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Envelopes of spacecraft trajectories with a single impulse

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Abstract This paper analyzes reachable domains of spacecraft using a single impulsive maneuver. In particular, expressions are obtained of the envelopes of spacecraft trajectories in closed form, in both cases of either radial or tangential impulse. Suitable bounds are enforced on the magnitude of the velocity variation in order to obtain an elliptic transfer trajectory after the maneuver with a pericenter radius greater than the primary body's radius. Three different cases are investigated: 1) the impulse point is fixed and the velocity variation may be varied; 2) a

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fixed impulse magnitude and free maneuver point; 3) both free impulse point and magnitude. Finally, some mission scenarios are analyzed to show the effectiveness of the proposed method.

Keywords Impulsive maneuver · Reachable domain · Trajectory envelopes · Radial impulse · Tangential impulse

Nomenclature

A, B	=	auxiliary variables, see Eq. (12)
e	=	eccentricity
h	=	angular momentum, $[\mathrm{km^2/s}]$
O	=	primary's center of mass
p	=	semilatus rectum, $[\mathrm{km}]$
r	=	primary-spacecraft distance, $[\mathrm{km}]$
v	=	spacecraft orbital velocity, $[\mathrm{km/s}]$
Δu	=	radial velocity variation, $[\mathrm{km/s}]$
Δv	=	tangential velocity variation, $[\mathrm{km/s}]$
α	=	auxiliary variable, see Eq. (13)
θ	=	polar angle, $[\mathrm{rad}]$
μ	=	primary's gravitational parameter, $[\mathrm{km^3/s^2}]$
ν_m	=	true anomaly of maneuver point, $[\mathrm{rad}]$
ξ	=	auxiliary variable, see Eq. (6)

Subscripts

0	=	value just before the maneuver
1	=	value just after the maneuver
max	=	maximum allowable value
min	=	minimum allowable value

1. Introduction

The transfer problem of a spacecraft using conventional chemical thrusters is usually addressed assuming impulsive maneuvers, which consists of instantaneous velocity variations applied to the spacecraft [1, 2, 3, 4]. The spacecraft may be inserted into a desired trajectory by suitably varying its magnitude, direction and instant of application of the impulse. The set of all the possible trajectories (and therefore, of all the points in the space) that the spacecraft can reach is referred to as reachable domain [5, 6]. This concept has been thoroughly investigated in the literature, with the first studies dating back to 1960 with the paper by Beckner [7], who investigated envelopes of the accessible regions of ballistic weapons.

The computation of the reachable domain may be a powerful tool for the rapid analysis of collision possibility [6, 8, 9] or to verify the existence of a solution to a two-point boundary value problem [10]. Xue et al. [8] determined the reachable domain with a single impulse of fixed magnitude, considering both the cases of fixed or arbitrary maneuver direction. Wen and Gurfil [11] extended the concept of reachable domain to the study of spacecraft relative motion. Zhang et al. [12] developed a method to obtain the envelopes of spacecraft trajectories (which define the boundary of the reachable domain) when a single impulsive maneuver is applied tangent to the initial elliptic orbit. The authors considered different possible cases in which the impulse point and magnitude are either fixed or free variables.

This paper proposes a new approach to solve the problem discussed in Ref. [12], by exploiting the fact that the spacecraft trajectory after the impulsive maneuver is described by the mathematical model proposed in Ref. [13]. This approach is shown to provide a simpler and compact solution to the problem. Moreover, the case of single radial impulse is also investigated, and different mission scenarios are analyzed. In particular, the true anomaly along the initial orbit in correspondence of which the impulse is applied and the magnitude of the velocity variation, are either constrained or left free to vary. In this study, the velocity variation is assumed to be chosen such that the transfer trajectory remains elliptical and does not intersect the primary body.

This paper is organized as follows. The next section shows the method used to compute the envelopes of spacecraft trajectories, first for the case of radial impulse and then for a tangential impulse. The cases of fixed or variable maneuver point and impulse magnitude are discussed. Section 3 shows some mission scenarios, and, finally, the last section draws the conclusion of this study.

2. Envelopes calculation

Consider a spacecraft that initially covers a Keplerian elliptic orbit of semilatus rectum p_0 and eccentricity e_0 around a primary body of gravitational parameter μ. Introduce a two-dimensional polar reference frame $\mathcal{T}(O; r, \theta)$, of which the origin coincides with the primary center of mass O, r is the primary-spacecraft distance, and θ is the polar angle measured counterclockwise from the pericenter of the initial orbit.

The spacecraft can perform a single impulsive maneuver either along the radial or along the tangential direction, so that its orbital plane is not affected by the maneuver. Just after the application of the impulse, the vehicle is inserted into a new Keplerian orbit, whose parameters depend on the magnitude of the impulsive velocity variation and on the position along the initial orbit at which the impulse is applied. The set of all spacecraft trajectories that can be generated as a result of the application of the impulsive maneuver defines a space region, which is called the reachable domain. When suitable bounds are assigned to the value of the velocity variation, the reachable domain is a closed set, of which the boundaries can be computed in analytical form as is discussed in the following sections.

2.1. Single radial impulse

Consider first the case in which the spacecraft can perform a single impulsive maneuver along the radial direction. As stated before, the family of spacecraft trajectories can be parameterized as a function of the impulsive velocity variation Δu and of the true anomaly $\nu_m \in [0, 2\pi)$ rad along the initial orbit [13], viz.

$$r(\theta, \Delta u, \nu_m) = \frac{p_0}{1 + e_0 \cos\theta - \sqrt{p_0/\mu}\,\Delta u\, \sin(\theta - \nu_m)} \tag{1}$$

Note that Δu is positive when the impulse is applied along the radial outward direction, it is instead negative when applied along the radial inward direction.

Consider now the case in which the maneuver point (and therefore ν_m) is fixed, whereas Δu is free to vary within a given interval $[\Delta u_{\min}, \Delta u_{\max}]$. The latter is chosen such that the trajectory after the maneuver remains elliptical and its pericenter radius is higher than the primary body's radius. The expression of the envelopes can be obtained by either maximizing or minimizing Eq. (1) with respect to Δu [8]. This is equivalent to minimizing or maximizing the denominator of Eq. (1), which is a linear function of Δu. Therefore, in this case, the envelopes coincide with the spacecraft trajectories such that $\Delta u = \Delta u_{\min}$ or $\Delta u = \Delta u_{\max}$. Note that, as long as $\sin(\theta - \nu_m) > 0$, the outer envelope is given by Eq. (1) with $\varDelta u = \varDelta u_{\max}$, whereas when $\sin(\theta - \nu_m) < 0$ the outer envelope is obtained with $\varDelta u = \varDelta u_{\min}$. The opposite situation occurs for the inner envelope.

On the other hand, if the value of $\varDelta u$ is fixed and the maneuver point can be freely changed, the envelopes can be obtained by imposing that

$$\frac{\partial}{\partial \nu_m}\left[1 + e_0 \cos\theta - \sqrt{p_0/\mu}\,\varDelta u\, \sin(\theta - \nu_m)\right] = \sqrt{\frac{p_0}{\mu}}\varDelta u \cos(\theta - \nu_m) = 0 \tag{2}$$

of which the solution is

$$\theta - \nu_m = \frac{\pi}{2} + k\pi \quad \text{with} \quad k \in \mathbb{Z} \tag{3}$$

By substituting Eq. (3) into Eq. (1), the expressions of the envelopes are obtained as

$$r(\theta, \varDelta u) = \frac{p_0}{1 + e_0 \cos\theta \pm \sqrt{p_0/\mu}\,\varDelta u} \tag{4}$$

Note that, if the direction of the impulsive maneuver is inverted (that is, the sign of $\varDelta u$ is changed), the envelopes remain the same; see Eq. (4). Finally, if both ν_m and $\varDelta u$ are free variables, the envelopes are given by Eq. (4) where $\varDelta u = \max\{|\varDelta u_{\min}|, |\varDelta u_{\max}|\}$.

2.2. Single tangential impulse

In the case of tangential impulsive maneuver, the family of curves describing the trajectories after the application of the tangential velocity change $\varDelta v$ can be expressed as [13]

$$r(\theta, \varDelta v, \nu_m) = \frac{p_0}{1 + e_0 \cos\theta + (1/\xi - 1)\left[1 - \cos(\theta - \nu_m)\right]} \tag{5}$$

with

$$\xi = \left(\frac{h_1}{h_0}\right)^2 \tag{6}$$

where h_1 (or h_0) is the spacecraft specific angular momentum just after (or just before) the application of the impulsive maneuver. Since the impulsive maneuver is tangential to the initial orbit, Eq. (6) can also be rewritten as

$$\xi = \left(1 + \frac{\varDelta v}{v_0}\right)^2 \tag{7}$$

where v_0 represents the spacecraft velocity just before the application of the maneuver, and can be expressed as a function of the true anomaly ν_m by exploiting the energy equation and the polar equation of the initial orbit, viz.

$$v_0 = \sqrt{\frac{\mu}{p_0}}\sqrt{1 + 2e_0 \cos\nu_m + e_0^2} \tag{8}$$

Note that, when the impulsive velocity variation is applied opposite to the spacecraft velocity vector, Δv is assumed negative.

Consider now the case in which the value of ν_m is fixed, whereas Δv may vary within a given interval $[\Delta v_{\min}, \Delta v_{\max}]$. As in the radial impulse case, the magnitude of Δv is constrained to assume values such that the trajectory after the maneuver is still elliptical and its pericenter radius is greater than the primary body's radius. The envelopes of the spacecraft trajectories can be obtained by looking for the stationary points of Eq. (5) with respect to Δv [8]. In particular, Eq. (5) is maximized (or minimized) when its denominator is minimized (or maximized), viz.

$$\frac{\partial}{\partial \Delta v}\left[1 + e_0 \cos\theta + (1/\xi - 1)\left[1 - \cos(\theta - \nu_m)\right]\right] = \frac{2v_0^2[\cos(\theta - \nu_m) - 1]}{(v_0 + \Delta v)^3} = 0 \tag{9}$$

of which the solution is when $\Delta v \to \pm\infty$. However, because the velocity variation cannot assume an infinite value, the envelopes are obtained when Δv is equal to its extremal values $\Delta v_{\min}$ or $\Delta v_{\max}$.

When the case of fixed Δv and unconstrained ν_m is investigated, the envelopes can be computed from

$$\frac{\partial}{\partial \nu_m}\{1 + e_0 \cos\theta + (1/\xi - 1)\left[1 - \cos(\theta - \nu_m)\right]\} = 0 \tag{10}$$

By computing the derivative in Eq. (10), the following equation is obtained

$$B - B\cos(\theta - \nu_m) + A\sin(\theta - \nu_m) = 0 \tag{11}$$

where

$$A = -(1/\xi - 1), \quad B = -\frac{2\mu e_0 \Delta v}{p_0 (v_0 + \Delta v)^3}\sin\nu_m \tag{12}$$

Divide Eq. (11) by $A^2 + B^2$, and introduce the auxiliary variable α, defined such that

$$\sin\alpha = -\frac{B}{A^2 + B^2}, \quad \cos\alpha = \frac{A}{A^2 + B^2} \tag{13}$$

Accordingly, Eq. (11) can be rewritten as

$$-\sin\alpha + \sin\alpha\cos(\theta - \nu_m) + \cos\alpha\sin(\theta - \nu_m) = 0 \tag{14}$$

which, using well known trigonometric identities, can be also expressed as

$$\sin(\alpha + \theta - \nu_m) = \sin\alpha \tag{15}$$

Equation (15) has two solutions, or

$$\theta - \nu_m = 0 \tag{16}$$

$$\theta - \nu_m = \pi - 2\alpha \tag{17}$$

Substituting Eq. (16) into Eq. (5), it can be verified that one of the envelopes coincides with the initial orbit. Instead, Eq. (17) gives the value of θ on the envelope as a function of the true anomaly ν_m at which the impulse is performed, that is, $\theta = \theta(\nu_m)$. Therefore, the expression for the second envelope can be given as a function of ν_m by substituting Eq. (17) into Eq. (5), viz.

$$r(\nu_m, \Delta v) = \frac{p_0}{1 - e_0 \cos(\nu_m - 2\alpha) + (1/\xi - 1)\,[1 + \cos(2\alpha)]} \tag{18}$$

For each value of $\nu_m \in [0,\, 2\pi)$ rad a point on the envelope is obtained, given by the pair $(\theta,\, r)$, through Eqs. (17)-(18).

Bearing in mind the definition of ξ given in Eq. (7), when both Δv and ν_m are free variables, three different cases may occur:

1. if both $\Delta v_{\min} > 0$ and $\Delta v_{\max} > 0$, the inner envelope is given by the initial orbit, whereas the outer envelope is given by Eq. (18) where $\Delta v = \Delta v_{\max}$;
2. if both $\Delta v_{\min} < 0$ and $\Delta v_{\max} < 0$, the outer envelope coincides with the initial orbit, and the inner envelope is given by Eq. (18) where $\Delta v = \Delta v_{\min}$;
3. if $\Delta v_{\min} < 0$ and $\Delta v_{\max} > 0$, both envelopes are given by Eq. (18): the inner envelope is obtained when $\Delta v = \Delta v_{\min}$, and the outer envelope is obtained when $\Delta v = \Delta v_{\max}$.

Some mission scenarios are analyzed in the following section, which shows the effectiveness of the proposed mathematical model.

3. Mission applications

In this section some mission scenarios are considered, and the method described in the previous section is used to compute the envelopes of spacecraft trajectories in the cases of radial or tangential impulse. The spacecraft is assumed to cover an initial orbit around the Earth ($\mu = 398600\,\mathrm{km^3/s^2}$) with $p_0 = 2\,\mathrm{DU}$ (where 1 DU is equal to the Earth's mean equatorial radius, i.e. $1\,\mathrm{DU} \simeq 6378\,\mathrm{km}$) and $e_0 = 0.3$. Recall that the range of acceptable values for the velocity variation are chosen such that the spacecraft trajectory after the impulse remains elliptic, and does not intersect the Earth's surface (i.e., its pericenter radius is larger than the Earth's radius).

3.1. Radial impulse case

Consider first the case of radial maneuver. Assume the impulse to be performed when $\nu_m = \pi/3$ rad, whereas $\Delta u \in [0.5,\ 2.5]$ km/s. In this case, the envelopes coincide with the trajectories obtained when $\Delta u = \Delta u_{\min} = 0.5$ km/s and $\Delta u = \Delta u_{\max} = 2.5$ km/s, and are represented in Fig. 1. Instead, when the magnitude of the velocity variation is

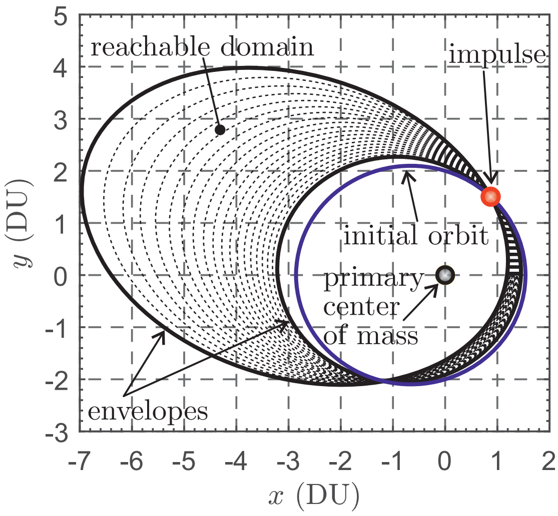

Fig. 1 Envelopes and reachable domain in the case of radial impulse when $\nu_m = \pi/3$ rad and $\Delta u \in [0.5,\ 2.5]$ km/s.

fixed and equal to $\Delta u = 1$ km/s, and ν_m is left unconstrained, the expression of the envelopes are given by Eq. (4), and the obtained results are shown in Fig. 2. Finally, when both ν_m and $\Delta u \in [-2.5, 2.5]$ km/s are free to vary, the

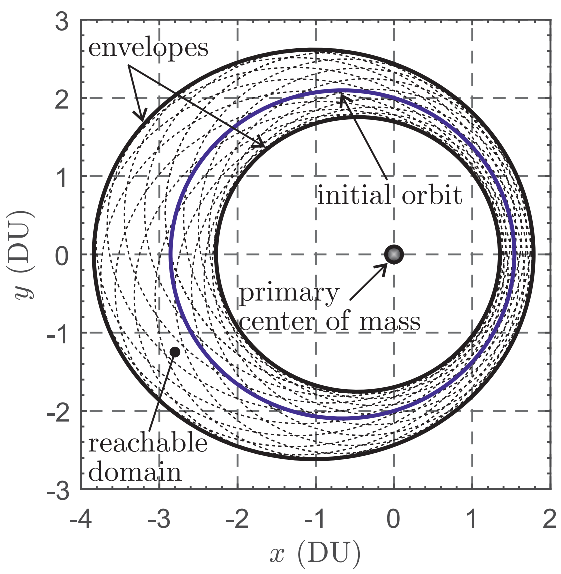

Fig. 2 Envelopes and reachable domain in the case of radial impulse with free ν_m and $\Delta u = 1$ km/s.

innermost and the outermost envelopes are obtained when $\Delta u = \max\{|\Delta u_{\min}|, |\Delta u_{\max}|\} = 2.5\,\text{km/s}$. The envelopes obtained for $\Delta u = \{0.5,\ 1,\ 1.5,\ 2,\ 2.5\}\,\text{km/s}$ (when ν_m is unconstrained) are shown in Fig. 4. Note that, as explained

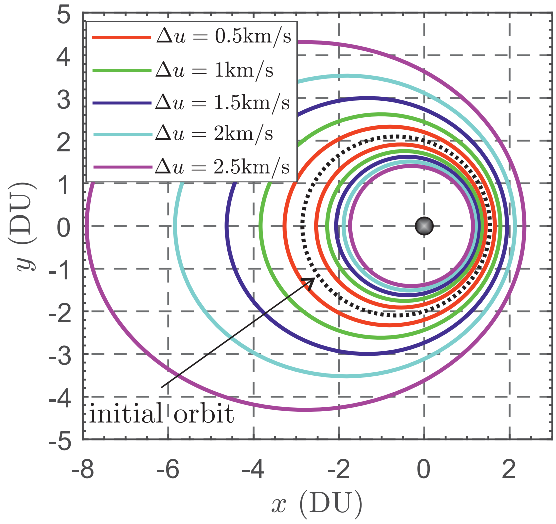

Fig. 3 Envelopes and reachable domain in the case of radial impulse with free ν_m and $\Delta u = \{0.5,\ 1,\ 1.5,\ 2,\ 2.5\}\,\text{km/s}$.

in the previous section (see also Eq. (4)), in the case of $\Delta u = \{-2.5,\ -2,\ -1.5,\ -1,\ -0.5\}\,\text{km/s}$ the same envelopes as those shown in Fig. 3 would be obtained.

3.2. Tangential impulse case

Consider now the case of single tangential impulse. When the impulse point is fixed by selecting $\nu_m = \pi/3\,\text{rad}$ and $\Delta v \in [-0.5,\ 1]\,\text{km/s}$, the envelopes coincide with the trajectories obtained when $\Delta v = \Delta v_{\min} = -0.5\,\text{km/s}$ and $\Delta v = \Delta v_{\max} = 1\,\text{km/s}$; see Fig. 4. On the other hand, if the velocity variation is fixed to $\Delta v = 1\,\text{km/s}$ and the maneuver point position ν_m is unconstrained, one of the envelope is given by the initial orbit. As far as the second one is concerned, Eqs. (17)-(18) may be used to obtain the points $(r,\ \theta)$ of the envelope as a function of $\nu_m \in [0,\ 2\pi)\,\text{rad}$, as shown in Fig. 5.

Finally, if both ν_m and $\Delta v \in [-0.5,\ 1]\,\text{km/s}$ can be freely varied, because in this case $\Delta v_{\max} > 0$ and $\Delta v_{\min} < 0$, the envelopes are given by Eq. (18) where $\Delta v = \Delta v_{\min} = -0.5\,\text{km/s}$ and $\Delta v = \Delta v_{\max} = 1\,\text{km/s}$. This result is also shown in Fig. 6, whereas Fig. 7 shows the function $\theta = \theta(\nu_m)$ computed using Eq. (17) when $\Delta v = \{-0.5,\ -0.2,\ 0.2,\ 0.5,\ 1\}\,\text{km/s}$.

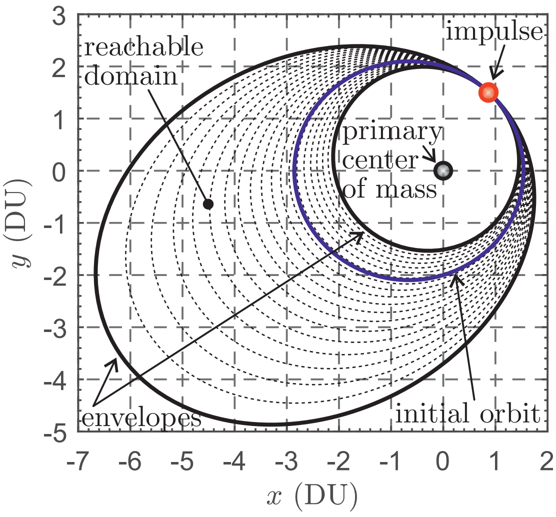

Fig. 4 Envelopes and reachable domain in the case of tangential impulse when $\nu_m = \pi/3$ rad and $\varDelta v \in [-0.5,\ 1]$ km/s.

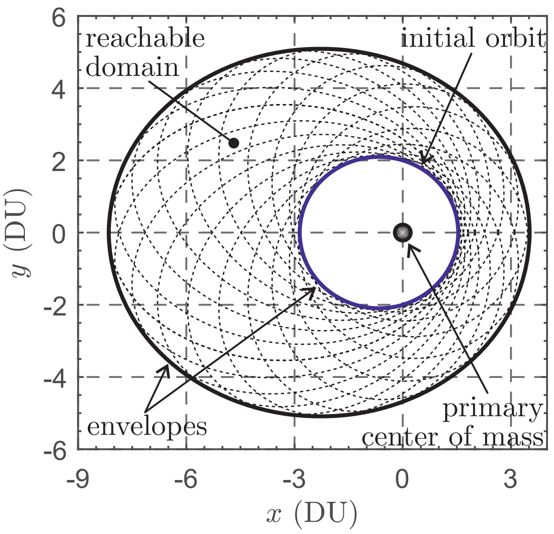

Fig. 5 Envelopes and reachable domain in the case of tangential impulse when ν_m is free and $\varDelta v = 1$ km/s.

3.3. Comparison with literature results

The case of single tangential impulse was already addressed in Ref. [12], but in that paper a simpler expression is provided for the computation of the envelopes. In order to validate our method, this section shows a comparison of the results presented in this work with those obtained in one of the mission scenarios discussed in Ref. [12]. In particular, consider an initial orbit around the Earth with pericenter altitude $h_p = 1000$ km and eccentricity $e_0 = 0.7$ (therefore the semilatus rectum is $p_0 \simeq 12543$ km $\simeq 1.97$ DU). When the value of ν_m can be freely varied and $\varDelta v \in [-0.5,\ 0.5]$ km/s, the envelopes are given by Eq. (18) where $\varDelta v = \varDelta v_{\min} = -0.5$ km/s and $\varDelta v = \varDelta v_{\max} = 0.5$ km/s, as shown in Fig. 8. Note that this result is equal to that reported in Ref. [12].

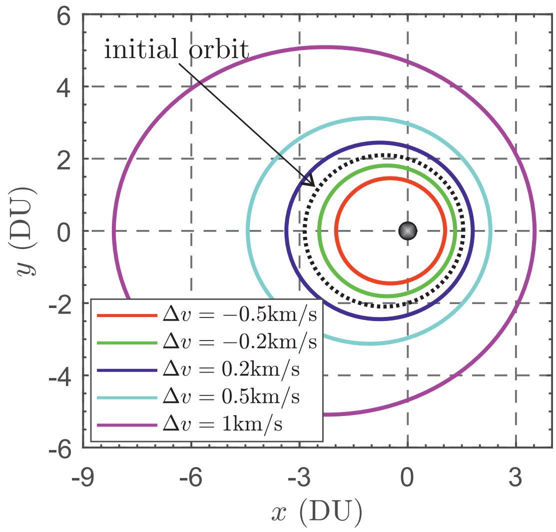

Fig. 6 Envelopes and reachable domain in the case of tangential impulse when ν_m is free to vary and $\Delta v = \{-0.5, -0.2, 0.2, 0.5, 1\}$ km/s.

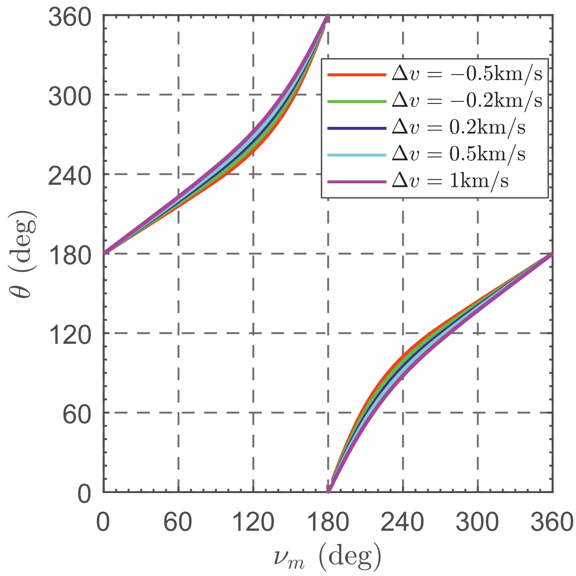

Fig. 7 $\theta = \theta(\nu_m)$ computed using Eq. (17) when $\Delta v = \{-0.5, -0.2, 0.2, 0.5, 1\}$ km/s.

4. Conclusions

In this paper, the problem of calculating the envelopes of spacecraft trajectories after a single impulsive velocity variation has been addressed. In particular, analytical expressions of the boundaries of the reachable domain have been obtained for both cases of either radial or tangential impulse. The cases of fixed or variable maneuver point and impulse magnitude have been analyzed, enforcing suitable bounds on the value of the velocity variation in order to obtain elliptical transfer trajectories after the maneuver and to avoid intersections with the primary body. Finally,

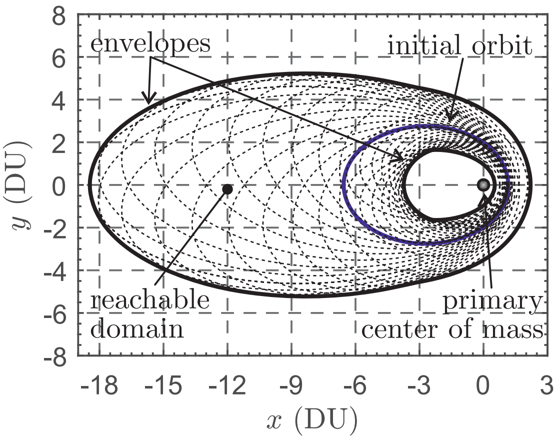

Fig. 8 Envelopes and reachable domain in the case of tangential impulse for the scenario examined in Ref. [12].

the method proposed in this study has been applied to some Earth-centered mission scenarios, and a comparison with the results of previous works has shown its effectiveness.

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