INFINITE-DIMENSIONAL CARNOT GROUPS AND GÂTEAUX DIFFERENTIABILITY

ENRICO LE DONNE, SEAN LI, AND TERHI MOISALA

Abstract. This paper contributes to the generalization of Rademacher's differentiability result for Lipschitz functions when the domain is infinite dimensional and has nonabelian group structure. We introduce an infinite-dimensional analogue of Carnot groups that are metric groups equipped with dilations (which we call metric scalable groups) admitting a dense increasing sequence of finite-dimensional Carnot subgroups. For such groups, we show that every Lipschitz function has a point of Gâteaux differentiability. As a step in the proof, we show that a certain σ-ideal of sets that are null with respect to this sequence of subgroups cannot contain open sets. We also give a geometric criterion for when such Carnot subgroups exist in metric scalable groups and provide examples of such groups. The proof of the main theorem follows the work of Aronszajn [Aro76] and Pansu [Pan89].

Contents

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1. Introduction

Rademacher's theorem states that Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}$ are differentiable almost everywhere. This result has far-reaching consequences in Geometric Measure Theory and has been generalized in many ways over the past few decades. In the case of considering domains more general than $\mathbb{R}^n$, there have been two distinct branches. On the one side, extensions of Rademacher's theorem have been studied in infinite-dimensional vector spaces, where there does not exist a Lebesgue-like measure. On the other side, there has been interest in removing the vector-space assumption but preserving the structure of metric measure space. Extensions to more general target spaces have also been considered, but is not the focus of this paper.

Our goal is to extend the theorem to domains that are nonabelian and infinite dimensional. We will concentrate on $\mathbb{R}$-valued functions, although the results will hold for more general targets like RNP Banach spaces. We now quickly review previous results, discuss the issues present in both branches, and provide some references.

For the case of Banach space domain X, derivatives of a function f are linear mappings. However, in infinite-dimensional case there are two ways this may be interpreted. A function is Gâteaux differentiable at $x_0 \in X$ if there exists a linear function $T : X \to \mathbb{R}$ satisfying

$$T(v) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

for every $v \in X$. Instead, a function is Fréchet differentiable at x_0 if the map T satisfies

$$f(x_0 + v) = f(x_0) + T(v) + o(\|v\|) \text{ as } \|v\| \to 0.$$

Thus, for Gâteaux differentiability, the rate of convergence as $t \to 0$ can depend on v whereas it only depends on $\|v\|$ for Fréchet differentiability. Fréchet differentiability clearly implies Gâteaux differentiability, but the opposite does not hold in general in the infinite dimensional setting. In fact, Lipschitz functions $f : X \to \mathbb{R}$ always have points of Gâteaux differentiability whereas they may lack any point of Fréchet differentiability [Aro76].

In infinite-dimensional settings, one also needs to find a good notion of "almost everywhere". One can reinterpret Rademacher's theorem as stating that the nondifferentiability points lie in the σ-ideal of Lebesgue null sets. Thus, one aims to prove that the nondifferentiability points of Lipschitz functions lie in some suitable σ-ideal $\mathcal{N}$. To guarantee at least one point of differentiability, the σ-ideal $\mathcal{N}$ should not contain open sets. Results of this type have been found for both Gâteaux differentiability and Fréchet differentiability, although the Fréchet differentiability results are far harder and less broad [Aro76, Man73, LP03, Pre90].

When considering domains without a linear structure, one typically works in a metric measure space where "almost everywhere" has natural meaning. But resolving what a derivative means becomes more involved, and one requires some additional structure on the domain. For general metric spaces, one needs a collection of Lipschitz charts —which may not, in general, exist— to differentiate the given function f against as it was done by Cheeger in [Che99]. Other works expanding on this theory of differentiation include [BL18, CK09, EB16, Sch16]. In the special case of Carnot group domains, there are in addition group structure as well as a family of scaling automorphism. This allows us to define a derivative as the limit of rescaled difference ratios converging to a homomorphism as it was done by Pansu in [Pan89].

This paper seeks to bridge the infinite-dimensional framework with the Carnot group setting. Specifically, we will define infinite-dimensional variants of Carnot groups and consider Gâteaux differentiability in this context. We will show that if G is a metric group with a family of dilations (or metric scalable group, as we will call them) and has a dense collection of finite-dimensional Carnot subgroups, then there is a nontrivial σ-ideal $\mathcal{N}$ so that the Gâteaux non-differentiability points of any Lipschitz function $f : G \to \mathbb{R}$ form an element in $\mathcal{N}$.

We remark that there have been previous studies of infinite-dimensional variants of Carnot groups. Notably, in [MR14], the authors defined so-called Banach homogeneous groups and showed that Lipschitz functions from $\mathbb{R}$ to these groups are almost everywhere differentiable in the notion of Pansu. Metric scalable groups include these groups as special cases, but they also contain other examples.

Our investigations leave open several natural questions. Most notably, one can ask how small the points of Fréchet nondifferentiability of Lipschitz functions are for metric scalable groups. Even in Banach space domains, this problem is very hard and depends on fine geometric properties of the norm, and so we leave this problem for the future. One can also ask if there are infinite-dimensional variants of Cheeger differentiability. Here, the question becomes more subtle as the differentiability charts must take value in an infinite-dimensional Banach space for which there is no canonical choice. Finally, one would like to know when a metric scalable group is generated by its finite-dimensional subgroups. Specifically, are there geometric properties (geodicity, for example) that tell us when this is the case?

We begin by introducing the notion of scalable group that is the underlying structure of the metric groups with which we will be concerned.

Definition 1.1 (Scalable group)**.** A *scalable group* is a pair (G, δ), where G is a topological group and $\delta \colon \mathbb{R} \times G \to G$ is a continuous map such that $\delta_\lambda := \delta(\lambda, \cdot) \in \mathrm{Aut}(G)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu}, \qquad \forall\, \lambda, \mu \in \mathbb{R}, \tag{1.2}$$

and $\delta_0 \equiv e_G$, where e_G is the identity element of G.

Property (1.2) can be rephrased as follows: for every $p \in G$, the map $\delta_{(\cdot)}(p) \colon (\mathbb{R} \setminus \{0\}, \cdot) \to \mathrm{Aut}(G)$ is a homomorphism. Hence it follows that δ_1 is the identity map of G.

In an obvious way, in the setting of scalable groups, one can consider the notion of scalable subgroups; a subgroup H of a scalable group (G, δ) is called a *scalable subgroup* if G if $\delta_\lambda(H) = H$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. We denote then $H < G$. In order to talk about Lipschitz functions, we will endow these groups with metrics that make the dilation automorphisms δ_λ metric scalings in the following sense.

Definition 1.3 (Metric scalable group)**.** A *metric scalable group* is a triple (G, δ, d) where (G, δ) is a scalable group and d is an admissible left-invariant distance on G such that

$$d(\delta_t(p), \delta_t(q)) = |t| d(p, q), \qquad \forall\, t \in \mathbb{R}.$$

By *admissible*, we mean that the metric induces the given topology.

Every Carnot group naturally has structure of a scalable group, where by Carnot group G we mean a simply connected Lie group whose Lie algebra $\mathrm{Lie}(G)$ is equipped with a stratification $\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_s$. The stratification is unique up to an isomorphism, see [LD17], and it defines a family of dilations on G. Indeed, one considers the Lie group homomorphisms corresponding to the Lie algebra scalings defined by $\delta^*_\lambda(X) = \lambda^k X$ for $X \in V_k$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Such a group can be metrized as a metric scalable group, and the metric is unique up to biLipschitz equivalence. Vice versa, we say that a scalable group (G, δ) has a Carnot group structure if there exists a Carnot group that is isomorphic to G as a topological group and whose dilations given by the stratification coincide with δ.

Given a group structure with a dilation, we can define derivatives as done by Pansu [Pan89]. First, for any $g \in G$, let $L_g : G \to G$ be the left multiplication operator. As mentioned before, in the infinite-dimensional case, we need to take care of the distinction between Gâteaux and Fréchet differentiability. Here, we define Gâteaux differentiability.

Definition 1.4 (Gâteaux differentiability)**.** Given two scalable groups G and H, a map $f : G \to H$ is *Gâteaux differentiable* at a point $p \in G$ if, as $\lambda \to 0$, the maps $\hat{f}_{p,\lambda} := \delta_{\frac{1}{\lambda}} \circ L^{-1}_{f(p)} \circ f \circ L_p \circ \delta_\lambda$ pointwise converge to a continuous homomorphism from G to H. We denote this map by Df_p and it is called the *Gâteaux differential* of f at p.

Notice that if Df_p exists, then it is 1-homogeneous in the sense that $Df_p(\delta_\lambda(u)) = \delta_\lambda(Df_p(u))$ for all $\lambda \in \mathbb{R}$ and $u \in G$.

We now introduce a notion requiring that our groups, which are possibly infinite dimensional, are generated by finite-dimensional Carnot subgroups. This will be needed to show that the σ-ideal we define later is not trivial.

Definition 1.5 (Filtration by Carnot subgroups)**.** We say that a scalable group G is *filtrated by Carnot subgroups* if there exists a sequence $(N_m)_{m\in\mathbb{N}}$ of scalable subgroups of G such that each N_m has a Carnot group structure, $N_m < N_{m+1}$, and G is the closure of $\cup_{m\in\mathbb{N}} N_m$. In this case, we say that the sequence $(N_m)_{m\in\mathbb{N}}$ is *a filtration by Carnot subgroups of the scalable group* G.

We have now all the necessary data to give the definition of infinite-dimensional Carnot group.

Definition 1.6. We call a complete metric scalable group that admits a filtration by Carnot subgroups an *infinite-dimensional Carnot group*.

Necessarily, a metric scalable group that admits a filtration by Carnot subgroups is separable. Note that an infinite-dimensional Carnot group G cannot be equal to its filtration $\cup_{m\in\mathbb{N}} N_m$ unless $G = N_m$ for some $m \in \mathbb{N}$. Indeed, each $N_m \subseteq G$ is nowhere dense and hence the union $\cup_{m\in\mathbb{N}} N_m$ is of first category in G. We define next what it means for a set to be null (that is, it lies in our σ-ideal).

Definition 1.7 (Filtration-negligible)**.** Given a filtration $(N_m)_{m\in\mathbb{N}}$ by Carnot subgroups of a scalable group G, we say that a Borel set $\Omega \subseteq G$ is $(N_m)_m$*-negligible* if Ω is the countable union of Borel sets Ω_m such that

$$\mathrm{vol}_{N_m}(N_m \cap (g\Omega_m)) = 0, \qquad \forall m \in \mathbb{N}, \forall g \in G,$$

where vol_{N_m} denotes any Haar measure on N_m.

We can now state the main theorem of this paper, which is the following generalization of Aronszajn's differentiability result [Aro76], and the one of Pansu [Pan89].

Theorem 1.8. *Let G be an infinite-dimensional Carnot group. If $f\colon G \to \mathbb{R}$ is a Lipschitz map, then there exists a Borel subset $\Omega \subseteq G$ that is $(N_m)_m$-negligible for every filtration $(N_m)_{m\in\mathbb{N}}$ by Carnot subgroups of G and such that for every $p \notin \Omega$ the map f is Gâteaux differentiable at p.*

Notice that the above statement is meaningful already within the class of metric scalable groups. However, be aware that a scalable group may not admit any filtration (for example, if the group is not separable), in which case the above theorem has no content, e.g., one can take $\Omega = G$. Nonetheless, there are large classes of scalable groups that admit filtrations (see Proposition 1.10 below for a general criterion and Section 5 for more examples). The first thing to clarify is that, as soon as there is one filtration, the whole scalable group cannot be negligible, as the next proposition states.

Proposition 1.9. *If $(N_m)_{m\in\mathbb{N}}$ is a filtration by Carnot subgroups of an infinite-dimensional Carnot group G and $\Omega \subseteq G$ is a Borel $(N_m)_m$-negligible set, then Ω has empty interior.*

As mentioned before, this allows us to conclude that, for groups admitting at least one filtration by Carnot subgroups, every Lipschitz function $f : G \to \mathbb{R}$ has at least one point of Gâteaux differentiability.

Finally, we would like to have geometric conditions that tell us when our group admits filtrations by Carnot subgroups. For a scalable group G define its *first layer* as

$$V_1(G) := \{p \in G : t \in \mathbb{R} \mapsto \delta_t(p) \text{ is a one-parameter subgroup}\},$$

where by one-parameter subgroup we mean that for all $t, s \in \mathbb{R}$,

$$\delta_{t+s}(p) = \delta_t(p)\delta_s(p).$$

Note that if $p \in V_1(G)$, then $\delta_r(p) \in V_1(G)$ for all $r \in \mathbb{R}$, since

$$\delta_{t+s}(\delta_r(p)) = \delta_{tr+sr}(p) = \delta_{tr}(p)\delta_{sr}(p) = \delta_t(\delta_r(p))\delta_s(\delta_r(p)),$$

We say that a set $A \subseteq G$ *generates* G *as a scalable group* or simply that A *generates* G if G is the closure of the group generated by $\{\delta_t(a) : a \in A,\, t \in \mathbb{R}\}$. Note that $V_1(G)$ is completely analogous to the generating first layer of a finite-dimensional Carnot group. Moreover, the following proposition holds.

Proposition 1.10. *Let G be a scalable group. If G admits a filtration by Carnot subgroups then $V_1(G)$ generates G as a scalable group. Vice versa, if G is nilpotent, $V_1(G)$ is separable, and $V_1(G)$ generates G as a scalable group, then G admits a filtration by Carnot subgroups.*

We point out that the nilpotency assumption in the previous proposition cannot be removed, since there exist scalable groups with generating first layer that do not admit filtrations (see Proposition 5.11). However, not every metric scalable group having filtrations is nilpotent, as shown in Proposition 5.10. We will discuss this relation in more detail in Section 2.

Relying on the result of Siebert, it is rather straightforward to show that scalable groups having Carnot group structure are exactly those scalable groups that are locally compact and have generating first layer (see Theorem 2.14 and the proof of Proposition 2.2). Therefore, keeping Proposition 1.10 in mind, our definition for infinite-dimensional Carnot groups (Definition 1.6) appears in this sense to be a natural non-locally compact generalization of Carnot groups.

We begin by proving Proposition 1.10 in Section 2. The crucial observation is that any nilpotent group generated by finitely many elements of $V_1(G)$ has structure of a Carnot group. In Section 3 we make a closer study of filtration-negligible sets and prove Proposition 1.9. Section 4 is devoted to the proof of Theorem 1.8 and finally in Section 5 we give examples and introduce a class of metric scalable groups that admit filtrations by Carnot subgroups.

2. Carnot groups generated

The aim of this section is to prove the following proposition, which easily implies Proposition 1.10.

Proposition 2.1. *Let G be a scalable group. The following are equivalent:*

i) *G admits a filtration by Carnot subgroups;*
ii) *there exists a sequence $(a_n)_n \subseteq V_1(G)$ such that $\{a_n\}_{n\in\mathbb{N}}$ generates G as a scalable group and the group generated by $\{a_1, \dots, a_m\}$ is nilpotent for every $m \in \mathbb{N}$.*

The challenging part is to prove that ii) implies i). In the core of the argument there is the following result, which we state as a proposition.

Proposition 2.2. *Let (G, δ) be a scalable group that is generated by $x_1, \dots, x_r \in V_1(G)$, with $r \in \mathbb{N}$. If G is nilpotent, then it has structure of a Carnot group.*

We give now a proof of Proposition 2.1 using Proposition 2.2 and devote the rest of the section for the proof of Proposition 2.2.

Proof of Proposition 2.1. Assume first that $(N_m)_m$ is a filtration by Carnot subgroups of G and denote by $\mathfrak{n}_m$ the corresponding Lie algebras. Since the groups $(N_m)_m$ are Lie subgroups of each others, we may define inductively a basis $\{e_1, \dots, e_{i_m}\}$ for $V_1(\mathfrak{n}_m)$ as an extension of the basis for $V_1(\mathfrak{n}_{m-1})$. By Chow-Rashevskii theorem, the set $\{\exp(e_1), \dots, \exp(e_{i_m})\}$ generates N_m as a scalable group, and since $\cup_m N_m$ is dense in G we may take $(\exp(e_n))_n$ as the desired sequence.

Next, let $(a_n)_{n\in\mathbb{N}} \subseteq V_1(G)$ be the sequence given by ii). This sequence generates a dense subgroup of G, and choosing N_m to be the scalable group generated by $\{a_1, \dots, a_m\}$ gives G a filtration by Carnot groups by Proposition 2.2. □

We begin by fixing the notation in Section 2.1. Analogously to Definition 1.1, one can consider $\mathbb{Q}$-*scalable groups* for which the dilation automorphism is defined on the rationals: $\delta\colon \mathbb{Q} \times G \to G$. In Section 2.2 we prove that if G is a nilpotent $\mathbb{Q}$-scalable group of step s that is generated by finitely many elements, then $G^{(s)}$ has structure of finite-dimensional $\mathbb{Q}$-vector space. Here $G^{(s)}$ is the last element of the lower central series of the nilpotent group G. Some of the simple commutator identities that we use are proved in Appendix A.

In Section 2.3 we use the result of Section 2.2 to show that under the assumption that G is a nilpotent scalable group generated by finitely many elements, the last layer $G^{(s)}$ is a real finite-dimensional topological vector space, and in particular it is locally compact. Consequently, see Theorem 2.11, also G is locally compact. The proof of Proposition 2.2 is concluded by the result of Siebert (Theorem 2.14), which says that any connected, locally compact, contractible group is a positively gradable Lie group. Namely, we find a gradation $\bigoplus_{t>0} V_t$ of the Lie algebra Lie(G) such that V_1 generates Lie(G), and hence $\bigoplus_{t>0} V_t$ is a stratification of G.

2.1. **Notation.** For a group G and elements $g, h \in G$ we define the group commutator by

$$[g, h] := ghg^{-1}h^{-1}.$$

The elements of lower central series are defined by $G^{(1)} = G$ and $G^{(k)}$ is the group generated by $[G, G^{(k-1)}]$. We say that G is nilpotent of step s if $G^{(s+1)} = \{e\}$ but $G^{(s)} \neq \{e\}$. Notice that in this case $G^{(s)}$ is an abelian subgroup of G. We denote by $Z(G)$ the center of G.

We follow the terminology of [Khu98] and define recursively *commutators of weight* k for $k \in \mathbb{N}$ in the variables $x_1, x_2, \ldots$ as formal bracket expressions. The letters $x_1, x_2 \ldots$ are commutators of length one; inductively, if c_1, c_2 are commutators of weight k_1 and k_2, then $[c_1, c_2]$ is a commutator of weight $k_1 + k_2$. We also call the commutator of the form $[x_1, [x_2, \ldots, [x_{k-1}, x_k] \ldots]]$ a *simple commutator* of $x_1, \ldots, x_k$.

During this section, it is useful to keep in mind the following lemma. We remark that in [Khu98] the definition of commutator is related to our notation by $[a, b]_{\mathrm{K}hu} = [a^{-1}, b^{-1}]$. However, since in the following lemma the generating set can equivalently be taken symmetric, it applies in our case without modifications.

Lemma 2.3 (Lemma 3.6(c) in [Khu98])**.** *Let G be a group and $M \subseteq G$ a subset of G. If M generates G as a group, then $G^{(k)}$ is generated by simple commutators of weight $\geq k$ in the elements $m^{\pm 1}$, $m \in M$.*

We also write down the definition of vector space to ease the discussion later on.

Definition 2.4. Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*vector space* is an abelian group G equipped with an operation $\sigma \colon \mathbb{K} \times G \to G$ satisfying

(i) $\sigma(q, \sigma(p, g)) = \sigma(qp, g)$
(ii) $\sigma(q, g)\sigma(p, g) = \sigma(q + p, g)$
(iii) $\sigma(1, g) = g$
(iv) $\sigma(q, g)\sigma(q, h) = \sigma(q, gh)$,

for all $q, p \in \mathbb{K}$ and $g, h \in G$. We denote the map $\sigma(q, \cdot)$ by σ_q.

2.2. **$\mathbb{Q}$-scalable groups.** In this section, G will always denote a nilpotent $\mathbb{Q}$-scalable group of step s with dilations δ_t, generated by $x_1, \ldots, x_r \in V_1(G)$. We will show that the last element $G^{(s)}$ of the lower central series admits a structure of finite-dimensional $\mathbb{Q}$-vector space.

Lemma 2.5. *Let $m \in \mathbb{N}$ and $y \in G^{(k)}$ be a simple commutator of k elements of $V_1(G)$ for some $k \in \{1, \ldots, s\}$. Then $\delta_m(y) = hy^{m^k}$ for some $h \in G^{(k+1)}$.*

Proof. The proof is by induction on k. If $k = 1$, then $\delta_m(y) = y^m$ since $t \mapsto \delta_t(y)$ is a one-parameter subgroup. Assume that the claim holds for $k-1$ and let $y \in G^{(k)}$. Now $y = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(k-1)}$ is a simple commutator of $k-1$ elements of $V_1(G)$. Hence

$$\delta_m(y) = [\delta_m(x), \delta_m(w)] = [x^m, zw^{m^{k-1}}],$$

where $z \in G^{(k)}$. By Lemma A.1 and Corollary A.3, we get

$$\delta_m(y) = h_1[x^m, z][x^m, w^{m^{k-1}}] = h_1[x^m, z]h_2[x, w]^{mm^{k-1}} = h[x, w]^{m^k},$$

where $h = h_1[x^m, z]h_2 \in G^{(k+1)}$. □

Lemma 2.6. *The abelian group $G^{(s)}$ is a $\mathbb{Q}$-vector space with the scalar multiplication $\sigma_{\frac{n}{m}}(z) := \delta_m^{-1}(z^{nm^{s-1}})$. Moreover, if $z = [x, w] \in G^{(s)}$ with $x \in V_1(G)$ and $w \in G^{(s-1)}$, then $\sigma_q(z) = [\delta_q(x), w]$.*

Proof. If the step $s = 1$, the group $G^{(s)} = G$ and the $\mathbb{Q}$-vector space structure is given by the dilation automorphisms $\delta\colon \mathbb{Q} \times G \to G$, as the maps $t \mapsto \delta_t(x_i)$ are one-parameter subgroups.

For step $s \geq 2$, let first $z \in G^{(s)}$ be a simple commutator of s elements of $V_1(G)$. In particular, $z = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(s-1)}$ is a simple commutator of $s-1$ elements of $V_1(G)$. Define $\sigma\colon \mathbb{Q} \times G^{(s)} \to G^{(s)}$ for simple commutators by

$$\sigma_q([x, w]) = [\delta_q(x), w].$$

If z is a product of simple commutators $z_1, \ldots, z_k \in G^{(s)}$, we set

$$\sigma_q(z_1 \cdots z_k) = \sigma_q(z_1) \cdots \sigma_q(z_k).$$

By Lemma 2.3, this is enough to define the map σ for all $z \in G^{(s)}$.

We show next that

$$\sigma_{\frac{n}{m}}(z) = \delta_m^{-1}(z^{nm^{s-1}}),$$

which proves that the map is well defined. Let first $z = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(s-1)}$ is a simple commutator of $s-1$ elements of $V_1(G)$ and $q = \frac{n}{m} \in \mathbb{Q}_+$, $n, m \in \mathbb{N}$. Lemma 2.5 gives us that

$$\delta_m(\sigma_q([x, w])) = [\delta_m(\delta_{n/m}(x)), \delta_m(w)] = [x^n, hw^{m^{s-1}}] = [x^n, w^{m^{s-1}}],$$

where $h \in G^{(s)} \subseteq Z(G)$. Since $[x, w] \in Z(G)$ as well, we get by iterating Corollary A.2 that

$$\delta_m(\sigma_q([x, w])) = [x, w]^{nm^{s-1}} = z^{nm^{s-1}}.$$

If $q \in \mathbb{Q}_-$, we replace x by x^{-1} in the above calculation as $\delta_{-q}(x) = \delta_q(x^{-1})$ and use Lemma A.4, which gives

$$[x^{-1}, w] = [x, w]^{-1},$$

since now $[x^{-1}, [w, x]] = e_G$.

If $z \in G^{(s)}$ is a product of simple commutators $z_1, \dots, z_k \in G^{(s)}$,

$$\begin{aligned}\delta_m(\sigma_q(z_1 \cdots z_k)) &= \delta_m(\sigma_q(z_1)) \cdots \delta_m(\sigma_q(z_k)) \\ &= z_1^{nm^{s-1}} \cdots z_k^{nm^{s-1}} \\ &= (z_1 \cdots z_k)^{nm^{s-1}} \\ &= z^{nm^{s-1}}\end{aligned}$$

since $z_i \in Z(G)$ for all i.

Finally, let $z = [x, w]$ be such that $x \in V_1(G)$ and w is an arbitrary element of $G^{(s-1)}$. Then, by Lemma 2.3 there exist simple commutators $v_1, \dots, v_l$ of length $s-1$ such that $w = v_1 \cdots v_l$. By Corollary A.2,

$$\begin{aligned}\sigma_q([x, v_1 \cdots v_l]) = \sigma_q([x, v_1] \cdots [x, v_l]) &= \sigma_q([x, v_1]) \cdots \sigma_q([x, v_l]) \\ = [\delta_q(x), v_1] \cdots [\delta_q(x), v_l] &= [\delta_q(x), v_1 \cdots v_l].\end{aligned}$$

It remains to check that the map $\sigma\colon \mathbb{Q} \times G^{(s)} \to G^{(s)}$ satisfies the conditions (i)–(iv) in the Definition 2.4. Condition (iv) is true by construction. The conditions (i) and (iii) follow from the fact that $\delta\colon (\mathbb{Q}^*, \cdot) \to \mathrm{Aut}(G)$ is a group homomorphism:

$$\delta_{qp} = \delta_q \circ \delta_p \quad \text{and} \quad \delta_1 = id,$$

so

$$\sigma_q(\sigma_p([x, w])) = [\delta_q \circ \delta_p(x), w] = [\delta_{qp}(x), w] = \sigma_{qp}([x, w])$$

and

$$\sigma_1([x, w]) = [\delta_1(x), w] = [x, w].$$

Condition (ii) holds by Corollary A.2 and because $t \mapsto \delta_t(x)$ is a one-parameter subgroup for all $x \in V_1(G)$, namely

$$\sigma_{q+p}([x, w]) = [\delta_{q+p}(x), w] = [\delta_q(x)\delta_p(x), w] = [\delta_q(x), w][\delta_p(x), w] = \sigma_q([x, w])\sigma_p([x, w]).$$

Hence the map σ defines a $\mathbb{Q}$-vector space structure on $G^{(s)}$. □

Lemma 2.7. *The group $G^{(s)}$ equipped with the $\mathbb{Q}$-vector space structure of Lemma 2.6 is finite dimensional.*

Proof. The proof is by induction on the step s. If step $s = 1$, $G = V_1(G)$ is commutative and the set $\{x_1, \dots, x_r\}$ is a basis for $V_1(G)$. Suppose that the claim holds for any $\mathbb{Q}$-scalable group of step $s-1$. Let $K \coloneqq G/G^{(s)}$ and define

$$\hat{\delta}\colon \mathbb{Q} \times K \to K, \quad \hat{\delta}_q(gG^{(s)}) \coloneqq \delta_q(g)G^{(s)}.$$

This map is well defined since $\delta(G^{(s)}) = G^{(s)}$. Hence the group K is a $\mathbb{Q}$-scalable group of step $s-1$ and it is generated by $\{x_1G^{(s)}, \dots, x_rG^{(s)}\}$. Notice that

$$[xG^{(s)}, yG^{(s)}]_K = [x, y]_G G^{(s)}.$$

Let $\hat{\sigma}\colon \mathbb{Q} \times K \to K$ be the map from Lemma 2.6, which makes $K^{(s-1)}$ a $\mathbb{Q}$-vector space. By induction hypothesis, there exists a basis $\{k_1, \dots, k_l\}$ of $K^{(s-1)}$. Let $\pi\colon G \to K$ be the projection and choose $u_i \in \pi^{-1}(k_i) \subseteq G^{(s-1)}$ for all $1 \leq i \leq l$. We show that the set

$\{[x_i, u_j] : 1 \le i \le r,\ 1 \le j \le l\}$ spans $G^{(s)}$. Since $G^{(s)}$ commutes, it is enough to show that $\{[x_i, u_j]\}$ spans all the elements of the form $[x, u]$, where $x \in V_1(G)$ and $u \in G^{(s-1)}$.

Fix $z = [x, u] \in G^{(s)}$ such that $x \in V_1(G)$ and $u \in G^{(s-1)}$. There exist $q_1, \dots, q_l \in \mathbb{Q}$, $q_i = \frac{n_i}{m_i}$, such that

$$\begin{aligned}\pi(u) &= \hat{\sigma}_{q_1}(k_1) \cdots \hat{\sigma}_{q_l}(k_l) \\ &= \hat{\delta}_{m_1}^{-1}((u_1 G^{(s)})^{n_1 m_1^{s-2}}) \cdots \hat{\delta}_{m_l}^{-1}((u_l G^{(s)})^{n_l m_l^{s-2}}) \\ &= \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}}) \cdots \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}}) G^{(s)} \\ &=: v G^{(s)}.\end{aligned}$$

Hence there exists an element $h \in G^{(s)} \subseteq Z(G)$ such that $u = vh$. Therefore

$$\begin{aligned}[x, u] &= [x, vh] = [x, v] \\ &= [x, \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}}) \cdots \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}})] \\ &= [x, \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}})] \cdots [x, \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}})] \\ &= \delta_{m_1}^{-1}([x^{m_1}, u_1^{n_1 m_1^{s-2}}]) \cdots \delta_{m_l}^{-1}([x^{m_l}, u_l^{n_l m_l^{s-2}}]) \\ &= \delta_{m_1}^{-1}([x, u_1]^{n_1 m_1^{s-1}}) \cdots \delta_{m_l}^{-1}([x, u_l]^{n_l m_l^{s-1}}) \\ &= \sigma_{q_1}([x, u_1]) \cdots \sigma_{q_l}([x, u_l]),\end{aligned}$$

where we used Corollaries A.2 and A.3. Since $x \in V_1(G)$, there exists $q \in \mathbb{Q}$ and $i \in \{1, \dots, r\}$ such that $x = \delta_q(x_i)$. Thus, by the second part of Lemma 2.6,

$$\begin{aligned}[x, u] &= \sigma_{q_1}([\delta_q(x_i), u_1]) \cdots \sigma_{q_l}([\delta_q(x_i), u_l]) \\ &= [\delta_{q_1 q}(x_i), u_1] \cdots [\delta_{q_l q}(x_i), u_l] \\ &= \sigma_{q_1 q}([x_i, u_1]) \cdots \sigma_{q_l q}([x_i, u_l]).\end{aligned}$$

□

2.3. **Proof of Proposition 2.2.** Our first task is to prove that G is locally compact. To show this, we consider the $\mathbb{Q}$-scalable subgroup $G_{\mathbb{Q}}$ of G that by definition is generated as a group by $\{\delta_t(x_i) : t \in \mathbb{Q}, 1 \le i \le r\} =: V_{\mathbb{Q}}$. Let $\sigma \colon \mathbb{Q} \times G_{\mathbb{Q}}^{(s)} \to G_{\mathbb{Q}}^{(s)}$ be the continuous map from Lemma 2.7 which makes $G_{\mathbb{Q}}^{(s)}$ a k-dimensional $\mathbb{Q}$-vector space for some $k \in \mathbb{N}$. We use the following facts about topological groups to show that $G^{(s)}$ is a finite-dimensional real topological vector space.

Theorem 2.8 (Theorem 1.22 in [Rud91])**.** *A Hausdorff topological vector space is locally compact if and only if it is finite dimensional.*

Lemma 2.9. *Every locally compact subgroup of a topological group is closed.*

Proof. This proof is adapted from a Mathematics Stack Exchange post by Eric Wofsey [hw]. Let H be a topological group and let K be a locally compact subgroup of H. Then $\overline{K}$ is also a subgroup of H, and K is dense in $\overline{K}$. We claim that every locally compact dense subset of a Hausdorff space is open. Indeed, let S be a locally compact dense subset of a Hausdorff space X and take $x \in S$. Let also U be open in S such that $x \in U$, $\overline{U} \subseteq S$, and $\overline{U}$ is compact. Take then an open set $V \subseteq X$ such that $V \cap S = U$. Since X is Hausdorff, $\overline{U}$ is closed in X and therefore $V \setminus \overline{U}$ is open in X. But

$$(V \setminus \overline{U}) \cap S = (V \cap S) \setminus \overline{U} = U \setminus \overline{U} = \emptyset,$$

and hence $V \setminus \overline{U} = \emptyset$ as S is dense in X. We conclude that $V \subseteq S$, which proves the claim.

Hence, by the previous claim K is open in $\overline{K}$. Recall that every open subgroup of a topological group is closed since the complement K^c of an open subgroup K is the union of open sets; $K^c = \cup_{x \in K^c} xK$. Hence K is closed in $\overline{K}$ and therefore also in H. □

Lemma 2.10. *$G^{(s)}$ equals to $\overline{G_{\mathbb{Q}}^{(s)}}$ and it is a k-dimensional real topological vector space.*

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for $G_{\mathbb{Q}}^{(s)}$. We claim that since $G_{\mathbb{Q}}^{(s)} \subseteq Z(G)$, we may assume that each v_i is of the form $[x_i, w_i]$ with $x_i \in V_{\mathbb{Q}}$ and $w_i \in G_{\mathbb{Q}}^{(s-1)}$. Indeed, recall that by Lemma 2.3 any element of $G_{\mathbb{Q}}^{(s)}$ is a product of simple commutators of elements of $V_{\mathbb{Q}}$ of weight s, which proves the claim. Let

$$W := \{[\delta_{t_1}(x_1), w_1] \cdots [\delta_{t_k}(x_k), w_k] \mid t_i \in \mathbb{R}\},$$

which is a group by Corollary A.2 and since $t \mapsto \delta_t(x_i)$ is a one-parameter subgroup for each $i \in \{1, \dots, k\}$. Now $G_{\mathbb{Q}}^{(s)} \subseteq W$ by definition of σ and $W \subseteq \overline{G_{\mathbb{Q}}^{(s)}}$ by continuity of dilations. We define $\tilde{\sigma} \colon \mathbb{R} \times W \to W$ by

$$\tilde{\sigma}_\lambda([\delta_{t_1}(x_1), w_1] \cdots [\delta_{t_k}(x_k), w_k]) = [\delta_{\lambda t_1}(x_1), w_1] \cdots [\delta_{\lambda t_k}(x_k), w_k].$$

This map is continuous and it defines an $\mathbb{R}$-vector space structure on W: since $\tilde{\sigma}$ is a continuous extension of σ, it is easy to show that $\tilde{\sigma}$ fulfills the conditions in Definition 2.4. Hence W is a k-dimensional real topological vector space. Therefore, by Theorem 2.8 and Lemma 2.9, W is closed and so $W = \overline{G_{\mathbb{Q}}^{(s)}}$. We conclude the proof by noting that $G = \overline{G_{\mathbb{Q}}}$, and hence $G^{(s)} = \overline{G_{\mathbb{Q}}}^{(s)} = \overline{G_{\mathbb{Q}}^{(s)}}$, where the last equality follows form the continuity of the group operation. □

The following statement on topological groups will allow us to conclude that G is locally compact.

Theorem 2.11 ([MZ74] p. 52)**.** *If a topological group G has a closed subgroup H such that H and the coset-space G/H are locally compact, then G is locally compact.*

Lemma 2.12. *Let (G, δ) be a nilpotent scalable group that is generated by $x_1, \dots, x_r \in G$ as a scalable group over $\mathbb{R}$. Then G is locally compact.*

Proof. The proof is again by induction on the step s. If $s = 1$, the group G is a real topological vector space with basis $\{x_1, \dots, x_r\}$ and hence locally compact by Theorem 2.8. Assume that the claim holds for step $s - 1$ and consider $K := G/G^{(s)}$, which is generated by $x_1 G^{(s)}, \dots, x_r G^{(s)}$ with dilations $\hat{\delta}_t(xG^{(s)}) := \delta_t(x) G^{(s)}$. Now K is indeed an $\mathbb{R}$-scalable topological group, since $G^{(s)}$ is a closed normal subgroup of G by Lemma 2.10. Hence K is locally compact by the induction hypothesis, and by Theorem 2.8 the group $G^{(s)}$ is locally compact as well. Finally Theorem 2.11 proves the claim. □

To prove Proposition 2.2, we use the result of Siebert below.

Definition 2.13. Let G be a topological group. A continuous automorphism ζ of G is said to be *contractive* if $\lim_{n\to\infty} \zeta^n(x) = e_G$ for all $x \in G$. A group that admits a contractive continuous automorphism is called *contractible.*

Theorem 2.14 (Corollary 2.4 in [Sie86])**.** *A topological group G is a positively gradable Lie group if and only if it is connected, locally compact and contractible. In particular, if $\zeta \in \mathrm{Aut}(G)$ is contractive, then the gradation $\bigoplus_{t>0} V_t$ given by ζ is such that*

$$\{X \in \mathrm{Lie}(G) \mid (\mathrm{d}\zeta - \alpha\,\mathrm{id})X = 0\} \subseteq V_{-\ln|\alpha|}$$

Proof of Proposition 2.2. We proved in Lemma 2.12 that the group G is locally compact. It is also connected, since the map $\gamma_x\colon [0,1] \to G$, $\gamma_x(t) = \delta_t(x)$ is a continuous path between e_G and x for every $x \in G$. Additionally, the group G is contractible as the automorphisms δ_t are contractive for all $t \in (0,1)$; for a fixed $t \in (0,1)$,

$$\lim_{n\to\infty} \delta_t^n(x) = \lim_{n\to\infty} \delta_{t^n}(x) = \delta_0(x) = e_G$$

for all $x \in G$. Hence by Theorem 2.14 the group G is a Lie group and each δ_t, $t \in (0,1)$, defines a positive gradation for $\mathrm{Lie}(G)$. We claim that, in order to prove that G admits a structure of Carnot group, it is enough to find a gradation of $\mathrm{Lie}(G)$ such that V_1 generates the whole of $\mathrm{Lie}(G)$. Indeed, a stratification of a Lie algebra $\mathrm{Lie}(G)$ is equivalent to a positive gradation whose degree-one layer generates $\mathrm{Lie}(G)$ as a Lie algebra.

Let us consider the gradation given by $\delta_{1/e}$. By Theorem 2.14,

$$\{X \in \mathrm{Lie}(G) \mid (\mathrm{d}\delta_{1/e} - \frac{1}{e}\,\mathrm{id})X = 0\} \subseteq V_{-\ln(1/e)} = V_1.$$

Let $x \in \{x_1, \ldots, x_r\}$. Since the map $t \mapsto \delta_t(x)$, $t \in \mathbb{R}$, is now a one-parameter subgroup of a Lie group, there exists $X \in \mathrm{Lie}(G)$ such that

$$\delta_t(x) = \exp(tX)$$

for all $t \in \mathbb{R}$. Additionally, on the one hand, since $\exp\colon \mathrm{Lie}(G) \to G$ is a global diffeomorphism,

$$\log(\delta_t(x)) = tX.$$

On the other hand

$$\log(\delta_t(x)) = \log(\delta_t(\exp(X))) = \log(\exp(\mathrm{d}\delta_t(X))) = \mathrm{d}\delta_t(X).$$

Hence

$$\mathrm{d}\delta_{1/e}(X) = \frac{1}{e}X$$

and $X \in V_1$. Therefore $\log(x_i) \in V_1$ for all $i \in \{1, \ldots, r\}$. Notice that, for any $Y \in \mathrm{Lie}(G)$, we have for some $l \in \mathbb{N}$ and $i_l \in \{1, \ldots, r\}$ that

$$\exp(Y) = \delta_{t_1}(x_{i_1}) \cdots \delta_{t_l}(x_{i_l}) = \exp(t_1 \log(x_{i_1})) \cdots \exp(t_l \log(x_{i_l})).$$

Hence $\{\log(x_1), \ldots, \log(x_r)\}$ generates $\mathrm{Lie}(G)$ as a Lie algebra by the Baker-Campbell-Hausdorff formula and $V_1 = \mathrm{span}(\log(x_1), \ldots, \log(x_r))$. Thus the gradation given by $\delta_{1/e}$ is a stratification and G has structure of a Carnot group.

We still need to verify that the one-parameter family $(\delta_t)_{t \in \mathbb{R}}$ of Lie group automorphisms are the Carnot group dilations given by the stratification. The Carnot group dilation of factor $t \neq 0$ is by definition the unique map $\zeta_t \in \mathrm{Aut}(G)$ such that

$$\mathrm{d}\zeta_t(X) = t^k X \quad \text{for all } X \in V_k, \tag{2.15}$$

and $\mathrm{d}\zeta_0$ is the zero map. Obviously $\mathrm{d}\delta_0 = 0$, so consider the case $t \neq 0$. Recall that each V_k is spanned by simple commutators of $\log(x_i)$, $i \in \{1, \ldots, r\}$ that span the first layer. Since $\mathrm{d}\delta_t$ is a Lie algebra homomorphism, we get for these elements that

$$\begin{aligned} \mathrm{d}\delta_t([\log(x_{i_k}), \ldots, [\log(x_{i_2}), \log(x_{i_1})]]) &= [\mathrm{d}\delta_t(\log(x_{i_k})), \ldots, [\mathrm{d}\delta_t(\log(x_{i_2})), \mathrm{d}\delta_t(\log(x_{i_1}))]] \\ &= [t\log(x_{i_k}), \ldots, [t\log(x_{i_2}), t\log(x_{i_1})]] \\ &= t^k[\log(x_{i_k}), \ldots, [\log(x_{i_2}), \log(x_{i_1})]]. \end{aligned}$$

By linearity of $\mathrm{d}\delta_t$ we conclude that the maps $\mathrm{d}\delta_t$ satisfy condition (2.15). Hence the scalable group (G, δ) is a Carnot group and the dilations δ_t, $t \in \mathbb{R}$, are the unique Carnot group dilations given by the stratification. □

It would be interesting to find geometric conditions that allow us to conclude that $V_1(G)$ generates G. Indeed, in the case of simply connected nilpotent Lie groups admitting dilations, geodicity implies that the first layer generates the entire group since rectifiable curves can be approximated by horizontal line segments. Thus, we make the following conjecture.

Conjecture 2.16. *If G is a metric scalable group that is separable and nilpotent and its distance is geodesic, then its first layer $V_1(G)$ generates G.*

3. Negligible sets of metric scalable groups

In this section (G, d, δ) denotes a metric scalable group (according to Definition 1.3). We begin by giving some auxiliary lemmas and then prove Proposition 1.9, which states that filtration-negligible sets always have empty interior. After that we introduce another notion of null-sets following [Aro76] and prove that it agrees with the definition of filtration-negligible sets. This result is formulated in Theorem 3.6.

3.1. Elementary properties of metric scalable groups.

Lemma 3.1. *For each $v \in V_1(G)$, $v \neq e$, the map $t \mapsto \delta_t(v)$ is a homothetic embedding from $\mathbb{R}$ to G.*

Proof. Let $c = d(0, \delta_1(v)) > 0$. We claim that $d(\delta_\alpha(v), \delta_\beta(v)) = c|\alpha - \beta|$. Indeed, as $\delta_t(v)$ is a one parameter subgroup, we get by left-invariance that

$$d(\delta_\alpha(v), \delta_\beta(v)) = d(0, \delta_{\beta-\alpha}(v)) = |\beta - \alpha| d(0, v) = c|\beta - \alpha|. \qquad \square$$

Lemma 3.2. *Let $K \subset G$ be a totally bounded set and $\epsilon > 0$. There exists $\delta > 0$ so that*

$$d(hk, k) < \epsilon, \qquad \forall h \in B(0, \delta), k \in K.$$

Proof. As K is totally bounded, there is a finite number of points $\{y_1, ..., y_n\} \in G$ so that $K \subseteq \bigcup_{j=1}^n B(y_j, \epsilon/4)$. Choose δ small enough so that for any $h \in B(0, \delta)$, we have $\max_{1\le j\le n} d(y_j, hy_j) < \epsilon/4$. Now let $k \in K$ and y_i be so that $d(k, y_i) < \epsilon/4$. Then for any $h \in B(0, \delta)$, we get

$$d(hk, k) \le d(hk, hy_i) + d(hy_i, y_i) + d(y_i, k) = 2d(y_i, k) + d(hy_i, y_i) < \epsilon,$$

where we used the left-invariance of the metric. □

Lemma 3.3. *Let G be a complete metric scalable group. For every $i \in \mathbb{N}$, let $\psi_i : \mathbb{R} \to G$ be continuous such that $\psi_i(0) = e_G$. Then for every non-empty open set U containing e_G there exists a sequence of positive numbers $\alpha_1, \alpha_2, ... > 0$ so that the map*

$$\begin{aligned}\phi : \prod_{i=1}^{\infty}[0, \alpha_i] &\to G\\ (t_1, t_2, ...) &\mapsto \cdots \psi_2(t_2)\psi_1(t_1)\end{aligned}$$

is well defined and has range in U.

Proof. We may assume that U contains the unit ball at e_G. Note that for each $k \in \mathbb{N}$, $K_k := \phi(\prod_{i=1}^k [0, \alpha_i] \times (0, 0, ...))$ is a compact set in G. We can construct α_i recursively. First choose $\alpha_1 > 0$ small enough so that $K_1 \subset B(e_G, 1/2)$. Now having chosen α_i, we choose $\alpha_{i+1} > 0$ so that

$$\sup_{g\in K_i}\ \sup_{t\in[0,\alpha_{i+1}]} d(g, \psi_{i+1}(t)g) < 2^{-i-1}.$$

This is possible by Lemma 3.2 and the fact that $\psi_{i+1}(0) = e_G$ is continuous at 0. Then each sequence defining a $\phi(t_1, t_2, ...)$ is Cauchy and so the limit exists. The fact that the image is in U also follows immediately. □

Here, ψ_i can be anything, but in the application of Lemma 3.3, we will take each $\psi_i(t)$ to be $\exp(t \log(v_i))$ for some $v_i \in V_1(G)$.

3.2. Non-negligibility of open sets: Proof of Proposition 1.9.

Let $(N_m)_{m\in\mathbb{N}}$ be a filtration of G by Carnot groups. Assume for contradiction that an $(N_m)_m$-negligible set Ω contains an open non-empty set U.

For every $m \in \mathbb{N}$, choose $\{v_{k_{m-1}+1}, ..., v_{k_m}\} \subset V_1(G)$ so that $\{v_1, \dots, v_{k_m}\}$ generate N_m as a basis. Let $\psi_i : \mathbb{R} \to G$ be $\psi_i(t) := \exp(t \log(v_i))$. With the above choice of U, let (α_m) and ϕ be as in Lemma 3.3. Notice that the maps $\tilde{\phi}_m : \mathbb{R}^{k_m} \to N_m$, $\tilde{\phi}_m(t_1, t_2, \dots, t_{k_m}) := \psi_{k_m}(t_{k_m}) \cdots \psi_2(t_2)\psi_1(t_1)$, are diffeomorphisms.

Let then

$$\phi_m := \tilde{\phi}_m|_{\prod_{i=1}^{k_m}[0,\alpha_i]}.$$

Let μ be the measure on G that is the pushforward via ϕ of the probability measure on $\prod_{i=1}^{\infty}[0, \alpha_i]$ that is the product of the rescaled probability Lebesgue measure on each of the $[0, \alpha_i]$.

Since ϕ has image contained in U, $\mu(U) = 1$ and hence $\mu(\Omega) = 1$. However, we shall show that $\mu(\Omega) = 0$, which will be our contradiction. Since the set Ω is $(N_m)_m$-negligible, then $\Omega = \cup_{m\in\mathbb{N}}\Omega_m$ for some Ω_m such that for each m,

$$\mathrm{vol}_{N_m}(N_m \cap g\Omega_m) = 0, \qquad \forall g \in G. \tag{3.4}$$

It is enough to show that $\mu(\Omega_m) = 0$ for any arbitrary m. For doing so, fix $m \in \mathbb{N}$ and let ν_1 and ν_2 denote the product probability measures (again with respect to the rescaled Lebsgue probability measures) on $C_1 = \prod_{i=1}^{k_m}[0, \alpha_i]$ and $C_2 = \prod_{i=k_m+1}^{\infty}[0, \alpha_i]$, respectively. Notice that $(\phi_m)_\#(\nu_1)$ is a smooth measure on some open set of N_m and hence it is absolutely continuous with respect to vol_{N_m}. In conjunction with (3.4), we get for any $\mathbf{t}_2 \in C_2$,

$$\begin{aligned}
\int_{C_1} \chi_{\phi^{-1}(\Omega_m)}(\mathbf{t}_1, \mathbf{t}_2)\, d\nu_1(\mathbf{t}_1) &= \int_{C_1} \chi_{\Omega_m}(\phi(\bar{0}, \mathbf{t}_2)\phi_m(\mathbf{t}_1))\, d\nu_1(\mathbf{t}_1) \\
&= \int_{C_1} \chi_{\phi_m^{-1}(\phi(\bar{0},\mathbf{t}_2)^{-1}\Omega_m)}(\mathbf{t}_1)\, d\nu_1(\mathbf{t}_1) \\
&= \nu_1(\phi_m^{-1}(\phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m)) \\
&= (\phi_m)_\#(\nu_1)(N_m \cap \phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m) \\
&\preceq \mathrm{vol}_{N_m}(N_m \cap \phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m) = 0.
\end{aligned}$$

Thus, $\mu(\Omega_m) = \int_{C_2}\int_{C_1} \chi_{\phi^{-1}(\Omega_m)}\, d\nu_1\, d\nu_2 = 0$. □

Remark 3.5. Note that the statement of Proposition 1.9 makes sense for scalable groups without any metric. Indeed, the notion of filtrations (and thus also negligibility) only relies on the topology. Thus, it may be possible that the result is true for all scalable groups although we have not verified this.

In the rest of this section we make a closer study of filtration-negligible sets of metric scalable groups. Below we define an *exceptional class* of null sets analogously to [Aro76] and prove that it is equivalent to our notion of filtration-negligible sets.

3.3. **The exceptional class $\mathcal{U}$.** Let G be a scalable group with identity element denoted by e_G and let $\mathcal{B}(G)$ be the Borel sets of G. For every $a \in V_1(G)$, with $a \neq e_G$ set

$$\mathcal{U}(a) := \{A \in \mathcal{B}(G) : \forall g \in G, |A \cap (g \cdot \mathbb{R}a)| = 0\},$$

where we denote by $\mathbb{R}a$ the image of the curve $t \in \mathbb{R} \mapsto \delta_t a$ and by $|\cdot|$ the 1-dimensional Lebesgue measure on the curve. In other words,

$$|A \cap (g \cdot \mathbb{R}a)| = |\{t \in \mathbb{R} : g\delta_t a \in A\}|.$$

For every countable set I and $\{a_n\}_{n\in I} \subset V_1(G) \setminus \{e_G\}$, define

$$\mathcal{U}(\{a_n\}_n) := \{A \in \mathcal{B}(G) : A = \cup_{n\in I} A_n, A_n \in \mathcal{U}(a_n)\}.$$

Finally, set $\mathcal{U}$ to be the intersection of all $\mathcal{U}(\{a_n\}_n)$ among all dense sequences $\{a_n\}_n \subseteq V_1(G) \setminus \{e_G\}$.

Recall a class of sets $\mathcal{F}$ is hereditary if $A \subset B$ and $B \in \mathcal{F}$ implies that $A \in \mathcal{F}$. The classes $\mathcal{U}(a), \mathcal{U}(\{a_n\}_n)$, and $\mathcal{U}$ are σ-additive, hereditary, and do not contain any open non-empty set (see the theorem below). Moreover, we have the property:

$$\{a'_n\} \subseteq \{a_n\} \implies \mathcal{U}(\{a'_n\}) \subseteq \mathcal{U}(\{a_n\}).$$

Theorem 3.6. *Let G be a metric scalable group and let $\{a_n\} \subset V_1(G) \setminus \{e_G\}$ be a dense sequence such that the group N_m generated by $\{a_1, \dots, a_m\}$ is nilpotent for all $m \in \mathbb{N}$. Then a set $\Omega \subseteq G$ is in the class $\mathcal{U}(\{a_n\})$ if and only if it is $(N_m)_{m\in\mathbb{N}}$ -negligible.*

Note that by Proposition 2.2, each N_m in the theorem above is a Carnot group and the statement makes sense. The proof of the theorem will be a straightforward consequence of Proposition 3.12. The proof of Proposition 3.12 needs some preparation, and we postpone it to the end of this section.

Lemma 3.7. *Let $A \subset G$ be a bounded Borel set and choose a $v \in V_1(G)$. Then the function*

$$f_A(x) = |A \cap (x \cdot \mathbb{R}v)|$$

is Borel.

Proof. Let $R > 0$ be arbitrary and let $\mathcal{A}$ denote the set of all $A \subseteq B(0,R)$ that satisfy the conclusion. We will prove that $\mathcal{A}$ contains the Borel sets of $B(0,R)$. We first prove that the open sets in $B(0,R)$ are in $\mathcal{A}$. Indeed, let A be open and $t \in \mathbb{R}$. We will show $A' = f_A^{-1}((t,\infty))$ is open.

As f_A is nonnegative, we may suppose without loss of generality that $t \geq 0$. Let $g \in f_A^{-1}((t,\infty))$ and $\delta = f_A(g) - t > 0$. Let $E = \{s \in \mathbb{R} : g\delta_s(v) \in A\}$, which is a bounded set by boundedness of A and Lemma 3.1. For each $s \in E$ define $d(s) = d(g\delta_s(v), A^c)$, a positive continuous function on E. We can choose $\epsilon > 0$ small enough so that

$$E' = \{s \in \mathbb{R} : d(g\delta_s(v)) > \epsilon\},$$

satisfies $|E'| > f_A(g) - \frac{\delta}{2}$.

Note that E' is totally bounded. By Lemma 3.1, the set $\{g\delta_s(v) : s \in E'\}$ is the isometric image of the totally bounded set E' and so it also is totally bounded. Thus by Lemma 3.2, there exists $\eta_0 > 0$ so that

$$\sup_{h\in B(0,\eta_0)} \sup_{s\in E'} d(hg\delta_s(v), g\delta_s(v)) < \epsilon.$$

Since G is topological, there exists some $\eta > 0$ small enough so that $B(g,\eta) \subseteq B(0,\eta_0)g$. This then gives that

$$\sup_{h\in B(g,\eta)} \sup_{s\in E'} d(h\delta_s(v), g\delta_s(v)) < \epsilon.$$

This shows that $h\delta_s(v) \in A$ when $h \in B(g,\eta)$ and $s \in E'$ and so

$$f_A(h) \geq |E'| > f_A(g) - \frac{\delta}{2} > t,$$

which proves $B(g,\eta) \subseteq f_A^{-1}((t,\infty))$ and so $f_A^{-1}((t,\infty))$ is open.

We now show that $\mathcal{A}$ is a monotone class of sets, which will prove that $\mathcal{A}$ contains all Borel sets. Let $\{E_i\}$ be an increasing sequence in $\mathcal{A}$ and $E = \bigcup_i E_i$. Then $E \cap (x \cdot \mathbb{R}v) = \bigcup_i (E_i \cap (x \cdot \mathbb{R}v))$, which is also an increasing family and so by monotone convergence theorem we get

$$f_E(x) = \lim_{i\to\infty} f_{E_i}(x).$$

Thus, f_E, the increasing pointwise limit of f_{E_i}, must be Borel and so $E \in \mathcal{A}$. Similarly, let $\{E_i\}$ be a decreasing sequence in $\mathcal{A}$ and let $E = \bigcap_i E_i$. Then $E \cap (x \cdot \mathbb{R}v) = \bigcap_i (E_i \cap (x \cdot \mathbb{R}v))$, which is another decreasing sequence. As E_1 is bounded, $f_{E_1}(x) < \infty$ and so by dominated convergence theorem we conclude $f_E(x) = \lim_{i \to \infty} f_{E_i}(x)$. Thus, $E \in \mathcal{A}$, which proves the monotonicity property of $\mathcal{A}$. □

Lemma 3.8. *Let $A \subseteq G$ be any Borel set and $v \in V_1(G)$. Then the set*

$$\{g \in A : |A \cap (g \cdot \mathbb{R}v)| > 0\}$$

is Borel.

Proof. Let $A_n = A \cap B(0, n)$. By monotone convergence theorem, the set in question is equal to $\bigcup_{n=0}^{\infty} \{g \in A_n : |A_n \cap (g \cdot \mathbb{R}v)| > 0\} = \bigcup_{n=0}^{\infty} (f_{A_n}^{-1}((0, \infty)) \cap A_n)$, which, by the previous lemma, is a countable union of Borel sets. □

3.4. **Null decomposition.** Let G be a Carnot group with $\dim V_1 = n$ and suppose G is homeomorphic to $\mathbb{R}^m$. We let $X_1, ..., X_n$ be the vector fields in $\mathbb{R}^m$ that are given by left translation of a basis in V_1.

Lemma 3.9. *Let M be an analytic manifold in $\mathbb{R}^m$ of dimension less than m. Then for vol_M-almost every $p \in M$, there exists an open neighborhood $U \subseteq M$ of p and an index $i \in \{1, ..., n\}$ for which $X_i(q) \notin T_q M$ for any $q \in U$.*

Proof. Let $k = \dim M < m$. For each $i \in \{1, ..., n\}$, let $A_i = \{p \in M : X_i(p) \in T_p M\}$, which are closed subsets of M. We claim that $A = \bigcap_i A_i$ has measure zero.

Suppose not. Let $f : U \to V$ be the inverse of an analytic chart map where $U \subset \mathbb{R}^k$ and $V \subset M$. We pushforward the basis vector fields of $\mathbb{R}^k$ via f to get vector fields $Y_1, ..., Y_k$ that form a basis of TV. As f is analytic, these are analytic vector fields.

Note that $A_i \cap V$ are precisely the points of V for which $X_i(p)$ is in the span of the $Y_j(p)$'s. This is the same as the being in the zero set of the function $g_i(p) = |X_i(p) - P_{\langle Y_1(p), ..., Y_k(p) \rangle} X_i(p)|^2$ where P is the orthogonal projection map onto the span of the $Y_j(p)$'s. Note that each g_i is an analytic function as projection is a combination of matrix multiplication and inverses. Thus, A is the zero set of the product function $g = g_1 \cdots g_k$, also an analytic function. Finally, we consider the function $g \circ f : U \to \mathbb{R}$, another analytic function. If A has positive measure, then $f^{-1}(A)$ has positive measure and so $g \circ f$ is identically zero [Mit15]. This means $A \cap U = U$. By definition of the A_i's, this means U is an integral manifold. We now derive a contradiction.

This means that TU contains the vector fields $\{X_i|_U\}$. If vector fields X, Y are tangent to U, then so is $[X, Y]$. As $\{X_i\}$ are tangent to U and generate all of $\mathbb{R}^m$ under Lie brackets, this means that $T_x U = \mathbb{R}^m$ for all $x \in U$. However, this is a contradiction as $\dim U = k < m$.

We have established that A^c is a full measure open set. Let $p \in A^c$. Then $p \in A_i^c$ for some i. As A_i^c is open, there then exists an open neighborhood $p \in U \subseteq A_i^c$. This neighborhood satisfies the conclusion of the lemma with X_i. □

Given a Borel set $A \subseteq \mathbb{R}^m$ and $i \in \{1, ..., n\}$ we define

$$A_i := \{p \in A : |A \cap (p \cdot \mathbb{R}X_i)| > 0\}.$$

By $p \cdot \mathbb{R}X_i$, we mean the 1-dimensional $\mathbb{R}$-flow of the vector field X_i that passes through $p \in \mathbb{R}^m$. Note that this is an analytic submanifold.

Given a word w written in the alphabet $\{1, ..., n\}$, we define $A_w = (A_{w'})_i$ where $w = w'i$ and $A_\emptyset = A$. Note that $A_w \subseteq A_{w'}$. Let w denote the word $123 \cdots n$, the concatenation of all the letters. Define the word w^k to be the k-fold concatenation of w (so w^k is kn letters long).

Lemma 3.10. *If $A \subset \mathbb{R}^m$ is a measure zero set, then $A_{w^m} = \emptyset$.*

Proof. Suppose otherwise. There then exists a point $p \in A_{w^m} = (A_{w'})_n$ and so

$$|A_{w'} \cap (p \cdot \mathbb{R}X_n)| > 0.$$

We let H_1 denote the analytic manifold $p \cdot \mathbb{R}X_n$, which has dimension 1. As $A_{w'} \subseteq A_{w^{m-1}}$, $|A_{w^{m-1}} \cap H_1| > 0$.

Now suppose we have a k-dimensional analytic manifold H_k that intersects $A_{w^{m-k}}$ in a positive measure set (based on the surface area of H_k). Thus, we can find a density point of $A_{w^{m-k}} \cap H_k$ satisfying the previous lemma, i.e., there exists a density point p of $A_{w^{m-k}} \cap H_k$, an open neighborhood $U \subseteq H_k$ of p, and an index $i \in \{1, ..., n\}$ so that $X_i \notin T_q H_k$ for all $q \in U$.

Since $A_{w^{m-k}} \subseteq (A_{w^{m-k-1}})_{1\cdots(i+1)}$, by definition of the set $(A_{w^{m-k-1}})_{1\cdots(i+1)}$, for any $q \in A_{w^{m-k}} \cap H_k$,

$$|(A_{w^{m-k-1}})_{1\cdots i} \cap (q \cdot \mathbb{R}X_i)| > 0.$$

Since $(A_{w^{m-k-1}})_{1\cdots i} \subseteq A_{w^{m-k-1}}$, we get for all $q \in A_{w^{m-k}} \cap H_k$ that

$$|A_{w^{m-k-1}} \cap (q \cdot \mathbb{R}X_i)| \geq |(A_{w^{m-k-1}})_{1\cdots i} \cap (q \cdot \mathbb{R}X_i)| > 0.$$

Let $H_{k+1} = \bigcup_{q \in U} (q \cdot \mathbb{R}X_i)$. As $X_i(q) \notin T_p U$, we conclude that H_{k+1} is a $k+1$-dimensional analytic manifold and $|H_{k+1} \cap A_{w^{m-k-1}}| > 0$.

We repeat until we obtain an m-dimensional analytic manifold for which $|H_m \cap A_\emptyset| = |H_m \cap A| > 0$. But since H_m has the same dimension as $\mathbb{R}^m$, it follows that $|A| > 0$, contradicting our assumption. ☐

Proposition 3.11. *Let $A \subset \mathbb{R}^m$ be a Borel set of zero measure. Then there exists a decomposition $A = C_1 \cup ... \cup C_n$ into Borel sets*

$$C_i = \bigcup_{k=0}^{m-1} (A_{w^k 1\cdots(i-1)} \backslash A_{w^k 1\cdots i})$$

so that for each $i \in \{1, ..., n\}$,

$$|C_i \cap (x \cdot \mathbb{R}X_i)| = 0, \qquad \forall x \in \mathbb{R}^m.$$

Proof. Let $B_1 = \{p \in A_\emptyset : |A_\emptyset \cap (p \cdot \mathbb{R}X_1)| = 0\}$. Then $A = B_1 \cup A_1$ where A_1 and B_1 are both Borel by Lemma 3.8, and so

$$|B_1 \cap (p \cdot \mathbb{R}X_1)| = 0, \qquad \forall p \in \mathbb{R}^m.$$

By induction, we obtain a Borel decomposition

$$A = B_1 \cup B_{12} \cup B_{123} \cup ... \cup B_{w^{m-1}1\cdots(n-1)} \cup A_{w^m} = B_1 \cup ... \cup B_{w^{m-1}1\cdots(n-1)}.$$

Note that for every $B_{w'i}$,

$$|B_{w'i} \cap (p \cdot \mathbb{R}X_i)| = 0, \qquad \forall p \in \mathbb{R}^m.$$

We take $C_i = \bigcup_{k=0}^{m-1} B_{w^k 1 \cdots i}$ to finish the proof of the proposition. □

Proposition 3.12. *Let H be a subgroup of G with a Carnot structure generated by $a_1, ..., a_k \in V_1(G)$ and $A \subset G$ be Borel. Then $\mathrm{vol}_H(H \cap gA) = 0$ for all $g \in G$ if and only if $A \in \mathcal{U}(\{a_1, ..., a_k\})$.*

Proof. The backwards direction is clear by Fubini. We will prove the forwards direction. Let H be homeomorphic to $\mathbb{R}^m$. We will reuse the notation of the previous section where for each Borel set $E \subset G$ and word w, we define $E_{wi} = \{g \in E_w : |E_w \cap (g \cdot \mathbb{R}a_i)| > 0\}$. Lemma 3.8 yields that these are Borel sets whenever E is. By construction, $E_{wi} \in \mathcal{U}(a_i)$.

We claim that $A = C_1 \cup ... \cup C_k$ where $C_i = \bigcup_{j=0}^{k-1} \left(A_{w^j 1 \cdots (i-1)} \backslash A_{w^j 1 \cdots i}\right)$, from which the proposition easily follows. To prove the claim, we observe that for any $g \in A$, we have by assumption that $\mathrm{vol}_H(H \cap g^{-1}A) = 0$. As $e_H \in H \cap g^{-1}A$, by Proposition 3.11, there exists some i so that

$$e_H \in \bigcup_{j=0}^{k-1} \left((H \cap g^{-1}A)_{w^j 1 \cdots (i-1)} \backslash (H \cap g^{-1}A)_{w^j 1 \cdots i}\right) \subseteq \bigcup_{j=0}^{k-1} \left(g^{-1}A_{w^j 1 \cdots (i-1)} \backslash g^{-1}A_{w^j 1 \cdots i}\right).$$

This means that $g \in C_i$. □

4. Differentiability of Lipschitz maps

We prove now our main result, Theorem 1.8. Notice that if $f \colon G \to H$ is a Lipschitz map between metric scalable groups for which Df_g exists for some $g \in G$, then it is Lipschitz as a function from G to H, with the same Lipschitz constant as f.

Lemma 4.1. *Let $f : G \to \mathbb{R}$ be Lipschitz and $(N_m)_{m \in \mathbb{N}}$ be a filtration of G by Carnot subgroups. Then there exists an $(N_m)_m$-negligible set $\Omega \subset G$ so that if $p \notin \Omega$ then for every N_m, the limit $\lim_{\lambda \to 0} \hat{f}_{p,\lambda}(u)$ exists for all $u \in N_m$ and the resulting map on N_m is a homomorphism.*

Proof. Fix N_m and let A denote the set of points $p \in G$ for which the limit $\lim_{\lambda \to 0} \hat{f}_{p,\lambda}(u)$ does not exist or the limit map is not a homomorphism. We will show $\mathrm{vol}_{N_m}(gA \cap N_m) = 0$ for all g. This will prove the lemma.

Fix a $g \in G$ and let $p \in gA \cap N_m$. If $F_g(u) := f(g^{-1}u)$ as a map defined on N_m, then $gp \in N_m$ is a nondifferentiability point of F. However, by Pansu's theorem [Pan89], F is differentiable almost everywhere with respect to the Haar measure on N_m. Thus, $\mathrm{vol}_{N_m}(gA \cap N_m) = 0$, which proves the lemma. □

With the previous lemma we get our differentiability result.

Proof of Theorem 1.8. As the theorem is vacuous if G does not admit a filtration by Carnot subgroups, we may assume that there is a filtration $(N_m)_m$. By the previous lemma, Df_g exists and is a homomorphism when restricted to any N_m for g outside of an (N_m)-negligible set. Take such a g. We first claim that Df_g exists on all of G. Indeed, this follows from the fact that the maps

$$u \mapsto n(f(g\delta_{1/n}(u)) - f(g))$$

are uniformly Lipschitz and converge, by assumption, on the dense subset $\bigcup_m N_m$.

As Df_g is a homomorphism when restricted to every N_m, an easy density argument yields that Df_g is also a homomorphism. This proves the theorem. □

5. Examples

In this final section we show that our derivative existence result does not generalize to general metric scalable groups and provide a constructive way to define infinite-dimensional Carnot groups. We start by constructing a metric scalable group G that does not admit filtrations by Carnot groups. We also construct a Lipschitz function $f : G \to \mathbb{R}^2$ that is nowhere differentiable. We then introduce L^p-sums of metric spaces when the indexing set is an abstract measure space. After that we restrict the discussion to ℓ_p-sequences of topological groups equipped with left-invariant metrics, and prove that this object is a topological group whenever $p \in [1, \infty)$ (see Proposition 5.3). Finally, we prove that an ℓ_p-sum of Carnot groups is an infinite-dimensional Carnot group for every $p \in [1, \infty)$ and give some detailed examples.

5.1. **A nowhere differentiable function on a metric scalable group.** An example of metric scalable groups not admitting filtrations by Carnot subgroups is the group $(\mathbb{R}, +)$ endowed with the metric $d^\gamma(x, y) = |x - y|^\gamma$ and scaling $\delta_\lambda(x) = \lambda^{1/\gamma}x$ where $\gamma = \frac{\log 3}{\log 4}$. One can also construct a Lipschitz function $f : G \to \mathbb{R}^2$ that is not differentiable anywhere. Indeed, let $K \subset \mathbb{R}^2$ be the Koch snowflake built from an equilateral triangle of sidelength 1. We can first define a map g from $([0, 1], d^\gamma)$ to one side of the Koch snowflake so that, for each $k \in \{0, 1, 2, 3\}$, $g|_{[0,1]}$ is equivalent to $3g_{[k/4,(k+1)/4]}$ up to postcomposition with an affine isometry.

We now prove nowhere differentiability of g. Recall that the derivative of g at x_0 is the pointwise limit of

$$h_r(y) := \frac{g(x_0 + r^{1/\gamma}y) - g(x_0)}{r} \tag{5.1}$$

as $r \to 0$. If g were differentiable, then h_r must converge to a homomorphism $\mathbb{R} \to \mathbb{R}^2$.

For every $n \geq 0$, there exists some $k \in \{0, ..., 4^{-n} - 1\}$ such that x_0 resides in $[k4^{-n}, (k + 1)4^{-n}]$. Then $h_{3^{-n}}|_{[k-4^n x_0, k+1-4^n x_0]}$ is equivalent to g up to postcomposition by an affine isometry as $3^n g|_{[k4^{-n},(k+1)4^{-n}]}$ is from self-similarity. Note that the length 1 interval $[k - 4^n x_0, k + 1 - 4^n x_0]$ lies in $[-1, 1]$. As g is not affine, we then get that h_r cannot converge to a homomorphism and so g is not differentiable anywhere.

To extend g to a nowhere differentiable function f on all of G, one can simply "wrap" g periodically around K so that $f|_{[n,n+1]} = f|_{[n+3,n+4]}$ for all $n \in \mathbb{N}$.

5.2. **L^p- and ℓ_p-sums.** Let $\Omega = (\Omega, \mu)$ be a measure space, e.g., the natural numbers $\mathbb{N}$ with the counting measure. Fix $p \in [1, \infty)$. For each $\omega \in \Omega$ fix a pointed metric space $X_\omega = (X_\omega, d_\omega, \star_\omega)$. We first define the collection $\mathcal{M}(\Omega, (X_\omega)_\omega)$ of *measurable sequences* as the set of those sequences $(x_\omega)_{\omega\in\Omega}$ with $x_\omega \in X_\omega$ such that the function $\omega \in \Omega \mapsto d(x_\omega, \star_\omega) \in \mathbb{R}$ is measurable. Then we define

$$\mathcal{L}^p((X_\omega)_\omega) := \{(x_\omega)_\omega \in \mathcal{M}(\Omega, (X_\omega)_\omega) : \int d_\omega(x_\omega, \star_\omega)^p \mathrm{d}\mu(\omega) < \infty\}.$$

and further $L^p((X_\omega)_\omega) \coloneqq \mathcal{L}^p/N$ with

$$N \coloneqq \{\{(x_\omega)_\omega \in \mathcal{M}(\Omega, (X_\omega)_\omega) : \int d_\omega(x_\omega, \star_\omega)^p \mathrm{d}\mu(\omega) = 0\}.$$

We write $L^p(\Omega; X)$ for $L^p((X_\omega)_\omega)$ if $X_\omega = X$ for all $\omega \in \Omega$.

The *distance function* on $L^p((X_\omega)_\omega)$ between $(x_\omega)_{\omega\in\Omega}, (y_\omega)_{\omega\in\Omega} \in L^p((X_\omega)_\omega)$ is

$$d((x_\omega)_{\omega\in\Omega}, (y_\omega)_{\omega\in\Omega}) := \left(\int d_\omega(x_\omega, y_\omega)^p \mathrm{d}\mu(\omega)\right)^{1/p}.$$

Proposition 5.2. *The set $L^p((X_\omega)_\omega)$ is naturally a pointed metric space, which is geodesic if all X_ω are geodesic.*

Proof. The fact that d is a metric for $L^p((X_\omega)_\omega)$ follows from the usual proof of Minkowski inequality for the norm $\|(x_\omega)_{\omega\in\Omega}\|_p \coloneqq d((x_\omega)_{\omega\in\Omega}, (\star_\omega)_{\omega\in\Omega})$.

Let us then show that $L^p((X_\omega)_\omega)$ is geodesic if X_ω is geodesic for each $\omega \in \Omega$. Let $(x_\omega)_\omega, (y_\omega)_\omega \in L^p((X_\omega)_\omega)$. Now for all $\omega \in \Omega$ there exists a curve $\gamma_\omega \colon [0, 1] \to X_\omega$ taking x_ω to y_ω such that $d(x_\omega, y_\omega) = L(\gamma_\omega)$. We may assume that γ_ω are parametrized by constant speed.

Let $\gamma \colon [0, 1] \to L^p((X_\omega)_\omega)$, $\gamma(t) = (\gamma_\omega(t))_\omega$. The curve γ is well defined, since for all $t \in [0, 1]$,

$$d(\gamma(t), (x_\omega)_\omega)^p = \int d(\gamma_\omega(t), x_\omega)^p \mathrm{d}\mu(\omega) \leq \int d(y_\omega, x_\omega)^p \mathrm{d}\mu(\omega) = d((y_\omega)_\omega, (x_\omega)_\omega)^p$$

and so

$$d(\gamma(t), (\star_\omega)_\omega) \leq d(\gamma(t), (x_\omega)_\omega) + d((x_\omega)_\omega, (\star_\omega)_\omega) \leq d((y_\omega)_\omega, (\star_\omega)_\omega) + 2d((x_\omega)_\omega, (\star_\omega)_\omega) < \infty.$$

Let then $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0, 1]$. Since γ_ω are geodesics with constant speed,

$$d(\gamma_\omega(t_{i-1}), \gamma_\omega(t_i)) = (t_i - t_{i-1}) d(x_\omega, y_\omega)$$

for all $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) &= \sum_{i=1}^{n} \Big(\int d(\gamma_\omega(t_{i-1}), \gamma_\omega(t_i))^p \mathrm{d}\mu(\omega) \Big)^{1/p} \\ &= \sum_{i=1}^{n} \Big(\int (t_i - t_{i-1})^p d(x_\omega, y_\omega)^p \mathrm{d}\mu(\omega) \Big)^{1/p} \\ &= \sum_{i=1}^{n} (t_i - t_{i-1})\, d((x_\omega)_\omega, (y_\omega)_\omega) \\ &= d((x_\omega)_\omega, (y_\omega)_\omega).\end{aligned}$$

Hence

$$L(\gamma) = \inf_{\mathcal{P}} (\sum_{t_i \in \mathcal{P}} d(\gamma(t_{i-1}), \gamma(t_i))) = d((x_\omega)_\omega, (y_\omega)_\omega),$$

where the infimum is taken over all partitions $\mathcal{P}$ of $[0, 1]$. The proof is complete. □

Notice that if each X_ω admits a group structure we may define a group operation for $L^p((X_\omega)_\omega)$ element wise. We focus now on ℓ_p-sums of groups. For a countable family $\{G_n\}_{n\in\mathbb{N}}$ of groups we define $\ell_p((G_n)_n)$ by

$$\ell_p((G_n)_n) \coloneqq \{(x_n)_{n\in\mathbb{N}} : x_n \in G_n, \sum_{n\in\mathbb{N}} d(x_n, e_n)^p < \infty\},$$

$$(x_n)_n \cdot (y_n)_n \coloneqq (x_n y_n)_n.$$

We write $\ell_p(G)$ for $\ell_p((G_n)_n)$ if $G_n = G$ for all $n \in \mathbb{N}$.

Proposition 5.3. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of topological groups metrized by left-invariant metrics and let $p \in [1, \infty)$. Then $\ell_p((G_n)_{n\in\mathbb{N}})$ is a topological group.*

Proof. We first show that the right translations are continuous. Fix $(b_n)_n \in \ell_p((G_n)_{n\in\mathbb{N}})$, that is, $b_n \in G_n$ and $\sum_{n=1}^{\infty} |b_n|^p < \infty$, where $|b_n| := d(b_n, e)$ and d is the distance on G_n. Let $(a_{n,j})_n$ be a sequence in $\ell_p((G_n)_{n\in\mathbb{N}})$ converging to some $(a_n)_n$. Fix some $\epsilon > 0$. We take N large enough so that $\sum_{n=N+1}^{\infty} |b_n|^p < \epsilon$. Then, being N fixed and being the right translations $R_{b_1}, \ldots, R_{b_N}$ continuous, we take J large enough so that for all $j > J$ and all $n = 1, \ldots, N$

$$d((a_{n,j})_n, (a_n)_n) < \epsilon \tag{5.4}$$

$$d(R_{b_n}(a_{n,j}), R_{b_n}(a_n))^p < \epsilon/N. \tag{5.5}$$

Notice that consequently

$$\begin{aligned}\sum_{n=N+1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p &\leq \sum_{n=N+1}^{\infty} (d(a_{n,j}b_n, a_{n,j}) + d(a_{n,j}, a_n) + d(a_n, a_nb_n))^p \\ &= \sum_{n=N+1}^{\infty} (|b_n| + d(a_{n,j}, a_n) + |b_n|)^p \\ &\leq 2^p d((a_{n,j})_n, (a_n)_n) + 2^{p+1} \sum_{n=N+1}^{\infty} |b| \\ &\leq 2^p \cdot 3\epsilon,\end{aligned}$$

where we used the trick

$$\sum (a+b)^p \leq \sum 2^p \max\{a,b\}^p \leq 2^p(\sum a^p + \sum b^p).$$

Then for all $j > J$

$$\begin{aligned}d(R_{(b_n)_n}(a_{n,j})_n, R_{(b_n)_n}(a_n)_n) &= \sum_{n=1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p \\ &= \left(\sum_{n=1}^{N} d(R_{b_n}(a_{n,j}), R_{b_n}(a_n))^p\right) + \sum_{n=N+1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p \\ &\leq N\epsilon/N + 2^p \cdot 3\epsilon = (1 + 3 \cdot 2^p)\epsilon.\end{aligned}$$

Therefore, the multiplication in $\ell_p((G_n)_{n\in\mathbb{N}})$ is continuous since, if $(a_{n,j})_n \to (a_n)_n$ and $(b_{n,j})_n \to (b_n)_n$, as $j \to \infty$, then using left invariance we have

$$\begin{aligned}d((a_{n,j})_n(b_{n,j})_n, (a_n)_n(b_n)_n) &\leq d((a_{n,j})_n(b_{n,j})_n, (a_{n,j})_n(b_n)_n) + d((a_{n,j})_n(b_n)_n, (a_n)_n(b_n)_n) \\ &\leq d((b_{n,j})_n, (b_n)_n) + d(R_{(b_n)_n}(a_{n,j})_n, R_{(b_n)_n}(a_n)_n) \to 0.\end{aligned}$$

We then show that the inversion is also continuous. Let $(a_{n,j})_n \to (a_n)_n$. Take N large so that $\sum_{n=N+1}^{\infty} |a_n|^p < \epsilon$. Since the inversions in $G_1, \ldots, G_N$ are continuous, there exists J such that for all $j > J$ and all $n = 1, \ldots, N$,

$$d((a_{n,j})_n, (a_n)_n) < \epsilon \tag{5.6}$$

$$d(a_{n,j}^{-1}, a_n^{-1})^p < \epsilon/N. \tag{5.7}$$

Then for all $j > J$

$$\begin{aligned}d((a_{n,j}^{-1})_n, (a_n^{-1})_n) &= \sum_{n=1}^{N} d(a_{n,j}^{-1}, a_n^{-1})^p + \sum_{n=N+1}^{\infty} d(a_{n,j}a_n^{-1}, e)^p \\ &\leq N\epsilon/N + \sum_{n=N+1}^{\infty} (d(a_{n,j}a_n^{-1}, a_{n,j}) + d(a_{n,j}, a_n) + d(a_n, e))^p \\ &\leq \epsilon + 2^p d((a_{n,j})_n, (a_n)_n) + 2^{p+1} \sum_{n=N+1}^{\infty} |a_n|^p \\ &\leq (1 + 3 \cdot 2^p)\epsilon.\end{aligned}$$

□

Remark 5.8. In a similar manner as for Proposition 5.3, one can also show that

$$c_0((G_n)_{n\in\mathbb{N}}) \coloneqq \{(x_n)_n \in \ell_\infty((G_n)_{n\in\mathbb{N}}) : \lim_{n\to\infty} d(x_n, e_{G_n}) = 0\}$$

is a topological group.

5.3. **Examples of metric scalable groups.** Using the previous subsection, we can build examples of metric scalable groups starting with arbitrary sequences of Carnot groups equipped with homogeneous distances.

Proposition 5.9. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of metric scalable groups and let $p \in [1,\infty)$. Then $\ell_p((G_n)_{n\in\mathbb{N}})$ is a metric scalable group. Moreover, if each G_n is complete and admits a filtration by Carnot subgroups, then $\ell_p((G_n)_{n\in\mathbb{N}})$ is complete and admits a filtration by Carnot subgroups.*

Proof. We define the scaling map $\delta \colon \mathbb{R} \times \ell_p((G_n)_n) \to \ell_p((G_n)_n)$ element wise using the scalings of each scalable group G_n. By the previous proposition, $\ell_p((G_n)_n)$ is a topological group. Hence it remains to see that δ satisfies the conditions of a scalable group as in Definition 1.1 and that the metric is homogeneous with respect to δ, which is straightforward to check. The proof for the fact that $\ell_p((G_n)_{n\in\mathbb{N}})$ is complete assuming that each G_n is complete, is analogous to the proof of completeness of the classical ℓ_p spaces. Assume then that $(N^n_m)_m$ is a filtration by Carnot subgroups for each G_n. Then letting

$$N_m = N^1_m \times N^2_{m-1} \times \cdots \times N^m_1 \times \{e\}^{\mathbb{N}}$$

for each $m \in \mathbb{N}$ defines a filtration by Carnot subgroups for $\ell_p((G_n)_{n\in\mathbb{N}})$. Indeed, each N_m is isomorphic to a finite product of Carnot groups, and the union $\cup_m N_m$ is dense in $\ell_p((G_n)_{n\in\mathbb{N}})$ as the set of finite sequences is dense in $\ell_p((G_n)_{n\in\mathbb{N}})$. □

Proposition 5.9 gives us a simple way to construct many different noncommutative and infinite-dimensional metric scalable groups that admit filtrations by Carnot subgroups. Indeed, we may consider examples where each G_n is a Carnot group, like $G_n = \mathbb{H}^1$ or $G_n = \mathbb{H}^n$, where $\mathbb{H}^n$ is the n-th Heisenberg group equipped with a homogeneous distance. We stress that the last result does not require any bound on the nilpotency step of $(G_n)_{n\in\mathbb{N}}$, in case they are Carnot groups. In fact, an interesting example is when G_n is the free Carnot group of step n and rank 2, which we denote by $\mathbb{F}_{2,n}$. We state this example as a result.

Proposition 5.10. *Even though $\ell_2((\mathbb{F}_{2,n})_n)$ is not nilpotent, it is a metric scalable group that is complete and admits a filtration by Carnot groups. Moreover, the subset $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ generates $\ell_2((\mathbb{F}_{2,n})_n)$ and is separable.*

Proof. The space $\ell_2((\mathbb{F}_{2,n})_n)$ is a metric scalable group by Proposition 5.9 and the filtration is simply given by

$$N_m = \mathbb{F}_{2,1} \times \cdots \times \mathbb{F}_{2,m}.$$

The first layer $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ is given by $\ell_2((V_1(\mathbb{F}_{2,n}))_n)$ as we defined the dilation map on $\ell_2((\mathbb{F}_{2,n})_n)$ component wise. Indeed, a sequence $(x_n)_n \in \ell_2((\mathbb{F}_{2,n})_n)$ is a one-parameter subgroup if and only if each $x_n \in \mathbb{F}_{2,n}$ is a one-parameter subgroup. The fact that $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ generates follows from Proposition 1.10. Moreover, $V_1(\ell_2((\mathbb{F}_{2,n})_n)) = \ell_2((V_1(\mathbb{F}_{2,n}))_n)$ is separable since now each $V_1(\mathbb{F}_{2,n})$ is separable. □

The property of having a filtration by Carnot subgroups is not, however, stable under taking subgroups, as shown by the following example in $\ell_2((\mathbb{F}_{2,n})_n)$. It also proves that the assumption of nilpotency cannot be removed in Proposition 1.10.

Proposition 5.11. *There exists a scalable subgroup of $\ell_2((\mathbb{F}_{2,n})_n)$ that is generated by its first layer but which does not admit a filtration by Carnot subgroups.*

Proof. Denote for every $n \in \mathbb{N}$ by $X_1^{(n)}, X_2^{(n)}$ the two generators of $\mathbb{F}_{2,n}$. Let $x = (\frac{1}{n}X_1^{(n)})_n$ and $y = (\frac{1}{n}X_2^{(n)})_n$ and consider the (non-nilpotent) scalable group H generated by x and y. Now both $x, y \in V_1(H)$ but H does not admit a filtration by Carnot groups. Indeed, any scalable group having Carnot group structure is generated by its first layer, but the only nilpotent subgroups of H generated by one-parameter subgroups are one-dimensional. □

If $\mathbb{H}^1$ is the first Heisenberg group, then by Proposition 5.9, for all $p \in [1, \infty)$, the space $\ell_p(\mathbb{H}^1)$ is a metric scalable group admitting filtration by Carnot subgroups. The space $\ell_2(\mathbb{H}^1)$ has the extra property of being a Banach Lie group. Indeed, it can be modelled on $\ell_2(\mathbb{R}^2) + \ell_1(\mathbb{R})$, following [MR14]. However, we shall show that there are metric scalable groups, e.g. $\ell_1(\mathbb{H}^1)$, admitting filtrations by Carnot groups that are not Banach manifolds. Hence the notion of metric scalable group strictly extends the one of Banach homogeneous group as defined in [MR14] and studied later in [MPS17].

Proposition 5.12. *The topological group $\ell_1(\mathbb{H}^1)$ is not a Banach Lie group.*

Proof. Suppose by contradiction that $\ell_1(\mathbb{H}^1)$ is a Banach Lie group and let Z be the center of $\mathbb{H}$. One sees that

$$Z = \{(\exp(\alpha_i Z_i))_i : \alpha_i \in \mathbb{R}\},$$

where Z_i is the center of the ith Heisenberg Lie algebra. As Z is a closed subgroup of a Banach Lie group, Z is a Banach Lie group as well. However, recall that the center of the Heisenberg group is isometric to $(\mathbb{R}, \sqrt{d_E})$ and therefore

$$\ell_1(Z) = \{(a_n)_{n\in\mathbb{N}} : \sum_n \sqrt{|a_n|} < \infty\} = \ell_{1/2}(\mathbb{R}).$$

Hence also $\ell_{1/2}(\mathbb{R})$ has a structure of a Banach Lie group and its Lie algebra is a Banach space. Since $\ell_{1/2}(\mathbb{R})$ is a vector space, the exponential map $\exp\colon \mathrm{Lie}(\ell_{1/2}(\mathbb{R})) \to \ell_{1/2}(\mathbb{R})$ is a linear isomorphism. But this is a contradiction as $\ell_{1/2}(\mathbb{R})$ is not even locally convex (to be proven) and so not a normed space.

That $\ell_{1/2}(\mathbb{R})$ is not locally convex follows simply from the fact that the convex hull of any ball is unbounded. Indeed, consider any ball around the origin

$$B(0, r) = \{(a_1, a_2, ...) : \sum_n \sqrt{|a_n|} < r\}.$$

Then $x_n = (r/2n, r/2n, ..., r/2n, 0, ...)$ is in the convex hull of $B(0, r)$ (where the first n coordinates are nonzero), but

$$d(x_n, 0) = n\sqrt{\frac{r}{2n}} = \sqrt{\frac{nr}{2}}$$

diverges as $n \to \infty$. □

Appendix A. Some useful commutator identities

Lemma A.1. *Let G be a group and $x, y, z \in G$. Then*

$$[xy, z] = [x, [y, z]][y, z][x, z] \quad \text{and} \quad [z, xy] = [z, x][z, y][[y, z], x] = h[z, x][z, y],$$

where h is a product of commutators of x, y, z of weight ≥ 3.

Proof. For the first equation,

$$[y, z][x, z] = [y, z]xzx^{-1}z^{-1} = [[y, z], x]xyzy^{-1}z^{-1}zx^{-1}z^{-1} = [[y, z], x][xy, z].$$

Since $[a, b] = [b, a]^{-1}$,

$$[xy, z] = [x, [y, z]][y, z][x, z].$$

Using this and the identity $[a, b] = [b, a]^{-1}$,

$$[z, xy] = [z, x][z, y][[y, z], x].$$

The last equation follows by reordering the terms, which produces some higher order commutators into h. □

Corollary A.2. *If $[y, z] \in Z(G)$, then*

$$[xy, z] = [x, z][y, z] \quad \text{and} \quad [z, xy] = [z, x][z, y].$$

Corollary A.3. *Let $n, m \in \mathbb{N}$. Then*

$$[x^n, y^m] = h[x, y]^{nm},$$

where h is a product of commutators of x and y of weight ≥ 3.

Proof. The proof is by iterating Lemma A.1 for nm times and reordering the terms, which produces some additional higher order commutators into h. □

Lemma A.4. *Let G be a group, $x, y \in G$. Then*

$$[x^{-1}, y] = [x^{-1}, [y, x]][x, y]^{-1}.$$

Proof. The statement follows from

$$[x^{-1}, [y, x]] = x^{-1}yxy^{-1}x^{-1}x[y, x]^{-1} = [x^{-1}, y][x, y].$$

□

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