, and on a L^p-regularity result by Amrouche and Girault [1] about the steady Stokes problems, valid in $C^{k,\alpha}$ domains. In [22] it is then proved that there exist $\boldsymbol{\sigma}^{(\mathrm{R})} \in L^{5/3}(\Omega)^9$ and $\overline{p} \in W^{1,5/4}(\Omega)/\mathbb{R}$ such that, up to a sub-sequence, when $t \to +\infty$, $M_t(\mathbf{v})$ converges to a field $\overline{\mathbf{v}} \in \mathbf{W}^{2,5/4}(\Omega)^3$, which satisfies in the sense of the distributions the closed Reynolds equations:

$$\begin{cases} (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} + \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{1.7}$$

Moreover, it is also proved that $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative on the mean flow, namely

$$0 \le (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}), \tag{1.8}$$

which is weaker than (1.5).

The main part of the present study is in continuation of [22], making substantial improvements. The novelty is that we are considering time dependent source terms $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$, which –to the best of our knowledge– have never been considered before for this problem. Moreover, we do not need extra regularity assumption on the domain $\Omega \subset \mathbb{R}^3$ and on $\mathbf{f}$. The first main result of this paper, Theorem 2.3 below, is close to that proved in [22]. Roughly speaking, we will show that $M_t(\mathbf{v})$ given by (1.2), converges to some $\overline{\mathbf{v}}$ (up to sub-sequences) and that there are $\overline{p}$ and $\boldsymbol{\sigma}^{(\mathrm{R})}$ such that (1.7) holds, at least in $\mathcal{D}'(\Omega)$, in which $\mathbf{f}$ is replaced by $\overline{\mathbf{f}}$.

One key point is the determination of a suitable class for the source term $\mathbf{f}$ which is enough to show existence of Leray-Hopf weak solutions with uniform estimates, allowing also the passage to the limit. For this reason, throughout the paper, we will take $\mathbf{f} : \mathbb{R}_+ \to V'$, for which there is a constant $C > 0$ such that

$$\forall\, t \in \mathbb{R}_+ \qquad \int_t^{t+1} \|\mathbf{f}(s)\|_{V'}^2 \, ds \le C.$$

In this respect, we observe that our results improve the previous ones also in terms of regularity of the force, not only because we consider a time-dependent one. The main building block of our work is the derivation of a uniform estimate for the L^2 norm of $\mathbf{v}$, for $\mathbf{f}$ as above (see (2.1) and its Corollary (2.2)). This allows to prove the existence of a weak solution on $[0, \infty)$ to the NSE and to pass to the limit in the equation satisfied by $M_t(\mathbf{v})$, when $t \to +\infty$.

Moreover, we are able to generalise Foias' result (1.5) by proving that the turbulent dissipation ε is bounded by the sum of the work of the Reynolds stress on the mean flow and of the averaged external turbulent flux, namely

$$\varepsilon = \nu \overline{\|\nabla \mathbf{v}'\|} \leq (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}) + \overline{< \mathbf{f}', \mathbf{v}' >}, \tag{1.9}$$

which is one of the main features of our result. Note that physically, it is expected that (1.9) becomes an equality for strong solutions. Furthermore, we also prove that when $\mathbf{f}$ is "attracted" (in some sense, see (2.5) below) by a stationary force $\widetilde{\mathbf{f}} = \widetilde{\mathbf{f}}(\mathbf{x})$ as $t \to +\infty$, then the average turbulent flux $\overline{< \mathbf{f}', \mathbf{v}' >}$ in (1.9) vanishes, so that (1.5) is restored. In particular, the question whether the stress tensor $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative or not remains an open problem for a generic unsteady $\mathbf{f}$, as those for which we still have global existence of weak solutions.

In the second part of the paper, we also consider ensemble averages, often used in practical experiments. This consists in considering $n \in \mathbb{N}$ realizations of the flow, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and in evaluating the arithmetic mean

$$\langle \mathbf{v} \rangle := \frac{1}{n} \sum_{k=1}^{n} \mathbf{v}_k.$$

Layton [20] considered such ensemble averages, by introducing the corresponding Reynolds stress, written as

$$R(\mathbf{v}, \mathbf{v}) := \langle \mathbf{v} \otimes \mathbf{v} \rangle - \langle \mathbf{v} \rangle \otimes \langle \mathbf{v} \rangle.$$

He proved that for a fixed stationary source term $\mathbf{f}$ and n *strong* solutions of the NSE, then $R(\mathbf{v}, \mathbf{v})$ is dissipative on the ensemble average, in the long-time average. More specifically it holds

$$\liminf_{t \to +\infty} M_t[(\nabla \cdot R(\mathbf{v}, \mathbf{v}), \langle \mathbf{v} \rangle)] \geq 0.$$

In order to remove the additional assumption used in [20, 18] of having strong solutions, we will carry out a similar approach, but performing the averaging in a reversed order: We first take the long-time average of the realizations, for a sequence of time independent source terms $\{\mathbf{f}_k\}_{k \in \mathbb{N}}$, identifying fields $\{\overline{\mathbf{v}}_k\}_{k \in \mathbb{N}}$, which are weak solutions of the corresponding Reynolds equations. Then, we form the ensemble average

$$\mathbf{S}^n := \frac{1}{n} \sum_{k=1}^{n} \overline{\mathbf{v}}_k.$$

Under suitable (but very light) regularity assumption about the $\mathbf{f}_k$'s, we show the weak convergence of $\{\mathbf{S}^n\}_{n\in\mathbb{N}}$, to some $\langle\mathbf{v}\rangle \in V$ satisfying the closed Reynolds equations, and such that dissipativity still holds true

$$0 \leq (\nabla \cdot \langle\boldsymbol{\sigma}^{(\mathrm{R})}\rangle, \langle\mathbf{v}\rangle).$$

See the specific statement in Theorem 2.4, which holds in the natural setting of weak solutions.

Plan of the paper. The paper is organized as follows. We start by giving in Sec. 2 the specific technical statements of the results we prove. Then, in Sec. 3 we give some results about the functional spaces we are working with and we prove in detail the main energy estimates (3.4) and (3.5) for source terms $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+, V')$, which yields an existence result of global weak solutions to the NSE for such $\mathbf{f}$. Sec. 4 is devoted to the Reynolds problem and to develop additional properties of the time average operator M_t. Finally, we give the proofs of the two main results on time averages in Sec. 5 and Sec. 6, respectively.

2 Main results

2.1 On the source term and an existence result

Since we aim to consider long-time averages for the NSE, we must consider solutions which are global-in-time (defined for all positive times). Due to the well-known open problems related to the NSE, this enforces us to restrict to weak solutions. Within a natural setting, we take the initial datum $\mathbf{v}_0 \in H$, where H is defined by (1.3).

The classical Leray-Hopf results of existence (but without uniqueness) of a global weak solution $\mathbf{v}$ to the NSE holds when $\mathbf{f} \in L^2(\mathbb{R}_+; V')$, and the velocity $\mathbf{v}$ satisfies

$$\mathbf{v} \in L^2(\mathbb{R}_+, V) \cap L^\infty(\mathbb{R}_+, H),$$

where V is defined by (1.6) and V' denotes its topological dual. We will also denote by $< \cdot, \cdot >$ the duality pairing[1] between V' and V.

Source terms $\mathbf{f} \in L^2(\mathbb{R}_+; V')$ verify $\int_t^\infty \|\mathbf{f}(s)\|^2_{V'}\, ds \to 0$ when $t \to +\infty$. Therefore, a turbulent motion cannot be maintained for large t, which is not relevant for our purpose. The choice adopted in the previous studies on Reynolds equations was that of a constant force, and we also observe that many estimates could have been easily extended to a uniformly bounded $\mathbf{f} \in L^\infty(\mathbb{R}_+; V')$. On the other hand, we consider a broader class for the source terms. According to the usual folklore in mathematical analysis, we decided to consider the space $L^2_{uloc}(\mathbb{R}_+; V')$ made of all strongly measurable vector fields $\mathbf{f} : \mathbb{R}_+ \to V'$ such that

$$\|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')} := \left[\sup_{t\geq 0} \int_t^{t+1} \|\mathbf{f}(s)\|^2_{V'}\, ds\right]^{1/2} < +\infty.$$

[1] Generally speaking and when no risk of confusion occurs, we always denote by $<\,,\,>$ the duality pairing between any Banach space X and its dual X', without mentioning explicitly which spaces are involved.

We will see in the following, that the above space, which strictly contains both $L^2(\mathbb{R}_+; V')$ and $L^\infty(\mathbb{R}_+; V')$, is well suited for our framework. We will prove the following existence result, in order to make the paper self-contained.

Theorem 2.1. *Let $\mathbf{v}_0 \in H$, and let $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$. Then, there exists a weak solution $\mathbf{v}$ to the NSE (1.1) global-in-time, obtained by Galerkin approximations, such that*

$$\mathbf{v} \in L^2_{loc}(\mathbb{R}_+; V) \cap L^\infty(\mathbb{R}_+; H),$$

and which satisfies for all $t \geq 0$,

$$\|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}_0\|^2 + \left(3 + \frac{C_\Omega}{\nu}\right)\frac{\mathcal{F}^2}{\nu}, \tag{2.1}$$

$$\nu \int_0^t \|\nabla \mathbf{v}(s)\|^2 ds \leq \|\mathbf{v}_0\|^2 + ([t]+1)\frac{\mathcal{F}^2}{\nu}, \tag{2.2}$$

where $\mathcal{F} := \|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$.

Remark 2.2. *The weak solution $\mathbf{v}$ shares most of the properties of the Leray-Hopf weak solutions, with estimates valid for all positive times. Notice that we do not know whether or not this solution is unique. Anyway, $\mathbf{v}$ will not necessarily get "regular" as $t \to +\infty$, which is the feature of interest for our study. As usual by "regular" solution we mean a strong solution, with the L^2-norm of the gradient (locally) bounded for all positive times.*

2.2 Long-time averaging

This section is devoted to state the main results of the paper about long-time and ensemble averages.

Theorem 2.3. *Let be given $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and let $\mathbf{v}$ a global-in-time weak solution to the NSE (1.1). Then, there exist*

a) a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} t_n = +\infty$;

b) a vector field $\overline{\mathbf{v}} \in V$;

c) vector field $\overline{\mathbf{f}} \in V'$;

d) a vector field $\mathbf{B} \in L^{3/2}(\Omega)^3$;

e) a second order tensor field $\boldsymbol{\sigma}^{(\mathrm{R})} \in L^3(\Omega)^9$;

such that it holds:

i) when $n \to +\infty$,

$$\begin{aligned}
&M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}} \qquad \text{in } V,\\
&M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}} \qquad \text{in } V',\\
&M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} \qquad \text{in } L^{3/2}(\Omega)^3,\\
&M_{t_n}(\mathbf{v}'\otimes\mathbf{v}') \rightharpoonup \boldsymbol{\sigma}^{(\mathrm{R})} \qquad \text{in } L^3(\Omega)^9;\\
&M_{t_n}(<\mathbf{f},\mathbf{v}>) \to <\overline{\mathbf{f}},\overline{\mathbf{v}}> + \overline{<\mathbf{f}',\mathbf{v}'>},
\end{aligned}$$

where $\mathbf{v}' = \mathbf{v} - \overline{\mathbf{v}}$, *and* $\mathbf{f}' = \mathbf{f} - \overline{\mathbf{f}}$;

ii) the closed Reynolds equations (2.3) *hold true in the weak sense:*

$$\begin{cases} (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} + \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma; \end{cases} \tag{2.3}$$

iii) the following equality $\mathbf{F} = \mathbf{B} - (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}$ *is valid in* $\mathcal{D}'(\Omega)$;

iv) the following energy balance holds true

$$\nu \|\nabla \overline{\mathbf{v}}\|^2 + \int_\Omega \mathbf{F} \cdot \overline{\mathbf{v}}\, d\mathbf{x} = < \overline{\mathbf{f}}, \overline{\mathbf{v}} >;$$

v) the turbulent dissipation ε *is bounded by the sum of the work of the Reynolds stress on the mean flow and the averages of external turbulent fluxes,*

$$\varepsilon = \nu \overline{\|\nabla \mathbf{v}'\|^2} \leq \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \overline{< \mathbf{f}', \mathbf{v}' >}; \tag{2.4}$$

vi) if in addition the source term $\mathbf{f}$ *verifies:*

$$\exists\, \widetilde{\mathbf{f}} \in V', \quad \text{such that} \quad \lim_{t \to +\infty} \int_t^{t+1} \|\mathbf{f}(s) - \widetilde{\mathbf{f}}\|^2_{V'}\, ds = 0, \tag{2.5}$$

then $\overline{\mathbf{f}} = \widetilde{\mathbf{f}}$ *and* $\overline{< \mathbf{f}', \mathbf{v}' >} = 0$; *in particular, the Reynolds stress* $\boldsymbol{\sigma}^{(\mathrm{R})}$ *is dissipative on the mean flow, that is* (1.8) *holds true.*

Our second result has to be compared with results in Layton *et al.* [19, 20], where the long-time averages are taken for an ensemble of solutions.

Theorem 2.4. *Let be given a sequence* $\{\mathbf{f}_k\}_{k\in\mathbb{N}} \subset L^q(\Omega)$ *converging weakly to some* $\langle \mathbf{f} \rangle$ *in* $L^q(\Omega)$, *with* $\frac{6}{5} < q < \infty$ *and let* $\{\overline{\mathbf{v}}^k\}_{k\in\mathbb{N}}$ *be the associated long-time average of velocities, whose existence has been proved in Theorem 2.3. Then, the sequence of arithmetic averages* $\{\langle \mathbf{v} \rangle^n\}_{n\in\mathbb{N}}$ *of the long-time limits* $\{\overline{\mathbf{v}}^k\}_{k\in\mathbb{N}}$, *which is defined as follows*

$$\langle \mathbf{v} \rangle^n := \frac{1}{n} \sum_{k=1}^n \overline{\mathbf{v}}^k,$$

converges weakly in V, *as* $n \to +\infty$, *to some* $\langle \mathbf{v} \rangle$, *which satisfies the following system of Reynolds type in the sense of distributions over* Ω

$$\begin{cases} \big(\langle \mathbf{v} \rangle \cdot \nabla\big)\, \langle \mathbf{v} \rangle - \nu \Delta \langle \mathbf{v} \rangle + \nabla \langle p \rangle + \nabla \cdot \langle \boldsymbol{\sigma}^{(\mathrm{R})} \rangle = \langle \mathbf{f} \rangle & \text{in } \Omega, \\ \nabla \cdot \langle \mathbf{v} \rangle = 0 & \text{in } \Omega, \\ \langle \mathbf{v} \rangle = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where $\langle \boldsymbol{\sigma}^{(\mathrm{R})} \rangle$ *is dissipative in average, that is more precisely it holds*

$$0 \leq \frac{1}{|\Omega|} \int_\Omega \left(\nabla \cdot \langle \boldsymbol{\sigma}^{(\mathrm{R})} \rangle \right) \cdot \langle \mathbf{v} \rangle \, d\mathbf{x}.$$

In this case we do not have a sharp lower bound on the dissipation as in Theorem 2.3, since here the averaging is completely different and the fluctuations are not those emerging in long-time averaging. Nevertheless, the main statement is in the same spirit of the first proved result.

3 Navier-Stokes equations with uniformly-local source terms

This section is devoted to sketch a proof of Theorem 2.1. Most of the arguments are quite standard and we will give appropriate references at each step, to focus on what seems (at least to us) non-standard when $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$; especially the proof of the uniform L^2-estimate (3.4), which is the building block for the results of the present paper. Before doing this, we introduce the function spaces we will use, and precisely define the notion of weak solutions we will deal with.

3.1 Functional setting

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary $\partial\Omega$. This is a sort of minimal assumption of regularity on the domain, in order to have the usual properties for Sobolev spaces and to characterize in a proper way divergence-free vector fields, see for instance Constantin and Foias [8], Galdi [15, 16], Girault and Raviart [17], Tartar [30].

We use the customary Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{1,p}(\Omega), \|\,.\,\|_{1,p})$. For simplicity, we denote the L^2-norm simply by $\|\,.\,\|$ and we write $H^1(\Omega) := W^{1,2}(\Omega)$. For a given sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, where $(X, \|\,.\,\|_X)$ is Banach space, we denote by $x_n \to x$ the strong convergence, while by $x_n \rightharpoonup x$ the weak one.

As usual in mathematical fluid dynamics, we use the following spaces

$$\begin{aligned}
\mathcal{V} &= \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3,\ \nabla \cdot \boldsymbol{\varphi} = 0\}, \\
H &= \left\{\mathbf{v} \in L^2(\Omega)^3,\ \nabla \cdot \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\}, \\
V &= \left\{\mathbf{v} \in H^1_0(\Omega)^3, \nabla \cdot \mathbf{v} = 0\right\},
\end{aligned}$$

and we recall that $\mathcal{V}$ is dense in H and V for their respective topologies [17, 30].

Let $(X, \|\,.\,\|_X)$ be a Banach space, we use the Bochner spaces $L^p(I; X)$, for $I = (0, T)$ (for some $T > 0$) or $I = \mathbb{R}_+ = (0, \infty)$, equipped with the norm

$$\|u\|_{L^p(I;X)} := \begin{cases} \left(\displaystyle\int_I \|u(s)\|_X^p \, ds \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \text{ess sup}_{s\in I} \|u(s)\|_X & \text{for } p = +\infty. \end{cases}$$

The existence of weak solutions for the NSE (1.1) is generally proved in the literature when $\mathbf{v}_0 \in H$ and the source term $\mathbf{f} \in L^2(I; V')$, or alternatively when the source term is a given constant element of V'. In order to study the long-time behavior of weak solutions of the NSE (1.1), we aim to enlarge the class of function spaces allowed for the source term $\mathbf{f}$, to catch a more complex behavior than that coming from constant external forces, as initially developed in [13, 22]. To do so, we deal with "uniformly-local" spaces, as defined below in a general setting.

Definition 3.1. *Let be given $p \in [1, +\infty[$. We define $L^p_{uloc}(\mathbb{R}_+; X)$ as the space of strongly measurable functions $f : \mathbb{R}_+ \to X$ such that*

$$\|f\|_{L^p_{uloc}(X)} := \left[\sup_{t\geq 0} \int_t^{t+1} \|f(s)\|_X^p ds\right]^{1/p} < +\infty.$$

It is easily checked that the spaces $L^p_{uloc}(\mathbb{R}_+; X)$ are Banach spaces strictly containing both the constant X-valued functions, and also $L^p(\mathbb{R}_+; X)$, as illustrated by the following elementary lemma.

Lemma 3.2. *Let be given $f \in C(\mathbb{R}_+; X)$ converging to a limit $\ell \in X$, when $t \to +\infty$. Then, for any $p \in [1, +\infty[$, we have that $f \in L^p_{uloc}(\mathbb{R}_+; X)$, and there exists $T > 0$ such that*

$$\|f\|_{L^p_{uloc}(X)} \leq \left[\sup_{t\in[0,T+1]} \|f(t)\|_X^p + 2^{p-1}(1 + \|\ell\|_X)^p\right]^{\frac{1}{p}}.$$

Proof. As $\ell = \lim_{t\to+\infty} f(t)$, there exists $T > 0$ such that: $\forall\, t > T$, $\|f(t) - \ell\|_X \leq 1$. In particular, it holds

$$\int_t^{t+1} \|f(s)\|_X^p \, ds \leq 2^{p-1}(1 + \|\ell\|_X)^p \qquad \text{for } t > T,$$

while for all $t \in [0, T]$,

$$\int_t^{t+1} \|f(s)\|_X^p \, ds \leq \sup_{t\in[0,T+1]} \|f(t)\|_X^p,$$

hence the result. ☐

However, it easy to find examples of discontinuous functions in $L^p_{uloc}(\mathbb{R}_+; X)$ which are not converging when $t \to +\infty$, and which are not belonging to $L^p(\mathbb{R}_+; X)$.

3.2 Weak solutions

There are many ways of defining weak solutions to the NSE (see also P.-L. Lions [24]). Since we are considering the incompressible case, the pressure is treated as a Lagrange multiplier. Following the pioneering idea developed by J. Leray [21], the NSE are projected over spaces of divergence-free functions. This is why when we talk about weak solutions the NSE, only the velocity $\mathbf{v}$ is mentioned, not the pressure.

As in J.-L. Lions [23], we give the following definition of weak solution, see also Temam [31, Ch. III].

Definition 3.3 (Weak solution)**.** *Given $\mathbf{v}_0 \in H$ and $\mathbf{f} \in L^2(I;V')$ we say that $\mathbf{v}$ is a weak solution over the interval $I = [0,T]$ if the following items are fulfilled:*

i) the vector field $\mathbf{v}$ has the following regularity properties

$$\mathbf{v} \in L^2(I;V) \cap L^\infty(I;H),$$

and is weakly continuous from I to H, while $\lim_{t\to 0^+} \|\mathbf{v}(t) - \mathbf{v}_0\|_H = 0$;

ii) for all $\boldsymbol{\varphi} \in \mathcal{V}$,

$$\frac{d}{dt}\int_\Omega \mathbf{v}(t,\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} - \int_\Omega \mathbf{v}(t,\mathbf{x}) \otimes \mathbf{v}(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} \\ + \nu \int_\Omega \nabla \mathbf{v}(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} = < \mathbf{f}(t), \boldsymbol{\varphi} >,$$

holds true in $\mathcal{D}'(I)$;

iii) the energy inequality

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}(t)\|^2 + \nu\|\nabla\mathbf{v}(t)\|^2 \leq < \mathbf{f}(t), \mathbf{v}(t) >, \tag{3.1}$$

holds in $\mathcal{D}'(I)$, where we write $\mathbf{v}(t)$ instead of $\mathbf{v}(t,\cdot)$ for simplicity.

When $\mathbf{f} \in L^2(0,T;V')$ and $\mathbf{v}$ is a weak solution in $I = [0,T]$, and this holds true for all $T > 0$, we speak of a "global-in-time solution", or simply a "global solution". In particular, ii) is satisfied in the sense of $\mathcal{D}'(0,+\infty)$.

There are several ways to prove the existence of (at least) a weak solution to the NSE. Among them, in what follows, we will use the Faedo-Galerkin method. Roughly speaking, let $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}}$ denote a Hilbert basis of V, and let, for $n \in \mathbb{N}$, $V_n := \operatorname{span}\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n\}$. By assuming $\mathbf{f} \in L^2(I;V')$, it can be proved by the Cauchy-Lipschitz theorem (see [23]) the existence of a unique $\mathbf{v}_n \in C^1(I;V_n)$ such that for all $\boldsymbol{\varphi}_k$, with $k = 1, \dots, n$, it holds

$$\frac{d}{dt}\int_\Omega \mathbf{v}_n(t,\mathbf{x}) \cdot \boldsymbol{\varphi}_k(\mathbf{x})\, d\mathbf{x} - \int_\Omega \mathbf{v}_n(t,\mathbf{x}) \otimes \mathbf{v}_n(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}_k(\mathbf{x})\, d\mathbf{x} \\ + \nu \int_\Omega \nabla \mathbf{v}_n(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}_k(\mathbf{x})\, d\mathbf{x} = < \mathbf{f}(t), \boldsymbol{\varphi}_k >, \tag{3.2}$$

and which naturally satisfies the energy balance (equality)

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}_n(t)\|^2 + \nu\|\nabla\mathbf{v}_n(t)\|^2 = < \mathbf{f}, \mathbf{v}_n > . \tag{3.3}$$

It can be also proved (always see again [23]) that from the sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ one can extract a sub-sequence converging, in an appropriate sense, to a weak solution to the NSE. When $I=\mathbb{R}_+$ we get a global solution.

However, if assume $\mathbf{f}\in L^2_{uloc}(\mathbb{R}_+;V')$, the global result of existence does not work so straightforward. Of course, for any given $T>0$, we have

$$L^2_{uloc}(\mathbb{R}_+;V')_{|[0,T]}\hookrightarrow L^2([0,T];V'),$$

where $L^2_{uloc}(\mathbb{R}_+;V')_{|[0,T]}$ denotes the restriction of a function in $L^2_{uloc}(\mathbb{R}_+;V')$ to $[0,T]$. Therefore, no doubt that the construction above holds over any time-interval $[0,T]$. In such case letting T go to $+\infty$ to get a global solution (with some uniform control of the kinetic energy) is not obvious, and we do not know any reference explicitly dealing with this issue, which deserves to be investigated more carefully. This is the aim of the next subsection, where we prove the most relevant a-priori estimates.

3.3 A priori estimates

Let be given $\mathbf{f}\in L^2_{uloc}(\mathbb{R}_+;V')$, and let $\mathbf{v}_n=\mathbf{v}$ be the solution of the Galerkin projection of the NSE over the finite dimensional space V_n. The function $\mathbf{v}$ satisfies (3.2) and (3.3) (we do not write the subscript $n\in\mathbb{N}$ for simplicity), is smooth, unique, and can be constructed by the Cauchy-Lipschitz principle over any finite time interval $[0,T]$. Hence, we observe that by uniqueness and the a priori estimate deriving from (3.3) the field $\mathbf{v}$ can be extended to $\mathbb{R}_+$. We then denote $\mathcal{F}:=\|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$ and then after a delicate manipulation of the energy balance combined with the Poincaré inequality, we get the following lemma.

Lemma 3.4. *For all $t\geq 0$ we have*

$$\|\mathbf{v}(t)\|^2\leq\|\mathbf{v}_0\|^2+\left(3+\frac{C_\Omega}{\nu}\right)\frac{\mathcal{F}^2}{\nu}, \tag{3.4}$$

as well as

$$\nu\int_0^t\|\nabla\mathbf{v}(s)\|^2\,ds\leq\|\mathbf{v}_0\|^2+([t]+1)\frac{\mathcal{F}^2}{\nu}, \tag{3.5}$$

where C_Ω denotes the constant in the Poincaré inequality $\|\mathbf{u}\|^2\leq C_\Omega\|\nabla\mathbf{u}\|^2$, valid for all $\mathbf{u}\in V$.

Proof. We focus on the proof of the a priori estimate (3.4), the estimate (3.5) being a direct consequence of the energy balance. By the Young inequality we deduce from the the energy inequality,

$$\forall\,\xi,\tau\in\mathbb{R}_+\quad\text{s.t.}\quad 0\leq\xi\leq\tau,$$

$$\|\mathbf{v}(\tau)\|^2+\nu\textstyle\int_\xi^\tau\|\nabla\mathbf{v}(s)\|^2\,ds\leq\|\mathbf{v}(\xi)\|^2+\frac{1}{\nu}\int_\xi^\tau\|\mathbf{f}(s)\|^2_{V'}\,ds.$$

In particular, when $0 \le \tau - \xi \le 1$,

$$\|\mathbf{v}(\tau)\|^2 + \nu \int_\xi^\tau \|\nabla \mathbf{v}(s)\|^2 \, ds \le \|\mathbf{v}(\xi)\|^2 + \frac{\mathcal{F}^2}{\nu}. \tag{3.6}$$

From this point, we argue step by step. The case $0 \le t \le 1$ is the first step, which is straightforward. The second step is the heart of the proof. The issue is that energy may increase, without control, when the time increases.

We will show that even if this happens, we can still keep the control on it, thanks to (3.6). The last step is carried out by induction on n writing $t = \tau + n$, for $\tau \in [0, 1]$.

<u>Step 1</u>. Fix $t \in [0, 1]$. Then, take in (3.6) $\xi = 0$ and $t = \tau \in [0, 1]$. It holds

$$\|\mathbf{v}(t)\|^2 \le \|\mathbf{v}_0\|^2 + \frac{\mathcal{F}^2}{\nu} \qquad \forall\, t \in [0, 1].$$

<u>Step 2</u>. We show that the following implication holds true

$$\|\mathbf{v}(t)\| \le \|\mathbf{v}(t+1)\| \;\Rightarrow\; \begin{cases} \|\mathbf{v}(t)\| \le \left(1 + \dfrac{C_\Omega}{\nu}\right) \dfrac{\mathcal{F}^2}{\nu}, \\ \|\mathbf{v}(t+1)\| \le \left(2 + \dfrac{C_\Omega}{\nu}\right) \dfrac{\mathcal{F}^2}{\nu}. \end{cases}$$

In the following we will set

$$\mathcal{C}^2 := \left(\|\mathbf{v}_0\|^2 + \frac{\mathcal{F}^2}{\nu}\right) + \left(2 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu} = \|\mathbf{v}_0\|^2 + \left(3 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}. \tag{3.7}$$

<u>Sub-Step 2.1.</u> By the energy inequality with $\xi = t$, and $\tau = t + 1$, we have, by using the hypothesis on the L^2-norm at times t and $t + 1$:

$$\nu \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds \le \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(t)\|^2 + \nu \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds \le \frac{\mathcal{F}^2}{\nu}.$$

Hence, by the Poincaré's inequality:

$$\int_t^{t+1} \|\mathbf{v}(s)\|^2 \, ds \le C_\Omega \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds \le \frac{C_\Omega \mathcal{F}^2}{\nu^2}, \tag{3.8}$$

<u>Sub-Step 2.2.</u> Let be given $\epsilon > 0$ and let $\xi \in [t, t+1]$ be such that

$$\|\mathbf{v}(\xi)\|^2 < \inf_{s \in [t, t+1]} \|\mathbf{v}(s)\|^2 + \epsilon \le \|\mathbf{v}(s)\|^2 + \epsilon \qquad \forall\, s \in [t, t+1].$$

Let us write:

$$\begin{aligned} \|\mathbf{v}(t)\|^2 \le \|\mathbf{v}(t+1)\|^2 &= \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 + \|\mathbf{v}(\xi)\|^2 \\ &= \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 + \int_t^{t+1} \|\mathbf{v}(\xi)\|^2 \, ds, \end{aligned}$$

being the integration with respect to the s variable.

To estimate the right-hand side we use the energy inequality (3.6) with $\tau = t+1$ to get

$$\|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 \leq \frac{\mathcal{F}^2}{\nu}.$$

Moreover, using the estimate (3.8) we get,

$$\int_t^{t+1} \|\mathbf{v}(\xi)\|^2 \, ds \leq \int_t^{t+1} (\|\mathbf{v}(s)\|^2 + \epsilon) \, ds \leq \frac{C_\Omega \mathcal{F}^2}{\nu^2} + \varepsilon,$$

therefore, letting $\varepsilon \to 0$ yields,

$$\|\mathbf{v}(t)\|^2 \leq \left(1 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}.$$

In addition, we get from the energy inequality

$$\|\mathbf{v}(t+1)\|^2 \leq \|\mathbf{v}(t)\|^2 + \frac{\mathcal{F}^2}{\nu} \leq \left(2 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}.$$

Step 3. Conclusion of the proof of (3.4). Any $t \geq 0$ can be decomposed as

$$t = n + \tau, \quad \text{with } n \in \mathbb{N} \text{ and } \tau \in [0, 1[.$$

We argue by induction on n. If $n = 0$, estimate (3.4) has been proved in Step 1. Assume that (3.4) is satisfied for $t := n + \tau$, for all $n \leq N$ that is,

$$\|\mathbf{v}(n+\tau)\| \leq \mathcal{C} \qquad n = 0, \dots, N,$$

where the constant $\mathcal{C}$ is defined by (3.7).

If $\|\mathbf{v}(N+1+\tau)\| < \|\mathbf{v}(N+\tau)\|$, then (3.4) holds at the time $t = N+1+\tau$, by the inductive hypothesis.

If $\|\mathbf{v}(N+\tau)\| \leq \|\mathbf{v}(N+1+\tau)\|$, then the inequality (3.4) is satisfied by Step 2 for $t = N+1+\tau$, ending the proof.

□

Once we have proved that the uniform (independent of $n \in \mathbb{N}$) L^2-estimate is satisfied by the Galerkin approximate functions, it is rather classical to prove that we can extract a sub-sequence that converges strongly in $L^2(0, T; H)$, weakly in $L^2(0, T; V)$, and weakly$*$ in $L^\infty(0, T; H)$ (for all positive T) to a weak solution to the NSE, which inherits the same bound. We refer to the references already mentioned for this point.

4 Reynolds decomposition and time-averaging

We sketch out the standard routine, concerning time-averaging, when used in turbulence modeling practice. In particular, we recall the Reynolds decomposition and the Reynolds rules, cf. [3]. Then, we give a few technical properties of the time-averaging operator M_t, defined by

$$M_t(\psi) := \frac{1}{t}\int_0^t \psi(s)\,ds,$$

for a given fixed time $t > 0$. We need to apply it not only to real functions of a real variable, but also to Banach valued functions, hence we need to deal with the Bochner integral.

We start with the following corollary of Bochner theorem (see Yosida [32, p. 132]).

Lemma 4.1. *Assume that, for some $t > 0$ we have $\psi \in L^p(0,t;X)$ (namely ψ is a Bochner p-summable function over $[0,t]$, with values in the Banach space X). Then, it holds*

$$\|M_t(\psi)\|_X \leq \frac{1}{t^{\frac{1}{p}}}\|\psi\|_{L^p(0,t;X)} \qquad \text{for all } t > 0. \tag{4.1}$$

Estimate (4.1) is the building block to give meaning to the long-time average defined as

$$\overline{\psi} := \lim_{t\to+\infty} M_t(\psi), \tag{4.2}$$

whenever the limit exists.

It is worth noting at this stage that the mapping μ behind the long-time average, defined on the Borel sets of $\mathbb{R}_+$ by

$$A \mapsto \lim_{t\to+\infty}\frac{1}{t}\lambda(A\cap[0,t]) := \lim_{t\to+\infty} M_t(\mathbb{1}_A) = \mu(A),$$

where λ the Lebesgue measure, is not –strictly speaking– a probability measure since it is not σ-additive[2]. Therefore, the quantity $\overline{\psi}$ is not rigorously a statistic, even if practitioners could be tempted to write it (in a suggestive and evocative meaningful way) as follows:

$$\overline{\psi}(\mathbf{x}) = \int_{\mathbb{R}_+}\psi(s,\mathbf{x})\,d\mu(s).$$

4.1 General setup of turbulence modeling

We recall that M_t is a linear filtering operator which commutes with differentiation with respect to the space variables (the so called Reynolds rules). In particular, one has the following result (its proof is straightforward), which is essential for our modeling process.

Lemma 4.2. *Let be given $\psi \in L^1(0,T;W^{1,p}(\Omega))$, then*

$$DM_t(\psi) = M_t(D\psi) \qquad \forall\, t > 0,$$

for any first order differential operator D acting on the space variables $\mathbf{x} \in \Omega$.

[2] The mapping μ satisfies $\mu(A\cup B) = \mu(A) + \mu(B)$ for $A\cap B = \emptyset$ but, on the other hand, we have $\sum_{n=0}^{\infty}\mu([n,n+1[) = 0 \neq 1 = \mu(\bigcup_{n=0}^{\infty}[n,n+1[)$.

By denoting the long-time average of any field ψ by $\overline{\psi}$ as in (4.2), we consider the fluctuations ψ' around the mean value, given by the *Reynolds decomposition*

$$\psi := \overline{\psi} + \psi'.$$

Observe that long-time averaging has many convenient *formal* mathematical properties, recalled in the following.

Lemma 4.3. *The following formal properties holds true, provided the long-time averages do exist.*

1. *The "bar operator" preserves the no-slip boundary condition. In other words, if $\psi_{|\Gamma} = 0$, then $\overline{\psi}_{|\Gamma} = 0$;*
2. *Fluctuation are in the kernel of the bar operator, that is $\overline{\psi'} = 0$;*
3. *The bar operator is idempotent, that is $\overline{\overline{\psi}} = \overline{\psi}$, which also yields $\overline{\overline{\psi}\,\varphi} = \overline{\psi}\overline{\varphi}$.*

Accordingly, the velocity components can be decomposed in the Reynolds decomposition as follows:

$$\mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}).$$

Let us determine (at least formally) the system of partial differential equations satisfied by $\overline{\mathbf{v}}$. From Lemma 4.2 it follows $\nabla \cdot \overline{\mathbf{v}} = 0$. Next, we use the above Reynolds rules to expand the nonlinear quadratic term into

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \overline{\mathbf{v}' \otimes \mathbf{v}'}, \tag{4.3}$$

which follows by observing that $\overline{\mathbf{v}' \otimes \overline{\mathbf{v}}} = \overline{\overline{\mathbf{v}} \otimes \mathbf{v}'} = \mathbf{0}$.

The above rules allow us to prove the following result showing a certain "orthogonality" between averages and fluctuations.

Lemma 4.4. *Let be given a linear space $X \subseteq L^2(\Omega)$ with a scalar product $(\,.\,,\,.\,)$. Let in addition be given a function $\psi : \mathbb{R}_+ \to X$ such that $\overline{\psi}$ is well defined. Then, it follows that*

$$\overline{(\psi', \psi')} = \overline{(\psi, \psi)} - (\overline{\psi}, \overline{\psi}),$$

provided all averages are well defined.

Proof. The proof follows by observing that $\psi' = \psi - \overline{\psi}$, hence

$$\overline{(\psi', \psi')} = \overline{(\psi - \overline{\psi}, \psi - \overline{\psi})} = \overline{(\psi, \psi)} - 2\overline{(\psi, \overline{\psi})} + \overline{(\overline{\psi}, \overline{\psi})},$$

and by the Reynolds rules $\overline{(\psi, \overline{\psi})} = (\overline{\psi}, \overline{\psi})$ and $\overline{(\overline{\psi}, \overline{\psi})} = (\overline{\psi}, \overline{\psi})$, from which it follows the thesis.

In particular, we will use it for the V scalar product showing that

$$\overline{\|\nabla \mathbf{u}'\|^2} = \overline{\|\nabla \mathbf{u}\|^2} - \|\nabla \overline{\mathbf{u}}\|^2, \tag{4.4}$$

for $\mathbf{u} : \mathbb{R}_+ \to V$, such that the long-time average exists. □

Remark 4.5. *Observe that, for weak solutions of the NSE* $\mathbf{v}$*, the average* $M_t(\|\nabla\mathbf{v}\|^2)$ *is bounded uniformly, by the result of Theorem 2.1 and –up to sub-sequences– some limit can be identified. Moreover, by using an argument similar to Lemma 4.1 it follows that*

$$\|M_t(\nabla\mathbf{v})\|^2 \leq M_t(\|\nabla\mathbf{v}\|^2),$$

which show that (up to sub-sequences) also the second term from the right-hand side of (4.4) can be properly defined. Consequently, also the average of the squared V*-norm of the fluctuations from the left-hand side is well defined, by difference.*

Long-time averaging applied to the Navier-Stokes equations (in a strong formulation) gives the following "equilibrium problem" for the long-time average $\overline{\mathbf{v}}(\mathbf{x})$,

$$\begin{cases} -\nu\Delta\overline{\mathbf{v}} + \nabla\cdot(\overline{\mathbf{v}\otimes\mathbf{v}}) + \nabla\overline{p} = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{4.5}$$

which we will treat in the next section, in connection with the closure problem.

4.2 Reynolds stress and Reynolds tensor

The first equation of system the averaged NSE (4.5) can be rewritten also as follows (by using the decomposition into averages and fluctuations)

$$-\nu\Delta\overline{\mathbf{v}} + \nabla\cdot(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}) + \nabla\overline{p} = -\nabla\cdot(\overline{\mathbf{v}'\otimes\mathbf{v}'}) + \overline{\mathbf{f}}, \tag{4.6}$$

called the Reynolds equations. Beside convergence issues, a relevant point is to characterize the average of product of fluctuations from the right-hand side, which is the divergence of the so called Reynolds stress tensor, defined as follows

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \overline{\mathbf{v}'\otimes\mathbf{v}'}. \tag{4.7}$$

The Boussinesq hypothesis, formalized in [5] (see also [7, Ch. 3 & 4], for a comprehensive and modern presentation) corresponds then to a closure hypothesis with the following linear constitutive equation:

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t\frac{\nabla\overline{\mathbf{v}} + \nabla\overline{\mathbf{v}}^T}{2} + \frac{2}{3}k\,\mathrm{Id}, \tag{4.8}$$

where $\nu_t \geq 0$ is a scalar coefficient, called turbulent viscosity or eddy-viscosity (sometimes called "effective viscosity"), and

$$k = \frac{1}{2}\overline{|\mathbf{v}'|^2},$$

is the turbulent kinetic energy, see [3, 7]. Formula (4.8) is a linear relation between stress and strain tensors, and shares common formal points with the linear constitutive equation valid for Newtonian fluids. In particular, this assumptions motivates the fact that $\boldsymbol{\sigma}^{(\mathrm{R})}$ must be dissipative[3] on the mean flow. Some recent results in the numerical verification of the hypothesis can be found in the special issue [4] dedicated to Boussinesq. Here, we show that, beside the validity of the modeling assumption (4.8), the Reynolds stress tensor $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative, under minimal assumptions on the regularity of the data of the problem.

4.3 Time-averaging of uniformly-local fields

We list in this section some technical properties of the operator M_t acting on uniform-local fields, and the corresponding global weak solutions to the NSE. The first result is the following

Lemma 4.6. *Let $1 \leq p < \infty$ and let be given $f \in L^p_{uloc}(\mathbb{R}_+; X)$. Then*

$$\forall\, t \geq 1, \qquad \|M_t(f)\|_X \leq 2\|f\|_{L^p_{uloc}(\mathbb{R}_+;X)}.$$

Proof. Applying (4.1) and some straightforward inequalities yields

$$\|M_t(f)\|_X \leq \frac{1}{t}\int_0^t \|f\|_X\,ds \leq \frac{1}{t}\int_0^{[t]+1} \|f\|_X\,ds \leq \frac{1}{t}\sum_{k=0}^{[t]}\int_k^{k+1} \|f\|_X\,ds.$$

Therefore by the Hölder inequality we get:

$$\begin{aligned}\|M_t(f)\|_X &\leq \frac{1}{t}\sum_{k=0}^{[t]}\left(\int_k^{k+1}\|f\|_X^p\,ds\right)^{1/p}\left(\int_k^{k+1} 1\,ds\right)^{1/p'}\\ &\leq \frac{[t]+1}{t}\|f\|^p_{L^p_{uloc}(X)} \leq 2\|f\|^p_{L^p_{uloc}(\mathbb{R}_+;X)},\end{aligned}$$

the last inequality being satisfied since $\frac{[x]+1}{x} \leq 2$ is valid for all $x \geq 1$. □

In the next section, we will focus on the case $p = 2$ and $X = V'$. We will need the following result, which is a consequence of Lemma 3.4.

Lemma 4.7. *Let be given $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and let $\mathbf{v}$ be a global weak solution to the NSE corresponding to the above data. Then, we have, $\forall\, t \geq 1$,*

$$M_t(\|\nabla \mathbf{v}\|^2) \leq \frac{\|\mathbf{v}_0\|^2}{\nu t} + 2\frac{\mathcal{F}^2}{\nu^2}, \tag{4.9}$$

$$M_t(\|\mathbf{f}\|^2_{V'}) \leq 2\mathcal{F}^2, \tag{4.10}$$

where we recall that $\mathcal{F} = \|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$.

[3]The sign adopted in (4.7) is a convention consistent with our mathematical approach. However, according to the analogy of the Reynolds stress with viscous forces, it is also common to define it as $\boldsymbol{\sigma}^{(\mathrm{R})} := -\overline{\mathbf{v}' \otimes \mathbf{v}'}$, which does not change anything.

Proof. It suffices to divide estimate (3.5) by νt. Therefore, it follows that

$$\frac{1}{t}\int_0^t \|\nabla \mathbf{v}(s)\|^2\,ds \le \frac{\|\mathbf{v}_0\|^2}{\nu t} + 2\frac{\mathcal{F}^2}{\nu^2}.$$

Estimate (4.10) is straightforward. □

In particular, the family $\{M_t(\mathbf{v})\}_{t\in\mathbb{R}_+}$ is bounded in V. Therefore, we can now state the following result, which will be stated in a more precise form in the next sections.

Corollary 4.1. *There exists $\overline{\mathbf{v}}$ such that -up to a sub-sequence- $M_t(\mathbf{v}) \to \overline{\mathbf{v}}$ as $t \to \infty$, in appropriate topologies.*

The following result is a direct consequence of (4.9) and (4.10), combined with Cauchy-Schwarz inequality.

Corollary 4.2. *The family $\{M_t(<\mathbf{f},\mathbf{v}>)\}_{t>0}$ is bounded uniformly in t, and it follows*

$$\big|M_t(<\mathbf{f},\mathbf{v}>)\big| \le \sqrt{2}\mathcal{F}\left(\frac{\|\mathbf{v}_0\|^2}{\nu t} + 2\frac{\mathcal{F}^2}{\nu^2}\right)^{\frac{1}{2}} \qquad \forall\, t \ge 1. \tag{4.11}$$

We finish this section with a last technical result, that we will need to prove Item vi) of Theorem 2.3.

Lemma 4.8. *Let $1 \le p < \infty$ and let be given $f \in L^p_{uloc}(\mathbb{R}_+;X)$, which satisfies in addition*

$$\exists\, \widetilde{f} \in X, \quad \textit{such that} \quad \lim_{t\to+\infty}\int_t^{t+1} \|f(s)-\widetilde{f}\|^p_X\,ds = 0. \tag{4.12}$$

Then, we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \|f(s)-\widetilde{f}\|^p_X\,ds = 0. \tag{4.13}$$

Moreover, $M_t(f)$ weakly converges to $\widetilde{f}$ in X, when $t \to +\infty$. In particular, we have $\widetilde{f} = \overline{f}$.

Proof. By the hypothesis (4.12), we have that

$$\forall\,\varepsilon > 0 \quad \exists\, M \in \mathbb{N}: \quad \int_t^{t+1} \|f(s)-\widetilde{f}\|^p_X\,ds < \frac{\varepsilon}{2} \qquad \forall\, t > M.$$

Hence, for $t \ge M$, then

$$\begin{aligned}\frac{1}{t}\int_0^t \|f(s)-\widetilde{f}\|^p_X\,ds &= \frac{1}{t}\int_0^M \|f(s)-\widetilde{f}\|^p_X\,ds + \frac{1}{t}\int_M^t \|f(s)-\widetilde{f}\|^p_X\,ds \\ &\le \frac{M}{t}\big(\|f\|^p_{L^p_{uloc}(X)} + \|\widetilde{f}\|^p_X\big) + \frac{[t]+1-M}{t}\frac{\varepsilon}{2}.\end{aligned}$$

It follows that one can choose M large enough such that

$$\frac{1}{t}\int_0^t \|f(s)-\widetilde{f}\|_X^p\, ds < \varepsilon \qquad \forall\, t > M,$$

hence, being this valid for arbitrary $\varepsilon > 0$, it follows (4.13).

It remains to prove the weak convergence of $M_t(f)$ to $\widetilde{f} \in X$ when $t \to +\infty$. To this end, let us fix $\varphi \in X'$. Then, we have

$$< \varphi, M_t(f) > - < \varphi, \widetilde{f} > = \frac{1}{t}\int_0^t < \varphi, f(s)-\widetilde{f} > \, ds,$$

which leads to

$$\big| < \varphi, M_t(f) > - < \varphi, \widetilde{f} > \big| \leq \frac{1}{t}\int_0^t \|\varphi\|_{X'}\|f(s)-\widetilde{f}\|_X\, ds,$$

and by Hölder inequality,

$$\Big| < \varphi, M_t(f) > - < \varphi, \widetilde{f} > \Big| \leq \|\varphi\|_{X'}\left(\frac{1}{t}\int_0^t \|f(s)-\widetilde{f}\|_X^p\, ds\right)^{\frac{1}{p}},$$

yielding, by (4.13), to $\lim\limits_{t\to+\infty} < \varphi, M_t(f) > = < \varphi, \widetilde{f} >$, hence concluding the proof. □

The following corollary will be particularly useful in the proof of Item vi) of Theorem 2.3.

Corollary 4.3. *Let be given $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$ that satisfies (4.12), and let $\mathbf{v}$ be a global weak solution to the NSE corresponding to the above data. Moreover let $\overline{\mathbf{v}}$ be such that $\lim\limits_{t\to+\infty} M_t(\mathbf{v}) = \overline{\mathbf{v}}$ in V (eventually up to a sub-sequence), then*

$$\lim_{t\to+\infty} M_t(< \mathbf{f}, \mathbf{v} >) = < \overline{\mathbf{f}}, \overline{\mathbf{v}} > .$$

Proof. Let us write the following decomposition:

$$\frac{1}{t}\int_0^t < \mathbf{f}, \mathbf{v} > \, ds = \frac{1}{t}\int_0^t < \mathbf{f}-\overline{\mathbf{f}}, \mathbf{v} > \, ds + \frac{1}{t}\int_0^t < \overline{\mathbf{f}}, \mathbf{v} > \, ds.$$

On one hand since $\overline{\mathbf{f}} \in V$ is independent of t, we obviously have

$$\frac{1}{t}\int_0^t < \overline{\mathbf{f}}, \mathbf{v} > \, ds \to < \overline{\mathbf{f}}, \overline{\mathbf{v}} > .$$

On the other hand, we have also

$$\begin{aligned}\left|\frac{1}{t}\int_0^t < \mathbf{f}-\overline{\mathbf{f}}, \mathbf{v} > \, ds\right| &\leq \frac{1}{t}\int_0^t \|\mathbf{f}-\overline{\mathbf{f}}\|_{V'}\|\nabla\mathbf{v}\|\, ds \\ &\leq \left(\frac{1}{t}\int_0^t \|\mathbf{f}-\overline{\mathbf{f}}\|_{V'}^2\, ds\right)^{1/2}\left(\frac{1}{t}\int_0^t \|\nabla\mathbf{v}\|^2\, ds\right)^{1/2}.\end{aligned} \tag{4.14}$$

Combining (4.13) with (4.9) shows that the right-hand side in (4.14) vanishes as $t \to +\infty$. □

Remark 4.9. *It is important to observe that*

$$M_t\big(< \mathbf{f}, \mathbf{v} > \big) \to < \overline{\mathbf{f}}, \overline{\mathbf{v}} >,$$

is –in some sense– an assumption on the (long-time) behavior of the "covariance" between the external force and the solution itself. Cf. Layton [20] for a related result in the case of ensemble averages.

The control of the (average/expectation of) kinetic energy in terms of the energy input is one of the remarkable features of classes of statistical solutions, making the stochastic Navier-Stokes equations very appealing in this context. See the review, with applications to the determination of the Lilly constant, in Ref. [2]. See also [10].

5 Proof of Theorem 2.3

In this section we have as before $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and $\mathbf{v}$ is a global weak solution to the NSE (1.1) corresponding to the above data. We split the proof of Theorem 2.3 into two steps. We first apply the operator M_t to the NSE, then we extract sub-sequences and take the limit in the equations. In the second step we make the identification with the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ and show that it is dissipative in average, at least when $\mathbf{f}$ satisfies in addition (2.5).

5.1 Extracting sub-sequences

We set:

$$\mathbf{V}_t(\mathbf{x}) := M_t(\mathbf{v})(\mathbf{x}).$$

Applying the operator M_t on the NSE we see that for almost all $t > 0$ and for all $\boldsymbol{\phi} \in V$, the field $\mathbf{V}_t$ is a weak solution of the following steady Stokes problem (where $t > 0$ is simply a parameter)

$$\begin{aligned}\nu \int_\Omega \nabla \mathbf{V}_t : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega M_t((\mathbf{v} \cdot \nabla)\, \mathbf{v}) \cdot \boldsymbol{\phi}\, d\mathbf{x} = &< M_t(\mathbf{f}), \boldsymbol{\phi} > \\ &+ \int_\Omega \frac{\mathbf{v}_0 - \mathbf{v}(t)}{t} \cdot \boldsymbol{\phi}\, d\mathbf{x}.\end{aligned} \tag{5.1}$$

The full justification of the equality (5.1) starting from the definition of global weak solutions can be obtained by following a very well-known path used for instance to show with a lemma by Hopf that Leray-Hopf weak solutions can be re-defined on a set of zero Lebesgue measure in $[0, t]$ in such a way that $\mathbf{v}(s) \in H$ for *all* $s \in [0, t]$, see for instance Galdi [15, Lemma 2.1]. In fact, by following ideas developed among the others by Prodi [27], one can take $\chi_{[a,b]}$ the characteristic function of an interval $[a, b] \subset \mathbb{R}$, and use as test function its regularization multiplied by $\boldsymbol{\phi} \in V$. Passing to the limit as the regularization parameter vanishes one gets (5.1).

The process of extracting sub-sequences, which is the core of the main result, is reported in the following proposition.

Proposition 5.1. *Let be given a global solution* $\mathbf{v}$ *to the NSE, corresponding to the data* $\mathbf{v}_0 \in H$ *and* $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$. *Then, there exist*

a) a sequence $\{t_n\}_{n\in\mathbb{N}}$ *that goes to* $+\infty$ *when* n *goes to* $+\infty$;

b) a vector field $\overline{\mathbf{f}} \in V'$;

c) a vector field $\overline{\mathbf{v}} \in V$;

d) a vector field $\mathbf{B} \in L^{3/2}(\Omega)^3$;

such that such that it holds when $n \to +\infty$:

$$\begin{aligned} &M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}} \qquad \text{in } V', \\ &M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}} \qquad \text{in } V, \\ &M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} \qquad \text{in } L^{3/2}(\Omega)^3 \subset V', \end{aligned}$$

and for all $\boldsymbol{\phi} \in V$

$$\nu \int_\Omega \nabla\overline{\mathbf{v}} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{B}\cdot\boldsymbol{\phi}\, d\mathbf{x} = <\overline{\mathbf{f}}, \boldsymbol{\phi}> . \tag{5.2}$$

Moreover, by defining

$$\mathbf{F} := \mathbf{B} - (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} \in L^{3/2}(\Omega)^3, \tag{5.3}$$

we can also rewrite (5.2) *as follows*

$$\nu \int_\Omega \nabla\overline{\mathbf{v}} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}}\cdot\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{F}\cdot\boldsymbol{\phi}\, d\mathbf{x} = <\overline{\mathbf{f}}, \boldsymbol{\phi}>; \tag{5.4}$$

e) writing $\mathbf{v}' = \mathbf{v} - \overline{\mathbf{v}}$, $\mathbf{f}' = \mathbf{f} - \overline{\mathbf{f}}$, *we also have*

$$M_{t_n}(<\mathbf{f}, \mathbf{v}>) \to <\overline{\mathbf{f}}, \overline{\mathbf{v}}> + \overline{<\mathbf{f}', \mathbf{v}'>}.$$

Proof of Proposition 5.1. As $\mathbf{f} \in L^2(\mathbb{R}_+; V')$, we deduce from Lemma 4.6 that $\{M_t(\mathbf{f})\}_{t>0}$ is bounded in V'. Hence, we can use weak pre-compactness of bounded sets in the Hilbert space V' to infer the existence of t_n and $\overline{\mathbf{f}} \in V'$ such that $M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}}$ in V'. Next, estimate (4.9) from Lemma 4.7, combined with estimate (4.1) from Lemma 4.1, leads to the bound

$$\exists\, c > 0: \qquad \|\nabla M_t(\mathbf{v})\| = \|M_t(\nabla\mathbf{v})\| \leq c \qquad \forall\, t > 0,$$

proving (up to a the extraction of a further sub-sequence from $\{t_n\}$, which we call with the same name) that $M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}}$ in V.

Then, we observe that, if $\mathbf{v} \in L^\infty(0,T;H) \cap L^2(0,T;V)$ by classical interpolation

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} \in L^r(0,T;L^s(\Omega)) \qquad \text{with} \quad \frac{2}{r}+\frac{3}{s}=4, \quad r\in[1,2].$$

In particular, we get

$$\|(\mathbf{v}\cdot\nabla)\,\mathbf{v}\|_{L^{3/2}(\Omega)} \leq \|\mathbf{v}\|_{L^6}\|\nabla\mathbf{v}\|_{L^2} \leq C_S\|\nabla\mathbf{v}\|^2,$$

where C_S is the constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$. Hence, by using the bounds (3.4)-(3.5) on the weak solution $\mathbf{v}$ we obtain that

$$\exists\, c: \qquad \|M_t\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big)\|_{L^{3/2}(\Omega)} \leq c, \qquad \forall\, t>0,$$

proving that, up to a further sub-sequence relabelled again as $\{t_n\}$,

$$M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} \qquad \text{in } L^{3/2}(\Omega)^3,$$

for some vector field $\mathbf{B}\in L^{3/2}(\Omega)^3$.

Next, we use (3.4) which shows that

$$\int_\Omega \frac{\mathbf{v}_0-\mathbf{v}(t)}{t}\cdot\boldsymbol{\phi}\,d\mathbf{x} \to 0 \qquad \text{as } t\to+\infty.$$

Then, writing the weak formulation and by using the results of weak convergence previously proved, we get (5.2). The identity (5.4) comes simply from the definition (5.3) of $\mathbf{F}$.

It remains to prove the last item. We know from (4.11) that the sequence $\{M_{t_n}(\langle \mathbf{f},\mathbf{v}\rangle)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. By extracting again a sub-sequence (still denoted by $\{t_n\}_{n\in\mathbb{N}}$), we can get a convergent sequence still denoted (after relabelling) by $\{M_{t_n}(\langle \mathbf{f},\mathbf{v}\rangle)\}_{n\in\mathbb{N}}$, and let $\overline{\langle \mathbf{f},\mathbf{v}\rangle}$ be its limit. Let us write the decomposition

$$\begin{aligned} &M_{t_n}(\langle \mathbf{f},\mathbf{v}\rangle) \\ &\quad =\langle \overline{\mathbf{f}},\overline{\mathbf{v}}\rangle + M_{t_n}(\langle \mathbf{f}',\overline{\mathbf{v}}\rangle) + M_{t_n}(\langle \overline{\mathbf{f}},\mathbf{v}'\rangle) + M_{t_n}(\langle \mathbf{f}',\mathbf{v}'\rangle). \end{aligned} \tag{5.5}$$

As $M_{t_n}(\langle \mathbf{f}',\overline{\mathbf{v}}\rangle) = \langle M_{t_n}(\mathbf{f}'),\overline{\mathbf{v}}\rangle$, we deduce from the results above that $M_{t_n}(\langle \mathbf{f}',\overline{\mathbf{v}}\rangle)\to 0$ as $n\to+\infty$. Similarly, we also have $M_{t_n}(\langle \overline{\mathbf{f}},\mathbf{v}'\rangle)\to 0$. Hence, we deduce from (5.5) that $\{M_{t_n}(\langle \mathbf{f}',\mathbf{v}'\rangle)\}_{n\in\mathbb{N}}$ is convergent, and if we denote by $\overline{\langle \mathbf{f}',\mathbf{v}'\rangle}$ its limit, the following natural decomposition holds true:

$$\overline{\langle \mathbf{f},\mathbf{v}\rangle} = \langle \overline{\mathbf{f}},\overline{\mathbf{v}}\rangle + \overline{\langle \mathbf{f}',\mathbf{v}'\rangle}, \tag{5.6}$$

concluding the proof. □

5.2 Reynolds stress, energy balance and dissipation

In the first step we have identified a limit $(\overline{\mathbf{v}}, \overline{\mathbf{f}})$ for the time-averages of both velocity and external force $(\mathbf{v}, \mathbf{f})$. We need now to recast this in the setting of the Reynolds equations, in order to address the proof of the Boussinesq assumption.

Proof of Theorem 2.3. Beside the results in Proposition 5.1, in order to complete the proof of Theorem 2.3, we have to prove the following facts:

1) the proper identification of the limits with the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$;

2) the energy balance for $\overline{\mathbf{v}}$;

3) to prove that (2.4) holds, namely

$$\varepsilon = \nu \overline{\|\nabla \mathbf{v}'\|^2} \leq \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \overline{< \mathbf{f}', \mathbf{v}' >}.$$

We proceed in the same order.

Item 1. Since $\mathbf{v} \in L^2(0,T;V) \subset L^2(0,T;L^6(\Omega)^3)$, it follows that $\mathbf{v} \otimes \mathbf{v} \in L^1(0,T;L^3(\Omega))$. Hence, the same argument as in the previous subsection shows that (possibly up to the extraction of a further sub-sequence) there exists a second order tensor $\boldsymbol{\theta} \in L^3(\Omega)^9$ such that

$$M_{t_n}(\mathbf{v} \otimes \mathbf{v}) \rightharpoonup \boldsymbol{\theta} \qquad \text{in } L^3(\Omega)^9.$$

Let us set

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \boldsymbol{\theta} - \overline{\mathbf{v}} \otimes \overline{\mathbf{v}}.$$

Since the operator M_t commutes with the divergence operator, the equation (5.1) becomes

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{V}_t : \nabla \boldsymbol{\phi}\, d\mathbf{x} - \int_\Omega M_t(\mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\phi}\, d\mathbf{x} = &< M_t(\mathbf{f}), \boldsymbol{\phi} > \\ &+ \int_\Omega \frac{\mathbf{v}_0 - \mathbf{v}(t)}{t} \cdot \boldsymbol{\phi}\, d\mathbf{x}. \end{aligned} \tag{5.7}$$

Then, by taking the limit along the sequence $t_n \to +\infty$ in (5.7), we get[4] the equality

$$\mathbf{F} = \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}.$$

Item 2. We use $\overline{\mathbf{v}} \in V$ in (2.3) as test function and we obtain the equality

$$\nu \|\nabla \overline{\mathbf{v}}\|^2 + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}}\, d\mathbf{x} = < \overline{\mathbf{f}}, \overline{\mathbf{v}} > . \tag{5.8}$$

[4] According to the formal decomposition (4.3), this suggests that $M_{t_n}(\mathbf{v}' \otimes \mathbf{v}') \to 0$, provided that one is able to give a rigorous sense and sufficiently strong bounds on $\mathbf{v}' \otimes \mathbf{v}'$, for the weak solution $\mathbf{v}$.

We observe that, due to the absence of the time-variable, the following identity concerning the integral over the space variables is valid

$$\int_\Omega (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} \cdot \overline{\mathbf{v}}\, d\mathbf{x} = \int_\Omega (\overline{\mathbf{v}} \cdot \nabla)\, \frac{|\overline{\mathbf{v}}|^2}{2}\, d\mathbf{x} = 0 \qquad \forall\, \overline{\mathbf{v}} \in V.$$

This is one of the main technical facts which are typical of the mathematical analysis of the steady Navier-Stokes equations and which allow to give precise results for the averaged Reynolds equations. We also recall that if $\mathbf{v}(t, \mathbf{x})$ is a non-steady (Leray-Hopf) weak solution, then the space-time integral

$$\int_0^T \int_\Omega (\mathbf{v} \cdot \nabla)\, \mathbf{v} \cdot \mathbf{v}\, d\mathbf{x}\, dt,$$

is not well defined and consequently the above multiple integral vanishes only formally.

Item 3. From now on, we assume that the assumption (2.5) in the statement of Theorem 2.3 holds true. We integrate the energy inequality (3.1) between 0 and t_n and we divide the result by $t_n > 0$, which leads to

$$\frac{\|\mathbf{v}(t)\|^2}{2t_n} + \frac{1}{t_n} \int_0^{t_n} \|\nabla \mathbf{v}(s)\|^2\, ds \leq \frac{\|\mathbf{v}_0\|^2}{2t_n} + \frac{1}{t_n} \int_0^{t_n} < \mathbf{f}, \mathbf{v} > \, ds. \tag{5.9}$$

Recall that by Lemma 3.4

$$\frac{\|\mathbf{v}(t)\|^2}{2t} \to 0 \quad \text{and} \quad \frac{\|\mathbf{v}_0\|^2}{2t} \to 0 \quad \text{as } t \to +\infty.$$

Therefore, we take the limit in (5.9) and we use (5.6), which yields

$$\nu \overline{\|\nabla \mathbf{v}\|^2} \leq \overline{< \mathbf{f}, \mathbf{v} >} = < \overline{\mathbf{f}}, \overline{\mathbf{v}} > + \overline{< \mathbf{f}', \mathbf{v}' >}.$$

By (5.8) we then have

$$\nu \overline{\|\nabla \mathbf{v}\|^2} \leq \nu \|\nabla \overline{\mathbf{v}}\|^2 + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \overline{< \mathbf{f}', \mathbf{v}' >},$$

which yields (2.4) by (4.4), and concluding the proof. □

6 On ensemble averages

In this section we show how to use the results of Theorem 2.3 to give new insight into the analysis of ensemble averages of solutions. In this case we study suitable averages of the long-time behavior and not the long-time behavior of statistics, as in Layton [20].

Since we first take long-time limits and then we average the Reynolds equations, the initial datum is not so relevant. In fact, due to the fact that it holds

$$\frac{\|\mathbf{v}_0\|^2}{t} \to 0 \qquad \text{as} \quad t \to +\infty,$$

then the mean velocity $\overline{\mathbf{v}}$ is not affected by the initial datum.

As claimed in the introduction, we consider now the problem of having several external forces, say a whole family $\{\mathbf{f}^k\}_{k\in\mathbb{N}} \subset V'$, all independent of time. We can think as different experiments with slightly different external forces, whose difference can be due to errors in measurement or to the uncertainty intrinsic in any measurement method. In particular, one can consider a given force $\mathbf{f}$ and the sequence $\{\mathbf{f}^k\}_{k\in\mathbb{N}}$ may represent small oscillations around it, hence we can freely assume that we have an uniform bound

$$\exists\, C > 0: \qquad \|\mathbf{f}^k\|_{V'} \leq C \qquad \forall\, k \in \mathbb{N}. \tag{6.1}$$

Having in mind this physical setting, we denote by $\overline{\mathbf{v}^k} \in V$ the long-time average of the solution corresponding to the external force $\mathbf{f}^k \in V'$ and, as explained before (without loss of generality) to the initial datum $\mathbf{v}_0 = \mathbf{0}$. For $k \in \mathbb{N}$ the vector $\overline{\mathbf{v}^k} \in V$ satisfies for all $\boldsymbol{\phi} \in V$ the following equivalent equalities

$$\begin{aligned}
&\nu \int_\Omega \nabla\overline{\mathbf{v}^k} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{B}^k \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \mathbf{f}^k, \boldsymbol{\phi} >,\\
&\nu \int_\Omega \nabla\overline{\mathbf{v}^k} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\overline{\mathbf{v}^k} \cdot \nabla)\, \overline{\mathbf{v}^k} \cdot \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{F}^k \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \mathbf{f}^k, \boldsymbol{\phi} >,
\end{aligned}$$

for appropriate $\mathbf{B}^k, \mathbf{F}^k \in L^3(\Omega)^{3/2}$. Since both V and V' are Hilbert spaces, by using (6.1) it follows that there exists $\langle \mathbf{f} \rangle \in V'$ and a sub-sequence (still denoted by $\{\mathbf{f}^k\}$) such that

$$\mathbf{f}^k \rightharpoonup \langle \mathbf{f} \rangle \qquad \text{in } V'.$$

Our intention is to characterize, if possible, the limit of $\{\overline{\mathbf{v}}^k\}_{k\in\mathbb{N}}$. If the forces are fluctuations around a mean value, then the field $\overline{\mathbf{v}^k}$ will remain bounded in V, but possibly without converging to some limit. From an heuristic point of view one can expect that averaging the sequence of velocities (which corresponds to averaging the result over different realizations) one can identify a proper limit, which retains the "average" effect of the flow.

Again, it comes an important idea at the basis of Large Scale methods: The average behavior of solutions seems the only quantity which can be measured or simulated.

It is well-known that one of the most used *summability technique* is that of Cesàro and consists in taking the mean values, hence we focus on the arithmetic mean of time-averaged velocities

$$\mathbf{S}^n := \frac{1}{n} \sum_{k=1}^{n} \overline{\mathbf{v}^k}.$$

It is a basic calculus result that if a real sequence $\{x_j\}_{j\in\mathbb{N}}$ converges to $x \in \mathbb{R}$, then also its Cesàro mean $S_n = \frac{1}{n}\sum_{j=1}^n x_j$ will converge to the same value x. On the other hand, the converse is false; sufficient conditions on the sequence $\{x_j\}_{j\in\mathbb{N}}$ implying that if the Cesàro mean converges, then the original sequence converges, are known in literature as Tauberian theorems. This is a classical topic in the study of divergent sequences/series. In the case of X-valued sequences $\{\mathbf{u}^k\}_{k\in\mathbb{N}}$ (the space X being an infinite dimensional Banach space) one has again that if a sequence converges strongly or weakly, then its Cesàro mean will converge to the same value, strongly or weakly in X, respectively.

The fact that averaging generally improves the properties of a sequence, is reflected also in the setting of Banach spaces even if with additional features coming into the theory. Two main results we will consider are two theorems known as Banach-Saks and Banach-Mazur.

Banach and Saks originally in 1930 formulated the result in $L^p(0,1)$, but it is valid in more general Banach spaces.

Theorem 6.1 (Banach-Saks)**.** *Let be given a bounded sequence $\{x_j\}_{j\in\mathbb{N}}$ in a reflexive Banach space X. Then, there exists a sub-sequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ such that the sequence $\{S_m\}_{m\in\mathbb{N}}$ defined by*

$$S_m := \frac{1}{m}\sum_{k=1}^{m} x_{j_k},$$

converges strongly in X.

The reader can observe that in some cases it is not needed to extract a subsequence (think of any orthonormal set in an Hilbert space, which is weakly converging to zero, and the Cesàro mean converges to zero strongly), but in general one cannot infer that the averages of the full sequence converge strongly. One sufficient condition is that of *uniform weak convergence*. We recall that $\{x_j\} \subset X$ *uniformly weakly* converges to zero if for any $\epsilon > 0$ there exists $j \in \mathbb{N}$, such that for all $\phi \in X'$, with $\|\phi\|_{X'} \leq 1$, it holds true that

$$\#\big\{j \in N:\ |\phi(x_j)| \geq \varepsilon\big\} \leq j.$$

See also Brezis [6, p. 168].

Another way of improving the weak convergence to the strong one is by a convex-combination theorem (cf. Yosida [32, p.120]).

Theorem 6.2 (Banach-Mazur)**.** *Let $(X, \|\,.\,\|_X)$ be a Banach space and let $\{x_j\} \subset X$ be a sequence such that $x_j \rightharpoonup x$ as $j \to +\infty$.*

Then, one can find for each $n \in \mathbb{N}$, real coefficients $\{\alpha_j^n\}$, for $j = 1, \dots, n$ with

$$\alpha_j^n \geq 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \alpha_j^n = 1,$$

such that

$$\sum_{j=1}^{n} \alpha_j^n x_j \to x \qquad \text{in } X, \quad \text{as} \quad n \to +\infty,$$

that is we can find a "convex combination" of $\{x_j\}$, which strongly converges to $x \in X$.

One basic point will be that of considering averages of the external forces, which we will denote by $\langle \mathbf{f} \rangle^n$ and considering the same type of averages of the solutions of the Reynolds equations $\langle \overline{\mathbf{v}} \rangle^n$. They are both bounded and hence, weakly converging (up to a sub-sequence) to some $\langle \mathbf{f} \rangle \in V'$ and $\langle \overline{\mathbf{v}} \rangle \in V$, respectively. Then, in order to prove that the dissipativity is preserved one has to handle the following limit of the products

$$\lim_{n \to +\infty} < \langle \mathbf{f} \rangle^n, \langle \overline{\mathbf{v}} \rangle^n >,$$

which cannot be characterized, unless (at least) one of the two terms converges strongly. This is why we have to use special means instead of the simple Cesàro averages

The first result of this section is then the following:

Proposition 6.1. *Let be given $\{\mathbf{f}^k\}_{k \in \mathbb{N}}$ uniformly bounded in V'. Then, one can find either a Banach-Saks sub-sequence or a convex combination of $\{\mathbf{v}^k\}_{k \in \mathbb{N}}$, which are converging weakly to some $\langle \mathbf{v} \rangle \in V$, which satisfies a Reynolds system (6.4), with an additional dissipative term.*

Proof of Theorem 6.1. We define $\langle \mathbf{f} \rangle^n$ and $\langle \mathbf{v} \rangle^n$ to be either

$$\langle \mathbf{f} \rangle^n := \frac{1}{n} \sum_{k=1}^{n} \mathbf{f}^{j_k} \qquad \text{and} \qquad \langle \mathbf{v} \rangle^n := \frac{1}{n} \sum_{k=1}^{n} \overline{\mathbf{v}}^{j_k},$$

or alternatively

$$\langle \mathbf{f} \rangle^n := \sum_{j=1}^{n} \alpha_j^n \mathbf{f}^j \qquad \text{and} \qquad \langle \mathbf{v} \rangle^n := \sum_{j=1}^{n} \alpha_j^n \overline{\mathbf{v}^j},$$

where the sub-sequence $\{j_k\}_{k \in \mathbb{N}}$ or the coefficients $\{\alpha_j^n\}_{j,\, n \in \mathbb{N}}$ are chosen accordingly to the Banach-Saks or Banach-Mazur theorems, in such a way that in both cases

$$\langle \mathbf{f} \rangle^n \to \langle \mathbf{f} \rangle \qquad \text{in } V'.$$

We define, accordingly to the same rules $\langle \mathbf{B} \rangle^n$, and we observe that, by linearity, we have $\forall\, n \in \mathbb{N}$

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle^n : \nabla \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega \langle \mathbf{B} \rangle^n \cdot \boldsymbol{\phi} \, d\mathbf{x} = < \langle \mathbf{f} \rangle^n, \boldsymbol{\phi} > \qquad \forall \boldsymbol{\phi} \in V. \tag{6.2}$$

Then, we can define $\langle \mathbf{F} \rangle^n := \langle \mathbf{B} \rangle^n - (\langle \mathbf{v} \rangle^n \cdot \nabla) \langle \mathbf{v} \rangle^n$, to rewrite (6.2) also as follows

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle^n : \nabla \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega \big(\langle \mathbf{v} \rangle^n \cdot \nabla \big) \langle \mathbf{v} \rangle^n \cdot \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega \langle \mathbf{F} \rangle^n \cdot \boldsymbol{\phi} \, d\mathbf{x} = < \langle \mathbf{f} \rangle^n, \boldsymbol{\phi} > . \tag{6.3}$$

By the uniform bound on $\|\mathbf{f}^k\|_{V'}$ and by results of Section 5.2 on the Reynolds equations it follows that there exists C such that $\|\overline{\mathbf{v}}^k\|_V \le C$, hence

$$\|\langle \mathbf{v} \rangle^n\|_V \le C \qquad \forall\, n \in \mathbb{N},$$

and we can suppose that (up to sub-sequences) we have the following weak convergences of the convex combinations

$$\begin{aligned} \langle \mathbf{v} \rangle^n &\rightharpoonup \langle \mathbf{v} \rangle \qquad \text{in } V, \\ \langle \mathbf{B} \rangle^n &\rightharpoonup \langle \mathbf{B} \rangle \qquad \text{in } L^{3/2}(\Omega)^3, \\ \langle \mathbf{F} \rangle^n &\rightharpoonup \langle \mathbf{F} \rangle \qquad \text{in } L^{3/2}(\Omega)^3, \end{aligned}$$

Hence, passing to the limit in (6.2), we obtain

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle : \nabla \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega \langle \mathbf{B} \rangle \cdot \boldsymbol{\phi} \, d\mathbf{x} = < \langle \mathbf{f} \rangle, \boldsymbol{\phi} > \qquad \forall \boldsymbol{\phi} \in V.$$

By the same reasoning used before we have, for $\langle \mathbf{F} \rangle := \langle \mathbf{B} \rangle - (\langle \mathbf{v} \rangle \cdot \nabla) \langle \mathbf{v} \rangle$, that

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle : \nabla \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega (\langle \mathbf{v} \rangle \cdot \nabla) \langle \mathbf{v} \rangle \cdot \boldsymbol{\phi} \, d\mathbf{x} + \int_\Omega \langle \mathbf{F} \rangle \cdot \boldsymbol{\phi} \, d\mathbf{x} = < \langle \mathbf{f} \rangle, \boldsymbol{\phi} > . \tag{6.4}$$

Then, if we take $\boldsymbol{\phi} = \langle \mathbf{v} \rangle$ in (6.4) we obtain

$$\nu \|\nabla \langle \mathbf{v} \rangle\|^2 + \int_\Omega \langle \mathbf{F} \rangle \cdot \langle \mathbf{v} \rangle \, d\mathbf{x} = < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle > . \tag{6.5}$$

On the other hand, if we take $\boldsymbol{\phi} = \langle \mathbf{v} \rangle^n$ in (6.3) and use the result of the previous section, we have

$$\nu \|\nabla \langle \mathbf{v} \rangle^n\|^2 \leq < \langle \mathbf{f} \rangle^n, \langle \mathbf{v} \rangle^n >,$$

hence passing to the limit, by using the strong convergence of $\langle \mathbf{f} \rangle^n$ in V' and the weak convergence of $\langle \mathbf{v} \rangle^n$ in V we have

$$\nu \|\nabla \langle \mathbf{v} \rangle\|^2 \leq \liminf_{n \to +\infty} \nu \|\nabla \langle \mathbf{v} \rangle^n\|^2 \leq < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle > .$$

If we compare with (6.5) we have finally the dissipativity

$$\frac{1}{|\Omega|} \int_\Omega (\nabla \cdot \langle \boldsymbol{\sigma}^{(\mathrm{R})} \rangle) \cdot \langle \mathbf{v} \rangle \, d\mathbf{x} = \frac{1}{|\Omega|} \int_\Omega \langle \mathbf{F} \rangle \cdot \langle \mathbf{v} \rangle \, d\mathbf{x} \geq 0,$$

that is a sort of ensemble/long-time Boussinesq hypothesis, cf. with the results from Ref. [19, 20]. □

In the previous theorem, we have a result which does not concern directly with the ensemble averages, but a selection of special coefficients is required. This is not completely satisfactory from the point of view of the numerical computations, where the full arithmetic mean should be considered. The main result can be obtained at the price of a slight refinement on the hypotheses on the external forces

To this end we recall a lemma, which is a sort of Rellich theorem in negative spaces (see also Galdi [16, Thm. II.5.3] and Feireisl [9, Thm. 2.8]).

Lemma 6.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let be given $1 < p < n$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence uniformly bounded in $L^q(\Omega)$ with $(p^*)' < q < \infty$, where $p^* = \frac{np}{n-p}$ is the exponent in the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Then, there exists a sub-sequence $\{f_{k_m}\}_{m\in\mathbb{N}}$ and $f \in L^q(\Omega)$ such that*

$$
\begin{aligned}
f_{k_m} &\rightharpoonup f && \text{in } L^q(\Omega),\\
f_{k_m} &\to f && \text{in } W^{-1,p'}(\Omega),
\end{aligned}
$$

or, in other words, the embedding $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p'}(\Omega)$ is compact.

We present the proof for the reader's convenience.

Proof of Lemma 6.3. Since by hypothesis $L^q(\Omega)$ is reflexive, by the Banach-Alaouglu-Bourbaki theorem we can find a sub-sequence f_{k_m} such that

$$
f_{k_m} \rightharpoonup f \qquad \text{in } L^q(\Omega),
$$

and by considering the sequence $\{f_{k_m} - f\}_{m\in\mathbb{N}}$ we can suppose that $f = 0$. We then observe that by the Sobolev embedding and duality we have the continuous embeddings

$$
W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \sim L^{(p^*)'}(\Omega) \hookrightarrow (W_0^{1,p}(\Omega))' \simeq W^{-1,p'}(\Omega),
$$

where $L^{p^*}(\Omega) \sim L^{(p^*)'}(\Omega)$ is the duality identification, while the second one $(W_0^{1,p}(\Omega))' \simeq W^{-1,p'}(\Omega)$ is the Lax isomorphism. This finally shows that $L^{(p^*)'}(\Omega) \subseteq W^{-1,p'}(\Omega)$.

Next, let be given a sequence $\{f_{k_m}\} \subset W^{-1,p'}(\Omega)$, then by reflexivity (since $1 < p < \infty$) there exists $\{\phi_{k_m}\} \subset W_0^{1,p}(\Omega)$ such that

$$
\|f_{k_m}\|_{W^{-1,p'}(\Omega)} = f_{k_m}(\phi_{k_m}) = < f_{k_m}, \phi_{k_m} >,
$$

with $\|\phi_{k_m}\|_{W_0^{1,p}(\Omega)} = \|\nabla \phi_{k_m}\|_{L^p(\Omega)} = 1$.

Hence, by using the classical Rellich theorem, we can find a sub-sequence $\{\phi_{k_j}\}_{j\in\mathbb{N}}$ such that

$$
\phi_{k_j} \to \phi \qquad \text{in } L^r(\Omega) \qquad \forall\, r < p^*.
$$

In particular, we fix $r = q'$ (observe that $q > (p^*)'$ implies $q' < p^*$) and we have

$$
\|f_{k_m}\|_{W^{-1,p'}(\Omega)} = < f_{k_m}, \phi_{k_m} - \phi > + < f_{k_m}, \phi > .
$$

The last term converges to zero, by the definition of weak convergence $f_{k_m} \rightharpoonup 0$, while the first one satisfies

$$
|< f_{k_m}, \phi_{k_m} - \phi >| \leq \|f_{k_m}\|_{W^{-1,p'}} \|\phi_{k_m} - \phi\|_{W_0^{1,p}},
$$

and since $\|f_{k_m}\|_{W^{-1,p'}}$ is uniformly bounded and $\|\phi_{k_m} - \phi\|_{W_0^{1,p}}$ goes to zero, then also this one vanishes as $j \to +\infty$. □

Proof of Theorem 2.4. The proof of this theorem can be obtained by following the same ideas of the Proposition 6.1. In fact, the main improvement is that the weak convergence $\mathbf{f}^j \rightharpoonup \langle \mathbf{f} \rangle$ in $L^q(\Omega)$ implies (without extracting sub-sequences) that

$$\mathbf{f}^k \to \mathbf{f} \qquad \text{in } V'.$$

This follows since from any sub-sequence we can find a further sub-sequence which is converging strongly, by Lemma 6.3. Then, by the weak convergence of the original sequence, the limit is always the same and this implies that the whole sequence $\{\mathbf{f}^k\}$ strongly converges to its weak limit.

Hence, we have

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{f}^k \to \mathbf{f} \qquad \text{in } V',$$

and then, since $\langle \mathbf{v} \rangle^n \rightharpoonup \langle \mathbf{v} \rangle$ in V, we can infer that

$$< \langle \mathbf{f} \rangle^n, \langle \mathbf{v} \rangle^n > = < \frac{1}{n}\sum_{k=1}^{n} \mathbf{f}^k, \frac{1}{n}\sum_{k=1}^{n} \overline{\mathbf{v}}^k > \to < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle >,$$

and the rest follows as in Proposition 6.1. □

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