

On the energy equality for the 3D Navier–Stokes equations

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Abstract

In this paper we study the problem of energy conservation for the solutions of the initial boundary value problem associated to the 3D Navier-Stokes equations, with Dirichlet boundary conditions.

First, we consider Leray-Hopf weak solutions and we prove some new criteria, involving the gradient of the velocity. Next, we compare them with the existing literature in scaling invariant spaces and with the Onsager conjecture.

Then, we consider the problem of energy conservation for very-weak solutions, proving energy equality for distributional solutions belonging to the so-called Shinbrot class. A possible explanation of the role of this classical class of solutions, which is not scaling invariant, is also given.

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1 Introduction

We consider the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, and vanishing Dirichlet boundary conditions

$$(1.1) \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \Omega \times (0, T), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases}$$

We study the problem in a bounded domain, but the results are valid also in more general domains, with minor changes in the proofs. For simplicity we treat the problem with unit viscosity and with vanishing external force, but both assumptions are unessential.

We recall that starting with the works of Leray [35] and Hopf [31] the notion of *kinetic energy* became of fundamental relevance in the analysis of the Navier-Stokes equations, since its boundedness represents the very basic *a priori* estimate, which allows to start the machinery which is used to construct the so-called Leray-Hopf weak solutions. It is well-known that solutions in this class exist for all positive times, but their regularity and even their uniqueness are still unproved at present. The balance of the kinetic energy formally follows by multiplying the equations by the solutions itself and by performing some integration by parts, once the needed computations

are allowed. On the other hand, the work of Leray and Hopf showed that the energy inequality below (opposed to the equality (1.5))

$$\frac{1}{2} \int_{\Omega} |u(t, x)|^2 dx + \int_0^t \int_{\Omega} |\nabla u(s, x)|^2 dx ds \leq \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx \quad \forall t \geq 0,$$

can be inferred for weak solutions in the class where globally in time existence holds. In the latter inequality the sign “less or equal” is due to the lack of regularity of weak solutions and it comes from a limiting process on smoother functions based on weak convergence. We note that under some restrictions on the domain also the *strong energy inequality* is valid, but this is inessential for the present research, see e.g. [26].

The first attempts to determine sufficient conditions implying the validity of the energy equality (1.5) in the class of weak solutions –especially in connection with the problem of uniqueness– came with a series of papers by Lions [37] and Prodi [41], where the criterion $u \in L^4(0, T; L^4(\Omega))$ has been identified. The connections with the 2D case and the results of Kiselev and Ladyzhenskaya [34] highlighted the similarities and differences between the two and three-dimensional case. A few years later, in the wake of the celebrated survey of Serrin [42], Shinbrot [43] derived the criterion

$$(1.2) \quad u \in L^r(0, T; L^s(\Omega)) \quad \text{with} \quad \frac{2}{r} + \frac{2}{s} = 1 \quad \text{for} \quad s \geq 4,$$

which contains the condition $L^4(0, T; L^4(\Omega))$ as a sub-case, and which extends to a wide range of exponents the condition for energy conservation.

Remark 1.1. *The condition derived by Shinbrot reads more precisely as $\frac{2}{r} + \frac{2}{s} \leq 1$. The most interesting is the limiting case and here we refer to condition (1.2), since the same results are straightforward if the sum is strictly smaller than one.*

It is extremely relevant to observe that the condition (1.2) does not depend on the space dimension n (in particular it is valid also for $n > 3$) and moreover –very remarkably– it is not scaling invariant, we will elaborate on this in the sequel. We observe that the results of Ladyzhenskaya, Prodi, and Serrin (popularized especially after the appearance of [42]) showed regularity of weak solutions (and hence validity of the energy equality) if the scaling invariant condition

$$(1.3) \quad u \in L^r(0, T; L^s(\Omega)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} = 1 \quad \text{for} \quad s > 3,$$

holds true (see also Sohr [45]). A nice survey of these classical results can be found in Galdi [26].

The fact that Shinbrot condition does not fit in with the framework of scaling invariance (and also that (1.2) reduces to (1.3) as $s \rightarrow +\infty$) makes the understanding and the explanation of (1.2) more involved. We will give later on in Section 2.3 a possible interpretation, which is based on a different scaling and which highlights the importance of the convective term as responsible of the non-linear character of the equations, which seems in this case more relevant than the non-local nature induced by the divergence-free constraint and by the pressure. Indeed, it seems that the “hyperbolic” part of the equations, plays a fundamental role in the redistribution of the energy, not at the Fourier level, but in a sort of regularity exchange, see Section 2.3.

The essential difference between the energy equality and the energy inequality is the presence of anomalous dissipation due to the presence of non-linearity. The dissipation phenomenon is expected to be connected with the possible roughness of the solutions. The importance of energy conservation/dissipation (especially in the limit of vanishing viscosity) came from Onsager’s

work [40]. Moreover, for the Navier-Stokes equations the possible connection between energy conservation and uniqueness of weak solutions still represent an interesting open problem. Furthermore, we observe that the energy equality, will imply in the case of Galerkin approximation to the solution the convergence of the norms $\|u^m\|_{L^2(0,T;H^1(\Omega))}$, which combined with the weak convergence will produce the strong convergence. This will have also relevant consequences when these methods are used in the numerical approximation of the solutions.

The famous Onsager conjecture for the Euler and Navier-Stokes equations predicts the threshold regularity for energy conservation. In this direction, energy conservation for $C^{0,1/3}$ -solutions of the Euler equations has been recently addressed in the work of Isett [32] and also Buckmaster, De Lellis, Székelyhidi, and Vicol [11]. On the other hand the role of Besov-Hölder continuous spaces in the energy conservation of the viscous Navier-Stokes equations was studied by Cheskidov, Constantin, Friedlander and Shvydkoy [12], who proved that weak solutions in the following class

$$u \in L^3(0, T; B_{3,\infty}^{1/3}(\mathbb{R}^3)),$$

conserve the energy. (Cf. also Constantin, E, and Titi [16] for the Euler equations) The latter result has been recently extended by Cheskidov and Luo [14] to the class

$$u \in L_w^\beta(0, T; B_{p,\infty}^{\frac{2}{\beta} + \frac{2}{p} - 1}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{1}{\beta} < 1 \quad \text{for} \quad 1 \leq \beta < p \leq \infty.$$

where L_w^β denotes the weak (Marcinkiewicz) space, while $B_{p,q}^s(\mathbb{R}^3)$ are the standard Besov spaces.

A similar approach, based on Fourier methods, allowed Cheskidov, Friedlander, and Shvydkoy [13] to prove the following sufficient condition for energy conservation (here A denotes the Stokes operator associated to the Dirichlet boundary conditions)

$$A^{5/12}u \in L^3(0, T; L^2(\Omega)),$$

which turns out to be –in terms of scaling– less strict than Shinbrot one (1.2). In fact, the latter criterion by Cheskidov, Friedlander, and Shvydkoy is *equivalent in terms of scaling* to $u \in L^3(0, T; L^{9/2}(\Omega))$. The comparison and discussion of the various results will be given in detail in Section 2.2.1. We also recall that Farwig [20] proved the following sufficient condition for energy conservation

$$A^{1/4}u \in L^3(0, T; L^{18/7}(\Omega)),$$

which turns out to have the same scaling.

Our first aim is to improve the above results by looking for regularity requirements sufficient to prove energy conservation and weaker than the existing ones. The improvements coming from our results can be measured in terms of scaling, see Theorem 1. Moreover, we will use tools of functional analysis accessible to an extremely wide range of researchers, since we mainly focus on showing conditions involving usual derivatives in Lebesgue spaces, without entering the technical difficulties of more complex functional settings. This complies with the spirit of looking for refinements of basic results in classical functional spaces. Possible further developments in slightly sharper, but more complicated spaces (but at the same level of scaling) are not excluded, but beyond our goals.

In order to present our first main result we recall, for the reader not acquainted with all the technicalities necessary when dealing with genuine Leray-Hopf weak solutions of the Navier-Stokes equations, that one main step in the analysis of the energy equality is to rigorously prove the following equality

$$(1.4) \quad \int_0^T \int_\Omega (u \cdot \nabla) u \cdot u \, dxdt = 0.$$

The latter equality is *formally* valid if u is a divergence-free vector field tangential to the boundary. The calculations are justified once u is smooth enough to ensure that the above space-time integral is finite, but this is not the case if u is a weak solution in the 3D case.

The novelty of our approach is to look for conditions involving the gradient of the velocity, instead of the velocity itself, which allow to conclude that the above integral is finite.

The first result of this paper is the following criterion.

Theorem 1. *Let $u_0 \in H$ and let u be a Leray-Hopf weak solution of (1.1) corresponding to u_0 as initial datum. Let us assume that one of the following conditions is satisfied*

$$(i) \quad \nabla u \in L^{\frac{q}{2q-3}}(0, T; L^q(\Omega)), \quad \text{for } \frac{3}{2} < q < \frac{9}{5};$$

$$(ii) \quad \nabla u \in L^{\frac{5q}{5q-6}}(0, T; L^q(\Omega)), \quad \text{for } \frac{9}{5} \leq q \leq 3;$$

$$(iii) \quad \nabla u \in L^{1+\frac{2}{q}}(0, T; L^q(\Omega)), \quad \text{for } q > 3.$$

Then, the velocity u satisfies the energy equality

$$(1.5) \quad \frac{1}{2} \int_{\Omega} |u(t, x)|^2 dx + \int_0^t \int_{\Omega} |\nabla u(s, x)|^2 dx ds = \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx \quad \forall t \geq 0.$$

These results will be proved in Section 3, but the comparison with the previous ones known in literature will be discussed already in Section 2.2.1. The overall strategy is, to assume that $\nabla u \in L^q(\Omega)$ for some fixed q , and to employ all the available regularity coming from the definition of weak solutions, so to find the smaller exponent in the time variable which allows to show (1.4).

Remark 1.2. *A similar approach had already been implemented also by Farwig and Taniuchi [23], focusing on the degree k of (fractional) smoothness of u and on the correct definition of spaces in the case of general unbounded domains. In particular, their result for $k = 1$ has the same scaling as ours in the case $q = 9/5$. Very recently, our approach of working with levels of regularity for the gradient of the velocity has inspired the paper of Beirão da Veiga and Yang [4] where they treat Newtonian and non-Newtonian fluids.*

Remark 1.3. *Even if not explicitly stated, it is clear that any of the conditions on the gradient in Theorem 1 implies, directly by the standard theory of traces, an extra condition on the initial datum. To keep the paper self-contained and more understandable we do not elaborate on this technical point, which can be probably improved by using the theory of weighted estimates as in Farwig, Giga, and Hsu [21] or by dealing with conditions valid on any time interval of the type $[\varepsilon, T]$ (with $\varepsilon > 0$), as done in the recent paper by Maremonti [39].*

In the last part of the paper (cf. Section 4) we will consider the problem of energy conservation for solutions which are very-weak, see Section 2.1 for the precise definition. In this case there is not any available regularity on u , apart the solution being in $L^2_{loc}((0, T) \times \Omega)$. The interest for very-weak solutions dates back to Foias [25], who proved their uniqueness under condition (1.3). Later Fabes, Jones, and Rivière [19] proved the existence of very-weak solutions for the Cauchy problem, while the case of the initial-boundary value problem has been studied mainly starting from the work of Amann [1]. As usual when dealing with very-weak solutions a duality argument can be employed to show uniqueness, by using properties of the adjoint problem (which in case of the Navier-Stokes equations, is a backward Oseen type-problem). The connection with the

energy equality has been very recently developed by Galdi [28], who showed that very-weak solution in the Lions-Prodi class $L^4(0, T; L^4(\Omega))$, and with initial data in $L^2(\Omega)$, satisfy (1.5). It is relevant to observe that the duality argument is used to improve the known regularity of the solution, in order to use the previously established results for usual weak solutions. This has been used also in a different context in [7], but the approach we follow here takes also inspiration from a bootstrap argument as used in Lions and Masmoudi [38] for results concerning *mild solutions*.

The second result we prove is then the following.

Theorem 2. *Let $u \in L^2_{loc}(\Omega \times (0, T))$ be a distributional solution of (1.1). If the initial datum $u_0 \in H$ and if (1.2) holds true, then the energy equality (1.5) is satisfied.*

The interpretation of the latter result and its connection with the scaling of space and time variables will be given in Section 2.3. The techniques employed here are inspired by the approach followed in Galdi [28, 29].

Summarizing the results, the sufficient conditions required for the energy equality are stronger if the requested regularity of the solution u is weaker. In particular, the less strict requirement of the solution (being just a distributional solution) seems to require more restrictive conditions than those in Theorem 1, which are nevertheless the same as those classically found in [41, 37, 43] for Leray-Hopf weak solutions.

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2 Functional setting and comparison with previous results

In this section we first introduce the notation and the precise definitions of solutions we want to deal with, then we compare our results with the ones in the existing literature. We will use the customary Sobolev spaces $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}})$ and we denote the L^p -norm simply by $\|\cdot\|_p$ and since the Hilbert case plays a special role we denote the $L^2(\Omega)$ -norm simply by $\|\cdot\|$. For X a Banach space we will also denote the usual Bochner spaces of functions defined on $[0, T]$ and with values in X by $(L^p(0, T; X), \|\cdot\|_{L^p(X)})$. In the case $X = L^q(\Omega)$ we denote the norm of $L^p(0, T; L^q(\Omega))$ simply by $\|\cdot\|_{p,q}$.

2.1 On Leray-Hopf weak and very-weak solutions

For the variational formulation of the Navier-Stokes equations (1.1), we introduce as usual the space \mathcal{V} of smooth and divergence-free vectors fields, with compact support in Ω . We then denote the completion of \mathcal{V} in $L^2(\Omega)$ by H and the completion in $H^1_0(\Omega)$ by V . The Hilbert space H is endowed with the natural L^2 -norm $\|\cdot\|$ and inner product (\cdot, \cdot) , while V with the norm $\|\nabla v\|$ and inner product $((u, v)) := (\nabla u, \nabla v)$. As usual, we do not distinguish between scalar and vector valued functions. The dual pairing between V and V' is denoted by $\langle \cdot, \cdot \rangle$, and the dual norm by $\|\cdot\|_*$.

Definition 1 (Leray-Hopf weak solutions). *A vector field $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ is a Leray-Hopf weak solution to the Navier-Stokes equations (1.1) if*

(i) u is a solution of (1.1) in the sense of distributions, i.e.

$$\int_0^T (u, \partial_t \phi) - (\nabla u, \nabla \phi) - ((u \cdot \nabla) u, \phi) dt = -(u_0, \phi(0)),$$

for all $\phi \in C_0^\infty([0, T[\times \Omega)$ with $\nabla \cdot \phi = 0$;

(ii) u satisfies the global energy inequality

$$(2.1) \quad \frac{1}{2} \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \leq \frac{1}{2} \|u_0\|^2 \quad \forall t \geq 0;$$

(iii) the initial datum is attained in the strong sense of $L^2(\Omega)$

$$\|u(t) - u_0\| \rightarrow 0 \quad t \rightarrow 0^+.$$

Since the works of Leray and Hopf [35, 31] for the Cauchy problem and for the initial boundary value problem, it is well-known that for initial data in H , and for all $T > 0$ there exists at least a weak solution in the above sense. An outstanding open problem is that of proving (or disproving) the uniqueness and regularity of weak solutions, see Constantin and Foias [17] and Galdi [26].

In the last section of the paper we deal with another less-restrictive notion of solution, which allows for infinite energy.

Definition 2 (very-weak solutions). *A vector field $u \in L_{loc}^2((0, T) \times \Omega)$ is a very-weak solution to the Navier-Stokes equations if*

(i) *the following identity*

$$\int_0^T (u, \partial_t \phi) + (u, \Delta \phi) + (u, (u \cdot \nabla) \phi) dt = -(u_0, \phi(0)),$$

holds true for all $\phi \in \mathcal{D}_T$, where

$$\mathcal{D}_T := \left\{ \begin{array}{l} \phi \in C^\infty([0, T] \times \overline{\Omega}), \text{ with support contained in a compact set of } [0, T] \times \overline{\Omega}, \\ \text{such that } \nabla \cdot \phi = 0 \text{ in } \Omega, \phi = 0 \text{ on } \partial\Omega \text{ and } \phi(T) = 0 \text{ in } \Omega \end{array} \right\};$$

(ii) $\nabla \cdot u = 0$ in the sense of $\mathcal{D}'(\Omega)$, that in the sense of distributions, for a.e. $t \in [0, T]$.

As mentioned before, the existence of very-weak solutions in the case of the whole space has been treated in [19], but the notion in the 3D time-evolution case in a bounded domain (especially with analysis of the meaning of boundary conditions) is analyzed for instance in Amann [1] and Farwig, Kozono, and Sohr [22], with emphasis on the non-homogeneous problem, too.

2.2 Comparison with previous results

In this section we compare our results with the ones already present in literature. From the physical point of view the most relevant results are those related with large q , especially with $q > 3$, being 3 the space dimension.

2.2.1 Energy conservation and Onsager conjecture

As mentioned in the introduction, the validity of (1.5) has a strong connection with Onsager conjecture about the threshold regularity of $C^{1/3}$ -Hölder continuity of the velocity which allows for energy conservation. While for the 3D Euler equations the conjecture is basically solved in both directions, for the 3D Navier-Stokes equations much has still to be done. This is a strong motivation to investigate further the energy equality for the 3D Navier-Stokes equations and to understand the anomalous energy dissipation phenomenon in this case. Our result involves Sobolev regularity of the velocity, but this can be compared to Hölder regularity as we will explain in the following. In the range $q > 3$, standard Sobolev embedding theorems imply $W^{1,q}(\Omega) \subset C^{0,1-3/q}(\overline{\Omega})$, and $W^{1,q}(\Omega)$ is a *proper* subset of $C^{0,1-3/q}(\overline{\Omega})$. On the other hand, in order to compare our result with previous ones, we can consider the two spaces as if they would be equivalent in a certain sense. Indeed, Sobolev norms measure a sort of combination of three properties of a function: height (amplitude), width (measure of the support), and frequency (inverse wavelength). Roughly speaking, if a function has amplitude A , is supported on a set of volume V , and has oscillations with frequency N , then the $W^{k,p}$ - (homogeneous) norm is of the order of $AN^kV^{1/p}$. A key quantity is the “weight” $k - \frac{n}{p}$ of the Sobolev space $W^{k,p}(\mathbb{R}^n)$: If one scales the metric by a constant $R > 0$ then this transformations scales the $W^{k,p}$ -norm by $R^{n/p-k}$ and the $C^{m,\alpha}$ -norm by $R^{-(m+\alpha)}$, showing the relevance of the weights. Hence, from the point of view of Sobolev embedding and fractional regularity it is sound to consider as very close (and so almost equivalent) in the 3D case the spaces W^{s_1,p_1} and W^{s_2,p_2} , when $1/p_1 - s_1/3 = 1/p_2 - s_2/3$, and they both embed in $C^{0,\alpha}$, for $\alpha := s_i - 3/p_i$, whenever this quantity is positive. Also in terms of interpolation theory, these spaces behave in the same way, as can be seen with the DeVore diagrams [18]. Hence, we can also consider $C^{0,1-3/q}(\overline{\Omega})$ as a rough –but meaningful– measure of the classical regularity of $W^{1,q}(\Omega)$.

By embedding, results from Theorem 1 in the range $q > 3$ are very close to the condition

$$u \in L^{1+\frac{2}{q}}(0, T; C^{0,1-\frac{3}{q}}(\overline{\Omega})).$$

In particular, by taking $q = 9/2$ we obtain, as class of solutions conserving the energy, that with scaling comparable to

$$u \in L^{\frac{13}{9}}(0, T; C^{0,1/3}(\overline{\Omega})),$$

which improves the previously cited results

$$\begin{aligned} u &\in L^3(0, T; B_{3,\infty}^{1/3}(\mathbb{R}^3)) && \text{from Ref. [12];} \\ u &\in L_w^{\frac{3}{2}}(0, T; B_{\infty,\infty}^{1/3}(\mathbb{R}^3)) && \text{from Ref. [14], when } p = \infty, \text{ and } \beta = 3/2. \end{aligned}$$

In the space variables the two functional spaces are very close to $C^{0,1/3}(\overline{\Omega})$, but $3 > 3/2 = 1.5 > 13/9 = 1.\overline{4}$.

We also warn again the reader that our results are obtained by embedding, hence they are valid for a proper subset of $C^{0,1/3}(\overline{\Omega})$ and there is not any direct connection between Hölder and Sobolev regularity (recall that the Weierstrass function $\sum_{n \in \mathbb{N}} a^n \cos(b^n \pi x)$, $0 < a < 1$, is $-\log(a)/\log(b)$ -Hölder continuous but is not even of bounded variation). Nevertheless, in terms of scaling our results present a better behavior as compared with the previous ones present in literature. This suggests that there could be also room for further improvements.

2.2.2 The case $q < 3$

In order to explain the comparison (we will make later on) we also wish to recall the notion of scaling invariance for space-time functions. By interpolation one can show that Leray-Hopf

weak solution have the following regularity

$$u \in L^r(0, T; L^s(\Omega)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} = \frac{3}{2}, \quad \text{for} \quad 2 \leq s \leq 6.$$

Several results (starting again from the classical work of Ladyžhenskaya, Prodi, and Serrin) concern uniqueness and regularity with scaling invariant conditions on solutions. In particular, if a weak solution satisfies condition (1.3) (see [42] for example), then it becomes unique, strong (namely the gradient belongs to $L^\infty(0, T; L^2(\Omega))$), smooth, and it satisfies the energy equality.

Full regularity of weak solutions follows also under alternative assumptions on the gradient of the velocity ∇u . More specifically, if

$$(2.2) \quad \nabla u \in L^p(0, T; L^q(\Omega)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \text{for} \quad q > \frac{3}{2},$$

then weak solutions are regular, see Beirão da Veiga [3] and also [5] for the problem in a bounded domain. In \mathbb{R}^3 standard Sobolev embeddings imply that if $\nabla u \in L^p(0, T; L^q(\Omega))$ then $u \in L^p(0, T; L^{q^*}(\Omega))$ where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{3}$. We recall that for weak solutions ∇u is simply (x, t) -square-integrable and $\frac{2}{2} + \frac{3}{2} = \frac{5}{2} > 2$.

The class defined by (1.3) is important from the point of view of the relation between scaling invariance and partial regularity of weak solutions. In fact, if a pair (u, p) solves (1.1), then so does the family $\{(u_\lambda, p_\lambda)\}_{\lambda > 0}$ defined by

$$(2.3) \quad u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad p_\lambda(x, t) := \lambda^2 p(\lambda^2 t, \lambda x).$$

Scaling invariance means that $\|u_\lambda\|_{L^r(0, T/\lambda^2; L^s(\Omega_\lambda))} = \|u\|_{L^r(0, T; L^s(\Omega))}$ and this happens if and only if (r, s) satisfy (1.3). (Here $\Omega_\lambda := \{x/\lambda : x \in \Omega\}$.) Likewise, the scaling invariance of ∇u occurs if and only if (2.2) holds true. We will use these notions to compare classical and more recent results with the new ones from Theorem 1.

If $u \in L^4(0, T; L^4(\Omega))$ then, *in terms of scaling*, this regularity lies in between the class of existence and that of regularity since

$$1 = \frac{2}{4} + \frac{2}{4} < \frac{2}{4} + \frac{3}{4} = \frac{5}{4} < \frac{3}{2}.$$

We note that also the class (1.2) identified by Shinbrot shares the same property, even if in terms of scaling

$$1 < \frac{2}{r} + \frac{3}{s} = \frac{2}{r} + \frac{2}{s} + \frac{1}{s} = 1 + \frac{1}{s} < \frac{3}{2},$$

hence, as s increases, this class becomes closer and closer to that of regularity given by (1.3). A natural question is that if we can lower down the level of regularity needed for a weak solution, in order to satisfy the energy equality.

The new results we present are related with the aim of finding sufficient conditions for the energy equality involving the gradient of the velocity.

The fact that condition (1.2) is not scaling invariant makes it possible to conjecture that perhaps some threshold can be broken by a more precise inspection of the calculations. In fact, the aforementioned results in [13, 20] identified the sufficient conditions

$$A^{5/12}u \in L^3(0, T; L^2(\Omega)) \quad \text{or} \quad A^{1/4}u \in L^3(0, T; L^{18/7}(\Omega)).$$

which are –in terms of scaling– both comparable with

$$u \in L^3(0, T; L^{9/2}(\Omega)),$$

which is less restrictive than (1.2), since

$$1 < \frac{2}{3} + \frac{2}{9/2} = \frac{10}{9},$$

but at the same time the space $L^3(0, T; L^{9/2}(\Omega))$ it is still between the class of existence and that of regularity, being $1 < \frac{2}{3} + \frac{3}{9/2} = \frac{4}{3} < \frac{3}{2}$.

The ranges obtained in Theorem 1 have respectively the following properties, which turn out easy to be compared with the condition (2.2).

We have in fact that:

- (i) it holds $2 < \frac{2}{p} + \frac{3}{q} = 4 - \frac{3}{q} < \frac{5}{2}$, for $\frac{3}{2} < q < \frac{9}{5}$;
- (ii) it holds $2 < \frac{2}{p} + \frac{3}{q} = 2 + \frac{3}{5q} < \frac{5}{2}$, for $\frac{9}{5} \leq q < 3$;
- (iii) it holds $2 < \frac{2}{p} + \frac{3}{q} = \frac{2q}{q+2} + \frac{3}{q} < \frac{5}{2}$, for $3 \leq q < 6$.

Thus, we have that for $\frac{3}{2} < q < 6$ our conditions imply range of exponents which are not those of scaling invariance (2.2), hence not those implying full regularity of the solutions (and consequently also energy equality in a trivial way).

Remark 2.1. *In cases (i) and (ii), i.e. when $\frac{3}{2} < q < 3$, the standard Sobolev embedding tells us that $u \in L^p(0, T; L^{q^*}(\Omega))$ where $q^* = \frac{3q}{3-q}$ and p is given as a function of q by Theorem 1. This means that by case (i) and case (ii) the exponent of integrability q^* in the space variable for the solution u can range from 3 to $+\infty$.*

Recalling that $q^* = \frac{3q}{3-q}$, the ranges obtained in Theorem 1 have respectively the following properties:

- (i) it holds $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(5q-6)}{3q}$, for $\frac{12}{7} < q < \frac{9}{5}$;
- (ii) it holds $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(10q-3)}{15q}$, for $\frac{9}{5} \leq q < 3$;
- (iii) it holds $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(2q-3)}{3q}$, for $q \geq 3$.

Thus, we showed that our range of exponents improves those in Shinbrot condition (1.2). We observe that Shinbrot condition for the space integrability of u (exponent ≥ 4) corresponds to $q > \frac{12}{7}$ in our classification.

Remark 2.2. *The “best exponent,” where best is measured in terms of giving the quantity $\frac{2}{p} + \frac{2}{q}$ as large as possible, turns out to be $q = 9/5$. This implies that by embedding $(9/5)^* = 9/2$ which gives as sufficient condition*

$$u \in L^3(0, T; W^{1,9/5}(\Omega)) \subset L^3(0, T; L^{9/2}(\Omega)),$$

that is at the same level of scaling of [13, 20], even if the various conditions are not directly comparable each other.

We further remark that, in the range $q > \frac{12}{7}$, our result improves also the ranges obtained by Leslie and Shvydkoy in [36]. Indeed, they prove (see [36, Theorem 1.1]) the validity of the energy equality for $u \in L^r(0, T; L^s(\Omega))$ with $3 \leq r \leq s$ and

$$\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2},$$

while, for $q > \frac{12}{7}$ we are assuming $\nabla u \in L^p(0, T; L^q(\Omega))$ and hence $u \in L^p(0, T; L^{q^*}(\Omega))$ with $p \leq q^*$ and

$$\frac{1}{p} + \frac{1}{q^*} > \frac{1}{2}.$$

However, authors in [36] studied also the case $s < 3$ corresponding in our case to $q < 3/2$ which is not covered here. Moreover, for the case $3 \leq s < r$, they can prove the energy equality with exponents r and s satisfying $\frac{1}{r} + \frac{1}{s} < \frac{1}{2}$: this matches our interval $3/2 \leq q < 12/7$, where $\frac{1}{p} + \frac{1}{q^*} < \frac{1}{2}$, thus showing that we are not improving [36] in this last range of exponents.

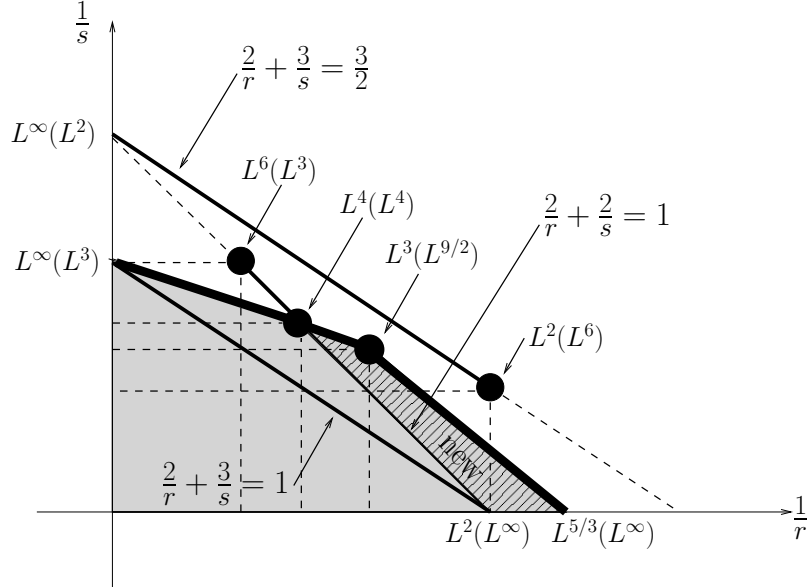


Figure 1: The shaded area corresponds to the ranges of Theorem 1 and in particular the dashed one corresponds to new ranges of exponents not obtained before in the literature.

2.3 On the Shinbrot condition (1.2) and very-weak solutions

In the case of very-weak solutions, the sufficient condition for energy equality we find is the same as (1.2), hence showing that it has a universal role, since it applies to distributional solutions which can be outside of any Lebesgue space, hence not the common Leray-Hopf weak solutions.

We observe that the condition (1.2) breaks all the standard theory of scaling invariance, even if we consider different families of scaling transformation. In fact, beside the standard parabolic scaling (2.3), it is well-known that useful different approaches are those concerning invariance of the equation under the following space-time transformation, indexed by $\alpha \in \mathbb{R}$

$$(2.4) \quad u_{\lambda, \alpha}(t, x) := \lambda^\alpha u(\lambda^{\alpha+1}t, \lambda x) \quad \text{and} \quad p_{\lambda, \alpha}(t, x) := \lambda^{2\alpha} p(\lambda^{\alpha+1}t, \lambda x).$$

Note that the transformation in (2.3) corresponds to $\alpha = 1$. On varying α , with these transformations it is possible to extract some heuristic and useful information from the equations. In particular, they keep for all $\alpha \in \mathbb{R}$ the material derivative operator $\frac{D}{Dt} := \partial_t + u \cdot \nabla$ unchanged, while the viscosity changes from ν to $\lambda^{\alpha-1}\nu$. This explains the relevance of these transformation in the study of Euler equations or also in presence of very small viscosities. We point

out that the case $\alpha = -1/3$ determine the so-called *Kolmogorov scaling*, which has connections with the conservation of energy input for stochastic statistically stationary solutions, stochastic equations, and Large Eddy Simulation for turbulence modeling. See [6, 8], Flandoli, Gubinelli, Hairer, and Romito [24], and Kupiainen [33]. If we try to consider the scaling of norms under the transformation (2.4) we obtain that

$$\|u_{\lambda,\alpha}\|_{L^r(0,T/\lambda^2;L^s(\Omega_\lambda))} = \lambda^{\alpha - \frac{3}{s} - \frac{\alpha+1}{r}} \|u\|_{L^r(0,T;L^s(\Omega))}$$

hence the norms cannot be invariant for the relevant values of α , since $\alpha - \frac{3}{s} - \frac{\alpha+1}{r}$ never vanishes for both $\alpha = 1$ and $\alpha = -1/3$. A possible explanation or justification of the condition (1.2) will come from a different analysis.

In particular, in condition (1.2) the space and time variables have the same “strength,” as in hyperbolic equations, while in our studies the viscosity and hence the parabolic part play a fundamental role. To understand in which situation the hyperbolic nature of the convection plays a major role we have to consider the maximal regularity type results for the Stokes problem, which will be used to handle the adjoint problem. Coming back to Solonnikov [46] (see also Giga and Sohr [30]) it is well-known that for a wide class of domains the following result holds true.

Theorem 3. *Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded (or exterior or the full-space or even the half-space) and let $\mathcal{F} \in L^\alpha(0, T; L^\beta(\Omega))$, with $1 < \alpha, \beta < \infty$. Then, the boundary initial value problem associated to the linear Stokes system*

$$(2.5) \quad \begin{aligned} \partial_t v - \Delta v + \nabla \pi &= \mathcal{F} && \text{in } (0, T) \times \Omega, \\ \nabla \cdot v &= 0 && \text{in } (0, T) \times \Omega, \\ v(t, x) &= 0 && \text{on } (0, T) \times \partial\Omega, \\ v(0, x) &= 0 && \text{in } \Omega, \end{aligned}$$

has a unique solution (v, π) such that

$$(2.6) \quad \exists c > 0 \quad \|\partial_t v\|_{L^\alpha(L^\beta)} + \|P\Delta v\|_{L^\alpha(L^\beta)} + \|\nabla \pi\|_{L^\alpha(L^\beta)} \leq c \|\mathcal{F}\|_{L^\alpha(L^\beta)},$$

where the constant c depends on T, α, β , and on the domain, while P is the Leray projection operator on divergence-free and tangential to the boundary vector fields.

For our purposes, a very interesting application consists in using this result to find improved regularity for the solutions of a linear problem in which $\mathcal{F} = (u \cdot \nabla) v$, where u is an assigned function with a given space-time summability as for instance (1.2).

We have the following lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded (or exterior or the full-space or even the half-space) and let $u \in L^r(0, T; L^s(\Omega))$, such that $\nabla \cdot u = 0$ in $\mathcal{D}'(\Omega)$, for a.e. $t \in [0, T]$. Then, the boundary initial value problem*

$$(2.7) \quad \begin{aligned} \partial_t v - \Delta v + \nabla \pi &= -(u \cdot \nabla) v && \text{in } (0, T) \times \Omega, \\ \nabla \cdot v &= 0 && \text{in } (0, T) \times \Omega, \\ v(t, x) &= 0 && \text{on } (0, T) \times \partial\Omega, \\ v(0, x) &= 0 && \text{in } \Omega, \end{aligned}$$

has a unique solution such that

$$v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

Moreover, if the solution v satisfies also

$$\nabla v \in L^\alpha(0, T; L^\beta(\Omega)) \quad \text{for some } 1 \leq \alpha, \beta \leq \infty,$$

then the solution v itself satisfies also the following improved estimate

$$\exists c > 0 \quad \|\partial_t v\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} + \|P\Delta v\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} + \|\nabla \pi\|_{L^{\frac{r\alpha}{r+\alpha}}(L^{\frac{s\beta}{s+\beta}})} \leq c \|u\|_{L^r(L^s)} \|\nabla v\|_{L^\alpha(L^\beta)},$$

for some non negative constant $c = c(T, r, s, \alpha, \beta, \Omega)$.

In order to show the requested regularity which will make the calculations justified in the proof of Theorem 2 we need to apply the above result several (but a finite number of) times, making possible a sort of bootstrap. This is the main difference with the proof of the result for $r = s = 4$ from [28], where a single step was enough.

In our case we need to apply an extremely sharp interpolation result, which has been kindly provided to us by Prof. Amann [2].

Lemma 2.2. *Let $\phi \in W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q} \cap W_0^{1,q}(\Omega))$, with $\phi(0) = 0$, for $1 < p, q < \infty$. Then, it follows that*

$$\phi \in L^{p_1}(0, T; W_0^{1,q}(\Omega)) \quad \text{for all } p_1 \leq p_*, \quad \text{where } \frac{1}{p_*} := \frac{1}{p} - \frac{1}{2}.$$

Remark 2.3. *We observe that in this embedding we have also compactness for any given $p_1 < p_*$ and this is a classical, e.g., see Simon [44, Corollary. 8]). We note that the case $r = s = 4$ can be treated directly, without Lemma 2.2 and also we observe that if instead of the sharp continuous embedding one uses the classical one (not valid in the endpoint case) the same result of Theorem 2 can be obtained, under the more restrictive condition $u \in L^r(0, T; L^s(\Omega))$ with $\frac{2}{r} + \frac{2}{s} < 1$, for $s > 4$.*

The interpolation type inequality from Lemma 2.2 allows us to infer the following result.

Proposition 2.3. *Let $u \in L^r(0, T; L^s(\Omega))$ be given, with $\frac{1}{r} + \frac{1}{s} = \gamma \leq \frac{1}{2}$. If the unique solution of (2.7) satisfies $\nabla v \in L^\alpha(0, T; L^\beta(\Omega))$, then in addition*

$$(2.8) \quad \nabla v \in L^{\alpha_1}(0, T; L^{\beta+s}(\Omega)) \quad \forall \alpha_1 \leq \left(\frac{\alpha r}{\alpha + r}\right)_*.$$

We observe that since by definition $1/\left(\frac{\alpha r}{\alpha + r}\right)_* = \frac{\alpha + r}{\alpha r} - \frac{1}{2}$, then the following equalities hold true

$$\frac{\alpha + r}{\alpha r} - \frac{1}{2} + \frac{\beta + s}{\beta s} = \frac{1}{r} + \frac{1}{s} - \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\beta} = \gamma - \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\beta},$$

hence the value $\gamma = \frac{1}{2}$ is exactly the one which keeps the mixed space-time regularity unchanged. Since we will need this to iterate the procedure to infer further regularity for the solution of the linear problem, we give now the precise regularity which is obtained by using repeatedly the same argument. To this aim let $(\alpha_1, \beta_1) = (2, 2)$ (this because the starting point is the energy space, that is $\nabla v \in L^2((0, T) \times \Omega)$), we define then by recurrence

$$\alpha_{n+1} := \left(\frac{\alpha_n r}{\alpha_n + r}\right)_* \quad \text{and} \quad \beta_{n+1} := \frac{\beta_n s}{\beta_n + s}.$$

We observe that

$$(2.9) \quad \frac{1}{\alpha_{n+1}} + \frac{1}{\beta_{n+1}} = \gamma - \frac{1}{2} + \frac{1}{\alpha_n} + \frac{1}{\beta_n} \quad \forall n \in \mathbb{N},$$

and taking into account that the iteration stops if the limiting values 1 or $+\infty$ for either α_{n+1} or β_{n+1} are reached.

Remark 2.4. *By direct computations using the recurrence definition, it follows that $\beta_{n+1} \leq \beta_n$, and consequently $\alpha_{n+1} \geq \alpha_n$. Hence, the use of the maximal regularity applied to the linear problem decreases the available regularity in the space variables, but increases the one of the time variable.*

Concerning the linear Stokes problem we have the following result

Proposition 2.4. *Let $u \in L^r(0, T; L^s(\Omega))$ be given with $\frac{1}{r} + \frac{1}{s} = \gamma \leq \frac{1}{2}$. Then, the unique solution v of (2.7) satisfies*

$$(2.10) \quad \nabla v \in L^{\tilde{\alpha}_n}(0, T; L^{\beta_n}(\Omega)),$$

for all $\tilde{\alpha}_n \leq \alpha_n$, where the couple (α_n, β_n) is defined as in (2.9). Hence, by Hölder inequality we have also

$$(2.11) \quad (u \cdot \nabla) v, \nabla \pi \in L^{\frac{\tilde{\alpha}_n r}{\tilde{\alpha}_n + r}}(0, T; L^{\frac{\beta_n s}{\beta_n + s}}(\Omega)),$$

and

$$\frac{1}{\frac{\tilde{\alpha}_n r}{\tilde{\alpha}_n + r}} + \frac{1}{\frac{\beta_n s}{\beta_n + s}} \leq \gamma + 1.$$

Observe that, while in the time variable we can consider by Hölder inequality also exponents smaller than α_n , in the space variables, this is not possible when the domain Ω is unbounded, and with infinite Lebesgue measure.

In terms of scaling the result of Proposition 2.4 can be viewed as a modulated and controlled exchange of regularity between space-and time (which resembles some of the estimates obtainable with Fourier analysis and has connections also with the early estimates for the gradient of the vorticity in $L^{4/3+\varepsilon}((0, T) \times \Omega)$ from [15]). The above result shows that the standard theory of linear parabolic equations can be used to obtain a full scale of results with similar regularity drain from some variables towards others. This will be enough for our purposes of applications to very-weak solutions.

3 On the energy equality for Leray-Hopf weak solutions.

Before proceeding with the proof of Theorem 1, let us recall that we mainly need to show that the conditions in the statement of Theorem 1 are sufficient to show that the double integral $\int_0^T (u \cdot \nabla u, u) dt$ is finite, hence that (1.4) holds true. The calculations leading to (1.5) can be justified by approximating u by the family of smoother function $(u)_\varepsilon = k_\varepsilon *_t u$ and then taking the limit for $\varepsilon \rightarrow 0^+$. The (time) mollification operator, denoted in the sequel by $(\cdot)_\varepsilon$, is defined for a space-time function $\Phi : (0, T) \times \Omega \rightarrow \mathbb{R}^n$ by

$$(3.1) \quad (\Phi)_\varepsilon(t, x) := \int_0^T k_\varepsilon(t - \tau) \Phi(\tau, x) d\tau \quad \text{for } 0 < \varepsilon < T.$$

It is a standard Friederichs mollification with respect to the time variable, where k is a $C_0^\infty(\mathbb{R})$, real-valued, non-negative even function, supported in $[-1, 1]$ with $\int_{\mathbb{R}} k(s) ds = 1$ and, as usual, $k_\varepsilon(t) := \varepsilon^{-1} k(t/\varepsilon)$.

Before starting the proof we recall the following property of weak solutions, which is a special case of a result by Hopf [31].

Lemma 3.1. *Let $u_0 \in H$ and let u be a Leray-Hopf weak solution of (1.1). Then u can be redefined on a set of zero Lebesgue measure in such a way that $u(t) \in L^2(\Omega)$ for all $t \in [0, T]$ and it satisfies the identity*

$$(3.2) \quad (u(s), \phi(s)) = (u_0, \phi(0)) + \int_0^T \left[\left(u, \frac{\partial \phi}{\partial t} \right) - ((u \cdot \nabla) u, \phi) - (\nabla u, \nabla \phi) \right] d\tau,$$

for all $\phi \in C_0^\infty([0, T]; C_0^\infty(\Omega))$, with $\nabla \cdot \phi = 0$ and all $0 \leq s < T$.

We observe that from the above Lemma it also follows that u is continuous as a map $[0, T]$ with values in $L^2(\Omega)$, if endowed with the weak topology, that is

$$\forall t_0 \in [0, T] \quad \lim_{t \rightarrow t_0} (u(t) - u(t_0), \phi) = 0 \quad \forall \phi \in L^2(\Omega).$$

The procedure of mollifying and taking the limits clearly explained for instance in Galdi [26, Sec. 4], even if we need to make some changes and adapt to our setting the results. Anyway, one has mainly to use the following two facts

$$(3.3) \quad \begin{aligned} \int_0^T (\nabla u, \nabla (u)_\varepsilon) dt &= \int_0^T (\nabla u, (\nabla u)_\varepsilon) dt \xrightarrow{\varepsilon \rightarrow 0^+} \int_0^T \|\nabla u\|^2 dt, \\ (u(t), (u)_\varepsilon(t)) &= \frac{\|u(t)\|^2}{2} + \mathcal{O}(\varepsilon). \end{aligned}$$

which come from the choice of the mollifier and the first relies on the fact that $u \in L^2(0, T; V)$, while the second comes from the weak- L^2 -continuity of u as a Leray-Hopf weak solution.

Next, the probably most relevant point is to show that

$$\int_0^T ((u \cdot \nabla) u, u_\varepsilon) dt \xrightarrow{\varepsilon \rightarrow 0^+} 0,$$

under the assumption of the theorem, since this step cannot be deduced –at present– from the properties of weak solution.

We can now give the proof of the first result of this paper.

Proof of Theorem 1. The proof follows some rather standard arguments but relies also on some new estimates for the nonlinear term. Let $0 < T < +\infty$ be given and let us fix t_0 with $0 < t_0 \leq T$. Now, let $\{u_m\} \subset C_0^\infty([0, T]; C_0^\infty(\Omega))$ be a sequence converging to u in $L^2(0, T; V) \cap L^p(0, T; W_0^{1,q}(\Omega))$, whose existence comes from standard density results. We choose in (3.2) $s = t_0$ and as legitimate test function $\phi = (u_m)_\varepsilon$, for some $0 < \varepsilon < t_0$. In this way we get the identity

$$(3.4) \quad \begin{aligned} (u(t_0), (u_m)_\varepsilon(t_0)) &= (u_0, (u_m)_\varepsilon(0)) \\ &+ \int_0^{t_0} \left[\left(u, \frac{\partial (u_m)_\varepsilon}{\partial t} \right) - ((u \cdot \nabla) u, (u_m)_\varepsilon) - (\nabla u, (\nabla (u_m)_\varepsilon)) \right] dt. \end{aligned}$$

Our first goal is to study in the previous equality, taking first the limit for $m \rightarrow \infty$, with $\varepsilon > 0$ fixed, and then the limit as $\varepsilon \rightarrow 0^+$. As usual, in passing to the limit the term which requires more care is the nonlinear one. So we focus on the following

$$\int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon) dt = \int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) ((u(t) \cdot \nabla) u(t), u_m(\tau)) d\tau dt.$$

For this purpose we split it into three terms as follows

$$(3.5) \quad \int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon) dt = \int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon - (u)_\varepsilon) dt \\ + \int_0^{t_0} ((u \cdot \nabla) u, (u)_\varepsilon - u) dt + \int_0^{t_0} ((u \cdot \nabla) u, u) dt,$$

and show that: a) the first term from the right-hand side converges to zero, as $m \rightarrow +\infty$, with $\varepsilon > 0$ fixed; b) the second term from the right-hand side converges to zero, as $\varepsilon \rightarrow 0^+$; c) the last term one is exactly zero, provided that u is a weak solutions satisfying one of the conditions of Theorem 1. We note that the splitting is a little bit different from the usual one employed in [41, 37, 43], since the hypotheses are now on the gradient of the velocity, instead of the velocity itself. For this reason we give detailed proofs of the steps which are not the same as in the previous papers.

We start by handling the point c) regarding the third term and proving that

$$(3.6) \quad \int_0^{t_0} ((u \cdot \nabla) u, u) dt = 0,$$

distinguishing among the three ranges given by Theorem 1.

Let $\{u_m\} \subset C_0^\infty([0, T[; C_0^\infty(\Omega))$ be a sequence converging to u in the space $L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^p(0, T; \dot{W}^{1,q}(\Omega))$, where $\dot{W}^{1,q}(\Omega)$ denotes as usual the homogeneous space, endowed with the semi-norm $\|\nabla u\|_{L^q}$. Since the field u_m is smooth, integrating by parts and since is redefined in such a way that $u(t) \in H$ for all $t \in [0, T)$, we get

$$\int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt = 0.$$

Thus, if we are able to show that

$$\int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt \rightarrow \int_0^{t_0} ((u \cdot \nabla) u, u) dt,$$

this would imply (3.6). To this end let us we handle the absolute value of the difference as follows

$$\left| \int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt - \int_0^{t_0} ((u \cdot \nabla) u, u) dt \right| \\ \leq \left| \int_0^{t_0} ((u \cdot \nabla) u_m, (u_m - u)) dt \right| + \left| \int_0^{t_0} ((u \cdot \nabla) (u_m - u), u) dt \right|.$$

and we will show the convergence of the two terms from the right hand side to zero in the three different cases.

In the sequel we will use the notation $a \lesssim b$ to denote that there exists a constant $C >$, depending only on p, q, Ω, T through the various inequalities valid in Lebesgue and Sobolev spaces, but not depending on the solution u itself, such that $a \leq C b$.

- (i) In the case $\frac{3}{2} < q < \frac{9}{5}$ we have $\nabla u \in L^p(0, T; L^q(\Omega))$ with $p = \frac{q}{2q-3}$. Let us note that in this range we have $q^* < 2q' < 6$, where $1/q' + 1/q = 1$ and q' is the conjugate exponent of q . Hence, we can interpolate and write that $\tilde{q} := 2q'$ satisfies $1/\tilde{q} = \theta/q^* + (1 - \theta)/6$,

which gives us $\theta := (3 - 2q)/(3(q - 2))$. Now, we can estimate, by the Sobolev embedding theorems, the space integral, obtaining

$$\begin{aligned}
\left| \int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \right| &\leq \left| \int_0^{t_0} \|u\|_{\tilde{q}} \|\nabla(u_m - u)\|_q \|u\|_{\tilde{q}} dt \right| \\
&\leq \left| \int_0^{t_0} \|u\|_{q^*}^{2\theta} \|u\|_6^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\
&\lesssim \left| \int_0^{t_0} \|\nabla u\|_q^{2\theta} \|\nabla u\|_2^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\
&\lesssim \|\nabla u\|_{p,q}^{2\theta} \|\nabla u\|_{2,2}^{2(1-\theta)} \|\nabla(u_m - u)\|_{p,q},
\end{aligned}$$

where we have applied in the last step Hölder inequality in the time variable with exponents x, y, z such that $2(1-\theta)x = 2$, $z = p$, and $2\theta y = p$, and thus satisfying $1/x + 1/y + 1/z = 1$. This shows that $\int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \rightarrow 0$, as $m \rightarrow +\infty$.

Analogously, we have also the following estimates

$$\begin{aligned}
&\left| \int_0^{t_0} ((u \cdot \nabla) u_m, (u_m - u)) dt \right| \\
&\leq \left| \int_0^{t_0} \|u\|_{\tilde{q}} \|\nabla u_m\|_q \|u_m - u\|_{\tilde{q}} dt \right| \\
&\leq \left| \int_0^{t_0} \|u\|_{q^*}^\theta \|u\|_6^{(1-\theta)} \|\nabla u_m\|_q \|u_m - u\|_{q^*}^\theta \|u_m - u\|_6^{(1-\theta)} dt \right| \\
&\lesssim \left| \int_0^{t_0} \|\nabla u\|_q^\theta \|\nabla u\|_2^{(1-\theta)} \|\nabla u_m\|_q \|\nabla(u_m - u)\|_q^\theta \|\nabla(u_m - u)\|_2^{(1-\theta)} dt \right| \\
&\lesssim \|\nabla u\|_{p,q}^\theta \|\nabla u\|_{2,2}^{(1-\theta)} \|\nabla u_m\|_{p,q} \|\nabla(u_m - u)\|_{p,q}^\theta \|\nabla(u_m - u)\|_{2,2}^{(1-\theta)},
\end{aligned}$$

where we used Hölder inequality in the time variable with exponents $\tilde{x} = 2x$ for both the terms $\|\nabla u\|_2^{(1-\theta)}$ and $\|\nabla(u_m - u)\|_2^{(1-\theta)}$, with $\tilde{y} = 2y$ for both the terms $\|\nabla u\|_q^\theta$ and $\|\nabla(u_m - u)\|_q^\theta$, and with $\tilde{z} = z$ for the term $\|\nabla u_m\|_q$. This implies that, as $m \rightarrow +\infty$ it holds $\int_0^{t_0} ((u \cdot \nabla) u_m, (u_m - u)) dt \rightarrow 0$.

The case (ii) is better handled when split into two sub-cases.

- (ii)₁ In the case $\frac{9}{5} \leq q < \frac{12}{5}$ we have $\nabla u \in L^p(0, T; L^q(\Omega))$ with $p = \frac{5q}{5q-6}$. Let us note that in this range we have $2 < 2q' < q^*$, where $1/q' + 1/q = 1$. Hence, we can interpolate and write that $\tilde{q} := 2q'$ satisfies $1/\tilde{q} = \theta/2 + (1-\theta)/q^*$, which implies $\theta := (5q - 9)/(5q - 6)$. Now, by using the Sobolev embedding theorems, we show that

$$\begin{aligned}
\left| \int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \right| &\leq \left| \int_0^{t_0} \|u\|_{\tilde{q}} \|\nabla(u_m - u)\|_q \|u\|_{\tilde{q}} dt \right| \\
&\leq \left| \int_0^{t_0} \|u\|_2^{2\theta} \|u\|_{q^*}^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\
&\lesssim \left| \int_0^{t_0} \|u\|_2^{2\theta} \|\nabla u\|_q^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\
&\lesssim \|u\|_{\infty,2}^{2\theta} \|\nabla u\|_{p,q}^{2(1-\theta)} \|\nabla(u_m - u)\|_{p,q},
\end{aligned}$$

thanks to Hölder inequality in the time variable with exponents γ_1, γ_2 such that $2(1-\theta)\gamma_1 = p$, $\gamma_2 = p$, with $1/\gamma_1 + 1/\gamma_2 = 1$. This shows that $\int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \rightarrow 0$. The term $\left| \int_0^{t_0} ((u \cdot \nabla) u_m, (u_m - u)) dt \right|$ can be treated in a very similar way, so that to conclude that $\int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt \rightarrow \int_0^{t_0} ((u \cdot \nabla) u, u) dt$.

(ii)₂ In the case $\frac{12}{5} \leq q \leq 3$ we have $\nabla u \in L^p(0, T; L^q(\Omega))$ with $p = \frac{5q}{5q-6}$. Let us denote by r the exponent such that $1/r + 1/q = 1/2$. Let us note that in this range we have $r < q^*$. Hence, we can use the convex interpolation between 2 and q^* to write that r satisfies $1/r = \theta/2 + (1-\theta)/q^*$, which gives us $\theta := (5q - 12)/(5q - 6)$. Now, by the Sobolev embedding theorems, we can estimate the integral as follows

$$\begin{aligned} \left| \int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \right| &\leq \left| \int_0^{t_0} \|u\|_2 \|\nabla(u_m - u)\|_q \|u\|_r dt \right| \\ &\leq \left| \int_0^{t_0} \|u\|_2^{1+\theta} \|\nabla(u_m - u)\|_q \|u\|_{q^*}^{1-\theta} dt \right| \\ &\lesssim \left| \int_0^{t_0} \|u\|_2^{1+\theta} \|\nabla(u_m - u)\|_q \|\nabla u\|_q^{1-\theta} dt \right| \\ &\lesssim \|u\|_{\infty, 2}^{1+\theta} \|\nabla(u_m - u)\|_{p, q} \|\nabla u\|_{p, q}^{1-\theta}, \end{aligned}$$

where we have applied Hölder inequality with respect to the time variable with exponents α_1, α_2 such that $(1-\theta)\alpha_1 = p$ and $\alpha_2 = p$, with $1/\alpha_1 + 1/\alpha_2 = 1$. For the other term we can argue similarly and thus this shows that $\int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt \rightarrow \int_0^{t_0} (u \cdot \nabla u, u) dt$ in this case, as well.

The case $q = 3$ can be treated in a mostly similar way, by observing that $W^{1,3}(\Omega) \subset L^r(\Omega)$, for all finite $3 \leq r$, while the rest of the proof remains unchanged.

(iii) In case $q > 3$ we have $\nabla u \in L^p(0, T; L^q(\Omega))$ with $p = 1 + \frac{2}{q}$. Let us denote by q' the conjugate exponent of q , i.e. $1/q' + 1/q = 1$ and by $1 < \tilde{q} := 2q' < \infty$. We estimate the $L^{\tilde{q}}(\Omega)$ -norm from convex interpolation between $L^2(\Omega)$ and $L^\infty(\Omega)$, with $1/\tilde{q} = \theta/2$, which gives us $\theta := 1 - \frac{1}{q}$. Now, by Sobolev embedding theorems

$$\begin{aligned} \left| \int_0^{t_0} ((u \cdot \nabla)(u_m - u), u) dt \right| &\leq \left| \int_0^{t_0} \|u\|_{\tilde{q}} \|\nabla(u_m - u)\|_q \|u\|_{\tilde{q}} dt \right| \\ &\lesssim \left| \int_0^{t_0} \|u\|_2^{2\theta} \|u\|_\infty^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\ &\lesssim \left| \int_0^{t_0} \|u\|_2^{2\theta} \|\nabla u\|_q^{2(1-\theta)} \|\nabla(u_m - u)\|_q dt \right| \\ &\lesssim \|u\|_{\infty, 2}^{2\theta} \|\nabla u\|_{p, q}^{2(1-\theta)} \|\nabla(u_m - u)\|_{p, q}, \end{aligned}$$

the last inequality being justified by use of the Hölder inequality in the time variable with exponents β_1, β_2 such that $2(1-\theta)\beta_1 = p$, and $\beta_2 = p$, such that $1/\beta_1 + 1/\beta_2 = 1$. As in the previous cases, we can treat also the other term in a similar way and conclude that $\int_0^{t_0} ((u \cdot \nabla) u_m, u_m) dt \rightarrow \int_0^{t_0} (u \cdot \nabla u, u) dt$, as $m \rightarrow +\infty$.

Collecting the above estimates we conclude the proof of (3.6), which is the most crucial step. The other terms will be treated below, but their analysis follows more classical path-lines, already outlined in previous works.

Next, for the sake of completeness, we show that the first two terms from the right hand side of (3.5) converge to zero. This follows easily using the usual properties of mollifiers together with similar estimates as above, distinguishing among the three different ranges. In case (i), going through the same lines of the preceding estimates and with same notation, indeed we can bound the first integral from the right hand side as follows

$$\begin{aligned}
& \left| \int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon - (u)_\varepsilon) dt \right| \\
& \leq \int_0^{t_0} \|u\|_{\tilde{q}} \|\nabla u\|_q \|(u_m)_\varepsilon - (u)_\varepsilon\|_{\tilde{q}} dt \\
& \leq \int_0^{t_0} \|u\|_{q^*}^\theta \|u\|_6^{1-\theta} \|\nabla u\|_q \|(u_m)_\varepsilon - (u)_\varepsilon\|_{q^*}^\theta \|(u_m)_\varepsilon - (u)_\varepsilon\|_6^{1-\theta} dt \\
& \lesssim \int_0^{t_0} \|\nabla u\|_q^{1+\theta} \|\nabla u\|_2^{1-\theta} \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_q^\theta \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_2^{1-\theta} dt \\
& \lesssim \|\nabla u\|_{p,q}^{1+\theta} \|\nabla u\|_{2,2}^{1-\theta} \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_{p,q}^\theta \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_{2,2}^{1-\theta},
\end{aligned}$$

where we have applied Hölder inequality in the time variable with exponents η_1, η_2, η_3 such that $(1+\theta)\eta_1 = p$, $(1-\theta)\eta_2 = 2$, and $\theta\eta_3 = p$, and satisfying $1/\eta_1 + 2/\eta_2 + 1/\eta_3 = 1$. Now, Lemma 2.5 in [27] ensures that

$$\lim_{m \rightarrow \infty} \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_{p,q} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \|\nabla((u_m)_\varepsilon - (u)_\varepsilon)\|_{2,2} = 0,$$

which, together with the previous estimate, implies that for each fixed $\varepsilon > 0$ it holds.

$$\lim_{m \rightarrow +\infty} \int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon - (u)_\varepsilon) dt = 0.$$

Cases (ii) and (iii) can be treated similarly and we skip the proof which is now straightforward.

The very same approach allows to bound the integral $\int_0^{t_0} ((u \cdot \nabla) u, (u)_\varepsilon - u) dt$ as follows

$$\left| \int_0^{t_0} ((u \cdot \nabla) u, (u)_\varepsilon - u) dt \right| \lesssim \|\nabla u\|_{p,q}^{1+\theta} \|\nabla u\|_{2,2}^{1-\theta} \|\nabla(u)_\varepsilon - u\|_{p,q}^\theta \|\nabla(u)_\varepsilon - u\|_{2,2}^{1-\theta},$$

which converge to zero as $\varepsilon \rightarrow 0^+$ once more thanks to Lemma 2.5 in [27].

By collecting all results we finally get that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{m \rightarrow +\infty} \int_0^{t_0} ((u \cdot \nabla) u, (u_m)_\varepsilon) dt = 0.$$

Passing to the limit as $m \rightarrow +\infty$ in (3.4) we then obtain

$$(u(t_0), (u)_\varepsilon(t_0)) = (u_0, (u)_\varepsilon(0)) + \int_0^{t_0} \left[\left(u, \frac{\partial(u)_\varepsilon}{\partial t} \right) - ((u \cdot \nabla) u, (u)_\varepsilon) - (\nabla u, (\nabla u)_\varepsilon) \right] dt.$$

and we now let $\varepsilon \rightarrow 0^+$ and the convective term vanishes. The term involving the time derivative of k_ε , i.e. $\int_0^{t_0} \left(u, \frac{\partial(u_m)_\varepsilon}{\partial t} \right)$ vanishes identically since k_ε is even. The usual properties of mollifiers from (3.3) imply that in the limit $\varepsilon \rightarrow 0^+$ we get

$$\|u(t_0)\|^2 + 2 \int_0^{t_0} \|\nabla u(s)\|^2 ds = \|u_0\|^2,$$

which is (1.5) for $t = t_0$. By the arbitrariness of t_0 , the proof of Theorem 1 is complete. \square

4 On the energy equality for distributional solutions.

In this section we prove Theorem 2 concerning the energy equality for a class of solution which do not satisfy a priori the energy inequality. The proof follows by a duality argument as done in [28], who treated the case $r = s = 4$. Here, we cover the full range of exponents by following a similar technique, but performing a more detailed analysis of the regularity of the solution of the adjoint problem. We point out that also in our case, the main point is to show that a very-weak solution (in the sense of Definition 2) with initial datum in H and such that (1.2) holds true becomes a Leray-Hopf weak solution, hence satisfies the energy equality by the classical results from [43]. From the proof and from the analysis performed in Sec. 2.3 it turns out that less stringent hypotheses (as those considered in the Theorem 1) are not sufficient to conclude the proof, hence condition (1.2) seems not improvable for very weak solutions, at present.

By following the same notation from [7, 28], given $g : (0, T) \times \Omega \rightarrow \mathbb{R}^3$ we define the function \tilde{g} as follows

$$\tilde{g}(t, x) := g(T - t, x).$$

In the sequel we will need again to approximate in a suitable way functions defined on $(0, T) \times \Omega \rightarrow \mathbb{R}^3$, mollifying also in the space variables.

Remark 4.1. *In the case of the Cauchy problem the smoothing can be done simply by mollifying also in the space variables (as in [28]) while in the case of an exterior domain one can use practically the same approach we propose, by using the result by Borchers and Sohr [10] in the appropriate setting.*

For this purpose we define

$$\Omega_\varepsilon := \left\{ x \in \Omega \subset \mathbb{R}^3 : |x| < \frac{1}{\varepsilon}, d(x, \partial\Omega) \geq 2\varepsilon \right\},$$

with $d(x, A)$ the Euclidean distance from $x \in \mathbb{R}^3$ to the closed set $A \subset \mathbb{R}^3$. For small enough $\varepsilon > 0$, the set Ω_ε turns out to be bounded and non-empty. Then, by taking $\rho \in C_0^\infty(\mathbb{R}^3)$, non-negative, radial, and such that $\int_{\mathbb{R}^3} \rho(x) dx = 1$, we define as usual the scaled $\rho_\varepsilon(x) = \varepsilon^{-3} \rho(x\varepsilon^{-1})$. Moreover, if χ_A denotes the indicator function of the measurable set $A \subset \mathbb{R}^3$, we define, for any $v \in (L_{loc}^1(\Omega))^3$, the function ${}^\varepsilon v(x) \in (C_0^\infty(\Omega))^3$ as follows

$${}^\varepsilon v(x) := [(v \chi_{\Omega_\varepsilon}) * \rho_\varepsilon](x).$$

It follows that ${}^\varepsilon v$ is smooth and with compact support by the standard properties of mollifiers, but $\nabla \cdot ({}^\varepsilon v) \neq 0$. For the energy estimates used later on we need a divergence-free approximation, hence one possibility is to use the Bogovskii [9] operator $\mathcal{B}[f]$ defined as follows

$$\mathcal{B}[f](x) := \int_\Omega f(y) \left[\frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi \left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy,$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$, with support in the unit ball and $\int_{\mathbb{R}^3} \psi(y) dy = 1$, when Ω contains the unit ball. (Some changes with scaling and translations are needed if this is not satisfied).

It is well-known that if Ω is smooth and bounded and if $f \in C_0^\infty(\Omega)$, then $\mathcal{B}[f] \in C_0^\infty(\Omega)$ and

$$\nabla \cdot \mathcal{B}[f](x) = f(x) - \frac{1}{|\Omega|} \int_{\Omega} f(y) dy.$$

In addition, if $f \in L^s(\Omega)$, with $1 < s < \infty$, then $\mathcal{B}[f] \in W_0^{1,s}(\Omega)$ and moreover it holds (for fixed ψ) that $\|f\|_{W^{1,s}} \leq C(s, \Omega) \|f\|_{L^s}$, see for instance the review in Galdi [27]. We thus define,

$$v^{(\varepsilon)}(x) := \varepsilon v(x) - \mathcal{B}[\nabla \cdot (\varepsilon v)](x),$$

and since $\int_{\Omega} \nabla \cdot (\varepsilon v) dx = \int_{\partial\Omega} \varepsilon v \cdot n dS = 0$, it follows that if $v \in L^s(\Omega)$, then we have the following relevant properties

- i) $v^{(\varepsilon)}(x) \in C_0^\infty(\Omega)$, with $\nabla \cdot v^{(\varepsilon)} = 0$;
- ii) $\|v^{(\varepsilon)}(x)\|_{L^s} \leq C(s, \Omega) \|v\|_{L^s}$, uniformly in $\varepsilon > 0$;
- iii) $v^{(\varepsilon)} \rightarrow v$ in $L^s(\Omega)$, as $\varepsilon \rightarrow 0^+$.

We apply this procedure to the initial datum $u_0 \in L^2(\Omega)$ and we define the family of functions $\{u_0^{(\varepsilon)}\}_{\varepsilon>0} \subset C_0^\infty(\Omega)$ in such a way that

$$\nabla \cdot u_0^{(\varepsilon)} = 0 \quad \text{and also} \quad \lim_{\varepsilon \rightarrow 0^+} \|u_0^{(\varepsilon)} - u_0\| = 0.$$

We also need to apply a similar smoothing procedure to space-time functions. Hence, given $u \in L^r(0, T; L^s(\Omega))$ we define $u^{(\varepsilon)}$ with the above technique obtaining a family $u^{(\varepsilon)}$ of functions which are divergence-free and smooth in the space variables. To regularize also in the time variable we use the same Friederichs mollifier as introduced in (3.1) and we define, for $0 < \varepsilon < T$

$$u_{(\varepsilon)}(t, x) := \int_0^T k_\varepsilon(t - \tau) u^{(\varepsilon)}(\tau, x) d\tau,$$

in such a way that

$$(4.1) \quad \begin{aligned} i) & \quad u_{(\varepsilon)}(t, x) \in L^\infty((0, T) \times \Omega), \\ ii) & \quad \nabla \cdot u_{(\varepsilon)} = 0, \\ iii) & \quad u_{(\varepsilon)} \rightarrow u \text{ in } L^r(0, T; L^s(\Omega)), \text{ as } \varepsilon \rightarrow 0^+. \end{aligned}$$

It turns out that $u_{(\varepsilon)}$ and also $\widetilde{u_{(\varepsilon)}}$ (that one with reversed time) are then suitable' vector fields to be used as a convection term in Oseen-type systems, in order to obtain global solution in the energy space, that is in $L^\infty(0, T; H) \cap L^2(0, T; V)$.

With this regularization at disposal, we can in fact start by considering $(u^\varepsilon, p^\varepsilon)$ solution of the linear Oseen problem

$$(4.2) \quad \begin{aligned} \partial_t u^\varepsilon - \Delta u^\varepsilon + (u_{(\varepsilon)} \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon &= 0 & \text{in } (0, T) \times \Omega, \\ \nabla \cdot u^\varepsilon &= 0 & \text{in } (0, T) \times \Omega, \\ u^\varepsilon(t, x) &= 0 & \text{on } (0, T) \times \partial\Omega, \\ u^\varepsilon(0, x) &= u_0^{(\varepsilon)} & \text{in } \Omega, \end{aligned}$$

and by the standard Galerkin method (combined with the method of invading domains if the domain is unbounded), it turns out that there exists a unique (distributional) solution $(u^\varepsilon, p^\varepsilon)$ such that

$$u^\varepsilon \in W^{1,2}(0, T; H) \cap L^2(0, T; H^2 \cap V) \subset C(0, T; H_0^1(\Omega)) \quad \text{and} \quad \nabla p^\varepsilon \in L^2((0, T) \times \Omega).$$

The main point is that $((u_{(\varepsilon)} \cdot \nabla) u^\varepsilon, u^\varepsilon) = 0$ and $(\nabla p^\varepsilon, u^\varepsilon) = 0$, coming from the divergence free constraint on both $u_{(\varepsilon)}$ and u^ε ; hence, one can show that in the sense of $\mathcal{D}'(0, T)$

$$\frac{1}{2} \frac{d}{dt} \|u^\varepsilon\|^2 + \nu \|\nabla u^\varepsilon\|^2 = 0,$$

and from this one can prove a first a priori estimate and then use the standard theory for linear equation systems of Stokes type to prove existence and regularity of the solution.

By the definition of very-weak solution we obtain that the following identity is satisfied by the difference $u - u^\varepsilon$

$$(4.3) \quad \int_0^T (u - u^\varepsilon, \partial_t \phi + \Delta \phi + (u_{(\varepsilon)} \cdot \nabla) \phi) \, ds = \int_0^T ((u - u_{(\varepsilon)}) \cdot \nabla \phi, u) - (u_0 - u_0^{(\varepsilon)}, \phi(0)),$$

for all $\phi \in \mathcal{D}_T$.

Let us consider now, for a given $f \in C_0^\infty((0, T) \times \Omega)$ the solution $(w^\varepsilon, \pi^\varepsilon)$ of the linear Oseen problem

$$(4.4) \quad \begin{aligned} \partial_t w^\varepsilon - \Delta w^\varepsilon - (\widetilde{u_{(\varepsilon)}} \cdot \nabla) w^\varepsilon + \nabla \pi^\varepsilon &= -\widetilde{f} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot w^\varepsilon &= 0 & \text{in } (0, T) \times \Omega, \\ w^\varepsilon(t, x) &= 0 & \text{on } (0, T) \times \partial\Omega, \\ w^\varepsilon(0, x) &= 0 & \text{in } \Omega, \end{aligned}$$

and define

$$\Psi^\varepsilon(t, x) := \widetilde{w^\varepsilon}(t, x) = w^\varepsilon(T - t, x) \quad \text{and} \quad \Xi^\varepsilon(t, x) := \widetilde{\pi^\varepsilon}(t, x) = \pi^\varepsilon(T - t, x),$$

in such a way that they have the same regularity as $(w^\varepsilon, \pi^\varepsilon)$ and they solve the *final value problem*

$$(4.5) \quad \begin{aligned} \partial_t \Psi^\varepsilon + \Delta \Psi^\varepsilon + (u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon - \nabla \Xi^\varepsilon &= f & \text{in } (0, T) \times \Omega, \\ \nabla \cdot \Psi^\varepsilon &= 0 & \text{in } (0, T) \times \Omega, \\ \Psi^\varepsilon(t, x) &= 0 & \text{on } (0, T) \times \partial\Omega, \\ \Psi^\varepsilon(T, x) &= 0 & \text{in } \Omega. \end{aligned}$$

By the standard theory of Galerkin methods it follows again that

$$\Psi^\varepsilon \in W^{1,2}(0, T; H) \cap L^2(0, T; H^2 \cap V) \subset C(0, T; H_0^1(\Omega)) \quad \text{and} \quad \nabla \Xi^\varepsilon \in L^2((0, T) \times \Omega),$$

and we would like to use Ψ^ε as test function in (4.3), instead of $\phi \in \mathcal{D}_T$. This can be done by using standard density arguments to approximate Ψ^ε in \mathcal{D}_T with uniform bounds in $\varepsilon > 0$, provided that Ψ^ε is smooth enough independently of ε to make all integrals well-defined.

Proof of Theorem 2. The proof follows by considering (4.3) with test function given by Ψ^ε . Formally (calculations would be rigorous by using a smooth approximation of Ψ^ε and this can be constructed by standard density arguments, as recalled in the appendix of [28]) we obtain that

$$(4.6) \quad \int_0^T (u - u^\varepsilon, f + \nabla \Xi^\varepsilon) \, ds = \int_0^T ((u - u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon, u) \, ds - (u_0 - u_0^{(\varepsilon)}, \Psi^\varepsilon(0)),$$

but the argument will be completely rigorous after we have proved that $(\Psi^\varepsilon, \Xi^\varepsilon)$ is smooth enough. This will come by a proper use of the maximal regularity arguments for the Stokes system and by standard density arguments

First, we need to prove that the identity

$$\int_0^T (u - u^\varepsilon, \nabla \Xi^\varepsilon) \, ds = 0,$$

is valid, showing enough regularity on Ξ^ε and using the fact that $\nabla \cdot (u - u^\varepsilon) = 0$ in $\mathcal{D}'(\Omega)$ a.e $t \in (0, T)$. Next, by taking the limit as $\varepsilon \rightarrow 0^+$, the uniform bound in $L^2(\Omega)$ of $\Psi^\varepsilon(0)$ implies that

$$\lim_{\varepsilon \rightarrow 0^+} (u_0 - u_0^{(\varepsilon)}, \Psi^\varepsilon(0)) = 0.$$

Since $u \in L^r(0, T; L^s(\Omega))$, then $u - u_{(\varepsilon)} \rightarrow 0$ in $L^r(0, T; L^s(\Omega))$ and

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^T ((u - u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon, u) \, ds = 0,$$

provided that $|\int_0^T ((u - u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon, u) \, ds| < \infty$, and it is at this point that we will use the assumption (1.2) and its consequences on the regularity of the Oseen system.

In particular, we have to show that

$$\nabla \Xi^\varepsilon, ((u - u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon) \in L^{r'}(0, T; L^{s'}(\Omega))$$

or equivalently, by recalling the link between r and s from (1.2),

$$(4.7) \quad \nabla \Xi^\varepsilon, ((u - u_{(\varepsilon)} \cdot \nabla) \Psi^\varepsilon) \in L^{\frac{2s}{2+s}}(0, T; L^{\frac{s}{s-1}}(\Omega)),$$

the second coming from the condition $\nabla \Psi^\varepsilon \in L^{\frac{s}{2}}(0, T; L^{\frac{s}{s-2}}(\Omega))$ by using the Hölder inequality.

Once the above steps are done, one can approximate Ξ^ε and Ψ^ε by a family of smooth compactly supported functions $\Theta_\eta^\varepsilon \in \mathcal{D}_T$ and $\Upsilon_\eta^\varepsilon \in C_0^\infty((0, T) \times \Omega)$, with uniform bounds in $\eta > 0$ in the spaces $L^{r'}(0, T; W^{1, s'}(\Omega))$ and $L^{\frac{s}{2}}(0, T; W^{1, \frac{s}{s-2}}(\Omega))$, respectively. These smooth functions Θ_η^ε and $\Upsilon_\eta^\varepsilon$, can be then employed as legitimate test functions in the definition of very-weak solution. We do not explicitly write this since it will be clear to the reader how to proceed, hence we simply show the requested regularity.

Once this step is performed, the uniform bounds of u^ε in $L^\infty(0, T; H) \cap L^2(0, T; V)$ imply –by standard results about weak compactness– the existence of $U \in L^\infty(0, T; H) \cap L^2(0, T; V)$ such that, along some sub-sequence $\{\varepsilon_n\}$ converging to 0^+ as $n \rightarrow +\infty$

$$\begin{aligned} u^{\varepsilon_n} &\overset{*}{\rightharpoonup} U && \text{weakly}^* \text{ in } L^\infty(0, T; H) \subset L^\infty(0, T; L^2(\Omega)), \\ u^{\varepsilon_n} &\rightharpoonup U && \text{weakly in } L^2(0, T; V) \subset L^2(0, T; H_0^1(\Omega)). \end{aligned}$$

hence the taking the limit as $n \rightarrow +\infty$ in (4.6) considered with $\varepsilon = \varepsilon_n$ we obtain

$$\int_0^T (u - U, f) ds \xrightarrow{n \rightarrow +\infty} \int_0^T (u - u^{\varepsilon_n}, f) ds \xrightarrow{n \rightarrow +\infty} 0 \quad \forall f \in C_0^\infty((0, T) \times \Omega),$$

hence $\int_0^T (u - U, f) ds = 0$ and, by the arbitrariness of the smooth f , this shows that u can be identified with $U \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$. This proves that u falls into the class of Leray-Hopf weak solutions, for which the classical results from [43] imply the energy equality.

In order to conclude the proof we need now to justify the method and in particular the fact that the integrals involving Ψ^ε and Ξ^ε are well-defined, with the available regularity of u from the assumption (1.2). To this end, observe that the starting point of our analysis is the fact that $\nabla w^\varepsilon \in L^2(0, T; L^2(\Omega))$ and also $u \in L^r(0, T; L^s(\Omega))$ is a very-weak solution of (1.1) with (1.2).

By using Propositions 2.3 and 2.4, we observe that under the hypotheses of (1.2) (that is $\gamma = 1/2$) it follows that $(\alpha_0, \beta_0) = (2, 2)$ and the n -th iteration of the maximal regularity results from Proposition 2.4 shows that, for $n \geq 1$

$$(\widetilde{u_{(\varepsilon)}} \cdot \nabla) w^\varepsilon, \nabla \pi^\varepsilon \in L^{\alpha_n}(0, T; L^{\beta_n}(\Omega)) \quad \text{with } \beta_n = \frac{\beta_{n-1} s}{\beta_{n-1} + s} \quad \text{and } \frac{1}{\alpha_n} + \frac{1}{\beta_n} = \frac{3}{2},$$

which reduces, in terms of $\Phi^\varepsilon, \Xi^\varepsilon$, and more explicitly in terms of the parameter $s > 4$, to

$$\alpha_n = \frac{s}{s - n} \quad \text{and} \quad \beta_n = \frac{2s}{2n + s}$$

and to have then

$$(4.8) \quad (\widetilde{u_{(\varepsilon)}} \cdot \nabla) \Psi^\varepsilon, \nabla \Xi^\varepsilon \in L^{\frac{s}{s-n}}(0, T; L^{\frac{2s}{2n+s}}(\Omega)) \quad \text{uniformly in } \varepsilon > 0.$$

Observe that the norms of the above functions depend on the number $n \in \mathbb{N}$ of applications of the maximal parabolic regularity, on the domain Ω , on the value of $s > 4$, on the norm $\|u\|_{r,s}$ (and the other data of the problem), but are independent of the relevant quantities.

We note that the case $s = 4$ has been treated in [28], hence we will not study it (even if comes after a single simpler step).

We distinguish between two cases depending on the fact that $s \in \mathbb{R}$ is or not an even number.

In the case $s > 4$ is an even number, that is $s = 2m$, with $2 < m \in \mathbb{N}$, it follows that with $m - 1$ steps it holds

$$\alpha_{m-1} = \frac{2s}{2 + s} \quad \text{and} \quad \beta_{m-1} = \frac{s}{s - 1},$$

that is exactly the regularity from (4.7).

In the case $s > 4$ is not an even integer, we observe that β_n is monotonically strictly decreasing, hence we determine $N \in \mathbb{N}$ solving the inequalities

$$1 < \beta_{N+1} < \frac{s}{s - 1} \leq \beta_N,$$

obtaining

$$\frac{s}{2} - 2 \leq N < \frac{s}{2} - 1$$

and the corresponding value of $N \in \mathbb{N}$ is

$$N := \left[\frac{s}{2} \right] - 1,$$

where $[x]$ denotes as usual the integer part of $x \in \mathbb{R}$.

At this point we can interpolate to obtain

$$L^{\frac{s}{s-1}}(\Omega) = \left[L^{\beta_N}(\Omega), L^{\beta_{N+1}}(\Omega) \right]_{\theta} \quad \text{with} \quad \theta = \frac{1}{2} \left(2 \left[\frac{s}{2} \right] - s + 2 \right)$$

and consequently to obtain with the same $\theta = \frac{1}{2} \left(2 \left[\frac{s}{2} \right] - s + 2 \right)$

$$\nabla \Xi, ((u - u_{(\varepsilon)}) \cdot \nabla) \Psi^{\varepsilon} \in \left[L^{\alpha_N}(0, T; L^{\beta_N}(\Omega)), L^{\alpha_{N+1}}(0, T; L^{\beta_{N+1}}(\Omega)) \right]_{\theta} = L^{\frac{2s}{2+s}}(0, T; L^{\frac{s}{s-1}}(\Omega)),$$

since, by direct computation $\frac{2+s}{2s} = \frac{\theta}{\alpha_N} + \frac{1-\theta}{\alpha_{N+1}}$, ending the proof. \square

References

- [1] H. Amann. On the strong solvability of the Navier-Stokes equations. *J. Math. Fluid Mech.*, 2(1):16–98, 2000.
- [2] H. Amann. *Linear and quasilinear parabolic problems. Vol. II*. Monographs in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2019, to appear. Functions spaces and linear differential operators.
- [3] H. Beirão da Veiga. A new regularity class for the Navier-Stokes equations in \mathbf{R}^n . *Chinese Ann. Math. Ser. B*, 16(4):407–412, 1995. A Chinese summary appears in *Chinese Ann. Math. Ser. A* **16** (1995), no. 6, 797.
- [4] H. Beirão da Veiga and J. Yang. On the energy equality for solutions to Newtonian and non-Newtonian fluids. Technical Report 1901.02277, ArXiv, 2019.
- [5] L. C. Berselli. On a regularity criterion for the solutions to the 3D Navier-Stokes equations. *Differential Integral Equations*, 15(9):1129–1137, 2002.
- [6] L. C. Berselli and F. Flandoli. On a stochastic approach to eddy viscosity models for turbulent flows. In *Advances in mathematical fluid mechanics*, pages 55–81. Springer, Berlin, 2010.
- [7] L. C. Berselli and G. P. Galdi. On the space-time regularity of $C(0, T; L^n)$ -very weak solutions to the Navier-Stokes equations. *Nonlinear Anal.*, 58(5-6):703–717, 2004.
- [8] L. C. Berselli, T. Iliescu, and W. J. Layton. *Mathematics of Large Eddy Simulation of turbulent flows*. Scientific Computation. Springer-Verlag, Berlin, 2006.
- [9] M. E. Bogovskiĭ. Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980.
- [10] W. Borchers and H. Sohr. On the equations $\text{rot } \mathbf{v} = \mathbf{g}$ and $\text{div } \mathbf{u} = f$ with zero boundary conditions. *Hokkaido Math. J.*, 19(1):67–87, 1990.
- [11] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., and Vicol V. Onsager’s conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.

- [12] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager’s conjecture for the Euler equations. *Nonlinearity*, 21(6):1233–1252, 2008.
- [13] A. Cheskidov, S. Friedlander, and R. Shvydkoy. On the energy equality for weak solutions of the 3D Navier-Stokes equations. In *Contributions to current challenges in mathematical fluid mechanics*, Adv. Math. Fluid Mech., pages 171–175. Birkhäuser, Basel, 2010.
- [14] A Cheskidov and X. Luo. Energy equality for the Navier-Stokes equations in weak-in-time Onsager spaces. Technical Report 1802.05785, ArXiv, 2018.
- [15] P. Constantin. Navier-Stokes equations and area of interfaces. *Comm. Math. Phys.*, 129(2):241–266, 1990.
- [16] P. Constantin, W. E, and E.S. Titi. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.
- [17] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
- [18] Ronald A. DeVore. Nonlinear approximation. In *Acta numerica, 1998*, volume 7 of *Acta Numer.*, pages 51–150. Cambridge Univ. Press, Cambridge, 1998.
- [19] E. B. Fabes, B. F. Jones, and N. M. Rivière. The initial value problem for the Navier-Stokes equations with data in L^p . *Arch. Rational Mech. Anal.*, 45:222–240, 1972.
- [20] R. Farwig. On regularity of weak solutions to the instationary Navier-Stokes system: a review on recent results. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 60(1):91–122, 2014.
- [21] R. Farwig, Y. Giga, and P.-Y. Hsu. Initial values for the Navier-Stokes equations in spaces with weights in time. *Funkcial. Ekvac.*, 59(2):199–216, 2016.
- [22] R. Farwig, H. Kozono, and H. Sohr. Very weak solutions of the Navier-Stokes equations in exterior domains with nonhomogeneous data. *J. Math. Soc. Japan*, 59(1):127–150, 2007.
- [23] R. Farwig and Y. Taniuchi. On the energy equality of Navier-Stokes equations in general unbounded domains. *Arch. Math. (Basel)*, 95(5):447–456, 2010.
- [24] F. Flandoli, M. Gubinelli, M. Hairer, and M. Romito. Rigorous remarks about scaling laws in turbulent fluids. *Comm. Math. Phys.*, 278(1):1–29, 2008.
- [25] C. Foias. Une remarque sur l’unicité des solutions des équations de Navier-Stokes en dimension n . *Bull. Soc. Math. France*, 89:1–8, 1961.
- [26] G. P. Galdi. An introduction to the Navier-Stokes initial-boundary value problem. In *Fundamental directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., pages 1–70. Birkhäuser, Basel, 2000.
- [27] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2011.
- [28] G. P. Galdi. On the energy equality for distributional solutions to Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 2018. online first.

- [29] G. P. Galdi. On the relation between very weak and Leray-Hopf solutions to Navier–Stokes equations. Technical Report arXiv:1809.03991, ArXiv, 2018.
- [30] Y. Giga and H. Sohr. Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Funct. Anal.*, 102(1):72–94, 1991.
- [31] E. Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.
- [32] P. Isett. A proof of Onsager’s conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.
- [33] A. Kupiainen. Statistical theories of turbulence. In: *Advances in Mathematical Sciences and Applications*, Gakkotosho, Tokyo,, 2003.
- [34] O. A. Ladyžhenskaya and A. A. Kiselev. On the existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid. *Izv. Akad. Nauk SSSR Ser. Mat.*, 21:655–680, 1957.
- [35] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.
- [36] T. M. Leslie and R. Shvydkoy. Conditions implying energy equality for weak solutions of the Navier-Stokes equations. *SIAM J. Math. Anal.*, 50(1):870–890, 2018.
- [37] J.-L. Lions. Sur la régularité et l’unicité des solutions turbulentes des équations de Navier Stokes. *Rend. Sem. Mat. Univ. Padova*, 30:16–23, 1960.
- [38] P.-L. Lions and N. Masmoudi. Uniqueness of mild solutions of the Navier-Stokes system in L^N . *Comm. Partial Differential Equations*, 26(11-12):2211–2226, 2001.
- [39] P. Maremonti. A note on Prodi-Serrin conditions for the regularity of a weak solution to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 20(2):379–392, 2018.
- [40] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (9)*, 6(Supplemento, 2 (Convegno Internazionale di Meccanica Statistica)):279–287, 1949.
- [41] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.
- [42] J. Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., pages 69–98*. Univ. of Wisconsin Press, Madison, Wis., 1963.
- [43] M. Shinbrot. The energy equation for the Navier-Stokes system. *SIAM J. Math. Anal.*, 5:948–954, 1974.
- [44] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.
- [45] H. Sohr. Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes. *Math. Z.*, 184(3):359–375, 1983.
- [46] V. A. Solonnikov. Estimates of the solution of a certain initial-boundary value problem for a linear nonstationary system of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 59:178–254, 257, 1976. Boundary value problems of mathematical physics and related questions in the theory of functions, 9.