POINT VORTICES FOR INVISCID GENERALIZED SURFACE QUASI-GEOSTROPHIC MODELS

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Abstract. We give a rigorous proof of the validity of the point vortex description for a class of inviscid generalized surface quasi-geostrophic models on the whole plane.

1. Introduction

The main aim of the paper is to give a rigorous proof of the validity of the point vortex description for a class of inviscid generalized surface quasi-geostrophic (briefly, gSQG) models. This extends the connections, well known in the case of Euler equations [29, 30], between the point vortex theory and these models.

We deal with the following class of problems on $\mathbb{R}^2$,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ (-\Delta)^{\frac{m}{2}} \psi = \theta, \\ u = \nabla^\perp \psi. \end{cases} \tag{1.1}$$

The case $m = 2$ corresponds to the Euler equations, the case $m = 1$ corresponds to the inviscid surface quasi-geostrophic equations (SQG). In meteorology the inviscid SQG has been derived to model the production of fronts due to the tightening of temperature gradients, see [12, 23, 22], see also [14, 32] for the first mathematical and geophysical studies on the subject. The generalized version of the model examined in this paper bridges the cases of Euler and SQG and shares a series of common physical features namely the emergence of inverse cascades [35, 36, 37, 38], as well as deeper universal invariance properties [2, 3, 18]. In this paper we will consider the cases $m \in (1, 2)$.

From the mathematical point of view the generalized models share the same difficulties of SQG. Local existence and uniqueness holds for smooth enough initial

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data, see for instance [8]. It is not known if the generalized SQG, including the case $m = 1$, has a global solution. There is numerical evidence [15] of emergence of singularities in the generalized SQG, for $m \in [1, 2)$, as well as global stable solutions [16]. Regularity criteria are known, see [9]. Weak solutions are known for SQG in L^2 [31] and L^p, with $p > \frac{4}{3}$ [26], see [8] for weak solution in L^2 on the torus for $m < 1$. We will give our version of weak solutions in Section 2.

Point vortices for (1.1) represent profiles that are sharply concentrated around some points. Formally (1.1) is a transport equation, so we may believe that an initial profile given as the configuration of N points,

$$\theta(0) = \sum_{j=1}^{N} \gamma_j \delta_{x_j}, \tag{1.2}$$

where $\gamma_1, \gamma_2, \dots, \gamma_N$, are given numbers (that we will call the *intensities* of the point vortices), evolves as a measure of the same kind, with constant intensities (a generalized version of the conservation of circulation) and where the positions evolve according to the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k \neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \dots, N, \tag{1.3}$$

where G_m is the Green function of the fractional Laplacian $(-\Delta)^{\frac{m}{2}}$ on $\mathbb{R}^2$,

$$G_m(x, y) = G_m(x - y) = \frac{\Gamma(\frac{2-m}{2})}{2^{m/2}\pi|\Gamma(\frac{m}{2})|}|x - y|^{m-2}. \tag{1.4}$$

In our first main result (Theorem 3.1) we prove that the above system (1.3) has, for fixed N, a global solution for a. e. initial condition, under a generic (and necessary) assumption on the intensities.

Additionally, in our second main result (Theorem 3.2) we prove that point vortices provide an approximation of solutions to (1.1), namely if an initial condition is approximated, in the sense of measures, by point vortices (1.2) as $N \uparrow \infty$, then solutions to (1.1) are approximated, again in the sense of measures, by the evolution of the point vortex measure

$$\sum_{j=1}^{N} \gamma_j \delta_{X_j(t)}.$$

Unfortunately, again due to the singularity of the kernel $\nabla^\perp G_m$, the evolution of vortices corresponds to a regularization of the original dynamics. The regularized kernel converges though to the original kernel as $N \uparrow \infty$ (see Remark 3.5 for additional considerations).

For measure valued solution, one should interpret (1.1) in the sense of distributions. But, as in the case of Euler equations ($m = 2$), this is not enough to include measures with atoms. In the case of Euler equations a symmetrisation [17] (see also [33, 34]) allows to tame the singularity of the Biot-Savart kernel. In this context, writing the equation against a test function φ only in terms of θ yields

$$\begin{aligned} \int\int \theta(t,x)\varphi(t,x)\,dx\,dt &+ \\ &+ \int\iint k_m(x-y)\cdot(\nabla\varphi(t,x) - \nabla\varphi(t,y))\theta(t,x)\theta(t,y)\,dx\,dy\,dt = 0, \end{aligned}$$

where $k_m = \nabla^\perp G_m$. Unfortunately, in this more singular setting, the new kernel $k_m(x-y)\cdot(\nabla\varphi(t,x)-\nabla\varphi(t,y))$ is not bounded on the diagonal, and there is no hope to give a meaning to solutions to (1.1) with point masses. Nevertheless, in our third main result (Theorem 3.6) we are able to prove that for values of the parameter m not too small ($\sqrt{3} < m < 2$), a sequence of vortex blobs solutions to (1.1) converges, as the size of the blobs goes to 0, to the configuration of point masses that obeys to (1.3).

The intuitive reason, valid for Euler [30] but crucial in this setting, is that a single vortex does not move subject to the self-generated velocity field, but only according to the velocity field generated by all other vortices. The singular self-interaction, absent in (1.3), does not play a role, although it should due to singularity of the kernel k_m, at the level of the equation. In rigorous terms, we prove localisation of vortices (Proposition 3.8), namely, if θ is initially a vortex blob, then it remains a vortex blob of comparable size. Our proof of localisation fails when $m \leq \sqrt{3}$, but it may be a technical issue of the method used and it is not clear if the main theorem about convergence of vortex blobs to point vortices fails.

We conclude with a few additional comments. The first is that the extension of these results to the torus is straightforward, due to the absence of boundaries. In the presence of boundaries the problem is more delicate. We wish also to emphasise the possible connection with the evolution of vortex patches, namely solutions that take only two values, and where the main interest is about the evolution of the interface. See for instance [8, 24, 16] for relevant results. Finally we remark that the validation of the point vortex motion proved here bolsters the statistical mechanics of point vortices discussed in [20], where the authors extend results on Euler equations from [5, 6, 25, 4].

Upon completion of the paper, after submission to arXiv, we have been pointed out paper [7], that roughly contains similar results. In general our approach is rigorous, contains full and, we believe, correct statements and proofs, deals with the not so obvious problem of existence of solutions of (1.1), and therefore includes accurate statements related to the precise notion of solutions.

Contents. The paper is organized as follows. In Section 2 we prove existence of weak solutions with initial conditions in $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ (Theorem 2.6), since this is the class for vortex blobs. In principle, following [26], one could do better. For instance, if $\theta \in L^p(\mathbb{R}^2)$ and $p < \frac{2}{m-1}$, by the Hardy-Littlewood-Sobolev inequality $u = k_m \star \theta \in L^q(\mathbb{R}^2)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}(m-1)$ and $\theta\, u \in L^r(\mathbb{R}^2)$ for some r if $p \geq 4m+1$. The assumption $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ for the initial condition greatly simplifies the existence theorem. We point out though that, using probabilistic techniques, [19] are able to solve the equation with initial conditions in a space of much rougher functions.

In Section 3 we prove that under suitable generic assumptions on the intensities, the point vortex motion (1.3) has a global non-colliding solution for a. e. initial condition (Theorem 3.1). The approximation of solutions of (1.1) is proved in Theorem 3.2. With a well defined motion at hand, we are finally able to prove the main result about convergence of vortex blobs to the point vortex motion (Theorem 3.6) using localisation (Proposition 3.8).

Notations. First of all, we will name *pseudo-vorticity* the term θ in (1.1), in analogy with the Euler case $m = 2$, even though this may be inappropriate for instance in the context of SQG, where θ is a temperature.

We will denote by $B_r(x)$ the ball centred at x with radius r, by δ_x the measure concentrated at a point $x \in \mathbb{R}^2$, by $\star$ the convolution product, by $\nabla^\perp$ the vector $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We will denote by $\|\cdot\|_{L^p}$ the norm of the Lebesgue space $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$ and we will sometime use also the local version $L^p_{\text{loc}}(\mathbb{R}^2)$ of all functions whose p^{th} norm is integrable over all bounded set. We recall that G_m is the Green's function of the fractional Laplacian, see (1.4), and $k_m = \nabla^\perp G_m$ here plays the role of the Biot-Savart kernel. Finally we shall use the symbol $\lesssim$ for inequalities up to some constant that does not depend on the main parameters of the problem, and thus ultimately does not matter.

2. Existence of weak solutions

In this section we prove existence of weak solutions for (1.1) with initial condition in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. To this end we recast problem (1.1) as

$$\begin{cases} \partial_t \theta + \nabla\cdot(u\theta) = 0, \\ u = k_m \star \theta, \end{cases} \tag{2.1}$$

where $k_m = \nabla^\perp G_m$ and G_m is the Green function for the fractional Laplacian $(-\Delta)^{\frac{m}{2}}$ given in (1.4).

Definition 2.1 (Weak solution)**.** Given $\theta_0 \in L^1_{\text{loc}}(\mathbb{R}^2)$, a solution to (2.1) is a distribution such that for all $t > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} (\theta(t,x) - \theta_0(x))\varphi(x)\,dx - \int_0^t \int_{\mathbb{R}^2} \theta(s,x)u(s,x)\cdot\nabla\varphi(s,x)\,dx\,ds = 0,$$

where $u = k_m \star \theta$.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, define its *centre of pseudo-vorticity* as

$$C_f := \int_{\mathbb{R}^2} x\, f(x)\,dx, \tag{2.2}$$

whenever the integral is well defined, and its *moment of inertia*

$$J_f := \int_{\mathbb{R}^2} |x - c_f|^2 |f(x)|^2\,dx. \tag{2.3}$$

The proof of existence of weak solutions of (2.1) proceeds through a vanishing viscosity approximation. We will actually make a two-steps approximation to prove some conservation properties that will turn out to be crucial for the proof of Theorem 3.6.

We start by stating a classical inequality about the velocity $u = k_m \star \theta$, that will be useful for our purposes.

Lemma 2.2. *Let $m \in (1,2)$. Then*

$$\|k_m \star f\|_{L^q} \lesssim \|f\|_{L^p} \lesssim \|f\|_{L^1}^{\frac{1}{q}+\frac{1}{2}(m-1)} \|f\|_{L^\infty}^{\frac{1}{2}(3-m)-\frac{1}{q}}.$$

where $q > \frac{2}{3-m}$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{2}(m-1)$.

Moreover there is $c = c(\|f\|_{L^1}, \|f\|_{L^\infty})$ such that for all $x, y \in \mathbb{R}^2$,

$$|k_m \star f(x) - k_m \star f(y)| \le c(1 \wedge |x-y|)^{m-1}.$$

To prove the existence of weak solutions to (2.1) (see Theorem 2.6), we will suitably regularize the initial condition and the velocity. To this end, let $\rho \in C_c^\infty(\mathbb{R}^2)$ be symmetric, $0 \leq \rho \leq 1$, and $\int_{\mathbb{R}^2} \rho(x)\,dx = 1$, and set $\rho_\epsilon = \epsilon^{-2}\rho(x/\epsilon)$. Denote by k_m^ϵ the kernel $k_m^\epsilon = \rho_\epsilon \star k_m$, and, for $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, set $\theta_0^\epsilon = \rho_\epsilon \star \theta_0$.

2.1. **The viscous approximation.** Given $\epsilon > 0$ and $\nu > 0$, consider the problem

$$\begin{cases} \partial_t \theta + \nabla\cdot(u\theta) = \nu\Delta\theta, \\ u = k_m^\epsilon \star \theta. \end{cases} \tag{2.4}$$

Proposition 2.3. *Given $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\epsilon > 0$, $\nu > 0$, there is a unique classical solution $\theta_{\epsilon,\nu}$ of (2.4) with initial condition θ_0^ϵ. Moreover, for every $n \geq 0$, every $p \in [1,\infty)$ and every $T > 0$, there is a number $c = c(\epsilon,\nu,n,p,T,\theta_0)$ such that*

$$\sup_{[0,T]} \|D^\alpha \theta_{\epsilon,\nu}\|_{L^p} \leq c, \tag{2.5}$$

for all multi-indices α with $|\alpha| = n$. In particular, for all $t > 0$ and all $p \in [1,\infty]$,

$$\|\theta_{\epsilon,\nu}\|_{L^p} \leq \|\theta_0\|_{L^p}. \tag{2.6}$$

Proof. We give a sketch of the proof, and for simplicity we drop the subscript ${}_{\epsilon,\nu}$. Existence of a solution is standard, see for instance [11, 10]. We first show the conservation in L^p, $1 \leq p < \infty$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} |\theta(t,x)|^p\,dx &= p\int_{\mathbb{R}^2} |\theta|^{p-1}\operatorname{sgn}(\theta)(\nu\Delta\theta - \nabla\cdot(u\theta))\,dx \\ &= p\nu\int_{\mathbb{R}^2} |\theta|^{p-1}\operatorname{sgn}(\theta)\Delta\theta\,dx - p\int_{\mathbb{R}^2} u\cdot\nabla|\theta|^p\,dx. \end{aligned}$$

The first integral on the right hand side is non-positive, see [11, 13], the second integral is zero by integration by parts, since $\nabla\cdot u = 0$, and this proves that the derivative is non-positive. The case $p = \infty$ follows in the limit $p \uparrow \infty$.

Likewise, if $\eta = D^\alpha\theta$, then η solves

$$\partial_t\eta + \nabla\cdot(u\eta) - \nu\Delta\eta = (\partial_{x_1}u)\cdot\nabla D^{\alpha-(1,0)}\theta + (\partial_{x_2}u)\cdot\nabla D^{\alpha-(0,1)}\theta + F,$$

where F is bilinear in the derivatives of order at most n of u and of order at most $n-1$ of θ. Since by Lemma 2.2, for every multi-index β, $\|D^\beta u\|_{L^\infty} \leq c(\epsilon,\beta)\|k_m \star \theta\|_{L^\infty} \leq c(m,\epsilon,\theta_0)$, we have that,

$$\frac{d}{dt}\int_{\mathbb{R}^2}\sum_{|\alpha|=n}|D^\alpha\theta|^p\,dx \leq (1 + c(m,\epsilon,\theta_0))\int_{\mathbb{R}^2}\sum_{|\alpha|=n}|D^\alpha\theta|^p\,dx + \|F\|_{L^p}$$

The bound follows by an induction argument to estimate $\|F\|_{L^p}$ in terms of lower order derivatives of θ, and Gronwall's lemma.

To prove uniqueness, let θ_1, u_1 and θ_2, u_2 solutions to (2.4) with the same initial condition θ_0^ϵ, and set $\delta = \theta_1 - \theta_2$, $\gamma = u_1 - u_2$. Then

$$\partial_t\delta + \nabla\cdot(u_1\delta + \gamma\theta_2) = \nu\Delta\delta,$$

therefore

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2}\delta^2\,dx &= -\int_{\mathbb{R}^2} u_1\cdot\nabla\delta^2\,dx - 2\int_{\mathbb{R}^2}\delta\gamma\nabla\theta_2\,dx - 2\nu\int_{\mathbb{R}^2}|\nabla\delta|^2\,dx \\ &\leq -2\int_{\mathbb{R}^2}\delta\gamma\nabla\theta_2\,dx. \end{aligned}$$

By the Hölder inequality and Lemma 2.2,

$$\Big|\int_{\mathbb{R}^2} \delta\gamma\nabla\theta_2\,dx\Big| \leq \|\delta\|_{L^2}\|\gamma\|_{L^{\frac{2}{2-m}}}\|\nabla\theta_2\|_{L^{\frac{2}{m-1}}} \lesssim \|\nabla\theta_2\|_{L^{\frac{2}{m-1}}}\|\delta\|_{L^2}^2.$$

Uniqueness follows by the Gronwall lemma. □

2.2. **The inviscid approximation.** Given $\epsilon > 0$, consider the following inviscid problem with regularized velocity,

$$\begin{cases}\partial_t\theta + \nabla\cdot(u\theta) = 0,\\ u = k_m^\epsilon \star \theta.\end{cases} \tag{2.7}$$

Proposition 2.4. *Given $\theta_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and $\epsilon > 0$, there is a unique classical solution θ_ϵ of (2.7) with initial condition θ_0^ϵ. Moreover, for every $n \geq 0$, every $p \in (1,\infty)$ and every $T > 0$, there is a number $c = c(\epsilon,\nu,n,p,T,\theta_0)$ such that*

$$\sup_{[0,T]}\|D^\alpha\theta_\epsilon\|_{L^p} \leq c, \tag{2.8}$$

for all multi-indices α with $|\alpha| = n$. In particular, for all $t > 0$ and all $p \in [1,\infty]$,

$$\|\theta_\epsilon\|_{L^p} \leq \|\theta_0\|_{L^p}. \tag{2.9}$$

Finally, for all $t > 0$,

$$\int_{\mathbb{R}^2}\theta_\epsilon(t,x)\,dx = \int_{\mathbb{R}^2}\theta_0^\epsilon(x)\,dx, \tag{2.10}$$

and, if $\theta_0 \geq 0$, then $\theta_\epsilon(t,x) \geq 0$ for all $t > 0$.

Proof. In Proposition 2.3 we have seen that the sequence $(\theta_{\epsilon,\nu})_{\nu>0}$ of solutions of (2.4) is bounded in $L^\infty(0,T;W^{k,p}(\mathbb{R}^2))$ for all $p > 1$, all $T > 0$ and all $k \geq 1$, and we wish to use this sequence to construct a solution of (2.7). By a diagonal argument there is a sequence $(\nu_n)_{n\geq1}$ such that $(\theta_{\epsilon,\nu_n})_{n\geq1}$ weak-$\star$ converges in $L^\infty(0,T;W^{k,p}(\mathbb{R}^2))$ to a function θ_ϵ, for every $T > 0$, $k \geq 1$ and $p > 1$. In particular, (2.8) and (2.9) (for $p > 1$) hold.

The convergence of $\theta_{\epsilon,\nu}$, as well as of its derivatives and of their respective equations, goes in an analogous, even simpler, way as in the proof of Theorem 2.6, where all details will be given, and is therefore omitted here. The argument for uniqueness is the same as in Proposition 2.3, since the viscous term is not used in the proof.

Finally, to prove conservation of mass and conservation of sign, consider for each $x \in \mathbb{R}^2$ and each $t > 0$ the backward system (of characteristics),

$$\begin{cases}\frac{d}{ds}Y_s^{t,x} = u_\epsilon(s,Y_s^{t,x}),\\ Y_t^{t,x} = x.\end{cases} \tag{2.11}$$

The solution is well defined and global since u is continuous and globally Lipschitz in the space variable. It is standard to see that $x \mapsto Y_s^{t,x}$, with $0 \leq s \leq t$, are diffeomorphisms. Moreover, if $J(s,x)$ is the determinant of the Jacobian matrix of $x \mapsto Y_s^{t,x}$, then $J(t,x) = 1$ and

$$\dot J(s,x) = (\nabla\cdot u_\epsilon)(s,Y_s^{t,x})J(s,x),$$

therefore $J(s,x) = 1$ for all $s \in [0,t]$, since u_ϵ is divergence free. Finally, a simple computation shows that

$$\frac{d}{ds}\theta_\epsilon(s,Y_s^{t,x}) = 0.$$

These arguments, together with the simple remark that if $\theta_0 \geq 0$, then $\theta_0^\epsilon \geq 0$ (since the regularizing kernel is positive) prove conservation of mass and conservation of sign, as well as (2.9) for $p = 1$. □

The above existence and uniqueness result can be improved. Indeed, we can get rid of the regularization in the initial condition. To this end, denote by $|\cdot|_\bullet$ the (bounded) metric $|x-y|_\bullet = 1 \wedge |x-y|$, and let W_1 be the 1-Wasserstein distance on non-negative finite measures on $\mathbb{R}^2$. The Wasserstein distance can be extended to signed measure with equal positive and negative masses by $W_1(\mu,\nu) = W_1(\mu_+,\nu_+) + W_1(\mu_-,\nu_-)$. Since here W_1 is based on a bounded metric, convergence in W_1 is equivalent to the standard weak convergence of measures.

Corollary 2.5. *Given a finite measure θ_0 on $\mathbb{R}^2$, and $\epsilon > 0$, there is a unique solution θ on $[0,\infty)$ in the sense of distributions of (2.7).*

Moreover, if θ_0^1, θ_0^2 are two different measures with the same positive and negative masses, then for all $T > 0$,

$$\sup_{t\in[0,T]} W_1(\theta_1(t),\theta_2(t)) \leq C(c_\epsilon, T) W_1(\theta_0^1,\theta_0^2), \tag{2.12}$$

where θ_1, θ_2 are solutions of (2.7) with respective initial conditions θ_0^1, θ_0^2, and $c_\epsilon = 2\|k_m^\epsilon\|_{L^\infty} \vee \|\nabla k_m^\epsilon\|_{L^\infty}$.

Finally, if $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p},$$

for every $t \geq 0$ and $p \in [0,\infty]$.

Proof. The idea here is to consider in Proposition 2.4 above two regularization parameters: one for the velocity (ϵ), and one for the initial condition. The L^p conservation follows as in the previous Proposition 2.4. We prove here only (2.12), which in particular proves existence and uniqueness.

Let us notice first that if μ, ν are measures, then

$$\|k_m^\epsilon \star \mu - k_m^\epsilon \star \nu\|_{L^\infty} \leq c_\epsilon W_1(\mu,\nu). \tag{2.13}$$

This is immediate since k_m^ϵ is Lipschitz with respect to $|\cdot|_\bullet$ with Lipschitz constant c_ϵ, and by duality, for probability measures μ, ν,

$$W_1(\mu,\nu) = \sup \int_{\mathbb{R}^2} f\, d(\mu - \nu),$$

where the supremum is taken over all $|\cdot|_\bullet$-Lipschitz function with Lipschitz constant 1.

Set $u_i = k_m^\epsilon \star \theta_i$, $i = 1, 2$. We claim that the following inequality holds,

$$W_1(\theta_1(t),\theta_2(t)) \leq \mathrm{e}^{c_\epsilon t}\, W_1(\theta_1^0,\theta_2^0) + c_\epsilon\, \mathrm{e}^{c_\epsilon t} \int_0^t W_1(\theta_1(s),\theta_2(s))\, ds. \tag{2.14}$$

By Gronwall's lemma, (2.12) then follows. We turn to the proof of (2.14). Let $\mathbb{P}$ be a coupling of $|\theta_1^0|, |\theta_2^0|$, then the measure $\mathbb{P}_t$, defined as

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)\, \mathbb{P}_t(\,dx,\,dy) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(X_1^{t,x}(0), X_2^{t,y}(0))\, \mathbb{P}(dx,dy),$$

is a coupling of $|\theta_1(t)|, |\theta_2(t)|$, where $X_i^{t,x}$, $i = 1, 2$ are the back-to-label maps of (2.11) corresponding to u_1, u_2. By (2.13) we have that

$$\begin{aligned}|X_1^{t,x}(0) - X_2^{t,y}(0)|_\bullet &\leq |x-y|_\bullet + \int_0^t |u_1(t-s, X_1^{t,x}(s)) - u_2(t-s, X_2^{t,x}(s))|\,ds\\ &\leq |x-y|_\bullet + c_\epsilon \int_0^t |X_1^{t,x}(s) - X_2^{t,y}(s)|_\bullet\,ds +\\ &\quad + c_\epsilon \int_0^t W_1(\theta_1(s), \theta_2(s))\,ds.\end{aligned}$$

The Gronwall lemma yields

$$|X_1^{t,x}(0) - X_2^{t,y}(0)|_\bullet \leq \mathrm{e}^{c_\epsilon t}\,|x-y|_\bullet + c_\epsilon\,\mathrm{e}^{c_\epsilon t} \int_0^t W_1(\theta_1(s), \theta_2(s))\,ds.$$

By integrating with $\mathbb{P}$ we have

$$W_1(\theta_1(t), \theta_2(t)) \leq \mathrm{e}^{c_\epsilon t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-y|_\bullet\,\mathbb{P}(dx, dy) + c_\epsilon\,\mathrm{e}^{c_\epsilon t} \int_0^t W_1(\theta_1(s), \theta_2(s))\,ds,$$

and taking the infimum over all $\mathbb{P}$ yields (2.14). □

2.3. **The inviscid problem.** We first prove existence of a weak solution for problem (2.1).

Theorem 2.6. *Let $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then there is a solution θ of (2.1) on $[0,\infty)$ with initial condition θ_0, in the sense of Definition 2.1. Moreover,*

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \tag{2.15}$$

for every $p \in [1,\infty]$ and all $t > 0$.

Proof. The family $(\theta_\epsilon)_{\epsilon>0}$ of solutions to (2.7) is bounded in $L^\infty(0,T;L^p(\mathbb{R}^2))$ for all $p > 1$, and all $T > 0$. By a diagonal argument there is a sequence $(\epsilon_n)_{n\geq1}$ such that $(\theta_{\epsilon_n})_{n\geq1}$ weak-$\star$ converges in $L^\infty(0,T;L^p(\mathbb{R}^2))$ to a function θ, for every $T > 0$ and $p > 1$. In the rest of the proof we will set $\theta_n = \theta_{\epsilon_n}$, $\rho_n = \rho_{\epsilon_n}$, $u_n = \rho_n \star k_m \star \theta_n$, and $u = k_m \star \theta$.

Step 1: strong convergence of θ_n. Fix $\varphi \in C_c^\infty(\mathbb{R}^2)$. By the Hölder inequality,

$$\int_{\mathbb{R}^2} \varphi \partial_t \theta_n\,dx = \int_{\mathbb{R}^2} \theta_n u_n \cdot \nabla\varphi\,dx \leq \|\nabla\varphi\|_{L^2} \|\theta_n\|_{L^2} \|u_n\|_{L^\infty} \leq c\|\nabla\varphi\|_{L^2}.$$

Therefore $(\partial_t(\varphi\theta_n))_{n\geq1}$ is bounded in $L^\infty(0,T;H^{-1}(\operatorname{Supp}\varphi))$. Since $(\varphi\theta_n)_{n\geq1}$ is bounded in $L^\infty(0,T;L^2(\operatorname{Supp}\varphi))$, the Aubin-Lions lemma ensures that $(\varphi\theta_n)_{n\geq1}$ is compact in $C([0,T];L^2(\operatorname{Supp}\varphi))$. Thus $(\varphi\theta_n)_{n\geq1}$ converges strongly to $\varphi\theta$ in $C([0,T];L^2(\operatorname{Supp}\varphi))$. In conclusion, by (2.9), $(\theta_n)_{n\geq1}$ converges strongly to θ in $C([0,T];L^p_{\mathrm{loc}}(\mathbb{R}^2))$ for all $T > 0$ and all $p \in [1,\infty)$.

Step 2: Conservation of L^p norms. Formula (2.15) for $p < \infty$ follows from the previous step and (2.9). The case $p = \infty$ follows classically by the convergence of L^p norms to the L^∞ norms.

Step 3: strong convergence of u_n. We show that u_n converges to u strongly in $L^p(0,T;L^p_{\mathrm{loc}}(\mathbb{R}^2))$ for all $p \in [1,\infty)$. Since by (2.9) and (2.15) u_n and u are uniformly bounded, it is sufficient to prove that

$$\int_0^T |u_n(x) - u(x)|^p\,dx \longrightarrow 0,$$

for a. e. x, all $T > 0$ and all $p > \frac{2}{m-1}$. Now,

$$u_n(x) - u(x) = \rho_n \star k_m \star (\theta_n - \theta)(x) + \big(\rho_n \star u(x) - u(x)\big).$$

Our claim for the second term on the right hand side is standard, so we concentrate on the first term. Given $R > 0$, write $k_m^i = k_m \mathbb{1}_{B_R(0)}$ and $k_m^o = k_m \mathbb{1}_{B_R(0)^c}$. By the Hölder and Young inequalities, and (2.9) and (2.15),

$$\|\rho_n \star k_m^o \star (\theta_n - \theta)\|_{L^\infty} \leq \|k_m^o \star (\theta_n - \theta)\|_{L^\infty} \leq cR^{-\alpha},$$

for a number $\alpha > 0$, where c depends on θ_0. Since $\operatorname{Supp} \rho \subset B_1(0)$,

$$|\rho_n \star k_m^i \star (\theta_n - \theta)(x)| \leq \sup_{y \in B_{\epsilon_n}(x)} |k_m^i \star (\theta_n - \theta)(y)|.$$

Using the Hölder inequality with $q < \frac{2}{3-m}$ and $\frac{1}{p} + \frac{1}{q} = 1$ (therefore $p > \frac{2}{m-1}$),

$$\begin{aligned} |k_m^i \star (\theta_n - \theta)(y)| &\leq \|k_m^i\|_{L^q} \|\mathbb{1}_{B_R(y)}(\theta_n - \theta)\|_{L^p} \\ &\leq c_R \|\mathbb{1}_{B_{R+1}(x)}(\theta_n - \theta)\|_{L^p}, \end{aligned}$$

since $y \in B_{\epsilon_n}(x)$ and, for n large enough, $\epsilon_n \leq 1$. In conclusion,

$$\int_0^T |\rho_n \star k_m \star (\theta_n - \theta)(x)|^p \, dx \lesssim R^{-\alpha p} + c_R \int_0^T \|\mathbb{1}_{B_{R+1}(x)}(\theta_n - \theta)\|_{L^p}^p \, dt.$$

By first taking the lim sup in $n \to \infty$ (using the first step of the proof), and then the limit $R \uparrow \infty$, the claim follows.

Step 3: conclusion. The convergence properties in the first two steps allows immediately to prove that θ is a weak solution. □

In the analysis of the connection between solutions of (1.1) and the point vortex motion we will need some additional properties of solutions to (1.1).

Corollary 2.7. *Let $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and let θ be a solution to (2.1) with initial condition θ_0 obtained as in Theorem 2.6 above. Then the following statements hold.*

- *If $\theta_0 \geq 0$ a. e., then $\theta(t) \geq 0$ a. e. for all $t > 0$.*
- *If $\int_{\mathbb{R}^2} |x| \, |\theta_0(x)| \, dx < \infty$, then*

$$\sup_{[0,T]} \int_{\mathbb{R}^2} |x| \, |\theta(t,x)| \, dx < \infty, \qquad \textit{for all } T > 0,$$
$$\int_{\mathbb{R}^2} \theta(t,x) \, dx = \int_{\mathbb{R}^2} \theta_0(x) \, dx, \qquad \textit{for all } t \geq 0.$$

- *If $\int_{\mathbb{R}^2} |x|^2 \, |\theta_0(x)| \, dx < \infty$, then*

$$\sup_{[0,T]} \int_{\mathbb{R}^2} |x|^2 \, |\theta(t,x)| \, dx < \infty, \qquad \textit{for all } T > 0,$$
$$C_{\theta(t)} = \int_{\mathbb{R}^2} x \, \theta(t,x) \, dx = \int_{\mathbb{R}^2} x \, \theta_0(x) \, dx, = C_{\theta_0}, \qquad \textit{for all } t \geq 0,$$
$$J_{\theta(t)} \leq J_{\theta_0}$$

Proof. As in the proof of the previous theorem there is a sequence $(\theta_n)_{n \geq 1}$, with $\theta_n = \theta_{\epsilon_n}$ and $u_n = u_{\epsilon_n}$, of solutions to (2.7), with regularized initial condition, such that $\theta_n \to \theta$ and $u_n \to u$ as in the proof of Theorem 2.6. Positivity is straightforward by Proposition 2.4.

Assume $\int_{\mathbb{R}^2} |x|\,|\theta_0(x)|\,dx < \infty$. By integration by parts,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|\,|\theta_n|\,dx = \int_{\mathbb{R}^2} |x|\,\mathrm{sgn}(\theta_n)\partial_t\theta_n\,dx = -\int_{\mathbb{R}^2} |x| u_n\cdot\nabla|\theta_n|\,dx\\ = \int_{\mathbb{R}^2} |\theta_n| u_n\cdot\nabla|x|\,dx \le \int_{\mathbb{R}^2} |u_n|\,|\theta_n|\,dx,\end{aligned}$$

and the last term on the right hand side is uniformly bounded by a number that depends only on θ_0. This proves the first claim and that $(\theta_n)_{n\geq 1}$ is uniformly integrable. By (2.10) conservation of mass follows for θ.

Assume $\int_{\mathbb{R}^2} |x|^2\,|\theta_0(x)|\,dx < \infty$. By integration by parts,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2|\theta_n|\,dx = -\int_{\mathbb{R}^2} |x|^2 u_n\cdot\nabla|\theta_n|\,dx\\ = \int_{\mathbb{R}^2} |\theta_n| u_n\cdot\nabla|x|^2\,dx \le 2\|u_n\|_{L^\infty}\int_{\mathbb{R}^2} |x|\,|\theta_n|\,dx,\end{aligned}$$

and, by the previous considerations, the last term on the right hand side is uniformly bounded, on a finite time interval $[0,T]$, by a number that depends only on θ_0 and T. This estimates implies that $(x\mapsto x\,\theta_n(t,x))_{n\geq 1}$ is uniformly integrable, therefore $C_{\theta_n(t)}\to C_{\theta(t)}$ for all t. Actually, uniform convergence holds, since for $T>0$ and $t\in[0,T]$,

$$\begin{aligned}|C_{\theta_n(t)} - C_{\theta(t)}| \le \int_{B_R(0)} |x|\,|\theta_n(t)-\theta(t)|\,dx + \int_{B_R(0)^c} |x|\,|\theta_n(t)-\theta(t)|\,dx\\ \le R\sup_{[0,T]}\|(\theta_n-\theta)\mathbb{1}_{B_R(0)}\|_{L^1} + \frac{1}{R}\int_{\mathbb{R}^2} |x|^2(|\theta_n(t)|+|\theta(t)|)\,dx.\end{aligned}$$

The second term on the right hand side is uniformly bounded in n and $t\in[0,T]$. By first taking the limit $n\to\infty$ and then $R\uparrow\infty$, uniform convergence follows. We have

$$\frac{d}{dt}C_{\theta_n(t)} = \int_{\mathbb{R}^2} x\,\partial_t\theta_n\,dx = -\int_{\mathbb{R}^2} x\,u_n\cdot\nabla\theta_n\,dx = \int_{\mathbb{R}^2}\theta_n u_n\,dx = 0,$$

since ρ_n is symmetric and k_m is anti-symmetric. This proves conservation of the centre of pseudo-vorticity for θ. Likewise,

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\theta_n\,dx = -\int_{\mathbb{R}^2} |x|^2 u_n\cdot\nabla\theta_n\,dx = 2\int_{\mathbb{R}^2}\theta_n u_n\cdot x\,dx = 0,$$

since ρ_n is symmetric, k_m is anti-symmetric, and $x\cdot k_m(x)=0$. By semicontinuity and the conservation of mass and centre proved before, we obtain $J_{\theta(t)}\le J_{\theta_0}$. □

In Section 3 we will single out the evolution of a single vortex blob and consider the velocity field generated by all other blobs as an external field. To this end the following slight modification of the previous results will be useful.

Corollary 2.8. *Let $\theta_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, and $F:[0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2$ be a bounded field. Then there is a solution to*

$$\begin{cases}\partial_t\theta + \nabla\cdot((u+F)\theta) = 0,\\ u = k_m\star\theta,\end{cases}$$

such that (2.15) holds. Moreover, the conclusions of Corollary 2.7 also hold, with the exception of the evolution of the vortex centre and the moment of inertia, that *are replaced by the following formulas,*

$$C_{\theta(t)} = C_{\theta_0} + \int_0^t \int_{\mathbb{R}^2} \theta(s,x) F(s,x)\,dx\,ds,$$
$$J_{\theta(t)} \le J_{\theta_0} + 2\int_0^t \int_{\mathbb{R}^2} \theta(s,x)(x - C_{\theta(s)}) \cdot F(s,x)\,dx\,ds.$$

3. The point-vortex motion

We turn to the main problem, the validation of the point-vortex motion system (1.3) in terms of solutions to (1.1).

For the Euler equations ($m = 2$ in our setting) these results have been already established and are somewhat classical, see [30].

We will look rigorously only at the case of the whole plane as a motivation for the validity of the system of evolution for point vortices. The extension of these results to the torus is straightforward, since conservation of the centre of pseudo-vorticity and of the moment of inertia still hold. The presence of boundaries makes the problem more difficult and it is not examined here.

3.1. **Global solutions for the point vortex motion.** Our first step to motivate the point vortex motion system (1.3), is to show that it gives a well defined dynamics, at least for a large enough set of initial conditions. Here we follow the approach used for the Euler equations in [28, 30].

The point vortex motion (1.3) is given by

$$\dot X_j = \sum_{k\neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \qquad j = 1, 2, \dots, N,$$

where G_m is the Green function of $(-\Delta)^{\frac{m}{2}}$ on the whole space, see (1.4). The motion is Hamiltonian, described by the Hamiltonian

$$H(X_1, X_2, \dots, X_N, \gamma_1, \gamma_2, \dots, \gamma_N) = \frac{1}{2}\sum_{j\neq k} \gamma_j\gamma_k G_m(X_j, X_k), \tag{3.1}$$

where $\gamma_1 \dots \gamma_N$ are vortex intensities, in the sense that the above system can be written as

$$\begin{cases} \gamma_j \dot X_{j,1} = \frac{\partial H}{\partial X_{j,2}}, \\ \gamma_j \dot X_{j,2} = -\frac{\partial H}{\partial X_{j,1}}, \end{cases} \qquad j = 1, 2, \dots, N.$$

Therefore the Hamiltonian H is conserved along the motion (1.3). Moreover, since the Hamiltonian is translation invariant and rotation invariant, the vortex centre

$$C = \sum_{j=1}^N \gamma_j X_j, \tag{3.2}$$

and the moment of inertia

$$J = \sum_{j=1}^N \gamma_j |X_j|^2.$$

are also conserved. Assume initially that all vortex intensities are positive (or all negative). Then by the conservation of the Hamiltonian there cannot be collapse. Additionally, by the conservation of the moment of inertia there cannot be explosion, namely that one or more vortices reach infinity in finite time. For the same reasons, even with vortices of different signs, there cannot be collapse or explosion for one or two vortices. For more than two vortices singularities are possible, see [1].

Our main assumption, the same in [30] for the case $m = 2$, for the existence of a global flow for almost every initial condition is

$$\sum_{j\in J}\gamma_j \neq 0 \qquad \text{for all } J \subset \{1, 2, \dots, N\}. \tag{3.3}$$

The main theorem is as follows. A version of this result on the torus can be found in [19].

Theorem 3.1. *Fix $1 < m < 2$ and assume (3.3). Then, outside a set of initial conditions of Lebesgue measure zero, the initial value problem associated to the vortex equation (1.3) has a global smooth solution.*

Proof. The theorem can be proved similarly to [30, Corollary 2.2, Ch. 4], We outline some of the main steps.

First of all we regularize the dynamics, to handle the singularity. Let G_m^ϵ be a $C^\infty(\mathbb{R}^2)$ function such that

- $G_m^\epsilon = G_m$ for $|x| \geq \epsilon$,
- $0 \leq G_m^\epsilon \leq G_m$,
- $|\nabla G_m^\epsilon| \lesssim |\nabla G_m|$.

The regularized dynamics X^ϵ defined by the Hamiltonian obtained by (3.1) by replacing G_m with G_m^ϵ is well defined and global. Moreover, as long as the particles in the regularized dynamics are at a distance of at least ϵ, their motion coincide with the original motion given by (1.3).

The first step is to prove a uniform estimate on non-collision. The following claim can be proved as Theorem 2.1 (chapter 4) of [30], with no substantial difference between the case with value $m = 2$ (discussed in the reference) and the case $1 < m < 2$.

> *There exists a number $c > 0$ independent of ϵ and of the initial condition, such that*
> $$\max_{1\leq j\leq N}\sup_{t\in[0,T]} |X_j^\epsilon(t) - X_j^\epsilon(0)| \leq c.$$

The proof is based essentially on the conservation of the vortex centre, defined as in (3.2). Here the assumption (3.3) is essential, while it is only required that $|\nabla G_m|$ goes to zero at infinity[1].

The previous claim implies that

> *For every $R, T > 0$, there is $R_\star > 0$ such that N vortices that start in $B_R(0)$ and evolve with the regularized dynamics, cannot leave $B_{R_\star}(0)$ within time T, for every initial data and every $\epsilon \in (0, 1)$.*

Define

$$D_T(x_1, \dots, x_N) = \min_{i\neq j}\min_{t\in[0,T]} |X_i(t) - X_j(t)|,$$

where $X(\cdot)$ is the dynamics (1.3) with initial condition $(x_1, \dots, x_N)$. Define similarly $D_T^\epsilon(x_1, \dots, x_N)$ for the regularized dynamics. To prove the theorem, it is sufficient to prove that the set $\{D_T(x_1, \dots, x_N) = 0\}$ has Lebesgue measure zero. To this end, it suffices to prove that the measure of $\{D_T(x_1, \dots, x_N) < \epsilon\} \cap B_R^N$ converges

[1] So in principle every $m \leq 2$ is allowed.

to 0 as $\epsilon \downarrow 0$ for all R, where B_R^N is the product of N-times the ball $B_R(0)$. But since

$$\{D_T(x_1,\dots,x_N) \geq \epsilon\} \cap B_R^N = \{D_T^\epsilon(x_1,\dots,x_N) \geq \epsilon\} \cap B_R^N$$

this is the same as proving that the measure of $\{D_T^\epsilon(x_1,\dots,x_N) < \epsilon\} \cap B_R^N$ goes to 0 as $\epsilon \downarrow 0$.

Define

$$\Phi_\epsilon(x_1,\dots,x_N) = \frac{1}{2}\sum_{i\neq j} G_m^\epsilon(x_i - x_j),$$

and let X^ϵ be the solution to the regularized dynamics with initial conditions $x_1,\dots,x_N$. A simple computation yields

$$\frac{d}{dt}\Phi_\epsilon(X_1^\epsilon(t),\dots,X_N^\epsilon(t)) \leq h(X_t^\epsilon) := \sum_{i\neq j, j\neq k, k\neq i} \frac{1}{|X_i^\epsilon - X_j^\epsilon|^{3-m}|X_i^\epsilon - X_k^\epsilon|^{3-m}},$$

where the most singular term has disappeared due to the product $\nabla\cdot\nabla^\perp$ being zero. Notice that since $m > 1$, Φ_ϵ and h are in L^1_{loc}, and the integral of Φ_ϵ and h over bounded sets is independent of ϵ. Using the invariance of the Lebesgue measure with respect to the regularized dynamics and the second claim above,

$$\begin{aligned}\int_{B_R^N} \sup_{[0,T]} |\Phi_\epsilon(X_t^\epsilon)|\,dx_1\dots\,dx_N &\leq \int_{B_R^N} |\Phi_\epsilon(x_1,\dots,x_N)|\,dx_1\dots\,dx_N \\ &\quad + T\int_{B_{R_\star}^N} h(x_1,\dots,x_N)\,dx_1\dots\,dx_N =: C(T,R_\star).\end{aligned}$$

Finally, $\{D_T^\epsilon(x_1,\dots,x_N) < \epsilon\} \cap B_R^N \subset \{\sup_{[0,T]} |\Phi_\epsilon(X_t^\epsilon)| \geq \frac{1}{2}\epsilon^{m-2}\}$, and the measure of the set on the right hand side, by the Chebychev inequality, is bounded by $2\epsilon^{2-m}C(T,R_\star)$ and thus converges to 0. □

3.2. **Vortex approximation.** In this section we prove that vortices provide an approximation of solutions to (1.1). These results are classical for $m = 2$, see [30], and have been recently proved for $m \in (2,3)$ in [21].

First, we set up the initial conditions for the approximation. Let $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, with $\int_{\mathbb{R}^2} |x|\,|\theta_0(x)|\,dx < \infty$. For every $N \geq 2$ consider $\gamma_1^N, \gamma_2^N, \dots, \gamma_N^N \in \mathbb{R}$ and $x_1^N, x_2^N, \dots, x_N^N$ such that

$$\sum_{j=1}^N (\gamma_j^N)_+ = \int_{\mathbb{R}^2} \theta_0(x)_+\,dx, \qquad \sum_{j=1}^N (\gamma_j^N)_- = \int_{\mathbb{R}^2} \theta_0(x)_-\,dx, \tag{3.4}$$

where $x_+ = x \vee 0$ and $x_- = (-x) \vee 0$, and set

$$\theta_0^N = \sum_{j=1}^N \gamma_j^N \delta_{x_j^N}.$$

For every $\epsilon > 0$ consider a smooth approximation k_m^ϵ of the kernel k_m[2], and consider the solution $(X_{\epsilon,j}^N)_{j=1,2,\dots,N}$ of the evolution

$$\dot X_{\epsilon,j}^N = \sum_{k=1}^N \gamma_k^N k_m^\epsilon(X_{\epsilon,j}^N - X_{\epsilon,k}^N), \tag{3.5}$$

[2]For instance, one can consider the smooth approximation $k_m^\epsilon = \rho_\epsilon \star k_m$ considered in Proposition 2.4, as well as $k_m^\epsilon = \eta_\epsilon k_m$, where η_ϵ is a radial function (so that k_m^ϵ is still divergence-free) which is 1 in $B_{2\epsilon}(0)^c$ and 0 in $B_\epsilon(0)$.

with initial conditions $x_1^N, x_2^N, \dots, x_N^N$. Set finally

$$\theta_\epsilon^N(t) = \sum_{j=1}^N \gamma_j^N \delta_{X_{\epsilon,j}^N(t)}, \qquad t \geq 0.$$

Theorem 3.2. *Let $m \in (1,2)$ and let θ_0, θ_0^N as above, and assume that*

$$W_1(\theta_0^N, \theta_0) \longrightarrow 0, \qquad \text{as } N \uparrow \infty.$$

Let θ be a solution of (1.1) given by Theorem 2.6. Then for every $T > 0$ there are two sequences $(\epsilon_n)_{n\geq 1}$ and $(N_n)_{n\geq 1}$ such that

$$\sup_{t\in[0,T]} W_1(\theta_{\epsilon_n}^{N_n}(t), \theta(t)) \longrightarrow 0, \qquad \text{as } n \to \infty.$$

Remark 3.3. Condition (3.4) is only technical in view of the evaluation in terms of the Wasserstein distance, since the Wasserstein distance can be infinite in case of measures with different masses. In case (3.4) holds only asymptotically, the solution is to compare θ_ϵ^N with a modification of θ_0 such that the equality of masses is re-established and the modification weakly converges to θ_0.

Remark 3.4. There are two oddities about the theorem above. The first is about the limit along a sequence of regularizations $(\epsilon)_{n\geq 1}$. In the analogous result on Euler equations ($m = 2$) this is not required, and this is due to the fact that uniqueness for initial conditions in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is not known when $m < 2$ (but see also Remark 3.5 in view of the uniqueness proof of Corollary 2.5).

The second issue regards the appearance of the regularized dynamics in place of the original dynamics (1.3). Consider for simplicity the regularization $k_m^\epsilon = \eta_\epsilon k_m$, where η_ϵ is smooth, bounded, radial, equal to 1 everywhere but in $B_\epsilon(0)$, and 0 in $B_{\epsilon/2}(0)$. With this choice it is immediate to see that the solutions to (1.3) and (3.5) are the same as long as

$$D_N := \min_{t\in[0,T]} \min_{i\neq j} |X_j^N(t) - X_i^N(t)|$$

is larger that ϵ, where X^N is the solution to (1.3). The problem is now apparent: in order to consider the true dynamics we need to have $D_{N_n} \ll \epsilon_n$, which in principle means N not too large. On the other hand, condition (3.7) in the proof requires to have N large, to compensate for the diverging constant.

Proof of Theorem 3.2. It is not difficult to see through Corollary 2.5 and the proof of Theorem 2.6 that there is a sequence $(\epsilon_n)_{n\geq 1}$ such that $\theta_{\epsilon_n} \to \theta$ strongly in $C([0,T]; L^p_{\text{loc}}(\mathbb{R}^2))$ for all $p \in [1, \infty)$, where θ_ϵ is the solution to (2.7) with initial condition θ_0.

Let us prove that

$$\sup_{t\in[0,T]} W_1(\theta_{\epsilon_n}(t), \theta(t)) \longrightarrow 0. \tag{3.6}$$

Indeed, it suffices to prove the same statement with the Wasserstein metric replaced by the L^1 metric. As in Corollary 2.7, we can prove that

$$c_0 := \sup_{n\geq 1} \sup_{t\in[0,T]} \int_{\mathbb{R}^2} |x|\, |\theta_{\epsilon_n}(t)|\, dx < \infty,$$

and this holds in the limit for θ. Therefore, if $R > 0$, for all $t \in [0,T]$,

$$\int_{B_R(0)^c} |\theta_{\epsilon_n}(t,x) - \theta(t,x)|\, dx \leq \frac{2c_0}{R},$$

and

$$\sup_{t\in[0,T]} \|\theta_{\epsilon_n}(t) - \theta(t)\|_{L^1} \le \sup_{t\in[0,T]} \|\mathbb{1}_{B_R(0)}(\theta_{\epsilon_n}(t) - \theta(t))\|_{L^1} + \frac{2c_0}{R}.$$

Claim (3.6) now follows by taking first the limit in $n \to \infty$ and then in $R \uparrow \infty$.

By Corollary 2.5,

$$\sup_{t\in[0,T]} W_1(\theta_{\epsilon_n}(t), \theta^N_{\epsilon_n}(t)) \le C(c_{\epsilon_n}, T) W_1(\theta^N_0, \theta_0).$$

Indeed, it is not difficult to check that θ^N_ϵ is a solution to (2.7).

Finally, choose $(N_n)_{n\ge1}$ so that

$$C(c_{\epsilon_n}, T) W_1(\theta^{N_n}_0, \theta_0) \longrightarrow 0, \tag{3.7}$$

as $n \to \infty$. Then

$$W_1(\theta^{N_n}_{\epsilon_n}(t), \theta(t)) \le W_1(\theta^{N_n}_{\epsilon_n}(t), \theta_{\epsilon_n}(t)) + W_1(\theta_{\epsilon_n}(t), \theta(t)).$$

and this proves the theorem. □

Remark 3.5. The assumption $\int_{\mathbb{R}^2} |x|\,|\theta_0(x)|\,dx$ seems a bit too strong. The same results holds without that assumption in the case $m = 2$, see [27]. A basic reason is that in this more singular case one does not expect to have well-defined characteristics. Indeed, by Lemma 2.2 we can expect a Hölder continuous velocity field. When $m = 2$ velocity is Lipschitz, up to a logarithmic correction, and this can be read as a contraction in Wasserstein distance. The same ideas do not work in this framework. Let us give a few details, and assume for simplicity of exposition that θ_0 is non-negative and of total mass one. Recall that the Wasserstein distance is an infimum over the transportation cost of the mass from one distribution to the other. Therefore it is sufficient to prove contraction with respect to a coupling. We first construct a suitable coupling of $\theta_\epsilon(t)$ and $\theta_\delta(t)$, for some $\epsilon \ge \delta > 0$, using the characteristics X^x_ϵ of (2.7), as in the proof of Corollary 2.5, namely

$$f \mapsto \int_{\mathbb{R}^2} f(X^x_\epsilon(t), X^x_\delta(t))\theta_0(x)\,dx$$

Set

$$\Psi(t) = \int_{\mathbb{R}^2} |X^x_\epsilon(t) - X^x_\delta(t)|_\bullet \theta_0(x)\,dx,$$

then using Lemma 2.2, eventually one gets,

$$\dot\Psi \le \epsilon^{m-1} + \Psi^{m-1}.$$

Since $m < 2$, the above differential inequality does not ensure that $\Psi \to 0$ as $\epsilon, \delta \to 0$, as it happens when $m \ge 2$ (with a logarithmic correction that does not change the result when $m = 2$).

3.3. **A derivation of the vortex model.** In this section we wish to prove conversely the connection between the vortex evolution (1.3) and the equation (1.1). Similar results for $m = 2$ can be found in [29], that we partially follow.

Fix $N \ge 1$, $\gamma_1, \gamma_2, \dots, \gamma_N \in \mathbb{R}$, and N points $x^0_1, x^0_2, \dots, x^0_N \in \mathbb{R}^2$. For every $\epsilon > 0$, consider a family of functions $\theta^\epsilon_{0,1}, \theta^\epsilon_{0,2}, \dots, \theta^\epsilon_{0,N}$ such that for all $j = 1, 2, \dots, N$,

- $\operatorname{Supp}\theta^\epsilon_{0,j} \subset B_\epsilon(x^0_j)$,
- $\theta^\epsilon_{0,j} \ge 0$ a. e.,
- [3] $|\theta^\epsilon_{0,j}| \lesssim \epsilon^{-2}$,

[3] More singularity may be allowed, namely a bound $\epsilon^{-2\eta}$ with $1 \le \eta < \frac{m(m-1)}{3-m}$.

- $\int_{\mathbb{R}^2} \theta^\epsilon_{0,j}(x)\,dx = 1$,
- $\sup_{\epsilon>0} \int_{\mathbb{R}^2} |x|^2 \theta^\epsilon_{0,j}(x)\,dx < \infty$.

A simple example of this setting is given by *vortex blobs*, namely we set $\theta^\epsilon_{0,j} = \epsilon^{-2}\eta_j((x-x_j^0)/\epsilon)$, where each η_j is non-negative, bounded, with support in $B_1(0)$, and with integral equal to 1 on $\mathbb{R}^2$.

Define

$$\theta_\epsilon(0,x) = \sum_{j=1}^N \gamma_j \theta^\epsilon_{0,j}(x), \tag{3.8}$$

where $\gamma_1,\dots,\gamma_N$ are the intensities of each vortex blob, $x_1^0,\dots,x_N^0$ are the centers, and ϵ is small enough that the balls $(B_\epsilon(x_j^0))_{j=1,\dots,N}$ are disjoint.

In the theorem below, we assume that the vortex evolution (1.3) with initial condition $(x_1^0, x_2^0, \dots, x_N^0)$ has a global solution. According to Theorem 3.1, this happens for a. e. choice of $(x_1^0, x_2^0, \dots, x_N^0)$ if the intensities are as in (3.3).

Theorem 3.6. *Assume $\sqrt{3} < m < 2$ and denote by θ_ϵ a solution to (1.1), according to Theorem 2.6, with initial condition $\theta_\epsilon(0)$ given by (3.8). Then for all $T > 0$,*

$$\lim_{\epsilon\to 0}\langle \theta_\epsilon(t), \phi\rangle = \sum_{j=1}^N \phi(X_i(t)), \qquad t \in [0,T],$$

where $(X_i)_{i=1,\dots,N}$ is the solution of the vortex evolution (1.3) with vortex intensities $\gamma_1, \gamma_2 \dots, \gamma_N$ and with initial conditions $(x_1^0, x_2^0, \dots, x_N^0)$.

The proof of this result follows broadly the proof of [29, Theorem 2.1]. It is based on a series of results that we prove in Section 3.3.1.

Remark 3.7. In principle an analogous result can be proved in the case of the evolution in bounded domains, with additional difficulties due to the boundary: there is non conservation of centre and moment of inertia, one should clarify in general the definition of fractional Laplacian in terms of the boundary conditions, the Green function one obtains is more singular also on the boundary, etc. In particular the presence of the boundary creates an effect of self-interaction on point vortices.

This does not happen on the torus, and there is no effect of self-interaction. We wish to discuss briefly and heuristically how the self-interaction term disappears in system (1.3) on the torus. First of all we notice that a structure theorem for the Green function G_m^{per} on the torus holds in terms of the Green function (1.4) on the whole space, namely $G_m^{per} = G_m + g_m^{per}$. By translation invariance, we have that $g_m^{per}(x,y) = g_m^{per}(x-y)$ and g_m^{per} is bounded.

Following [30], a heuristic motivation for the self-interaction term can be seen as follows. Consider a single vortex blob, as in (3.8), of intensity γ centred at $x_0 \in \mathbb{T}_2$, for instance $\theta_\epsilon(x) = \gamma\epsilon^{-2}\eta(x/\epsilon)$. Assume moreover that η is *radial*. By the decomposition discussed above,

$$u_\epsilon(x_0) = \int_{\mathbb{T}_2} \nabla_x^\perp G_m(x_0,y)\theta_\epsilon(y)\,dy + \int_{\mathbb{T}_2} \nabla^\perp g_m^{per}(x_0,y)\theta_\epsilon(y)\,dy$$

The first integral is zero by symmetry, and since $\theta_\epsilon \rightharpoonup \gamma\delta_0$, the second integral converges,

$$\int_D \nabla^\perp g_m^{per}(x_0,y)\theta_\epsilon(y)\,dy \longrightarrow \gamma\nabla^\perp g_m^{per}(x_0,x_0).$$

In conclusion $u_\epsilon(x_0) \to \gamma\nabla^\perp g_m^{per}(x_0, x_0)$, and $\gamma\nabla^\perp g_m^{per}(x_0, x_0)$ can be considered the velocity field generated by the vortex itself. By translation invariance this term is 0 and confirms the validity of the evolution (1.3). As a final remark, notice that this heuristic argument strongly depends on the symmetry of the vortex blob. If the blob shape is not symmetric, then the integrals above may diverge.

3.3.1. *Convergence of vortex blobs to point-vortices.* The following proposition is the version of [29, Theorem 3.1] in our setting, and proves Theorem 3.6 above for a single point vortex subject to an additional external velocity field. A major outcome of the proposition below is the property of localisation, namely the evolution of (1.1) started on a vortex blob stays around the centre of pseudo-vorticity.

First, we single out the initial condition from the setting of Theorem 3.6. Fix $x_0 \in \mathbb{R}^2$ and $T > 0$. Consider a family $(\theta_\epsilon(0))_{\epsilon>0}$ such that

- $\theta_\epsilon(0) : \mathbb{R}^2 \to \mathbb{R}$ non-negative functions,
- $\mathrm{Supp}(\theta_\epsilon(0)) \subset B_\epsilon(x_0)$,
- $|\theta_\epsilon(0, x)| \lesssim \epsilon^{-2}$, for all x,
- $\int_{\mathbb{R}^2} \theta_\epsilon(0, x)\, dx = 1$ and $\int_{\mathbb{R}^2} |x|^2\theta_\epsilon(0, x)\, dx < \infty$.

Moreover, consider a family $(F_\epsilon)_{\epsilon>0}$ such that

- $F_\epsilon : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ are continuous divergence-free vector fields,
- F_ϵ uniformly bounded (in t, x, ϵ),
- F_ϵ uniformly Lipschitz in the space variable, with common Lipschitz constant $M > 0$,
- there is $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that
$$\sup_{t\in[0,T],x\in\mathbb{R}^2} \|F_\epsilon - F\|_{L^\infty([0,T]\times\mathbb{R}^2)} \lesssim \epsilon,$$
- $\mathrm{Supp}(F_\epsilon(t))$ is contained in a ball centred at $c(t)$ and with radius of order $O(\epsilon)$, for every $t \in [0, T]$.

Here c is the solution to

$$\begin{cases} \dot{c} = F(t, c), \\ c(0) = x_0. \end{cases} \tag{3.9}$$

Proposition 3.8. *Let $\sqrt{3} < m < 2$ and consider x_0, T, $(\theta_\epsilon(0))_{\epsilon>0}$ and $(F_\epsilon)_{\epsilon>0}$, F as above. Denote by θ_ϵ a solution, according to Corollary 2.8 of*

$$\begin{cases} \partial_t\theta + \nabla\cdot((u + F_\epsilon)\theta) = 0, \\ u = k_m \star \theta, \end{cases}$$

with initial condition $\theta_\epsilon(0)$. Finally denote by c_ϵ the centre of pseudo-vorticity of $\theta_\epsilon(t)$ (see (2.2)). Then

1. *$c_\epsilon \to c$ uniformly in $t \in [0, T]$,*
2. *For every $\phi \in C_b^1(\mathbb{R}^2)$,*
$$\langle\theta_\epsilon(t), \phi\rangle \xrightarrow{\epsilon\to 0} \phi(c(t)), \qquad \textit{uniformly in } t \in [0, T].$$
3. *For every $R > 0$ there is $\epsilon_0 = \epsilon_0(R, T) > 0$ such that, if $\epsilon \le \epsilon_0$, then $\mathrm{Supp}\,\theta_\epsilon(t) \subset B_R(c_\epsilon(t))$ for $t \in [0, T]$.*

Proof. The proof follows the proof of [29, Theorem 3.1], with some non-trivial changes due to the more singular problem.

Step 1: The evolution of c_ϵ. As in Corollary 2.7,

$$\frac{dc_\epsilon}{dt} = \int_{\mathbb{R}^2} x\partial_t\theta_\epsilon\,dx = -\int_{\mathbb{R}^2} x\,\nabla\cdot((u_\epsilon + F_\epsilon)\theta_\epsilon)\,dx = \int_{\mathbb{R}^2}\theta_\epsilon(u_\epsilon + F_\epsilon)\,dx = \int_{\mathbb{R}^2}\theta_\epsilon F_\epsilon\,dx, \tag{3.10}$$

since $u_\epsilon = k_m \star \theta_\epsilon$ and k_m is anti-symmetric, so that

$$\int_{\mathbb{R}^2}\theta_\epsilon(t,x)u_\epsilon(t,x)\,dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_m(x-y)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dx\,dy = 0.$$

Moreover, by an elementary computation $|c_\epsilon(0) - x_0| \lesssim \epsilon$.

Step 2: The evolution of the moment of inertia. Set

$$J_\epsilon(t) := \int_{\mathbb{R}^2}|x - c_\epsilon(t)|^2\theta_\epsilon(t,x)\,dx = \int_{\mathbb{R}^2}|x|^2\theta_\epsilon(t,x)\,dx - c_\epsilon(t)^2.$$

By our assumptions, $J_\epsilon(0) \lesssim \epsilon^2$. Then, as in Corollary 2.7, by (3.10),

$$\begin{aligned}
\frac{dJ_\epsilon(t)}{dt} &= \int_{\mathbb{R}^2}|x|^2\partial_t\theta_\epsilon\,dx - 2c_\epsilon\dot{c}_\epsilon\\
&= -\int_{\mathbb{R}^2}|x|^2\,\nabla\cdot(u_\epsilon + F_\epsilon)\theta_\epsilon)\,dx - 2c_\epsilon\int_{\mathbb{R}^2}\theta_\epsilon F_\epsilon\,dx\\
&= 2\int_{\mathbb{R}^2}(x - c_\epsilon)\cdot F_\epsilon\theta_\epsilon\,dx + 2\int_{\mathbb{R}^2} x\cdot u_\epsilon\theta_\epsilon\,dx\\
&= 2\int_{\mathbb{R}^2}(x - c_\epsilon)\cdot F_\epsilon\theta_\epsilon\,dx.
\end{aligned}$$

Here $\int_{\mathbb{R}^2} x\cdot u_\epsilon\theta_\epsilon\,dx$ is zero since k_m is anti-symmetric, $x\cdot k_m(x) = 0$ and, by symmetrisation,

$$\int_{\mathbb{R}^2}\theta_\epsilon(t,x)u_\epsilon(t,x)\cdot x\,dx = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(x-y)\cdot k_m(x-y)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dx\,dy.$$

By the definition of centre of pseudo-vorticity,

$$\begin{aligned}
\frac{dJ_\epsilon(t)}{dt} &= 2\int_{\mathbb{R}^2}\theta_\epsilon(t,x)(x - c_\epsilon(t))\cdot(F_\epsilon(t,x) - F_\epsilon(t,c_\epsilon(t)))\,dx\\
&\leq 2M\int_{\mathbb{R}^2}|x - c_\epsilon(t)|^2\theta_\epsilon(t,x)\,dx = 2MJ_\epsilon(t),
\end{aligned}$$

therefore,

$$\sup_{t\in[0,T]} J_\epsilon(t) \lesssim \epsilon^2. \tag{3.11}$$

Step 3: Convergence of the centre of pseudo-vorticity. Consider the solution c of (3.9) and recall (3.10). Using the conservation of θ_ϵ,

$$\begin{aligned}
c_\epsilon(t) - c(t) = c_\epsilon(0) - c(0) &+\\
&+ \int_0^t\Big(F_\epsilon(s,c_\epsilon(s)) - F_\epsilon(s,c(s))\Big)\,ds + \int_0^t\Big(F_\epsilon(s,c(s)) - F(s,c(s))\Big)\,ds +\\
&+ \int_0^t\int_{\mathbb{R}^2}(F_\epsilon(s,x) - F_\epsilon(s,c_\epsilon(s)))\theta_\epsilon(s,x)\,dx\,ds,
\end{aligned}$$

therefore, by (3.11) and the assumptions on F_ϵ,

$$\begin{aligned}|c_\epsilon(t)-c(t)| &\lesssim \epsilon + M\int_0^t |c_\epsilon(s)-c(s)|\,ds + M\int_0^t\int_{\mathbb{R}^2}|x-c_\epsilon(s)|\theta_\epsilon(s,x)\,dx\,ds\\ &\le \epsilon + M\int_0^t |c_\epsilon(s)-c(s)|\,ds + M\int_0^t J_\epsilon(s)^{1/2}\,ds\\ &\lesssim \epsilon + M\int_0^t |c_\epsilon(s)-c(s)|\,ds.\end{aligned}$$

By Gronwall's Lemma,

$$\sup_{[0,T]}|c(t)-c_\epsilon(t)|\lesssim \epsilon.$$

Step 4: Convergence of the pseudo-vorticity. Let ϕ be a test function. From (3.11),

$$\begin{aligned}\int_{\mathbb{R}^2}\phi(x)\theta_\epsilon(t,x)\,dx - \phi(c_\epsilon(t)) &= \int_{\mathbb{R}^2}\theta_\epsilon(t,x)(\phi(x)-\phi(c_\epsilon(t)))\,dx\\ &\le \|\nabla\phi\|_\infty\int_{\mathbb{R}^2}\theta_\epsilon(t,x)|x-c_\epsilon(t)|\,dx \le \|\nabla\phi\|_\infty J_\epsilon(s)^{1/2}\lesssim\epsilon.\end{aligned}$$

Since by Step 3, $c_\epsilon\longrightarrow c$ uniformly, it follows that

$$\sup_{[0,T]}\Big|\int_{\mathbb{R}^2}\phi(x)\theta_\epsilon(t,x)\,dx - \phi(c(t))\Big|\lesssim\epsilon.$$

Step 5: Control of the support of the pseudo-vorticity. We use the idea of [29] to prove that the amount of pseudo-vorticity crossing the boundary of a small ball around c_ϵ is small. This allows to prove that the radial part of the velocity is also small and pseudo-vorticity cannot spread out away from c_ϵ.

Let $\delta\in(0,1)$ and consider a radial function $\phi_\delta\in C^\infty$ such that $0\le\phi_\delta\le 1$, and

$$\phi_\delta=\begin{cases}1, & |x|\le\delta,\\ 0, & |x|\ge 2\delta,\end{cases}\qquad |\nabla\phi_\delta|\lesssim\frac{1}{\delta},\qquad |D^2\phi_\delta|\lesssim\frac{1}{\delta^2}.$$

Let

$$\mu_\delta = 1-\int_{\mathbb{R}^2}\phi_\delta(c_\epsilon(t)-x)\theta_\epsilon(t,x)\,dx.$$

We have

$$\frac{d}{dt}\mu_\delta = \int_{\mathbb{R}^2}\nabla\phi_\delta(c_\epsilon(t)-x)\cdot(u_\epsilon+F_\epsilon)\theta_\epsilon\,dx - \int_{\mathbb{R}^2}\dot c_\epsilon(t)\cdot\nabla\phi_\delta(c_\epsilon(t)-x)\theta_\epsilon\,dx =: \boxed{1}+\boxed{2}.$$

By conservation of mass and definition of u_ϵ,

$$\begin{aligned}\boxed{1} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla\phi_\delta(c_\epsilon(t)-x)\cdot k_m(x-y)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dx\,dy\\ &\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla\phi_\delta(c_\epsilon(t)-x)\cdot F_\epsilon(t,x)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dx\,dy,\end{aligned}$$

and by (3.10),

$$\boxed{2} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla\phi_\delta(c_\epsilon(t)-x)\cdot F_\epsilon(t,y)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx,$$

so that,

$$\begin{aligned}\frac{d}{dt}\mu_\delta &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\phi_\delta(c_\epsilon(t)-x)\cdot(F_\epsilon(t,x)-F_\epsilon(t,y))\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx\\ &\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\phi_\delta(c_\epsilon(t)-x)\cdot k_m(x-y)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx =: \boxed{a}+\boxed{b}.\end{aligned}$$

To estimate $\boxed{a}$ and $\boxed{b}$ we set some positions and give some useful inequalities. Let

$$m_\delta(t) = \int_{B_\delta(c_\epsilon(t))^c} \theta_\epsilon(t,x)\,dx \tag{3.12}$$

be the amount of pseudo-vorticity outside the ball of radius δ centred at $c_\epsilon(t)$, and notice that

$$m_\delta(t) \le \frac{1}{\delta^2}\int_{B_\delta(c_\epsilon(t))^c} |c_\epsilon(t)-x|^2\theta_\epsilon(t,x)\,dx = \frac{1}{\delta^2}J_\epsilon(t) \lesssim \frac{\epsilon^2}{\delta^2}. \tag{3.13}$$

Let us start with the estimate of $\boxed{a}$ and split the integral in y into an integral over $B_\delta(c_\epsilon(t))^c$ and an integral over $B_\delta(c_\epsilon(t))$,

$$\boxed{a}|_{B_\delta(c_\epsilon(t))^c} \le 2\|F_\epsilon\|_\infty \int\int_{B_\delta(c_\epsilon(t))^c} |\nabla\phi_\delta(c_\epsilon(t)-x)|\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx \lesssim \frac{1}{\delta}m_\delta(t)^2,$$

since $\nabla\phi_\delta(c_\epsilon(t)-x)=0$ if $|c_\epsilon(t)-x|\le\delta$, and

$$\boxed{a}|_{B_\delta(c_\epsilon(t))} \le M\int_{B_{2\delta}(c_\epsilon(t))}\int_{B_\delta(c_\epsilon(t))} \frac{|x-y|}{\delta}\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx \lesssim m_\delta(t),$$

since $\nabla\phi_\delta(c_\epsilon(t)-x)\neq 0$ only if $\delta<|c_\epsilon(t)-x|<2\delta$. In conclusion

$$\boxed{a} \lesssim m_\delta(t) + \frac{1}{\delta}m_\delta(t)^2.$$

We turn to the analysis of $\boxed{b}$. Since the integrand is zero when $x\in B_\delta(c_\epsilon(t))$, it is sufficient to consider the integral only over $E_1\cup E_2$, where

$$\begin{aligned}E_1 &= B_\delta(c_\epsilon(t))^c\times B_{\delta'}(c_\epsilon(t)),\\ E_2 &= B_\delta(c_\epsilon(t))^c\times B_{\delta'}(c_\epsilon(t))^c \cup B_{\delta'}(c_\epsilon(t))^c\times B_\delta(c_\epsilon(t))^c,\end{aligned}$$

and $\delta'=\delta^5$. Denote for brevity by $\boxed{b.E_1}$ and $\boxed{b.E_2}$ the parts of the integral of $\boxed{b}$ over E_1 and E_2. Since ϕ_δ is radial, we have that $\nabla\phi_\delta(x)\cdot k_m(x)=0$, therefore,

$$\begin{aligned}\boxed{b.E_1} &= \iint_{E_1} (k_m(x-y)-k_m(x-c_\epsilon(t)))\cdot\nabla\phi_\delta(c_\epsilon(t)-x)\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx\\ &\lesssim \frac{1}{\delta}\iint_{E_1}\frac{|y-c_\epsilon(t)|}{\delta^{4-m}}\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx\\ &\lesssim \delta^m m_\delta(t).\end{aligned}$$

Here we have used conservation of mass and

$$|k_m(x)-k_m(y)| \lesssim \frac{|x-y|}{\delta^{4-m}}, \qquad |x|,|y|\gtrsim\delta. \tag{3.14}$$

We turn to $\boxed{b.E_2}$. The domain E_2 is symmetric in x, y, therefore by symmetrisation, and using that k_m is anti-symmetric,

$$\begin{aligned}\boxed{b.E2} &= \frac{1}{2}\iint_{E_2} k_m(x-y)\cdot(\nabla\phi_\delta(c_\epsilon(t)-x)-\nabla\phi_\delta(c_\epsilon(t)-y))\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx\\ &\lesssim \frac{1}{\delta^2}\iint_{E_2}|x-y|^{m-2}\theta_\epsilon(t,x)\theta_\epsilon(t,y)\,dy\,dx\\ &= \frac{2}{\delta^2}\int_{B_\delta(c_\epsilon(t)^c)}\theta_\epsilon(t,x)\Big(\int_{B_{\delta'}(c_\epsilon(t)^c)}|x-y|^{m-2}\theta_\epsilon(t,y)\,dy\Big)\,dx.\end{aligned}$$

By definition (3.12) and since $\theta_\epsilon(t,y)\leq\|\theta_\epsilon(0)\|_\infty\sim\epsilon^{-2}$, by the Hardy-Littlewood-Sobolev inequality,

$$\int_{B_{\delta'}(c_\epsilon(t)^c)}|x-y|^{m-2}\theta_\epsilon(t,y)\,dy\lesssim\epsilon^{-2(1-m/2)}m_{\delta'}(t)^{m/2},$$

therefore

$$\boxed{b.E_2}\lesssim\frac{\epsilon^{m-2}}{\delta^2}m_{\delta'}(t)^{m/2}m_\delta(t).$$

In conclusion

$$\boxed{b}\lesssim\delta^m m_\delta(t)+\frac{\epsilon^{m-2}}{\delta^2}m_{\delta'}(t)^{m/2}m_\delta(t),$$

and, using (3.13),

$$\begin{aligned}\frac{d}{dt}\mu_\delta(t)&\lesssim m_\delta(t)+\frac{1}{\delta}m_\delta(t)^2+\delta^m m_\delta(t)+\frac{\epsilon^{m-2}}{\delta^2}m_{\delta'}(t)^{m/2}m_\delta(t)\\ &\lesssim m_\delta(t)+\frac{\epsilon^4}{\delta^5}+\frac{\epsilon^{2m}}{\delta^{4+5m}}\lesssim m_\delta(t)+\frac{\epsilon^{2m}}{\delta^{14}}.\end{aligned}$$

Finally, since $\phi_{\delta/2}\leq\mathbb{1}_{B_\delta(0)}$ hence $m_\delta(t)\leq\mu_{\delta/2}(t)$, we have that for $t\in[0,T]$,

$$\mu_\delta(t)\leq\mu_\delta(0)+cT\frac{\epsilon^{2m}}{\delta^{14}}+c\int_0^t\mu_{\delta/2}(s)\,ds, \tag{3.15}$$

where c is a number that does not depend on the parameters ϵ,δ,T. By the assumptions on the initial condition, $\mu_\delta(0)=0$ as long as $\delta\geq\epsilon$.

Fix now ϵ,δ and choose $k\sim\log\epsilon^{-1}$ so that $\delta\geq\epsilon 2^k$. We can iterate inequality (3.15) for $2^{-k}\delta$, $2^{-k+1}\delta$, …, $2^{-1}\delta$, δ to obtain (recall that $\mu_\delta\leq 1$)

$$\mu_\delta(t)\leq\frac{(cT)^k}{k!}+T\sum_{j=0}^{k-1}\frac{(cT)^j}{j!}\frac{\epsilon^{2m}}{(2^{-j}\delta)^{14}}\leq\frac{(cT)^k}{k!}+T\frac{\epsilon^{2m}}{\delta^{14}}\,\mathrm{e}^{2^{14}cT}.$$

If we choose $\delta=\epsilon^a$ with a sufficiently small and $k/(log\epsilon^{-1})$ sufficiently small, we can deduce that

$$\mu_\delta(t)\leq c\epsilon^{2m-14a},$$

with $c=c(T,a)$.

Step 6: control of the velocity and conclusion. Let us compute the velocity outside the disc $D_2=B_{\epsilon^{a/4}}(c_\epsilon(t))$ centred at $c_\epsilon(t)$ and with radius $\epsilon^{a/4}$. To this end let $D_1=B_{\epsilon^a}(c_\epsilon(t))$. Fix $x\in\overline{D_2}^c$ and let $\vec{n}$ be the unit vector in the direction $c_\epsilon(t)-x$. The velocity at x can be decomposed in three components: the first, $u_{\epsilon 1}$, corresponds to the contribution of the pseudo-vorticity in D_1, the second, $u_{\epsilon 2}$, corresponds to the contribution of the vorticity in D_1^c, and the third, $u_{\epsilon 3}$, due to the external field. Since $\vec{n}\cdot k_m(x-c_\epsilon(t))=0$, by (3.14),

$$|u_{\epsilon 1}(t,x)\cdot\vec{n}| = \Big|\int_{D_1}\vec{n}\cdot(k_m(x-y)-k_m(x-c_\epsilon(t)))\theta_\epsilon(t,y)\,dy\Big| \lesssim \\ \lesssim \frac{\epsilon^a}{\epsilon^{\frac14 a(4-m)}}\int_{D_1}\theta_\epsilon(t,y)\,dy \leq \epsilon^{\frac14 am}.$$

Moreover, by the Hardy-Littlewood-Sobolev inequality, since $\theta_\epsilon(t,y)\lesssim\epsilon^{-2}$ and

$$\int_{D_1^c}\theta_\epsilon(t,y)\,dy = m_{\epsilon^a}(t)\lesssim\epsilon^{2m-14a},$$

we have that,

$$|u_{\epsilon 2}(t,x)\cdot\vec{n}| \leq \int_{D_1^c}\frac{1}{|x-y|^{3-m}}\theta_\epsilon(t,y)\,dy \\ \lesssim (\epsilon^{-2})^{\frac12(3-m)}(\epsilon^{2m-14a})^{\frac12(m-1)} = \epsilon^{m^2-3-7a(m-1)}.$$

We notice that the condition $m>\sqrt{3}$ is only necessary here to ensure that $u_{\epsilon 2}(t,x)$ is small. Finally, the velocity due to the external field is small by assumption.

In conclusion the velocity outside D_2 is arbitrarily small, so in a finite time T particles cannot go too far away from $c_\epsilon(t)$ and thus are contained in a ball around $c_\epsilon(t)$ with radius independent on $\epsilon\leq\epsilon_0$ (but dependent on T). □

3.4. **Proof of Theorem 3.6.** Given $N\geq 2$ and $T>0$, fix intensities $\gamma_1,\gamma_2,\dots,\gamma_N$ and initial vortex positions $x_1,x_2,\dots,x_N$, so that the vortex motion (1.3) has a solution in $[0,T]$ without collisions. Therefore the number

$$D=\min_{t\in[0,T],i\neq j}|X_i(t)-X_j(t)|,$$

is positive. Consider $\epsilon\ll D$, so that the initial blobs $(\theta_{0,j}^\epsilon)_{j=1,2,\dots,N}$ are disjoint (and separated by a distance comparable with D). We first consider the regularized equation (2.7) with regularisation size δ also much smaller than D, so that the regularisations of the initial blobs are still separated by a distance comparable with D. We denote by $\tilde\theta_\epsilon^\delta$ the solution to the regularized problem (here ϵ refers to the parameter in the initial condition and δ to the regularisation size).

It is not difficult to see that the supports of the regularized blobs remain disjoint (but not necessarily localized) using the diffeomorphisms (2.11) introduced in the proof of Proposition 2.4. Therefore we can single out the evolutions $(\tilde\theta_{\epsilon,j}^\delta)_{j=1,2,\dots,N}$ such that $\tilde\theta_{\epsilon,j}^\delta(0)$ is the regularisation of $\theta_{0,j}^\epsilon$, to obtain for $j=1,2,\dots,N$,

$$\begin{cases}\partial_t\tilde\theta_{\epsilon,j}^\delta+\nabla\cdot((\tilde u_{\epsilon,j}^\delta+\tilde F_{\epsilon,j}^\delta)\tilde\theta_{\epsilon,j}^\delta)=0,\\ \tilde u_{\epsilon,j}^\delta=k_m^\delta\star\tilde\theta_{\epsilon,j}^\delta,\\ \tilde F_{\epsilon,j}^\delta=\sum_{\ell\neq j}\tilde u_{\epsilon,\ell}^\delta.\end{cases}$$

Proposition 3.8 obviously holds also for the regularized system above and shows that the blobs stay localized. In particular, if for each j we localize, at each time, the velocity field $\tilde F_{\epsilon,j}^\delta$ generated by all other blobs around the support of $\tilde\theta_{\epsilon,j}^\delta$, again with a size much smaller than D, the evolution of each blob is unchanged.

In the limit as $\delta \to 0$, we obtain that the evolution of vortex blobs can be singled out to

$$\begin{cases} \partial_t \theta_{\epsilon,j} + \nabla\cdot((u_{\epsilon,j} + F_{\epsilon,j})\theta_{\epsilon,j}) = 0, \\ u_{\epsilon,j} = k_m \star \theta_{\epsilon,j}, \\ F_{\epsilon,j} = \sum_{\ell \neq j} u_{\epsilon,\ell}. \end{cases}$$

for each $j = 1, 2, \ldots, N$. We can finally apply Proposition 3.8 to each blob, and this finally concludes the proof of the theorem.

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