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FOCK SPACE REPRESENTATION OF THE CIRCLE QUANTUM GROUP

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ABSTRACT. In [SS17] we have defined quantum groups $\mathbf{U}_v(\mathfrak{sl}(\mathbb{R}))$ and $\mathbf{U}_v(\mathfrak{sl}(S^1))$, which can be interpreted as continuous generalizations of the quantum groups of the Kac-Moody Lie algebras of finite, respectively affine type A. In the present paper, we define the Fock space representation $\mathcal{F}_{\mathbb{R}}$ of the quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{R}))$ as the vector space generated by real pyramids (a continuous generalization of the notion of partition). In addition, by using a variant of the "folding procedure" of Hayashi-Misra-Miwa, we define an action of $\mathbf{U}_v(\mathfrak{sl}(S^1))$ on $\mathcal{F}_{\mathbb{R}}$.

CONTENTS

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1. INTRODUCTION

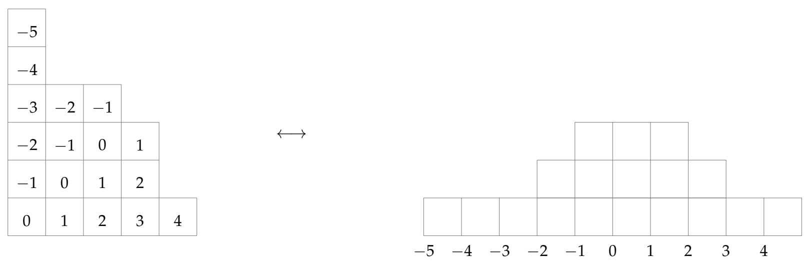
The present article continues the study of the *circle quantum group* $\mathbf{U}_v(\mathfrak{sl}(S^1))$ introduced in [SS17], where $S^1 := \mathbb{R}/\mathbb{Z}$. In that paper we constructed geometrically a family of representations $V_{g,r}$ indexed by a pair of positive integers (g,r)[1], with $V_{0,1}$ being the natural "vector" representation and $V_{0,r}$ an analog of the r-fold symmetric power of $V_{0,1}$. Our goal here is to define a Fock space representation $\mathcal{F}_\mathbb{R}$ of $\mathbf{U}_v(\mathfrak{sl}(S^1))$, which may be thought of as a limit of $V_{0,r}$ as r tends to infinity.

Just as the quantum group $\mathbf{U}_v(\mathfrak{sl}(S^1))$ is an uncountable colimit of $\mathbf{U}_v(\widehat{\mathfrak{sl}}(n))$ as n tends to infinity, the Fock space $\mathcal{F}_\mathbb{R}$ is an uncountable colimit of the standard Fock space representation of $\mathbf{U}_v(\widehat{\mathfrak{sl}}(n))$ due to Hayashi [Hay90]. Recall that this Fock space is a vector space with a basis is indexed by all partitions λ

$$\mathcal{F} := \bigoplus_\lambda \widetilde{\mathbb{Q}}|\lambda\rangle\ ,$$

where $\widetilde{\mathbb{Q}} := \mathbb{Q}[v, v^{-1}]$. The action is induced, via a "folding procedure" (see also Misra-Miwa [MM90]), by an action of the quantum enveloping algebra $\mathbf{U}_v(\mathfrak{sl}(\infty))$ on the same space $\mathcal{F}$.

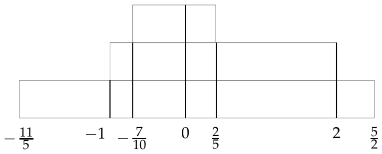
Main results. One main novelty of the limit which we consider is that instead of partitions, the Fock space $\mathcal{F}_\mathbb{R}$ has a basis indexed by what we call *(real) pyramids* (see Section 2). Integral pyramids are close cousins of Maya diagrams and are in bijection with partitions. For instance, on the left-hand-side there is the Young diagram of the partition $(5,4^2,3,1^2)$ with its standard contents written inside the corresponding boxes, and on the right-hand-side there is the corresponding integral pyramid:



Pictorially, we pass from Young diagrams to $\mathbb{Z}$-pyramids by tilting 45 degrees to the left (or writing partitions the russian way) and letting gravity act. Unlike partitions, pyramids admit a

[1]In the setting of [SS17], $\mathbf{U}_v(\mathfrak{sl}(S^1))$ is realized as (reduced) Drinfeld double of the spherical Hall algebra of parabolic torsion sheaves on a genus g curve. The representation $V_{g,r}$ arises from the Hall algebra of rank r parabolic vector bundles on the curve.

natural extensions to $\mathbb{Q}$ and $\mathbb{R}$. For example, the following is a rational pyramid:



Fix $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and let $\mathsf{Pyr}(\mathbb{K})$ be the set of all $\mathbb{K}$-pyramids. Our Fock space is defined as

$$\mathcal{F}_{\mathbb{K}} := \bigoplus_{p \in \mathsf{Pyr}(\mathbb{K})} \widetilde{\mathbb{Q}} |p\rangle \ .$$

In [SS17] we also defined a Lie algebra $\mathfrak{sl}(\mathbb{K})$ — now as a limit of the Lie algebras $\mathfrak{sl}(n)$ as n tends to ∞ — and a corresponding quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ (the *line quantum group*, cf. Definition 3.2). When $\mathbb{K} = \mathbb{Z}$, our quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{Z}))$ coincides with $\mathbf{U}_v(\mathfrak{sl}(\infty))$, but it is much bigger when $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$. Let 0 stand for the empty pyramid.

Theorem (cf. Theorem 3.6)**.** *There exists an action of* $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ *on* $\mathcal{F}_{\mathbb{K}}$ *making it an irreducible highest weight representation with highest weight vector* $|0\rangle$*.*

Let $S^1_{\mathbb{K}} := \mathbb{K}/\mathbb{Z}$ for $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Due to the absence of simple roots for the circle Lie algebra $\mathfrak{sl}(S^1_{\mathbb{K}})$ it is not obvious to adapt Hayashi's folding procedure. Indeed, the circle quantum group $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ is generated by elements E_J, F_J, K_J for J a $\mathbb{K}$-interval in $S^1_{\mathbb{K}}$ and the folding procedure involves breaking up J into finitely many smaller intervals in all possible ways, see Section 4.2. In order to obtain the correct action we use a geometric reinterpretation of the folding procedure due to Varagnolo and Vasserot [VV99] which is based on the theory of Hall algebras and which does pass to the limit.

Theorem (cf. Theorem 4.5)**.** *There exists a natural action of* $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ *on* $\mathcal{F}_{\mathbb{K}}$*, which strictly contains the irreducible highest weight representation generated by the vacuum vector* $|0\rangle$*.*

As opposed to the case of the Fock space of $\mathbf{U}_v(\mathfrak{sl}(n))$ (see e.g. [KMS95]), $\mathcal{F}_{\mathbb{K}}$ contains no highest weight vector other than the vacuum vector $|0\rangle$ when $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$. On the other hand, it is not a cyclic representation. It is, however, a cyclic representation of the *Hall algebra of* $S^1_{\mathbb{K}}$ (which strictly contains $\mathbf{U}_v^-(\mathfrak{sl}(S^1_{\mathbb{K}}))$). Its precise structure will be studied in a sequel to this paper.

Line and circle quantum groups vs. continuum quantum groups. In [ASS18], the authors, together with Andrea Appel, define a *continuum* generalization of the Kac–Moody Lie algebras, associated with a *topological* generalization of the notion of a quiver. In particular, the vertex set of a quiver is replaced by an Hausdorff topological space X, and the vertices of the quiver are replaced by connected intervals in X (we refer to *loc. cit.* for the relevant definitions). It is proved in *loc. cit.* that when $X = \mathbb{K}, S^1_{\mathbb{K}}$, the corresponding continuum Kac–Moody Lie algebras, denoted by $\mathfrak{g}_X$, coincide with $\mathfrak{sl}(\mathbb{K})$ and $\mathfrak{sl}(S^1_{\mathbb{K}})$ respectively. In [AS19], the first-named author, together with Andrea Appel, constructed a topological Lie bialgebra structure on the continuum Kac–Moody Lie algebra $\mathfrak{g}_X$ of any space X and defined algebraically the quantization $\mathbf{U}_v(\mathfrak{g}_X)$ of $\mathfrak{g}_X$, which is called the *continuum quantum group* of X. It is proved in *loc. cit.* that $\mathbf{U}_v(\mathfrak{g}_X)$ coincides with $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ and $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ when $X = \mathbb{K}, S^1_{\mathbb{K}}$ respectively. Finally, in [AKSS19], we are constructing the quantum group $\mathbf{U}_v(\mathfrak{g}_X)$ via the theory of Hall algebras. We expect to extend some of the techniques of this paper to such a more general setting, in particular the construction of a Fock space representation for any $\mathbf{U}_v(\mathfrak{g}_X)$ will be the subject of a sequel to this paper.

Outline. The paper is organized as follows. In Section 2, we recall the notions of partitions and Young diagrams, and describe their topological refinement as rational and real pyramids. In Section 3, we introduce our Fock space $\mathcal{F}_{\mathbb{K}}$ and the action of the quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$, for $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, on it, while in Section 4, we define the action of $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$, for $S^1_{\mathbb{K}} := \mathbb{K}/\mathbb{Z}$

and $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$, on $\mathcal{F}_{\mathbb{K}}$. The proof that such an action is well-defined is given in Section 6 and it is based on a variant of the "folding procedure" depending on the theory of Hall algebras associated with certain infinite quivers, as introduced in Section 5.

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Notation and convention. For any integer n, set

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}} \quad \text{and} \quad [n]! := [n]\,[n-1]\,\cdots\,[1]\,.$$

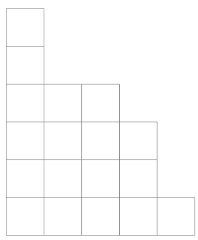
Set $\widetilde{\mathbb{Q}} := \mathbb{Q}[v, v^{-1}]$ and put $q = v^2$.

2. Partitions and pyramids

In this section, we introduce integral, rational, and real *pyramids* and establish their basic properties. Rational and real pyramids are some "continuous" generalization of the notion of partition, while we shall show that integral pyramids coincide with partitions.

2.1. **Recollection on partitions.** A *partition* of a positive integer n is a nonincreasing sequence of positive numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $|\lambda| := \sum_{a=1}^{\ell} \lambda_a = n$. We call $\ell = \ell(\lambda)$ the *length* of the partition λ. Another description of a partition λ of n uses the notation $\lambda = (1^{m_1}\, 2^{m_2} \cdots)$, where $m_i = \#\{a \in \mathbb{Z}_{\geq 0} \mid \lambda_a = i\}$ with $\sum_i i\, m_i = n$ and $\sum_i m_i = \ell(\lambda)$. We denote by $\Pi(n)$ the set of all partitions of n, and $\Pi := \bigcup_n \Pi(n)$. On the set Π of all partitions there is a natural partial ordering called *dominance ordering*: for two partitions μ and λ, we write $\mu \leq \lambda$ if and only if $|\mu| = |\lambda|$ and $\mu_1 + \cdots + \mu_a \leq \lambda_1 + \cdots + \lambda_a$ for all $a \geq 1$. We write $\mu < \lambda$ if and only if $\mu \leq \lambda$ and $\mu \neq \lambda$.

One can associate with a partition λ its *Young diagram*, which is the set $Y_\lambda = \{(x, y) \in \mathbb{Z}^2_{>0} \mid 1 \leq y \leq \ell(\lambda)\,, 1 \leq x \leq \lambda_y\}$. Then λ_y is the length of the y-th row of Y_λ; we write $|Y_\lambda| = |\lambda|$ for the *weight* of the Young diagram Y_λ. We shall identify a partition λ with its Young diagram Y_λ. For example, with the partition $\lambda = (5, 4^2, 3, 1^2)$ we associate the Young diagram Y_λ:



For a partition λ, the *transpose partition* λ' is the partition whose Young tableau is $Y_{\lambda'} := \{(b, a) \in \mathbb{Z}^2_{>0} \mid (a, b) \in Y_\lambda\}$.

Finally, we call *standard content* of $s = (x, y) \in Y_\lambda$ the quantity $c(s) := x - y$. We say that box s is *of color* i if $c(s) = i$. An *addable* i*-box* is a box of color i which can be added to Y_λ in such a way that the new diagram still comes from a partition, similarly a *removable* i*-box* is a box of color i which can be removed from Y_λ. For $i \in \mathbb{Z}$, define

$$n_i(\lambda) := \#\{\text{addable } i\text{-boxes of } Y_\lambda\} - \#\{\text{removable } i\text{-boxes of } Y_\lambda\}\,.$$

Remark 2.1. Note that

$$n_i(\lambda) = \begin{cases} 1 & \text{if there exists an addable box of color } i\,, \\ -1 & \text{if there exists a removable box of color } i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$\triangle$

2.2. **Integral pyramids.** We now provide another combinatorial realization of partitions very similar to that of Maya diagrams, which we call "integral pyramids". Below, 'increasing' and 'decreasing' are meant in the broad sense.

Definition 2.2. An *integral pyramid* is a function $p\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ satisfying the following properties:

i) $p(n) = 0$ for $|n| \gg 0$;

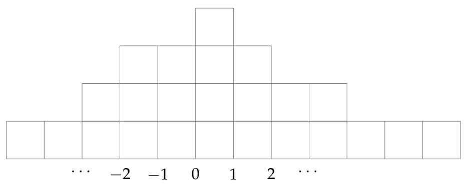
ii) $p(0) = \max\{p(n) \mid n \in \mathbb{Z}\}$;

iii) p is increasing on $\mathbb{Z}_-$ and decreasing on $\mathbb{Z}_+$;

iv) $|p(n) - p(n+1)| \leq 1$ for all $n \in \mathbb{Z}$.

⊘

We may represent a pyramid as a box diagram in which we draw $p(n)$ boxes over the integer n: for instance, we represent the pyramid such that $p(-5) = p(-4) = 1, p(-3) = 2, p(-2) = p(-1) = 3, p(0) = 4, p(1) = 3, p(2) = p(3) = 2, p(4) = p(5) = p(6) = 1$ and $p(n) = 0$ for $n < -5$ or $n > 6$ as follows:

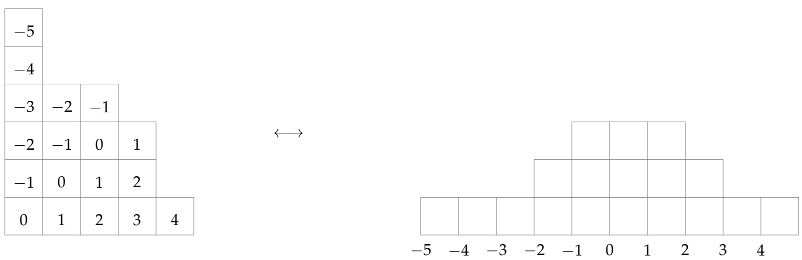


The set of pyramids with n boxes will be denoted $\mathsf{Pyr}(n)$.

Lemma 2.3. *There is a canonical bijection between* $\mathsf{Pyr}(n)$ *and the set* $\Pi(n)$ *of partitions of* n.

Proof. The bijection $\lambda \mapsto p_\lambda$ goes as follows. If λ is a partition and Y_λ its Young diagram, we fill the boxes of Y_λ with the standard content $c(s) = y - x$ if $s = (x, y)$. Then we set $p_\lambda(n) := \#\{s \in Y_\lambda \mid c(s) = n\}$. □

Example 2.4. For instance, to the partition $\lambda = (5, 4^2, 3, 1^2)$ we associate the pyramid $p_\lambda(-5) = p_\lambda(-4) = p_\lambda(-3) = 1, p_\lambda(-2) = 2, p_\lambda(-1) = p_\lambda(0) = p_\lambda(1) = 3, p_\lambda(2) = 2, p_\lambda(3) = p_\lambda(4) = 1$.



Pictorially, the bijection amounts to folding clockwise by 90 degrees the part of the pyramid which lies over $-\mathbb{Z}_{\geq 0}$ (and shifting accordingly each layer of the pyramid to the right by a number of boxes equal to its height). Conversely, starting from a partition, one may tilt its Young diagram 45 degrees to the left (so that it stands on its corner) and let gravity act to obtain the associated pyramid. △

It is easy to translate several classical notions from partitions to pyramids. For instance,

$$\ell(\lambda) = \sup\{n \mid p_\lambda(-n) \neq 0\}, \tag{2.1}$$

$$p_{\lambda'} = p_\lambda \circ \iota, \tag{2.2}$$

where λ' is the transpose partition of λ and $\iota\colon \mathbb{Z} \to \mathbb{Z}$ sends n to $-n$, and

$$|\lambda| = \int p_\lambda := \sum_n p(n) . \tag{2.3}$$

The dominance ordering is less pleasant. It translates into the following set of inequalities:

$$p \geq q \qquad \Leftrightarrow \qquad \int \inf(p, \kappa_n) \geq \int \inf(q, \kappa_n) \quad \text{for all } n , \tag{2.4}$$

where

$$\kappa_n(i) := \begin{cases} n & \text{if } i \geq 0 , \\ n+i & \text{if } -n < i < 0 , \\ 0 & \text{if } i \leq -n . \end{cases}$$

2.3. **Rational and real pyramids.** We now generalize the concept of pyramids to allow for "jumps" of the function p located at non-integral points. More precisely, let us introduce the following definition.

Definition 2.5. A *rational* (resp. *real*) *pyramid* is defined to be a function $p\colon \mathbb{Q} \to \mathbb{Z}_{\geq 0}$ (resp. a function $p\colon \mathbb{R} \to \mathbb{Z}_{\geq 0}$) satisfying the following properties (we set $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ accordingly):

i) $p(x) = 0$ for $|x| \gg 0$;
ii) $p(0) = \max\{p(x) \mid x \in \mathbb{K}\}$;
iii) p is increasing on $\mathbb{K}_-$ and decreasing on $\mathbb{K}_+$;
iv) p is right–continuous and piecewise constant, with finitely many points of discontinuity;
v) $|p_+(x) - p_-(x)| \leq 1$ for all $x \in \mathbb{K}$.

⊘

We extend the notions of length, transpose and size to rational or real pyramids using (2.1), (2.2), (2.3) respectively. The notion of dominance is also extended accordingly, where we now require the inequality (2.4) for any $n \in \mathbb{K}$.

We may still represent rational or real pyramids as diagrams (now simply the graph of the function p). On the other hand, it is not possible to represent rational or real pyramids as Young diagrams anymore, as the operation of "folding clockwise by 90 degrees" does not make sense. We denote by $\mathsf{Pyr}_{\mathbb{K}}$ the set of all $\mathbb{K}$-pyramids. We will sometimes also denote by $\mathsf{Pyr}_{\mathbb{K}}(u)$ the set of $\mathbb{K}$-pyramids with size u.

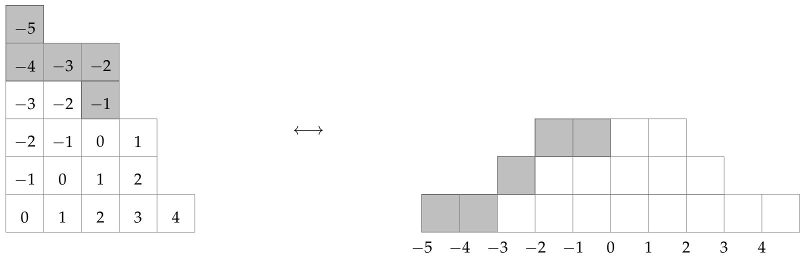
2.4. **Addable and removable intervals.** Fix $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

Definition 2.6. We call a $\mathbb{K}$-*interval* a closed-open interval of the form $J = [a, b[:= \{x \in \mathbb{R} \mid a \leq x < b\}$ with $a, b \in \mathbb{K}$. We denote by $\mathbb{1}_J$ the *characteristic function* of J. ⊘

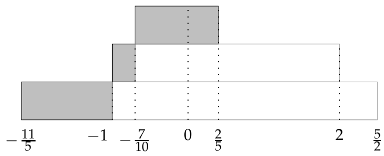
Definition 2.7. Let p be a $\mathbb{K}$-pyramid. A $\mathbb{K}$-interval J is called *addable* (resp. *removable*) if $p + \mathbb{1}_J$ (resp. $p - \mathbb{1}_J$) is still a $\mathbb{K}$-pyramid. ⊘

Remark 2.8. When $\mathbb{K} = \mathbb{Z}$, as proved in Lemma 2.3 a pyramid p corresponds to a partition λ_p. A length one $\mathbb{Z}$-interval J gives rise to a box $s := (1 + p(a), 1 + p(a) - a)$. Hence J is addable (resp. removable) if s is addable (resp. removable) in the sense defined in Section 2.1. More generally, when J is of arbitrary length, it corresponds to a connected strip. △

Example 2.9. Here is an example of an addable $\mathbb{Z}$-interval and the corresponding strip:



and here is an example of an addable $\mathbb{Q}$ (or $\mathbb{R}$)-interval (in this case, $J = [-\frac{11}{5}, \frac{2}{5}[$):



Note that we could not have added $[-\frac{11}{5}, \frac{5}{2}[$ since there would be a jump of 2 over $\frac{5}{2}$, nor could we have added the interval $[-\frac{11}{5}, 0[$ since 0 would not be the maximum anymore. △

Definition 2.10. Let p be a $\mathbb{K}$-partition. We put

$$D_{\mathbb{K}}(p) := \{y \in \mathbb{R} \mid y \text{ is a point of discontinuity of } p\} .$$

and call this the *set of discontinuities of p*. ⊘

Remark 2.11. Let $J = [a, b[$ be a $\mathbb{K}$-interval and let p be a $\mathbb{K}$-partition. Then $p + \mathbb{1}_J$ is a pyramid if and only if one of the following mutually exclusive cases occurs

- $a \in D_{\mathbb{K}}(p)$ if $0 < a$;
- $a, b \notin D_{\mathbb{K}}(p)$ if $a \leq 0 < b$;
- $b \in D_{\mathbb{K}}(p)$ if $b < 0$.

Similarly, $p - \mathbb{1}_J$ is a pyramid if and only if one the following mutually exclusive cases occurs

- $b \in D_{\mathbb{K}}(p)$ if $0 < a$;
- $a, b \in D_{\mathbb{K}}(p)$ if $a \leq 0 < b$;
- $a \in D_{\mathbb{K}}(p)$ if $b < 0$.

△

Let p be a $\mathbb{K}$-pyramid and let J be an addable interval. We define the *p-height* of J as

$$\mathsf{ht}_p(J) := \sup \{|p(x) - p(y)| \mid x, y \in J\} .$$

We will also consider the variants

$$\mathsf{ht}_p^{\pm}(J) := \sup \{|p(x) - p(y)| \mid x, y \in J \cap \mathbb{K}_{\pm}\} ,$$

so that $\mathsf{ht}_p(J) = \max\{\mathsf{ht}_p^+(J), \mathsf{ht}_p^-(J)\}$.

Example 2.12. In the last example above, we have, for $J = [-\frac{11}{5}, \frac{5}{2}[$,

$$\mathsf{ht}_p(J) = 2 , \quad \mathsf{ht}_p^-(J) = 2 , \quad \mathsf{ht}_p^+(J) = 0 .$$

△

3. Fock space representation of the quantum group of the line

Fix $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We denote by $\mathbb{F}(\mathbb{K})$ the algebra of piecewise constant, right–continuous functions $f\colon \mathbb{R} \to \mathbb{Z}$, with finitely many points of discontinuity, bounded support and whose points of discontinuity belong to $\mathbb{K}$. This means that $f \in \mathbb{F}(\mathbb{K})$ if and only if $f = \sum_J c_J \mathbb{1}_J$, where the sum runs over all intervals of $\mathbb{K}$ and $c_J \in \mathbb{Z}$ is zero for all but finitely many J. Given $f, g \in \mathbb{F}(\mathbb{K})$, we define

$$\langle f, g\rangle := \sum_x f_-(x)(g_-(x) - g_+(x)) \quad \text{and} \quad (f, g) := \langle f, g\rangle + \langle g, f\rangle\ , \tag{3.1}$$

where we have set $h_\pm(x) := \lim_{t\to 0, t>0} h(x \pm t)$. Let $\mathbb{F}(\mathbb{K})^+$ (resp. $\mathbb{F}(\mathbb{K})^-$) be the set of functions $f \in \mathbb{F}(\mathbb{K})$ such that $f(x) \geq 0$ (resp. $f(x) \leq 0$) for any $x \in \mathbb{R}$.

Remark 3.1. Let $J_i = [a_i, b_i[$ be a $\mathbb{K}$-interval for $i = 1, 2$. Then

- $\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2}\rangle = 1$ if one of the following conditions holds: $J_1 = J_2$, $a_1 = a_2$ and $b_2 < b_1$, $a_2 < a_1$ and $b_1 = b_2$, $a_2 < a_1 < b_2 < b_1$;
- $\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2}\rangle = 0$ if one of the following conditions holds: $\overline{J}_1 \cap \overline{J}_2 = \emptyset$, $b_2 = a_1$, $a_1 = a_2$ and $b_1 < b_2$, $a_1 < a_2$ and $b_1 = b_2$, $a_1 < a_2 < b_2 < b_1$, $a_2 < a_1 < b_1 < b_2$;
- $\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2}\rangle = -1$ if either $b_1 = a_2$ or $a_1 < a_2 < b_1 < b_2$.

Thus, the following holds:

- for any $\mathbb{K}$-interval J, we have $(\mathbb{1}_J, \mathbb{1}_J) = 2$;
- if J_1, J_2 are $\mathbb{K}$-intervals of the form $J_1 = [a, b[$ and $J_2 = [b, c[$, we get
$$(\mathbb{1}_{J_1}, \mathbb{1}_{J_2}) = (\mathbb{1}_{J_2}, \mathbb{1}_{J_1}) = -1;$$
- in any other case, (i.e. if $I \neq J$ are $\mathbb{K}$-intervals such that $\mathbb{1}_J + \mathbb{1}_I$ is not the characteristic function of any interval) then $(\mathbb{1}_I, \mathbb{1}_J) = 0$.

$\triangle$

3.1. The bialgebra $\mathbf{U}_\upsilon(\mathfrak{sl}(\mathbb{K}))$.

Definition 3.2 (cf. [SS17, Section 1.1]). Let $\mathbf{U}_\upsilon(\mathfrak{sl}(\mathbb{K}))$ be the topological $\widetilde{\mathbb{Q}}$-bialgebra generated by elements $E_J, F_J, K_J^{\pm 1}$, where J runs over all $\mathbb{K}$-intervals, modulo the following set of relations:

- *Drinfeld-Jimbo relations*:
  - for any intervals J, I, I_1, I_2,
$$[K_{I_1}, K_{I_2}] = 0\ ,$$
$$K_I\, E_J\, K_I^{-1} = \upsilon^{(\mathbb{1}_I, \mathbb{1}_J)}\, E_J\ ,$$
$$K_I\, F_J\, K_I^{-1} = \upsilon^{-(\mathbb{1}_I, \mathbb{1}_J)}\, F_J\ ;$$
  - if J_1, J_2 are intervals such that $J_1 \cap J_2 = \emptyset$,
$$[F_{J_1}, E_{J_2}] = 0\ ;$$
  - for any interval J,
$$[E_J, F_J] = \frac{K_J - K_J^{-1}}{\upsilon - \upsilon^{-1}}\ ;$$
- *join relations*: if J_1, J_2 are intervals of the form $J_1 = [a, b[$ and $J_2 = [b, c[$ then
$$K_{J_1}\, K_{J_2} = K_{J_1 \cup J_2}\ ;$$

and

$$E_{J_1 \cup J_2} = v^{1/2} E_{J_1} E_{J_2} - v^{-1/2} E_{J_2} E_{J_1} ,$$
$$F_{J_1 \cup J_2} = v^{-1/2} F_{J_2} F_{J_1} - v^{1/2} F_{J_1} F_{J_2} ;$$

- *nest relations*:
  - if J_1, J_2 are intervals such that $\overline{J_1} \cap \overline{J_2} = \emptyset$,
$$[E_{J_1}, E_{J_2}] = 0 \quad \text{and} \quad [F_{J_1}, F_{J_2}] = 0 ; \tag{3.2}$$
  - if J_1, J_2 are intervals such that $J_1 \subseteq J_2$,
$$v^{\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2} \rangle} E_{J_1} E_{J_2} = v^{\langle \mathbb{1}_{J_2}, \mathbb{1}_{J_1} \rangle} E_{J_2} E_{J_1} ,$$
$$v^{\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2} \rangle} F_{J_1} F_{J_2} = v^{\langle \mathbb{1}_{J_2}, \mathbb{1}_{J_1} \rangle} F_{J_2} F_{J_1} .$$

The coproduct is given by:

$$\Delta(K_J) = K_J \otimes K_J ,$$
$$\Delta(E_{[a,b[}) = E_{[a,b[} \otimes 1 + \sum_{a<c<b} v^{-1/2} (v - v^{-1}) E_{[a,c[} K_{[c,b[} \otimes E_{[c,b[} + K_{[a,b[} \otimes E_{[a,b[} ,$$
$$\Delta(F_{[a,b[}) = 1 \otimes F_{[a,b[} - \sum_{a<c<b} v^{-1/2} (v - v^{-1}) F_{[c,b[} \otimes F_{[a,c[} K_{[c,b[}^{-1} + F_{[a,b[} \otimes K_{[a,b[}^{-1} .$$

Here the sums on the right-hand-side run over all possible[2] $\mathbb{K}$-values $c \in [a, b[$. ⊘

Definition 3.3. Let $f = \sum_J c_J \mathbb{1}_J \in \mathbb{F}(\mathbb{K})$. Then we set $K_f := \prod_J K_J^{c_J}$. ⊘

Remark 3.4. Let J_1, J_2 be $\mathbb{K}$-intervals of the form $J_1 = [a, b[$ and $J_2 = [b, c[$ (so that $J_1 \cup J_2$ is again a interval). Then, by the join and the nest relations we derive:

$$E_{J_1} E_{J_2}^2 - [2]\, E_{J_2} E_{J_1} E_{J_2} + E_{J_2}^2 E_{J_1} = 0 \tag{3.3}$$
$$F_{J_1} F_{J_2}^2 - [2]\, F_{J_2} F_{J_1} F_{J_2} + F_{J_2}^2 F_{J_1} = 0$$

and similarly with J_1 and J_2 exchanged. In other words, the usual (type A) cubic Serre relations are implied by the join and nest relations. △

Definition 3.5. We define a $\mathbb{F}(\mathbb{K})$-gradation on $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ by setting

$$\deg(E_J) = \mathbb{1}_J , \quad \deg(F_J) = -\mathbb{1}_J \quad \text{and} \quad \deg(K_J) = 0$$

for any $\mathbb{K}$-interval J. ⊘

As usual, we define $\mathbf{U}_v^+(\mathfrak{sl}(\mathbb{K}))$ (resp. $\mathbf{U}_v^-(\mathfrak{sl}(\mathbb{K}))$) as the subalgebra of $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ generated by the E_J (resp. the F_J) for any $\mathbb{K}$-interval J. Let also $\mathbf{U}_v^0(\mathfrak{sl}(\mathbb{K}))$ be the subalgebra of $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ generated by the $K_J^{\pm 1}$ for any $\mathbb{K}$-interval J. Finally, set

$$\mathbf{U}_v^{\leqslant 0}(\mathfrak{sl}(\mathbb{K})) := \mathbf{U}_v^-(\mathfrak{sl}(\mathbb{K})) \cdot \mathbf{U}_v^0(\mathfrak{sl}(\mathbb{K})) \quad \text{and} \quad \mathbf{U}_v^{\geqslant 0}(\mathfrak{sl}(\mathbb{K})) := \mathbf{U}_v^0(\mathfrak{sl}(\mathbb{K})) \cdot \mathbf{U}_v^+(\mathfrak{sl}(\mathbb{K})) .$$

Then $\mathbf{U}_v^{\leqslant 0}(\mathfrak{sl}(\mathbb{K}))$ and $\mathbf{U}_v^{\geqslant 0}(\mathfrak{sl}(\mathbb{K}))$ may be endowed with the structure of topological $\widetilde{\mathbb{Q}}$-bialgebras.

3.2. **The Fock space $\mathcal{F}_{\mathbb{K}}$ of $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$.** Our aim in this section is to define explicitly an action of the quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ on the space

$$\mathcal{F}_{\mathbb{K}} := \bigoplus_{p \in \mathsf{Pyr}_{\mathbb{K}}} \widetilde{\mathbb{Q}} |p\rangle,$$

generalizing the standard Fock space representation of $\mathbf{U}_v(\mathfrak{sl}(\infty))$.

[2] $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ is a *topological* coalgebra, i.e. the comultiplication only takes values in a suitable completion, see [SS17]; we will not use the coproduct in this paper.

Theorem 3.6. *The following formulas define an action of the quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{K}))$ on $\mathcal{F}_\mathbb{K}$: for any $J = [a, b[$ and $p \in \mathsf{Pyr}_\mathbb{K}$*

$$E_J|p\rangle := \begin{cases} -v^{1/2}(-v)^{p(b)-p(a)}\,|p - \mathbb{1}_J\rangle & \text{if } J \text{ is a removable interval of } p\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

$$F_J|p\rangle := \begin{cases} v^{1/2}(-v)^{p(a)-p(b)}\,|p + \mathbb{1}_J\rangle & \text{if } J \text{ is an addable interval of } p\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

$$K_J|p\rangle := v^{n_J(p)}\,|p\rangle\,,$$

where

$$n_J(p) = \begin{cases} 0 & \text{if } J \text{ is neither addable or removable to } p\,,\\ 1 & \text{if } J \text{ is addable to } p\,,\\ -1 & \text{if } J \text{ is removable to } p\,.\end{cases}$$

The representation $\mathcal{F}_\mathbb{K}$ is highest weight and irreducible.

Proof. For simplicity, let us put

$$e_{J,p} := -v^{1/2}(-v)^{p(b)-p(a)} \quad \text{and} \quad f_{J,p} := v^{1/2}(-v)^{p(a)-p(b)}\,.$$

We will check the compatibility with all the defining relations directly.

Nest relations. Let's start by verifying the nest relations for F_J. We begin with the case of a pair of intervals I, J such that $\overline{I} \cap \overline{J} = \emptyset$. Then it is possible to successively add I and then J to p if and only if it is possible to successively add J and then I to p. Moreover, in this case we have $f_{J,p+\mathbb{1}_I} = f_{J,p}$ and $f_{I,p+\mathbb{1}_J} = f_{I,p}$. It follows that $F_I\, F_J|p\rangle = F_J\, F_I|p\rangle$ as wanted.

Now assume that $I = [a, b[\subseteq J = [a', b'[$. We claim that it is possible to add both I and J to p (in either order) only if $\overline{I} \subset J^\circ$, i.e., only if $a' < a < b < b'$. Indeed, suppose for instance that $a' = a$. Then $(p + \mathbb{1}_I + \mathbb{1}_J)(a) = p(a) + 2$ while $(p + \mathbb{1}_I + \mathbb{1}_J)_-(a) = p_-(a)$. But then p and $p + \mathbb{1}_I + \mathbb{1}_J$ cannot both be pyramids: if $a \leq 0$ then we have $p_-(a) \leq p(a)$ hence $(p + \mathbb{1}_I + \mathbb{1}_J)_-(a) \leq (p + \mathbb{1}_I + \mathbb{1}_J)(a) - 2$, violating condition iii) of pyramids; likewise, if $a > 0$ then $p_(a) \leq p(a) + 1$ hence $(p + \mathbb{1}_I + \mathbb{1}_J)_-(a) \leq (p + \mathbb{1}_I + \mathbb{1}_J)(a) - 1$, again violating condition iii) of pyramids. A very similar reasoning takes care of the case $b = b'$. We are thus left with the case $a' < a < b < b'$. In this situation, it is easy to see that either I and J are both addable to p, in which case they can be added in either order, or one of them is not addable to p, in which case the two can not be added (in either order). If both may be added, then we have

$$p(a) - p(b) = (p + \mathbb{1}_J)(a) - (p + \mathbb{1}_J)(b) \quad \text{and} \quad p(a') - p(b') = (p + \mathbb{1}_I)(a') - (p + \mathbb{1}_I)(b')\,,$$

from which it immediately follows that $f_{I,p+\mathbb{1}_J}\, f_{J,p} = f_{J,p+\mathbb{1}_I}\, f_{I,p}$, i.e., $F_I\, F_J|p\rangle = F_J\, F_I|p\rangle$ (note that $\langle \mathbb{1}_I, \mathbb{1}_J\rangle = \langle \mathbb{1}_J, \mathbb{1}_I\rangle = 0$, cf. Remark 3.1). The proof of the nest relations for the E_J follows similarly as above.

Join relations. Let's now move to verify the join relations for the F_J. Assume that $I = [a, b[, J = [b, c[$. There are three mutually exclusive possible situations: it is not possible to add both I and J to p (in either order) and neither $I \cup J$; it is possible to add I, then J, hence also $I \cup J$; it is possible to add J then I, hence also $I \cup J$. In the first case, we have $F_I\, F_J|p\rangle = 0 = F_J\, F_I|p\rangle$, hence the join relation is proved. In the second case, we have

$$(p + \mathbb{1}_I)(b) = p(b)\,, \quad (p + \mathbb{1}_I)(c) = p(c)\,,$$

hence

$$v^{-1/2} f_{J,p+\mathbb{1}_I}\, f_{I,p} = v^{1/2}(-v)^{p(a)-p(c)} = f_{I\cup J,p}\,.$$

In the last case, we have

$$(p+\mathbb{1}_J)(a) = p(a)\,, \quad (p+\mathbb{1}_J)(b) = p(b)+1\,.$$

Hence

$$-v^{1/2} f_{I,p+\mathbb{1}_J} f_{J,p} = -v^{3/2}(-v)^{p(a)-p(c)-1} = f_{I\cup J,p}\,.$$

In both cases, the join relation $F_{I\cup J}|p\rangle = v^{1/2}F_I\,F_J|p\rangle - v^{-1/2}F_J\,F_I|p\rangle$ is verified. Similarly, we verify the join relation for E_J.

Drinfeld-Jimbo type relations. Finally, let us verify the Drinfeld-Jimbo type relations. The only relation we have to address carefully is the commutation relation between the E_J and the F_J. By the same argument as at the beginning of this proof, it is not possible to both add and remove the same interval J to a pyramid p (otherwise it would be possible to add twice the interval J to $p-\mathbb{1}_J$). Thus either $F_J\,E_J|p\rangle = 0$ or $E_J\,F_J|p\rangle = 0$. There are three (mutually exclusive) possibilities:

- it is neither possible to add nor remove J. The assertion follows.
- it is possible to add $J = [a,b[$. Then
$$E_J\,F_J|p\rangle = -v^{1/2}(-v)^{(p+\mathbb{1}_J)(b)-(p+\mathbb{1}_J)(a)}v^{1/2}(-v)^{p(a)-p(b)}|p\rangle = -v(-v)^{-1}|p\rangle = |p\rangle\,.$$
Hence, $[E_J,F_J]|p\rangle = |p\rangle$, as wanted.
- it is possible to remove J. Then
$$F_J\,E_J|p\rangle = v^{1/2}(-v)^{(p-\mathbb{1}_J)(a)-(p-\mathbb{1}_J)(b)}(-v^{1/2})(-v)^{p(b)-p(a)}|p\rangle = -v(-v)^{-1}|p\rangle\,.$$
Hence, $[E_J,F_J]|p\rangle = -|p\rangle$, as expected.

The rest of the Drinfeld-Jimbo relations are easier to be verified and we leave the check to the interested reader.

To finish, let us check that $\mathcal{F}_{\mathbb{K}}$ is generated by $|0\rangle$ and is irreducible. Let p be a $\mathbb{K}$-pyramid. We may write p as a sum $p = \sum_{i=1}^{s}\mathbb{1}_{I_i}$ where $I_1,\dots I_s$ are strictly nested intervals, i.e. $I_1 \supset I_2 \supset \cdots \supset I_s$ and the endpoints of the I_i are all distinct. It is easy to see that, up to a constant, $F_{I_s}\cdots F_{I_1}\cdot|0\rangle = |p\rangle$, proving the first assertion. The irreducibility may be proved by reversing this argument: up to a constant, we have $E_{I_1}\cdots E_{I_s}\cdot|q\rangle = \delta_{p,q}|0\rangle$ if q is any pyramid such that $|p| = |q|$. □

Remark 3.7. Note that for $J = [a,b[$ we have

$$\langle\mathbb{1}_J, p\rangle = \sum_{x>a}^{b}(p_-(x)-p_+(x)) = p(a)-p(b), \quad \langle p,\mathbb{1}_J\rangle = p_-(b)-p_-(a)\,.$$

Hence

$$e_{J,p} = -v^{1/2}(-v)^{-\langle\mathbb{1}_J,p\rangle}\,, \quad f_{J,p} = v^{1/2}(-v)^{\langle\mathbb{1}_J,p\rangle}\,.$$

△

For completeness, we also state the following result.

Lemma 3.8. *Let J be a $\mathbb{K}$-interval and p be a $\mathbb{K}$-pyramid. Then*

$$n_J(p) = \delta_{0\in J} - (\mathbb{1}_J, p)\,.$$

Proof. Let $J = [a,b[$. Note that $(\mathbb{1}_J,p) = (p_-(b)-p(b)) - (p_-(a)-p(a))$. There are three (mutually exclusive) possibilities to consider:

- it is neither possible to add nor remove J. We will check that $\delta_{0\in J} - (\mathbb{1}_J,p) = 0$. There are two subcases to consider, according to whether $0\in J$ or $0\notin J$.
  (a’) $0\notin J$. In that case, either the two endpoints a and b of J do not lie on points of discontinuity of p or they both lie on points of discontinuity of p. Thus $(\mathbb{1}_J,p) = 0$.

(a″) $0 \in J$. In that case, exactly one of the endpoints a and b of J lie on a point of discontinuity of p. Moreover, $a \leq 0$ and $b > 0$ so that $p_-(a) \leq p(a)$ while $p_-(b) \geq p(b)$. It follows that $(\mathbb{1}_J, p) = 1$.

- it is possible to add J. Then we will show that $\delta_{0\in J} - (\mathbb{1}_J, p) = 1$. There are again two subcases to consider, according to wether $0 \in J$ or $0 \notin J$.
  (b′) $0 \notin J$. In that case, either $a > 0$ and a is a point of discontinuity of p but not b, or $b < 0$ and b is a point of discontinuity of p but not a. Hence $(\mathbb{1}_J, p) = -1$.

  (b″) $0 \in J$. In that case, neither endpoints a and b of J may lie over a point of discontinuity of p, and thus $(\mathbb{1}_J, p) = 0$.
- it is possible to remove J. Then we will show that $\delta_{0\in J} - (\mathbb{1}_J, p) = -1$. There are two subcases to consider, according to whether $0 \in I$ or $0 \notin J$.
  (c′) $0 \notin J$. Here, either $a < 0$ and a lies over a point of discontinuity of p (but not b), or $a > 0$ and b lies over a point of discontinuity of p (but not a). We deduce $(\mathbb{1}_J, p) = 1$.

  (c″) $0 \in J$. In that case, $a \leq 0, b > 0$ and both a and b lie over points of discontinuity of p, so that $(\mathbb{1}_J, p) = 2$.

□

3.3. **Generalisation to arbitrary discrete subsets.** Let $\alpha\colon \mathbb{Z} \to \mathbb{R}$ be any strictly increasing map. We may define a quantum group $\mathbf{U}_v(\mathfrak{sl}(\alpha(\mathbb{Z})))$ as the subalgebra (but not coalgebra) of $\mathbf{U}_v(\mathfrak{sl}(\mathbb{R}))$ generated by all elements E_J, F_J, K_J where the endpoints of J belong to $\alpha(\mathbb{Z})$. There is an obvious isomorphism $\gamma_\alpha\colon \mathbf{U}_v(\mathfrak{sl}(\mathbb{Z})) \xrightarrow{\sim} \mathbf{U}_v(\mathfrak{sl}(\alpha(\mathbb{Z})))$ mapping E_J, F_J, K_J to $E_{\alpha(J)}, F_{\alpha(J)}, K_{\alpha(J)}$ respectively. Let us now assume that $0 \in \mathsf{Im}(\alpha)$ and let $\mathcal{F}_{\alpha(\mathbb{Z})}$ be the linear subspace of $\mathcal{F}_{\mathbb{R}}$ spanned by pyramids p satisfying $D_{\mathbb{R}}(p) \in \mathsf{Im}(\alpha)$. It is clear that $\mathcal{F}_{\alpha(\mathbb{Z})}$ is stable under the action of $\mathbf{U}_v(\mathfrak{sl}(\alpha(\mathbb{Z})))$ and that the following square is commutative:

$$\begin{array}{ccc} \mathbf{U}_v(\mathfrak{sl}(\alpha(\mathbb{Z}))) & \longrightarrow & \mathsf{End}(\mathcal{F}_{\alpha(\mathbb{Z})}) \\ {\scriptstyle\gamma_\alpha}\uparrow & & {\scriptstyle\iota_\alpha}\uparrow \\ \mathbf{U}_v(\mathfrak{sl}(\mathbb{Z})) & \longrightarrow & \mathcal{F}_{\mathbb{Z}} \end{array} ,$$

where $\iota_\alpha\colon \mathcal{F}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{F}_{\alpha(\mathbb{Z})}$ is the linear isomorphism induced by α.

3.4. **Relation with $\mathbf{U}_v(\mathfrak{sl}(\infty))$.** Recall that the quantized enveloping algebra $\mathbf{U}_v(\mathfrak{sl}(\infty))$ of $\mathfrak{sl}(\infty)$ is the unital associative $\widetilde{\mathbb{Q}}$-Hopf algebra generated by $E_i, F_i, K_i^{\pm 1}$, for $i \in \mathbb{Z}$, subject to the Drinfeld-Jimbo type relations

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i , \quad K_i K_j = K_j K_i ,$$

$$K_i E_j K_i^{-1} = v^{a_{i,j}} E_j , \quad K_i F_j K_i^{-1} = v^{-a_{i,j}} F_j , \quad [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}} ,$$

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{i,j}-k)} = 0 = \sum_{k=0}^{1-a_{i,j}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{i,j}-k)} \quad \text{if } i \neq j ,$$

where

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j , \\ -1 & \text{if } |i-j| = 1 , \\ 0 & \text{otherwise} , \end{cases}$$

and

$$E_i^{(k)} := \frac{E_i^k}{[k]!} \quad \text{and} \quad F_i^{(k)} := \frac{F_i^k}{[k]!} .$$

The coproduct is

$$\Delta(K_i) = K_i \otimes K_i\,, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i\,, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i\,.$$

Following [VV99, Section 4], there is an action of $\mathbf{U}_v(\mathfrak{sl}(\infty))$ on $\mathcal{F}_{\mathbb{Z}}$ given by

$$E_i|\lambda\rangle := \begin{cases} |\nu\rangle & \text{if } Y_\lambda \setminus Y_\nu \text{ is a box with color } i\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$F_i|\lambda\rangle := \begin{cases} |\mu\rangle & \text{if } Y_\mu \setminus Y_\lambda \text{ is a box with color } i\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$K_i|\lambda\rangle := v^{n_i(\lambda)}|\lambda\rangle$$

for $i \in \mathbb{Z}$.

The quantum group $\mathbf{U}_v(\mathfrak{sl}(\mathbb{Z}))$ admits a minimal set of generators $\{E_{[i,i+1[}, F_{[i,i+1[}, K^{\pm 1}_{[i,i+1[} \mid i \in \mathbb{Z}\}$. Thanks to the nest relation (3.2) and the Serre relations (3.3), the assignment

$$E_i \mapsto E_{[i,i+1[}\,, \quad F_i \mapsto F_{[i,i+1[}\,, \quad K_i^{\pm 1} \mapsto K^{\pm 1}_{[i,i+1[}$$

defines an isomorphism of bialgebras $\mathbf{U}_v(\mathfrak{sl}(\infty)) \to \mathbf{U}_v(\mathfrak{sl}(\mathbb{Z}))$.

The action of $\mathbf{U}_v(\mathfrak{sl}(\mathbb{Z}))$ on $\mathcal{F}_{\mathbb{Z}}$ given in Theorem 3.6 reduces to

$$E_{[i,i+1[}|\lambda\rangle := \begin{cases} -v^{1/2}\,|\nu\rangle & \text{if } Y_\lambda \setminus Y_\nu \text{ is a box with color } i \text{ and } i < 0\,, \\ v^{-1/2}\,|\nu\rangle & \text{if } Y_\lambda \setminus Y_\nu \text{ is a box with color } i \text{ and } i \geq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$F_{[i,i+1[}|\lambda\rangle := \begin{cases} -v^{-1/2}\,|\mu\rangle & \text{if } Y_\mu \setminus Y_\lambda \text{ is a box with color } i \text{ and } i < 0\,, \\ v^{1/2}\,|\mu\rangle & \text{if } Y_\mu \setminus Y_\lambda \text{ is a box with color } i \text{ and } i \geq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$K_{[i,i+1[}|\lambda\rangle := v^{n_i(\lambda)}|\lambda\rangle$$

for $i \in \mathbb{Z}$. Here, we used the bijection between $\mathbb{Z}$-pyramids and partitions (cf. Lemma 2.3) and the identity

$$n_i(\lambda) = n_{[i,i+1[}(p_\lambda)\,,$$

where p_λ is the $\mathbb{Z}$-pyramid associated with λ. Thus the $\mathbf{U}_v(\mathfrak{sl}(\mathbb{Z}))$-action on $\mathcal{F}_{\mathbb{Z}}$ defined in Theorem 3.6 is identified with a suitable rescaling of the standard $\mathbf{U}_v(\mathfrak{sl}(\infty))$-action on the Fock space.

4. Fock space representation of the circle quantum group

In this section, we recall the definition of the circle quantum group and define an action of it on the Fock space $\mathcal{F}_{\mathbb{K}}$. We wil use the "folding procedure" of Hayashi-Misra-Miwa (see [Hay90, Section 6.2] and [MM90, Section 2]) as reinterpreted by Varagnolo-Vasserot in [VV99, Section 6]. This uses the realization of these quantum groups as Hall algebras. Let $\mathbb{K}$ be either $\mathbb{Q}, \mathbb{R}$ or $\alpha(\mathbb{Z})$ where $\alpha\colon \mathbb{Z} \to \mathbb{R}$ is a strictly increasing map whose image contains 0 and is invariant under integer translations. Set $S^1_{\mathbb{K}} := \mathbb{K}/\mathbb{Z}$ and denote by $\pi_{\mathbb{K}}\colon \mathbb{K} \to S^1_{\mathbb{K}}$ the projection map.

4.1. **The bialgebra $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.** The notion of (closed on the left, open on the right) intervals generalizes in a straightforward way to $S^1_{\mathbb{K}}$. We say that an interval J of $S^1_{\mathbb{K}}$ is *strict* if $J \neq S^1_{\mathbb{K}}$.

We denote by $\mathbb{F}(S^1_{\mathbb{K}})$ the algebra of piecewise constant, right–continuous, $\mathbb{Z}$-valued functions $f\colon S^1_{\mathbb{R}} \to \mathbb{R}$, with finitely many points of discontinuity, whose points of discontinuity belong to $S^1_{\mathbb{K}}$. There is an obvious map $\pi_{\mathbb{K}}\colon \mathbb{F}(\mathbb{K}) \to \mathbb{F}(S^1_{\mathbb{K}})$. Formula (3.1) defines bilinear forms $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ on $\mathbb{F}(S^1_{\mathbb{K}})$. One defines $\mathbb{F}(S^1_{\mathbb{K}})^{\pm}$ as before. There is an obvious map $\pi_{\mathbb{K}}\colon \mathbb{F}(\mathbb{K})^{\pm} \to \mathbb{F}(S^1_{\mathbb{K}})^{\pm}$.

Remark 4.1. Note that $(\mathbb{1}_{S^1}, f) = 0$ for any $f \in \mathbb{F}(S^1_{\mathbb{K}})$. In fact it is easy to see that the kernel of $(\cdot,\cdot)$ is equal to $\mathbb{Z}\,\mathbb{1}_{S^1}$. △

Define, for a strict interval $J \subset S^1_{\mathbb{K}}$,

$$\mathsf{Int}(J) := \big\{ I \subset \mathbb{K} \mid I = [a,b[\text{ is a } \mathbb{K}\text{-interval}\,, \pi_{\mathbb{K}}(I) = J \big\}\,.$$

Thus $\mathsf{Int}(J)$ consists of all integer translates of some (any) $\mathbb{K}$-interval $\tilde{J}$ such that $\pi_{\mathbb{K}}(\tilde{J}) = J$.

Definition 4.2. Let J and J' be strict intervals of $S^1_{\mathbb{K}}$. We say that J is *left adjacent* to J' if $J \cap J' = \emptyset$, $\overline{J} \cap J' \neq \emptyset$ and $J \cup J'$ is an interval of $S^1_{\mathbb{K}}$. We denote it by $J \to J'$. ⊘

We are now ready to give the definition of the circle quantum group.

Definition 4.3 ([SS17, Definition 1.1]). Let $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ be the topological $\widetilde{\mathbb{Q}}$-bialgebra generated by elements $E_J, F_J, K_{J'}^{\pm 1}$, where J (resp. J') runs over all strict intervals (resp. intervals) of $S^1_{\mathbb{K}}$, modulo the following set of relations:

- *Drinfeld-Jimbo relations*:
  - for any intervals I, I_1, I_2 and strict interval J,
$$[K_{I_1}, K_{I_2}] = 0\,, \tag{4.1}$$
$$K_I\, E_J\, K_I^{-1} = v^{(\mathbb{1}_I, \mathbb{1}_J)}\, E_J\,, \tag{4.2}$$
$$K_I\, F_J\, K_I^{-1} = v^{-(\mathbb{1}_I, \mathbb{1}_J)}\, F_J\,; \tag{4.3}$$
  - if J_1, J_2 are strict intervals such that $J_1 \cap J_2 = \emptyset$,
$$[E_{J_1}, F_{J_2}] = 0\,; \tag{4.4}$$
  - for any strict interval J,
$$[E_J, F_J] = \frac{K_J - K_J^{-1}}{v - v^{-1}}\,; \tag{4.5}$$
- *join relations*:
  - if J_1, J_2 are strict intervals such that J_1 is left adjacent to J_2,
$$K_{J_1}\, K_{J_2} = K_{J_1 \cup J_2}\,;$$
  - if J_1, J_2 are strict intervals such that J_1 is left adjacent to J_2 and $J_1 \cup J_2$ is again a strict interval,
$$E_{J_1 \cup J_2} = v^{1/2}\, E_{J_1}\, E_{J_2} - v^{-1/2}\, E_{J_2}\, E_{J_1}\,,$$
$$F_{J_1 \cup J_2} = v^{-1/2}\, F_{J_2}\, F_{J_1} - v^{1/2}\, F_{J_1}\, F_{J_2}\,;$$
- *nest relations*:
  - if J_1, J_2 are strict intervals such that $\overline{J_1} \cap \overline{J_2} = \emptyset$,
$$[E_{J_1}, E_{J_2}] = 0 \quad \text{and} \quad [F_{J_1}, F_{J_2}] = 0\,;$$

– if J_1, J_2 are strict intervals such that $J_1 \subseteq J_2$,

$$v^{\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2} \rangle} E_{J_1} E_{J_2} = v^{\langle \mathbb{1}_{J_2}, \mathbb{1}_{J_1} \rangle} E_{J_2} E_{J_1} \,,$$
$$v^{\langle \mathbb{1}_{J_1}, \mathbb{1}_{J_2} \rangle} F_{J_1} F_{J_2} = v^{\langle \mathbb{1}_{J_2}, \mathbb{1}_{J_1} \rangle} F_{J_2} F_{J_1} \,.$$

The coproduct is given by

$$\Delta(K_J) = K_J \otimes K_J \,,$$
$$\Delta(E_{[a,b[}) = E_{[a,b[} \otimes 1 + \sum_{a<c<b} v^{-1/2} (v - v^{-1}) E_{[a,c[} K_{[c,b[} \otimes E_{[c,b[} + K_{[a,b[} \otimes E_{[a,b[} \,,$$
$$\Delta(F_{[a,b[}) = 1 \otimes F_{[a,b[} - \sum_{a<c<b} v^{-1/2} (v - v^{-1}) F_{[c,b[} \otimes F_{[a,c[} K^{-1}_{[c,b[} + F_{[a,b[} \otimes K^{-1}_{[a,b[} \,.$$

Here the sums on the right-hand-side run over all possible $\mathbb{K}$-values $c \in [a, b[$. ⊘

Definition 4.4. We define a $\mathbb{F}(S^1_{\mathbb{K}})$-gradation on $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ by setting

$$\deg(E_J) = \mathbb{1}_J \,, \quad \deg(F_J) = -\mathbb{1}_J \quad \text{and} \quad \deg(K_{J'}) = 0$$

for any strict interval J and any interval J'. ⊘

As in Section 3.1, we define the subalgebras $\mathbf{U}^{\epsilon}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$, with $\epsilon = 0, \pm 1$, as well as the negative subalgebra $\mathbf{U}^{\leq 0}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ and the positive subalgebra $\mathbf{U}^{\geq 0}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.

4.2. **The Fock space $\mathcal{F}_{\mathbb{K}}$ of $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.** We will now define the main object of the present paper, namely the Fock space representation of the circle quantum group. As a vector space, this Fock space is again

$$\mathcal{F}_{\mathbb{K}} = \bigoplus_{p \in \mathsf{Pyr}_{\mathbb{K}}} \widetilde{\mathbb{Q}} |p\rangle \,.$$

For I a $\mathbb{K}$-interval and p a $\mathbb{K}$-pyramid, we set

$$n^{>}_I(p) = \sum_{m \geq 1} n_{\tau_m(I)}(p), \qquad n^{<}_I(p) = \sum_{m \geq 1} n_{\tau_{-m}(I)}(p), \qquad \overline{n}_I(p) = \sum_{m \in \mathbb{Z}} n_{\tau_m(I)}(p) \,,$$

where $n_J(p)$ is defined as in Theorem 3.6 and where $\tau_m(I)$ is the translation of I by $m \geq 1$ to the right, i.e. if $I = [a, b)$ then $\tau_m(I) = [a + m, b + m)$. If I, J are $\mathbb{K}$-intervals then we write $I < J$ if $I = [a, b), J = [c, d)$ and $b < c$.

Theorem 4.5. *The following formulas define an action of the quantum group $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ on $\mathcal{F}_{\mathbb{K}}$:*

$$E_J|p\rangle = \sum_{J'_1,\ldots,J'_\ell} v^{\frac{1-\ell}{2} - \sum_i n^{<}_{J'_i}(p)} (-v)^{-\sum_i \langle \mathbb{1}_{J'_i}, p \rangle} (v - v^{-1})^{\ell-1} |p - \sum_i \mathbb{1}_{J'_i}\rangle \,,$$

$$F_J|p\rangle = \sum_{J''_1,\ldots,J''_\ell} v^{\frac{\ell-1}{2} + \sum_i n^{>}_{J''_i}(p)} (-v)^{\sum_i \langle \mathbb{1}_{J''_i}, p \rangle} (v^{-1} - v)^{\ell-1} |p + \sum_i \mathbb{1}_{J''_i}\rangle \,, \tag{4.6}$$

$$K_J|p\rangle = v^{\overline{n}_J(p)} |p\rangle \,, \tag{4.7}$$

where the sums range over all tuples of removable $\mathbb{K}$-intervals $(J'_1, \ldots, J'_l)$ (resp. all tuples of addable $\mathbb{K}$-intervals $(J''_1, \ldots, J''_l)$) satisfying the conditions

a′) $J'_1 > J'_2 > \cdots > J'_\ell$,

b′) $\pi_{\mathbb{K}}(J'_1) \to \pi_{\mathbb{K}}(J'_2) \to \cdots \to \pi_{\mathbb{K}}(J'_\ell)$,

c′) $\pi_{\mathbb{K}}(J'_1) \sqcup \cdots \sqcup \pi_{\mathbb{K}}(J'_\ell) = J$.

(resp.

a″) $J''_1 < J''_2 < \cdots < J''_\ell$, *b″)* $\pi_{\mathbb{K}}(J''_1) \to \pi_{\mathbb{K}}(J''_2) \to \cdots \to \pi_{\mathbb{K}}(J''_\ell)$,

c″) $\pi_{\mathbb{K}}(J''_1) \sqcup \cdots \sqcup \pi_{\mathbb{K}}(J''_\ell) = J$.)

The proof of this Theorem occupies Sections 4.3 and 6. For now, we show that the above formulas are well-defined and we examine certain simple properties of this Fock space.

4.3. **Well-definedness.** Let us first check that E_J and F_J are well-defined, i.e. that the sums involved in their definition are in fact finite. We treat the case of the operators F_J, the case of E_J being similar.

Let us fix a $\mathbb{K}$-pyramid p. By definition, a collection of $\mathbb{K}$-intervals $J''_1, J''_2, \ldots, J''_\ell$ satisfying the conditions (a″), (b″), (c″) in Theorem 4.5 induce a subdivision $J_1 \to J_2 \to \cdots \to J_\ell$ of J, with $J_i = \pi_{\mathbb{K}}(J''_i)$. For any given interval $I \subset S^1_{\mathbb{K}}$ and any pyramid q, there are at most finitely many $I'' \in \mathsf{Int}(I)$ which are addable to q (because q is of compact support). Hence it is enough to prove that only finitely many subdivisions $J_1 \to J_2 \to \cdots \to J_\ell$ of J may give rise to a collection of addable intervals $(J''_1, \ldots, J''_\ell)$.

Write $J = [a, b)$ and $J_1 = [c_0 = a, c_1), J_2 = [c_1, c_2), \ldots, J_\ell = [c_{\ell-1}, c_\ell = b)$. We claim that if there exists addable $\mathbb{K}$-intervals $J''_1, \ldots, J''_\ell$ such that $J''_i \in \mathsf{Int}(J_i)$ for all i and $J''_1 < J''_2 < \cdots < J''_\ell$ then necessarily $c_1, c_2, \ldots, c_{\ell-1} \in \pi_{\mathbb{K}}(D_{\mathbb{K}}(p))$. Indeed, writing $J''_i = [a''_i, b''_i)$ for $i = 1, \ldots, \ell$ (so that $\pi_{\mathbb{K}}(a''_i) = c_{i-1}, \pi_{\mathbb{K}}(b''_i) = c_i$) we have by Remark 2.11

$$b''_i \in D_{\mathbb{K}}(p) \quad \text{if } b''_i < 0 \text{ and } i \neq \ell\ ,$$

$$a''_i \in D_{\mathbb{K}}(p) \cup \{0\} \quad \text{if } a''_i \geq 0 \text{ and } i \neq 1\ .$$

Since $D_{\mathbb{K}}(p)$ is finite, this implies the desired finiteness of possible tuples $(J_1, \ldots, J_\ell)$. The argument for E_J is similar, using the *reverse* condition $J'_1 > J'_2 > \cdots > J'_\ell$.

4.4. **Non-cyclicity.** In the finite setup (i.e., for Hayashi's Fock space of $\mathbf{U}_v(\widehat{\mathfrak{sl}}(n))$) the Fock space is not cyclic. The same holds here:

Proposition 4.6. *The subspace* $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}})) \cdot |0\rangle$ *of* $\mathcal{F}_{\mathbb{K}}$ *is strict.*

Proof. We claim that the element $|\mathbb{1}_{[0,1[}\rangle$ does not belong to $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}})) \cdot |0\rangle$. Let us argue by contradiction. Let $u := P(F_{J_1}, \ldots, F_{J_s})$ be a linear combination of monomials in generators $F_{J_1}, \ldots, F_{J_s}$ such that $u \cdot |0\rangle = \mathbb{1}_{[0,1[}$. Choose a finite subset $\overline{\alpha} \subset S^1_{\mathbb{K}}$ containing all the endpoints of the intervals J_i and let $\alpha \colon \mathbb{Z} \to \mathbb{K}$ be such that $\pi_{\mathbb{K}}(\alpha(\mathbb{Z})) = \overline{\alpha}$. There is a canonical embedding $\mathbf{U}_v(\mathfrak{sl}(S^1_{\alpha(\mathbb{Z})})) \to \mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ whose image contains u. Moreover, $\mathbf{U}_v(\mathfrak{sl}(S^1_{\alpha(\mathbb{Z})}))$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{sl}}(N))$, where $N = |\overline{\alpha}|$, and the restriction of $\mathcal{F}_{\mathbb{K}}$ to $\mathbf{U}_v(\mathfrak{sl}(S^1_{\alpha(\mathbb{Z})}))$ contains the Fock space $\mathcal{F}_{\alpha(\mathbb{Z})}$ as the subspace spanned by all pyramids p satisfying $D_{\mathbb{K}}(p) \in \alpha(\mathbb{Z})$. Hence it is enough to check that $|\mathbb{1}_{[0,1[}\rangle \notin \mathbf{U}_v(\mathfrak{sl}(S^1_{\alpha(\mathbb{Z})})) \cdot |0\rangle$ or equivalently that the element $|\lambda\rangle$ with $\lambda = (N)$ does not belong to the subspace $\mathbf{U}_v(\widehat{\mathfrak{sl}}(N)) \cdot |0\rangle$ in Hayashi's Fock space. This last statement may be checked by some simple direct calculations which we leave to the reader. □

4.5. **Highest weight vectors.** The vacuum vector $|0\rangle$ is an obvious highest weight vector. Somewhat surprisingly, it turns out to be the *only* highest weight vector in $\mathcal{F}_{\mathbb{K}}$ when $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$.

Proposition 4.7. *Assume that* $\mathbb{K} = \mathbb{Q}$ *or* $\mathbb{K} = \mathbb{R}$. *We have*

$$(\mathcal{F}_{\mathbb{K}})^{\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))} = \widetilde{\mathbb{Q}}|0\rangle\ .$$

Proof. Let $v := \sum_p \alpha_p |p\rangle$ be a highest weight vector. By homogeneity we may assume that all p for which $\alpha_p \neq 0$ are of the same size s. Assume that $s \neq 0$ (i.e., that $v \notin \widetilde{\mathbb{Q}}|0\rangle$). Because α_p is nonzero for only finitely many pyramids p, the set

$$D := \bigcup_{p,\, \alpha_p \neq 0} D_{\mathbb{K}}(p)$$

is nonempty and finite. Set $x = \max(D)$. Choose some small $\epsilon > 0$ such that $[x-\epsilon, x] \cap D = \{x\}$. Let p be a pyramid for which $\alpha_p \neq 0$ and $x \in D_{\mathbb{K}}(p)$ (hence the support of p is an interval of the form $[a, x]$). Set $J = [x - \epsilon, x)$. By construction, J is removable from p. Moreover p is the only pyramid which can contribute to $p - \mathbb{1}_J$ in $E_J \cdot v$; this follows from condition (a') in the definition of the action of E_J (cf. Theorem 4.5) and from the fact that $x - \epsilon \notin D_{\mathbb{K}}(q)$ for any pyramid q occuring in v. We deduce that $\langle p - \mathbb{1}_J | E_J \cdot v\rangle = \langle p - \mathbb{1}_J | E_J \cdot p\rangle \neq 0$, which is in contradiction with the assumption on v. □

Remark 4.8. Hayashi's Fock space decomposes into a direct sum of highest weight representations. In [S00], these highest weight vectors are obtained by the action of the center of the Hall algebra on the vacuum vector. In the setting of the infinite quivers $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ such a center only appears in a suitable completion (see [SS17]). This suggests that a better object to study would be a similar completion of our Fock space. We hope to return to this in the future. △

5. Hall algebra action on the Fock space

It is possible but very tedious to check directly that the operators in the statement of Theorem 4.5 satisfy the join and nest relations. We will use instead a more interesting approach, based on the identification of $\mathbf{U}_v^+(\mathfrak{sl}(S^1_{\mathbb{K}}))$ with the (spherical) Hall algebra of a certain category of representations of an infinite quiver.

5.1. **Representations of infinite quivers.** Let us begin with the line $\mathbb{K}$, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$ as before. We start by recalling the notion of persistence modules (see [BBCB18] [DEHH18, Section 2.3] for a recent report.

Definition 5.1. Let k be a field. A $\mathbb{K}$-*persistence module*[3] is a functor $F\colon (\mathbb{K}, \leq) \to \mathbf{Vect}_k$ from the poset $(\mathbb{K}, \leq)$ to the category $\mathbf{Vect}_k$ consisting of finite-dimensional vector spaces over k with linear maps between them. Explicitly, F is determined by:

i) a finite-dimensional k-vector space $F(t)$ for every $t \in \mathbb{K}$,
ii) a k-linear map $F(s \leq t)\colon V_s \to V_t$ for every pair of real numbers $s \leq t$ such that
   - $F(t \leq t)$ is the identity map from V_t to itself,
   - given real numbers $s \leq t \leq u$, one has $F(s \leq u) = F(t \leq u) \circ F(s \leq t)$.

⊘

Morphisms of $\mathbb{K}$-persistence modules are defined in the obvious way.

Remark 5.2. The poset $(\mathbb{K}, \leq)$ can be interpreted as a category, whose objects are points of $\mathbb{K}$ and the set of maps between two objects $s, t \in \mathbb{K}$ consists of exactly one map $\iota_{t,s}$ if $s \leq t$, otherwise is empty. Thanks to this interpretation, one can extend the definition above replacing $(\mathbb{K}, \leq)$ with any (small) category, as done e.g. in [BBCB18, Section 2]. △

Definition 5.3. A point $t \in \mathbb{K}$ is *regular* for a persistence module F if there exists an interval $I \subseteq \mathbb{K}$ where $t \in I$ and $F(a \leq b)$ is an isomorphism for all pairs $a, b \in I$. Otherwise we say that t is *critical*. A persistence module F is *tame* if it has finitely many critical values. ⊘

The set of all regular points will be called the *regular locus* and its complement the *critical locus*.

Example 5.4. Any $\mathbb{K}$-interval $J = [a, b[$ gives rise to a persistence module k_J, for which $k_J(t) = k$ if $t \in J$, otherwise $k_J(t) = \{0\}$, and for any pair $s, t \in \mathbb{K}$ one has that $k_J(s \leq t)$ is the identity map if $s, t \in J$, otherwise $k_J(s \leq t)$ is the zero map. △

From now on, we shall restrict to a smaller class of persistence modules.

[3]In the literature of persistent homology [Carl09, Oud15], this is called *pointwise finite-dimensional* persistence module.

Definition 5.5. We say that a persistence module F is *coherent* if it is tame and the map $t \mapsto \dim F(t)$ is right–continuous and compactly supported. ⊘

Remark 5.6. The above definition is somehow inspired by the notion of *finitely presented parabolic sheaves*; see e.g. [Tal18, Section 3.3]. △

Thanks to the fundamental theorem of persistent homology (see e.g. [BBCB18, Section 2.3]), we get that:

Lemma 5.7. *For any coherent* $\mathbb{K}$*-persistence module* F *there exists a partition*

$$\{x \in \mathbb{K} \mid F(x) \neq \{0\}\} = \bigsqcup_i J_i$$

into finitely many $\mathbb{K}$*-intervals* $J_i = [a_i, b_i[$ *such that* $F(v \leq u)$ *is an isomorphism for any pair* $v \leq u$ *of elements belonging to the same interval* J_i.

From now on, we shall denote the data of a coherent $\mathbb{K}$-persistence module in the following way: $V_t := F(t)$ and $x_{t,s} := F(s \leq t)$. Thus a coherent $\mathbb{K}$-persistence module is encoded by a collection of data $(V, x) := (V_t, x_{t,s})_{t,s}$. We shall also call a coherent persistence module a *representation of the infinite type A quiver* $\mathbb{K}$. Summarizing, a representation of $\mathbb{K}$ is a collection of data $(V, x) := (V_t, x_{t,s})_{t,s}$ with

i) V_t is a finite-dimensional k-vector space for every $t \in \mathbb{K}$,

ii) $x_{t,s} \colon V_s \to V_t$ is a k-linear map for every pair $s, t \in \mathbb{K}$ with $s \leq t$,

such that

a) the map $t \mapsto \dim(V_t)$ is right–continuous, compactly supported and with finitely many discontinuities,

b) we have $x_{t,r} = x_{t,s} \circ x_{s,r}$ for any triple $r \leq s \leq t$,

c) there exists a partition $\{t \in \mathbb{K} \mid V_t \neq \{0\}\} = \bigsqcup_i J_i$ into finitely many intervals $J_i = [a_i, b_i[$ such that $x_{u,v}$ is an isomorphism for any pair $v \leq u$ of elements belonging to the same interval J_i.

We denote by $\mathsf{Rep}_k\mathbb{K}$ the category of representations of $\mathbb{K}$. Then $\mathsf{Rep}_k\mathbb{K}$ is abelian. It may be realized as a certain colimit of categories of representations of (locally) finite type A quivers. More precisely, for any locally finite subset $S \subset \mathbb{K}$, let us denote by $\mathsf{Rep}_k^{(S)}\mathbb{K}$ the full subcategory of $\mathsf{Rep}_k\mathbb{K}$ consisting of representations $(V_t, x_{t,s})_{t,s}$ for which $x_{t,s}$ is an isomorphism for $s, t \in \mathbb{K} \setminus S$. Then $\mathsf{Rep}_k^{(S)}\mathbb{K}$ is equivalent to the category of representations of the quiver $\mathcal{Q}_S$, whose vertices are the maximal intervals in $\mathbb{K} \setminus S$ and whose arrows are given by the adjacency relation. It is clear that $\mathcal{Q}_S$ is either a finite type A quiver (if $|S| < \infty$) or a quiver of type A_∞ (if S is infinite).

The categories $\mathsf{Rep}_k^{(S)}\mathbb{K}$ form a direct system with respect to the inclusion $S \subset S'$ and we have

$$\mathsf{Rep}_k\mathbb{K} \simeq \varinjlim \mathsf{Rep}_k^{(S)}\mathbb{K} \ .$$

It follows that $\mathsf{Rep}_k\mathbb{K}$ is *hereditary*, i.e., that $\mathsf{Ext}^i(M, N) = \{0\}$ for any pair of representations (M, N) and any $i > 1$. By [BBL18, Proposition 2.2], the persistence module k_J introduced in Example 5.4 is a (unique up to isomorphism) indecomposable object associated to the interval J.

Let us next consider the case of the circle $S^1_{\mathbb{K}}$. Let Γ be the oriented fundamental groupoid of $S^1_{\mathbb{K}}$: its objects are homotopy classes of orientation preserving paths $[0,1] \cap \mathbb{K} \to S^1_{\mathbb{K}}$; these may be parametrized by triples (s, t, n) where s, t are (not necessarily distinct) elements $S^1_{\mathbb{K}}$ and $n \in \mathbb{N}$ is the winding number of the path. For $\gamma = (s, t, n)$ we will sometimes write $\gamma' = s, \gamma'' = t$.

By following Remark 5.2, we replace $(\mathbb{K}, \leq)$ in Definition 5.1 by the category whose objects are points $t \in S^1_{\mathbb{K}}$ and morphisms are given by $\gamma \in \Gamma$: thus, the notion of coherent persistence modules makes sense. We denote by $\mathsf{Rep}_k S^1_{\mathbb{K}}$ the category of these coherent persistence modules, which we shall call *representations of the infinite affine type A quiver* $S^1_{\mathbb{K}}$. Note that Lemma 5.7 holds also in this case, thanks to [BBL18, Theorem 1.1]. Summarizing, a representation of $S^1_{\mathbb{K}}$ is a collection of data $(V, x) := (V_t, x_\gamma)_{t,\gamma}$ with $t \in S^1_{\mathbb{K}}$ and $\gamma \in \Gamma$ with

i) V_t is a finite-dimensional k-vector space for every $t \in S^1_{\mathbb{K}}$,
ii) $x_\gamma \colon V_{\gamma'} \to V_{\gamma''}$ is a k-linear map for every $\gamma \in \Gamma$,

such that

a′) the map $t \mapsto \dim(V_t)$ is right–continuous and with finitely many discontinuities,
b′) we have $x_{\gamma_1} \circ x_{\gamma_2} = x_{\gamma_1 \cdot \gamma_2}$ for any composable pair γ_1, γ_2 of elements of Γ,
c′) there exists a partition $S^1_{\mathbb{K}} = \bigsqcup_i J_i$ into finitely many intervals $J_i = [a_i, b_i[$ such that x_γ is an isomorphism for any path γ contained in a single interval J_i.

Again, $\mathsf{Rep}_k S^1_{\mathbb{K}}$ is a hereditary abelian category. Like $\mathsf{Rep}_k \mathbb{K}$ it may be realized as a colimit $\mathsf{Rep}_k S^1_{\mathbb{K}} = \varinjlim \mathsf{Rep}_k^{(S)} S^1_{\mathbb{K}}$, where S ranges over all finite subsets of $S^1_{\mathbb{K}}$. Note that $\mathsf{Rep}_k^{(S)} S^1_{\mathbb{K}}$ is now equivalent to a category $\mathsf{Rep}_k \mathcal{Q}_S$ of representations of an affine (cyclically oriented) quiver of type A. For any strict interval $J = [a, b[$ of $S^1_{\mathbb{K}}$ there is a indecomposable object k_J of $\mathsf{Rep}_k S^1_{\mathbb{K}}$, and one obtains in this way all rigid indecomposables.

Taking the dimension function $f(V) \colon t \mapsto \dim(V_t)$ defines a map $\underline{\dim}$ from the Grothendieck groups $\mathsf{K}_0(\mathsf{Rep}_k \mathbb{K})$ and $\mathsf{K}_0(\mathsf{Rep}_k S^1_{\mathbb{K}})$ to $\mathbb{F}(\mathbb{K})$ and $\mathbb{F}(S^1_{\mathbb{K}})$ respectively.

5.2. **Hall algebras of infinite quivers.** When $k = \mathbb{F}_q$ is a finite field, $\mathsf{Rep}_k \mathbb{K}$ and $\mathsf{Rep}_k S^1_{\mathbb{K}}$ are finitary categories, i.e., the sets $\mathsf{Ext}^i(V, V')$ are finite for any V, V' and $i = 0, 1$. In this context we may consider the Hall algebras $\mathbf{H}_{\mathbb{K}}$ and $\mathbf{H}_{S^1_{\mathbb{K}}}$ of $\mathsf{Rep}_k \mathbb{K}$ and $\mathsf{Rep}_k S^1_{\mathbb{K}}$ respectively. We recall this construction very briefly and refer to [S12] for details.

Set $v := q^{1/2}$. Let $\mathcal{M}_{\mathbb{K}}, \mathcal{M}_{S^1}$ denote the set of all isomorphism classes of objects of $\mathsf{Rep}_k \mathbb{K}$ and $\mathsf{Rep}_k S^1_K$ respectively. As vector spaces we have

$$\mathbf{H}_{\mathbb{K}} := \{h \colon \mathcal{M}_{\mathbb{K}} \to \mathbb{C} \mid |\mathsf{supp}(f)| < \infty\} \quad \text{and} \quad \mathbf{H}_{S^1_{\mathbb{K}}} := \{h \colon \mathcal{M}_{S^1} \to \mathbb{C} \mid |\mathsf{supp}(f)| < \infty\} .$$

We will denote by $[V]$ the characteristic function of an object V. The function $\underline{\dim}$ induces a natural gradation

$$\mathbf{H}_{\mathbb{K}} = \bigoplus_{f \in \mathbb{F}(\mathbb{K})} \mathbf{H}_{\mathbb{K}}[f] \quad \text{and} \quad \mathbf{H}_{S^1_{\mathbb{K}}} = \bigoplus_{f \in \mathbb{F}(S^1_K)} \mathbf{H}_{S^1_{\mathbb{K}}}[f] . \tag{5.1}$$

The multiplication in $\mathbf{H}_{\mathbb{K}}$ is defined as follows. For $V \in \mathbf{H}_{\mathbb{K}}[f]$ and $V' \in \mathbf{H}_{\mathbb{K}}[f']$ we set

$$[V] \star [V'] := v^{\langle f, f' \rangle} \sum_W P^W_{V,V'}[W]$$

where W ranges over the (finite) set of extensions of V' by V and where

$$P^W_{V,V'} := |\{U \subset W \mid U \simeq V', W/U \simeq V\}| .$$

The multiplication in $\mathbf{H}_{S^1_{\mathbb{K}}}$ is defined in the same fashion. Note that the multiplication is graded with respect to the decomposition (5.1).

We let $\mathbf{H}^{\mathsf{sph}}_{S^1_{\mathbb{K}}}$ be the subalgebra of $\mathbf{H}_{S^1_{\mathbb{K}}}$ generated by the elements $[k_J]$ where J runs through the set of *strict* intervals in S^1.

Theorem 5.8.

(i) The assignment $[k_J] \mapsto v^{-1/2} E_J$ *for all intervals* J *defines an isomorphism of graded algebras* $\mathbf{H}_{\mathbb{K}} \simeq \mathbf{U}^+_v(\mathfrak{sl}(\mathbb{K}))$.

(ii) *The assignment* $[k_J] \mapsto v^{-1/2}E_J$ *for all strict intervals* J *defines an isomorphism of graded algebras* $\mathbf{H}^{\mathsf{sph}}_{S^1_{\mathbb{K}}} \simeq \mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.

Proof. Notice that any order-preserving bijection $\mathbb{K} \to]0,1[\cap\mathbb{K}$ induces both a fully faithful embedding $\mathsf{Rep}_k\mathbb{K} \hookrightarrow \mathsf{Rep}_k S^1_{\mathbb{K}}$ and an inclusion of algebras $\mathbf{U}^+_v(\mathfrak{sl}(\mathbb{K})) \hookrightarrow \mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$. Hence, by the compatibility between the Hall algebra construction and fully faithful embeddings (see e.g. [S12, Section 1.8]) it is enough to treat the case of $\mathsf{Rep}_k S^1_{\mathbb{K}}$ and $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$. By the same functorial properties of Hall algebras, $\mathbf{H}^{\mathsf{sph}}_{S^1_{\mathbb{K}}}$ is isomorphic to an inductive limit of Hall algebras $\mathbf{H}^{\mathsf{sph}}(\mathsf{Rep}^{(S)}_k(S^1_{\mathbb{K}}))$ as S ranges over all finite subsets of $S^1_{\mathbb{K}}$. On the other hand, denote by $\mathbf{U}^+_v(\mathfrak{sl}^{(S)}(S^1_{\mathbb{K}}))$ the subalgebra of $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ generated by the elements E_J for $J = [a,b[$ satisfying $a,b \in S$. Then $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ is the inductive limit of $\mathbf{U}^+_v(\mathfrak{sl}^{(S)}(S^1_{\mathbb{K}}))$ as S ranges over all finite subsets of $S^1_{\mathbb{K}}$. Hence it suffices to prove that the assignment $E_J \mapsto v^{1/2}[k_J]$ extends to an algebra isomorphism $\mathbf{U}^+_v(\mathfrak{sl}^{(S)}(S^1_{\mathbb{K}})) \xrightarrow{\sim} \mathbf{H}^{\mathsf{sph}}(\mathsf{Rep}^{(S)}_k(S^1_{\mathbb{K}}))$ for any S. Since S is finite, this reduces to the well-known identification between the spherical Hall algebra of a cyclic quiver with n vertices and $\mathbf{U}^+_v(\widehat{\mathfrak{sl}}(n))$ (see e.g. [SS17, Section 4.2]). □

We will need a slight variant of the above result, involving $\mathbf{U}^-_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ instead of $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.

Corollary 5.9.

(i) *The assignment* $[k_J] \mapsto -v^{-1/2}F_J$ *for all intervals* J *defines an isomorphism of graded algebras* $\mathbf{H}_{\mathbb{K}} \simeq \mathbf{U}^-_v(\mathfrak{sl}(\mathbb{K}))$.

(ii) *The assignment* $[k_J] \mapsto -v^{-1/2}F_J$ *for all strict intervals* J *defines an isomorphism of graded algebras* $\mathbf{H}^{\mathsf{sph}}_{S^1_{\mathbb{K}}} \simeq \mathbf{U}^-_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$.

Proof. This comes from the isomorphism $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}})) \simeq \mathbf{U}^-_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ given by $E_J \mapsto -F_J$. □

5.3. **Folding procedure.** To unburden the notation, let us simply denote by $f \mapsto \overline{f}$ the projection $\pi_{\mathbb{K}}\colon \mathbb{F}(\mathbb{K}) \to \mathbb{F}(S^1_{\mathbb{K}})$ and likewise $s \mapsto \overline{s}$ for the projection $\mathbb{K} \to S^1_{\mathbb{K}}$. We will denote by $s, t, \ldots$ elements of $\mathbb{K}$ and by $a, b, \ldots$ elements of $S^1_{\mathbb{K}}$.

Following [VV99, Section 6] we will now construct a family of maps

$$\gamma_f\colon \mathbf{H}_{S^1_{\mathbb{K}}}[\overline{f}] \to \mathbf{H}_{\mathbb{K}}[f]$$

for every $f \in \mathbb{F}(\mathbb{K})$. Fix a function $f \in \mathbb{F}(\mathbb{K})$ and a collection of vector spaces V_t, for $t \in \mathbb{K}$, such that $\dim(V_t) = f(t)$ for all t. Put:

$$V = \bigoplus_t V_t\ , \quad \overline{V} = \bigoplus_a \overline{V}_a\ , \quad \overline{V}_a = \bigoplus_{\overline{s}=a} V_s\ .$$

Thus, even if they may be canonically identified, V is a $\mathbb{K}$-graded vector space while $\overline{V}$ is $S^1_{\mathbb{K}}$-graded. Note that since $\mathsf{supp}(f)$ is compact, $\overline{V}_a$ is finite-dimensional for any a. For any $s \in \mathbb{K}$ we define

$$V_{\geq s} := \bigoplus_{t \geq s} V_t\ ,$$

and we denote by $V^\bullet_{\geq}$ the associated filtration of V. There is an induced filtration $\overline{V}^\bullet_{\geq}$ of $\overline{V}$ where

$$\overline{V}_{\geq s} = \bigoplus_a \overline{V}_{\geq s,\, a} \quad \text{and} \quad \overline{V}_{\geq s,\, a} = \bigoplus_{t \geq s,\, \overline{t}=a} V_t\ .$$

The filtrations $V^\bullet_>, \overline{V}^\bullet_>$ are defined in the same way, replacing $\geq$ by $>$. The associated graded space

$$\mathsf{gr}(\overline{V}) := \bigoplus_s (\overline{V}_{\geq s}/\overline{V}_{>s})$$

is canonically isomorphic to V as a $\mathbb{K}$-graded vector space.

Let us denote by $\mathcal{E}_V$ the groupoid of all representations of $\mathbb{K}$ in V: objects are collections of maps $x_{t,s}\colon V_s \to V_t$ for $s < t$ satisfying conditions (b) and (c) in Section 5.1 and morphisms are given by elements of $\prod_t \mathsf{GL}(V_t)$. We likewise denote by $\mathcal{E}_{\overline{V}}$ the groupoid of all representations of $S^1_{\mathbb{K}}$ in $\overline{V}$: objects are collections of maps $x_\gamma\colon \overline{V}_{\gamma'} \to \overline{V}_{\gamma''}$ for $\gamma \in \Gamma$ satisfying (b′) and (c′) and morphisms are elements of $\prod_a \mathsf{GL}(\overline{V}_a)$. Finally, let $\mathcal{E}_{V,\overline{V}}$ denote the groupoid of representations in $\mathcal{E}_{\overline{V}}$ preserving the filtration $\overline{V}^\bullet_\geq$: objects are collections x_γ as above such that for any $\gamma \in \Gamma$ and any $s \in \mathbb{K}$ satisfying $\overline{s} = \gamma'$ we have $x_\gamma(V_{\geq s}) \subseteq V_{\geq \gamma(s)}$ – here $\gamma(s)$ is the deck transformation of s associated to γ; morphisms are given by elements of $\prod_a P_a$ where $P_a \subset \mathsf{GL}(\overline{V}_a)$ is the parabolic subgroup of elements preserving the induced filtration $\overline{V}^\bullet_\geq \cap \overline{V}_a$.

There is an obvious functor $j\colon \mathcal{E}_{V,\overline{V}} \to \mathcal{E}_{\overline{V}}$. Passing to the associated graded we also get a functor $p\colon \mathcal{E}_{V,\overline{V}} \to \mathcal{E}_V$ defined as follows. For every $s, t \in \mathbb{K}$ with $s < t$, let γ correspond to the oriented path $s \mapsto t$ in $\mathbb{K}$; the map $x_\gamma\colon \overline{V}_{\overline{s}} \to \overline{V}_{\overline{t}}$ sends $\overline{V}_{\geq s}$ to $\overline{V}_{\geq t}$ and $\overline{V}_{>s}$ to $\overline{V}_{>t}$; we let $x_{t,s}$ be the induced map $V_s \to \overline{V}_{\geq s}/\overline{V}_{>s} \to \overline{V}_{\geq t}/\overline{V}_{>t} = V_t$.

Lemma 5.10. *The functor $p\colon \mathcal{E}_{V,\overline{V}} \to \mathcal{E}_V$ has essentially finite fibers.*

Proof. Let $\rho = (x_{t,s})_{s,t} \in \mathcal{E}_V$. Denote by $D_\rho \subset \mathbb{K}$ the set of critical points of ρ and let $\overline{D_\rho} \subset S^1_{\mathbb{K}}$ be its reduction mod $\mathbb{Z}$. Any object $\tilde{\rho} = (x_\gamma)_\gamma$ in the fiber $p^{-1}(\rho)$ has a critical set $D_{\tilde{\rho}}$ included in $\overline{D_\rho}$. Let S be a finite subset of $S^1_{\mathbb{K}}$ containing $\overline{D_\rho}$. Let $\mathcal{E}^{(S)}_V, \mathcal{E}^{(S)}_{\overline{V}}, \mathcal{E}^{(S)}_{V,\overline{V}}$ be the (full) subgroupoids of $\mathcal{E}_V, \mathcal{E}_{\overline{V}}, \mathcal{E}_{V,\overline{V}}$ whose objects are those representations whose critical sets are contained in $\pi^{-1}_{\mathbb{K}}(S)$ and S respectively. The functor $p\colon \mathcal{E}_{V,\overline{V}} \to \mathcal{E}_V$ restricts to a functor $p^{(S)}\colon \mathcal{E}^{(S)}_{V,\overline{V}} \to \mathcal{E}^{(S)}_V$ and there is a cartesian square

$$\begin{array}{ccc} \mathcal{E}^{(S)}_{V,\overline{V}} & \longrightarrow & \mathcal{E}_{V,\overline{V}} \\ \big\downarrow p^{(S)} & & \big\downarrow p \\ \mathcal{E}^{(S)}_V & \longrightarrow & \mathcal{E}_V \end{array} .$$

Since $\rho \in E^{(S)}_V$, it suffices to show that $p^{(S)}$ has essentially finite fibers. This is obvious since $\mathcal{E}^{(S)}_{V,\overline{V}}$ has finitely many objects up to isomorphism. □

For a groupoid X, we denote by $\mathbb{C}(X)$ the space of complex functions on $\mathsf{Obj}(X)$ which are invariant under isomorphism and have essentially finite support. We will identify $\mathbb{C}(\mathcal{E}_V)$ and $\mathbb{C}(\mathcal{E}_{\overline{V}})$ with $\mathbf{H}_{\mathbb{K}}[f]$ and $\mathbf{H}_{S^1_K}[\overline{f}]$ respectively. We set

$$\gamma_f := v^{-h(f)}\, p_! \circ j^*\colon \mathbf{H}_{S^1_{\mathbb{K}}}[\overline{f}] \to \mathbf{H}_{\mathbb{K}}[f] ,$$

where

$$h(f) := -\sum_{\ell<0} \langle f, \tau^\ell(f) \rangle ,$$

with $\tau(f)\colon t \mapsto f(t-1)$ being the translation operator in $\mathbb{F}(\mathbb{K})$, and where $p_!$ is taken in the sense of groupoids, i.e.,

$$p_!(u)(\rho) := \sum_{\tilde{\rho} \in \mathsf{Obj}(p^{-1}(\rho))/\sim} u(\tilde{\rho}) \frac{|\mathsf{Aut}(\rho)|}{|\mathsf{Aut}(\tilde{\rho})|} .$$

Proposition 5.11. *For any pair $h, h' \in \mathbb{F}(S^1_{\mathbb{K}})$ and any $f \in \mathbb{F}(\mathbb{K})$ such that $\overline{f} = h + h'$ we have*

$$\sum_{\substack{g+g'=f \\ \overline{g}=h, \overline{g'}=h'}} v^{(g, \sum_{\ell>0} \tau^\ell(g'))} \gamma_g(u) \star \gamma_{g'}(u') = \gamma_f(u \star u')$$

for any $u \in \mathbf{H}_{S^1_{\mathbb{K}}}[h]$, $u' \in \mathbf{H}_{S^1_{\mathbb{K}}}[h']$.

Proof. Let f, h, h' be as above and fix some $u \in \mathbf{H}_{S^1_{\mathbb{K}}}[h]$, $u' \in \mathbf{H}_{S^1_{\mathbb{K}}}[h']$. Let S be a finite subset of $S^1_{\mathbb{K}}$ containing the critical sets of all representations occurring in u, u' as well as all the discontinuities of f, h, h'. Then S contains also the critical sets of any representation occurring in $u \star u'$, as well as in $\gamma_g(u), \gamma_{g'}(u')$ and hence also of $\gamma_g(u) \star \gamma_{g'}(u')$. Arguing as in the proof of Lemma 5.10, we see that it is enough to prove the statement of the theorem when we replace everywhere $\mathcal{E}_V, \ldots$ by $\mathcal{E}_V^{(S)}, \ldots$. But in this case $\mathsf{Rep}_k^{(S)} S^1_{\mathbb{K}}$ and $\mathsf{Rep}_k^{(S)} \mathbb{K}$ are equivalent to categories of representations of a cyclic quiver and a bi-infinite linear quiver respectively and we are in the precise[4] setting of [VV99, Proposition 6.1], to which we refer for a proof. □

Using the above Proposition, we may define an algebra morphism from $\mathbf{H}_{S^1_{\mathbb{K}}}$ to a certain completion of $\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))$ that we are going to introduce now. Let $\mathbb{F}(\mathbb{K})^c$ be the space of piecewise constant, right–continuous functions $f\colon \mathbb{R} \to \mathbb{Z}$ whose points of discontinuity all belong to $\mathbb{K}$ and project onto a finite subset of $S^1_{\mathbb{K}}$. In other words, $\mathbb{F}(\mathbb{K})^c$ is an analogue of $\mathbb{F}(\mathbb{K})$ but without the bounded support condition. Let $\mathbf{U}_v^0(\mathfrak{sl}(\mathbb{K}))^c$ be the algebra generated by elements $K_f^{\pm}$ for $f \in \mathbb{F}(\mathbb{K})^c$ subject to the relations $K_f K_g = K_{f+g}$, for $f, g \in \mathbb{F}(\mathbb{K})^c$. We set

$$\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c := \bigoplus_{g \in \mathbb{F}(S^1_{\mathbb{K}})} \mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c[g]$$

$$\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c[g] := \prod_{f \in \mathbb{F}(\mathbb{K}), \overline{f}=g} \left\{ \mathbf{U}_v^0(\mathfrak{sl}(\mathbb{K}))^c \ltimes \mathbf{U}_v^-(\mathfrak{sl}(\mathbb{K}))[f] \right\} .$$

One can show that $\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c$ is an algebra. For $x \in \mathbf{H}_{S^1_{\mathbb{K}}}[g]$ we set

$$r(x) := \sum_{f, \overline{f}=g} K_{o(f)}\, \gamma_f(u) \in \mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c$$

where $o(f) := \sum_{\ell<0} \tau^\ell(f)$.

Corollary 5.12. *The map $r := \mathbf{H}_{S^1_{\mathbb{K}}} \to \mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c$ is an algebra homomorphism.*

Using the above Proposition, we can pull-back via r any weight representation of $\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))$ which extends to $\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))^c$; in particular we define the Fock space representation of $\mathbf{H}_{S^1_{\mathbb{K}}}$ or $\mathbf{U}_v^-(\mathfrak{sl}(S^1_{\mathbb{K}}))$ as the pullback by r of the Fock space representation $\mathcal{F}_{\mathbb{K}}$ of $\mathbf{U}_v^{\leq 0}(\mathfrak{sl}(\mathbb{K}))$. Let us compute explicitly the action of elements F_J.

Lemma 5.13. *Let J be a strict interval of $S^1_{\mathbb{K}}$. Then*

$$r(F_J) = \sum_{J_1, \ldots, J_\ell} v^{\frac{\ell-1}{2}} (v^{-1} - v)^{\ell-1} F_{J_1} \cdots F_{J_\ell} \prod_{i, \ell>0} K_{\mathbb{1}_{J_i+\ell}} \tag{5.2}$$

where the sum ranges over all tuples $J_1, J_2, \ldots, J_\ell$ such that $\mathbb{1}_J = \pi(\mathbb{1}_{J_1} + \cdots + \mathbb{1}_{J_\ell})$, $J_1 < J_2 < \cdots$ and $\pi_{\mathbb{K}}(J_1) \to \pi_{\mathbb{K}}(J_2) \to \cdots$.

[4] Note, however, that [VV99] use the opposite multiplication for Hall algebras and work over a field $\mathbb{F}_{q^2}$.

Proof. Let $f \in \mathbb{F}(\mathbb{K})$ be such that $\overline{f} = \mathbb{1}_J$. We claim that

$$\gamma_f(F_J) = v^{\frac{1-\ell}{2}}(v^{-1} - v)^{\ell-1} F_{J_1} \cdots F_{J_\ell}$$

if there exists $J_1, J_2, \ldots, J_\ell$ such that $f = \mathbb{1}_{J_1} + \cdots + \mathbb{1}_{J_\ell}$, $J_1 < J_2 < \cdots$ and $\pi_{\mathbb{K}}(J_1) \to \pi_{\mathbb{K}}(J_2) \to \cdots$ and $\gamma_f(F_J) = 0$ otherwise. Let us write $f = \sum_{i=1}^{\ell} \mathbb{1}_{J_i}$ with $J_1 < J_2 < \cdots J_\ell$. For $\gamma_f(F_J)$ to be nonzero, there must exist a filtration $M_1 \subset M_2 \subset \cdots \subset M_\ell = k_J$ of the indecomposable $S^1_{\mathbb{K}}$-module k_J such that $\dim(M_i/M_{i-1}) = \overline{\mathbb{1}_{J_i}}$ for $i = 1, \ldots, \ell$. This is possible if and only if $\pi_{\mathbb{K}}(J_1) \to \pi_{\mathbb{K}}(J_2) \to \cdots$ and moreover in that case we have $\bigoplus_i M_i/M_{i-1} \simeq \bigoplus_i k_{J_i}$. The claim now follows from the facts that $|\mathsf{Aut}(k_J)| = q-1$, while $|\mathsf{Aut}(\bigoplus_i I_{J_i})| = (q-1)^\ell$ and $h(f) = 1-\ell$, and finally from the easily checked relation $[\bigoplus_i k_{J_i}] = [k_{J_1}] \star \cdots \star [k_{J_\ell}]$. To obtain formula (5.2), observe that for $i < k$ we have $(\tau^\ell(\mathbb{1}_{J_k}), \mathbb{1}_{J_i}) = 0$ for all $\ell > 0$ while for $i > k$ there exists a unique ℓ such that $(\tau^\ell(\mathbb{1}_{J_k}), \mathbb{1}_{J_i}) = 1$ and $(\tau^\ell(\mathbb{1}_{J_k}), \mathbb{1}_{J_i}) = 0$ for all other values of ℓ. □

Using Formula (5.2) and Theorem 3.6 we deduce

$$F_J|p\rangle = \sum_{J_1,\ldots,J_\ell} v^{\frac{\ell-1}{2} + \sum_i n^{>}_{J_i}(p)} (-v)^{\sum_i \langle \mathbb{1}_{J_i}, p\rangle} (v^{-1} - v)^{\ell-1} |p + \sum_i \mathbb{1}_{J_i}\rangle \,.$$

This proves that operators (4.6) indeed define an action of $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ as wanted. To prove that the operators (4.7) define an action of $\mathbf{U}^-_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$, one may argue in a similar fashion — using Hall algebra of the opposite quivers with $v^{-2} = q$ — and show that the assignment

$$E_J \mapsto \sum_{J_1,\ldots,J_\ell} v^{\frac{1-\ell}{2}}(v - v^{-1})^{\ell-1} E_{J_1} \cdots E_{J_\ell} \prod_{i,\ell>0} K^{-1}_{\mathbb{1}_{J_i - \ell}}$$

where the sum ranges over all tuples $J_1, \ldots, J_\ell$ of intervals such that $\mathbb{1}_J = \pi(\mathbb{1}_{J_1} + \cdots + \mathbb{1}_{J_\ell})$, $J_1 > J_2 > \cdots$ and $\pi_{\mathbb{K}}(J_1) \to \pi_{\mathbb{K}}(J_2) \to \cdots$, defines an algebra morphism $\mathbf{U}^-_v(\mathfrak{sl}(S^1_{\mathbb{K}})) \to \mathbf{U}^-_v(\mathfrak{sl}(\mathbb{K}))^c$.

Remark 5.14. As we have seen, the Fock space is not a cyclic representation of $\mathbf{U}^+_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$. However, it is a cyclic representation of the Hall algebra $\mathbf{H}_{S^1_{\mathbb{K}}}$. This follows by reduction to the case of cyclic quivers which is treated in [VV99]. △

6. End of the proof of Theorem 4.5

6.1. **The Drinfeld-Jimbo relations.** It remains to check that the actions of $\mathbf{U}^{\pm}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ and of $\mathbf{U}^0_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$ glue together to form an action of $\mathbf{U}_v(\mathfrak{sl}(S^1_{\mathbb{K}}))$. Each of the relations (4.1 – 4.5) involves finitely many generators E_J, F_J, K_J acting on a pyramid p.

Let $\alpha\colon \mathbb{Z} \to \mathbb{R}$ be a strictly increasing map, whose image is invariant under integer translations, contains 0 as well as $D_{\mathbb{K}}(p)$ and the endpoints of all involved intervals J. Put $N = \#(\alpha^{-1}([0,1)))$. Arguing as in Section 3.3 we construct natural isomorphism $\mathbf{U}_v(\mathfrak{sl}(S^1_{\frac{1}{N}\mathbb{Z}})) \simeq \mathbf{U}_v(\mathfrak{sl}(S^1_{\alpha(\mathbb{Z})}))$ and $\mathcal{F}_{\frac{1}{N}\mathbb{Z}} \simeq \mathcal{F}_{\alpha(\mathbb{Z})}$ compatible with the formulas in Theorem 4.5. Thus it is enough to prove the theorem in the case $\mathbb{K} = \frac{1}{N}\mathbb{Z}$, which we assume from now on.

Relations (4.1), (4.2), (4.3) are obvious. Using the join relations, we may reduce relations (4.4), (4.5) to the cases where J_1, J_2, resp. J are of length $\frac{1}{N}$; this is clear for (4.1) and results from a lengthy but straightforward computation for (4.2). In the case of length $\frac{1}{N}$-intervals (i.e. of simple roots of $\widehat{\mathfrak{sl}}(N)$) these relations are well-known; we nevertheless derive them below for the reader's comfort: let p be a $\frac{1}{N}\mathbb{Z}$-pyramid, and let $J = [i, i+\frac{1}{N}[$ for some $i \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. Let p' be another $\frac{1}{N}\mathbb{Z}$ pyramid. By construction, we have $\langle p' \mid E_J F_J p\rangle = \langle p' \mid F_J E_J p\rangle = 0$ unless $p' = p - \mathbb{1}_{J'} + \mathbb{1}_{J''}$ for a pair of intervals J, J' such that $\pi(J') = \pi(J') = J$. Let us first show that if $p \neq p'$ then $\langle p' \mid F_J E_J p\rangle = \langle p' \mid E_J F_J p\rangle$. In that situation, J' and J'' are unique and we have

$$\langle p' \mid E_J F_J p\rangle = \langle p' \mid E_J(p + \mathbb{1}_{J''})\rangle \langle p + \mathbb{1}_{J''} \mid F_J p\rangle = v^{n^{>}_{J''}(p) - n^{<}_{J'}(p + \mathbb{1}_{J''})}\,,$$

$$\langle p' \mid F_J E_J p\rangle = \langle p' \mid F_J(p - \mathbb{1}_{J'})\rangle \langle p - \mathbb{1}_{J'} \mid E_J p\rangle = v^{n^{>}_{J''}(p-\mathbb{1}_{J'}) - n^{<}_{J'}(p)} .$$

There are two possibilities: $J'' > J'$ or $J'' < J'$. In the first one we have $n^{>}_{J''}(p) = n^{>}_{J''}(p - \mathbb{1}_{J'})$ and $n^{<}_{J'}(p + \mathbb{1}_{J''}) = n^{<}_{J'}(p)$, while in the second one we have $n^{>}_{J''}(p) = n^{>}_{J''}(p - \mathbb{1}_{J'}) + 1$ and $n^{<}_{J'}(p + \mathbb{1}_{J''}) = n^{<}_{J'}(p) - 1$. In both cases the desired equality follows.

Thus only when $p = p'$ may we have a contribution to the commutator $[E_J, F_J]$. Let $I_1 < I_2 < \cdots < I_c$ be the different addable or removable intervals of p which are congruent to J. We may partition $[1, c] = A \sqcup R$ with $A = \{i \mid I_i \text{ is addable}\}$ and $R = \{i \mid I_i \text{ is removable}\}$. Let us also write

$$a_{>h} = |A \cap [h+1, n]| , \ r_{>h} = |R \cap [h+1, n]| , \ a_{<h} = |A \cap [1, h-1]| , \ r_{<h} = |R \cap [1, h-1]| .$$

The contribution of the interval I_h to $\langle p \mid [E_J, F_J] p\rangle$ is equal to

$$\langle p \mid E_J(p + \mathbb{1}_{I_h})\rangle \langle p + \mathbb{1}I_h \mid F_J p\rangle = v^{(a_{>h} - r_{>h}) - (a_{<h} - r_{<h})}$$

if $h \in A$, while it is

$$-\langle p \mid F_J(p - \mathbb{1}_{I_h})\rangle \langle p - \mathbb{1}I_h \mid E_J p\rangle = -v^{(a_{>h} - r_{>h}) - (a_{<h} - r_{<h})}$$

if $h \in R$. All together we get

$$\langle p \mid [E_J, F_J] p\rangle = \sum_{h \in A} v^{(a_{>h} - r_{>h}) - (a_{<h} - r_{<h})} - \sum_{h \in R} v^{(a_{>h} - r_{>h}) - (a_{<h} - r_{<h})} . \tag{6.1}$$

It thus only remains to check that the r.h.s. of relation (6.1) coincides with

$$\langle p \mid (K_J - K_J^{-1}) p\rangle / (v - v^{-1}) = (v^{|A| - |R|} - v^{|R| - |H|}) / (v - v^{-1}) .$$

This is a purely combinatorial statement, which may be proved as follows. We first check it when $R = [1, u]$, $A = [u+1, n]$, and then we prove that the r.h.s. of relation (6.1) remains unchanged when we exchange the position of a pair of adjacent elements, one which belongs to A and the other to R. We leave the details to the reader.

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