Calibrations for minimal networks in a covering space setting

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Abstract

In this paper we define a notion of calibration for an approach to the classical Steiner problem in a covering space setting and we give some explicit examples. Moreover we introduce the notion of calibration in families: the idea is to divide the set of competitors in a suitable way, defining an appropriate (and weaker) notion of calibration. Then, calibrating the candidate minimizers in each family and comparing their perimeter, it is possible to find the minimizers of the minimization problem. Thanks to this procedure we prove the minimality of the Steiner configurations spanning the vertices of a regular hexagon and of a regular pentagon.

Keywords: Minimal partitions, Steiner problem, covering spaces, calibrations

Mathematics Subject Classification (2010): 49Q20, 49Q05, 57M10

1 Introduction

The classical Steiner problem, whose first modern formulation can be found in [17], can be stated as follows: given a collection S of m points of $\mathbb{R}^n$, find the connected set that contains S with minimal length, namely

$$\inf\{\mathcal{H}^1(K) : K \subset \mathbb{R}^n, \text{ connected and such that } S \subset K\} \tag{1.1}$$

(we refer to [20] for a survey on the topic). In its highest generality the problem can be stated replacing the ambient space $\mathbb{R}^n$ with any metric space [27]. From a computational point of view the Steiner problem is NP-hard, hence, it is interesting to attack it in new ways (for example by multiphase approximations à la Modica–Mortola [9, 11, 12, 16, 23]) in order to improve the rate of convergence of the algorithms. From a theoretical perspective the solution of the Steiner problem by variational methods has received an increasing interest starting from several results of the 90's, which establish an equivalence between the Steiner problem and the minimal partition problem [3, 4, 26, 28], and ending to more recent approaches, such as currents or vector valued BV functions defined on a covering space [2, 14], currents with coefficients in a group [24], rank-one tensor valued measures [11].

The first approach via covering space to the Steiner problem, and more in general to Plateau's type problems, is due to Brakke [14]. Having in mind a possible candidate minimizer (unoriented soap films, soap films with singularities, soap films touching only part of a knotted curve) for a certain Plateau's type problem he introduces a double covering called *pair covering space* chosen compatibly to the soap film he wants to obtain as a minimizer. Then

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he minimizes the mass of integral currents defined on it, and only in some special cases, via a calibration argument, he proves that the minimizer coincides with his candidate. As a consequence of this *ad hoc* approach, he did not give an explicit proof of the equivalence with the Steiner problem; however his setting allows to describe a great variety of different objects, for istance soap films in higher dimension. In [2] the authors revive Brakke's covering space approach, constructing an m–sheeted covering space Y_Σ of $\mathbb{R}^2 \setminus S$ and minimizing the total variation of vector valued BV functions on Y_Σ satisfying a certain constraint. In this setting the authors prove the equivalence between their minimization problem and the Steiner problem in the plane.

The first part of this paper is devoted to an improvement of the result in [2] reducing the vector valued problem in [2] to a scalar one: we minimize the *perimeter* in a family of finite perimeter sets in Y_Σ instead of the total variation of constrained vector valued BV functions. In particular we state the following minimization problem in Y_Σ:

$$\mathscr{A}_{constr}(S) = \inf \{P(E) : E \in \mathscr{P}_{constr}(Y_\Sigma)\} \ , \tag{1.2}$$

where $\mathscr{P}_{constr}(Y_\Sigma)$ is the space of sets of finite perimeter in Y_Σ satisfying a suitable constraint. Once proved the existence of minimizers we show an equivalence between our minimization problem and the classical Steiner problem (we refer the reader to the beginning of Section 2 for further explanations).

In the second part of the paper we introduce a notion of calibration suitable to our setting. In the context of minimal surfaces, a *calibration* for a k–dimensional oriented manifold in $\mathbb{R}^{n+1}$ is a closed k-form ω such that $|\omega| \leq 1$ (the so–called size condition) and $\langle \omega, \xi \rangle = 1$, where ξ is the unit k-vector orienting the manifold. It is easy to see that the existence of a calibration for a certain manifold implies that the manifold is area minimizing in its homology class. In Definition 3.3 we adapt this notion to our setting taking advantage of the theory of Null-Lagrangians [3, 4]: a calibration for $E \in \mathscr{P}_{constr}(Y_\Sigma)$ is a divergence-free vector field defined on the covering space Y_Σ such that $\int_{Y_\Sigma} \Phi \cdot D\chi_E = P(E)$ and a suitable size condition for Φ is fulfilled (see Remark 3.4). As for minimal surfaces, we show that the existence of a calibration for a set $E \in \mathscr{P}_{constr}(Y_\Sigma)$ implies that it is a minimizer to (1.2). In order to show the usefulness of our theory we give the explicit examples of calibrations for the Steiner minimal configuration connecting two points, three points located at the vertices of any triangle and for the four vertices of a square.

A notion of calibration for the partition problem was firstly introduced by Morgan and Lawlor in [22]: their *paired calibration* technique allows to prove the minimality of soap films among all the competitors that split the domain in a fixed number of regions. In the context of the Steiner problem the limit of this approach is that it can be applied only when the points of S belong to the boundary of a convex set. As mentioned previously, a notion of calibration, adapted to the covering space approach, is proposed also in [14]. Finally, Marchese and Massaccesi in [24] describe Steiner trees using currents with coefficient in groups and rephrase the Steiner problem as a minimum problem for the mass of currents. In this way they were able to introduce a related notion of calibration (see also [15] for a comparison of the different notions). Both [14] and [24] have a companion paper devoted to numerical results (see [13] and [25]). We underline that our approach, as the one introduced in [14, 24], does not require that the points of S lie on the boundary of a convex set.

The goal of the last part of the paper is to tackle the minimality of the Steiner minimal configurations for the vertices of a regular pentagon and of a regular hexagon using the theory of calibrations.
We remind that the explicit minimizers for the Steiner problem, if the points of S are the vertices of regular n-gon, are well known. In particular for $n \geq 6$ the Steiner minimal network is the polygon without an edge. The first proof for the cases $n \leq 6$ and $n \geq 13$ is due to Jarnik and Kössler in 1934 [21]. Fifty years later, Du, Hwang and Weng proved the remaining cases [18].

The main purpose of searching for a calibration is to have an easy argument to show the minimality of a certain candidate. Unfortunately finding a calibration is in general not an easy task and only very few example are known even for the Steiner problem. As already anticipated, in this work we propose a calibration argument to prove in a strikingly easy way the minimality for the Steiner minimal configuration connecting the vertices of the regular pentagon and of the regular hexagon.
The interest of our technique goes beyond these specific results because it can be generalized to arbitrary configurations of points in $\mathbb{R}^2$ and suggests how to "decompose" the Steiner problem in several simpler convex problems that can be solved (and calibrated) separately. Moreover the authors believe that a similar idea could be applied to different variational problems. The idea is to divide the set of competitors in different families, denoted by $\mathcal{F}(\mathcal{J})$, and define an appropriate notion of calibration in each family with a weaker size condition. To be more precise all the competitors that belong to the same family share a property related to the projection of their essential boundary onto the base set M: for certain couples of indices (i, j) in $\{1, \dots, m\} \times \{1, \dots, m\}$ the intersection of the projections onto M of the boundary of the part of the set E in the i–th sheet and in the j–th sheet is $\mathcal{H}^1$–negligible. As a consequence, the definition of calibration in a family does not require to verify the size condition for the pairs of sheets associated to these couples of indices (i, j). Once identified a candidate minimizer in each family, we calibrate them and in conclusion we compare their energy to find the explicit global minimizers of Problem (1.2). Thanks to this procedure we prove the minimality of the Steiner configurations spanning the vertices of a regular hexagon (the hexagon without one edge) and of the regular pentagon.

Finally we outline the structure of the paper: in the beginning of Section 2 we summarize the setting introduced in [2] describing the construction of the covering space and we define finite perimeter sets on it. In Subsection 2.3 we introduce the space $\mathscr{P}_{constr}(Y_\Sigma)$ and prove the existence of minimizers for Problem (1.2). Then, in Theorem 2.27, we present a regularity result for the essential boundary of minimizers proving a *local* equivalence with the problem of minimal partitions [3, 4, 26, 28]. We conclude Section 2 proving the equivalence between the classical Steiner problem and our minimization problem (1.2).
In Section 3, after giving the definition of calibration, we show that the existence of a calibration for $E \in \mathscr{P}_{constr}(Y_\Sigma)$ implies minimality of E with respect to (1.2). We construct explicit examples of calibration for the Steiner minimal configuration connecting two points, three points and the four vertices of a square.
In Section 4 we develop the notion of calibration in families and use this tool to prove the minimality of the Steiner minimal configurations spanning the vertices of a regular hexagon and of a regular pentagon.

Acknowledgements

The authors are warmly grateful to Giovanni Bellettini and Maurizio Paolini for several discussions and helpful conversations. Once this manuscript was already completed, the authors have been told by Frank Morgan that the idea about separating the competitors into classes for the hexagon problem was found independently by Gary Lawlor and presented at Lehigh University in 1998, but was not published even in preprint form. The authors would like to thank Frank Morgan and Gary Lawlor for their remarks about the history of the problem and for their kind mail correspondence.

2 Equivalence between Problem (1.2) and the Steiner problem

In this section we consider the construction of the m–sheeted covering space Y_Σ of $M := \mathbb{R}^2 \setminus S$ presented in [2] and we define sets of finite perimeter on the covering. In particular we define the space $\mathscr{P}_{constr}(Y_\Sigma)$ of the sets of finite perimeter E in Y_Σ satisfying a suitable boundary condition at infinity and such that for almost every x in the base space there exists exactly one point y of E such that $p(y) = x$, where p is the projection onto the base space (see Definition 2.13). Notice that for every set $E \in \mathscr{P}_{constr}(Y_\Sigma)$ it is possible to show a formula (Proposition 2.18) that relates its perimeter to the $\mathcal{H}^1$-measure of the projection onto the base space M of its essential boundary. In Subsection 2.3 we state the minimization problem (1.2) and we prove existence and non–triviality of the minimizers.
The last part of the section is devoted to the equivalence between our minimization problem and the classical Steiner problem. On one side it is enough to show that given a minimizer for Problem (1.2), the network obtained as the closure of the projection onto M of the essential boundary is a competitor for the Steiner problem. Roughly speaking given the m points of S in $\mathbb{R}^2$, the covering space of M is constructed in such a way that if we consider a closed curve γ (namely a loop) with index one with respect to at most $m-1$ points and with index zero with respect to at least one point of S, then $p^{-1}(\gamma)$ is connected. Combining this property of the covering with the constraint on the set it is possible to show that the set S is contained in a connected component of the closure of the projection of the essential boundary. On the other hand we describe a procedure to construct a set in $\mathscr{P}_{constr}(Y_\Sigma)$ from a minimal Steiner graph.

2.1 Construction of the covering space

For the rest of the paper we consider $S = \{p_1, \dots, p_m\}$ a finite set of points in $\mathbb{R}^2$ and $M := \mathbb{R}^2 \setminus S$. Moreover we fix an open, smooth and bounded set $\Omega \subset \mathbb{R}^2$ such that $\mathrm{Conv}(S) \subset \Omega$, where $\mathrm{Conv}(S)$ is the convex envelope of S.

Definition 2.1 (Admissible cuts)**.** We denote by $\mathrm{Cuts}(S)$ the set of all $\Sigma := \bigcup_{i=1}^{m-1} \Sigma_i \subset \Omega$ such that:

(a) for $i = 1, \dots, m-1$, Σ_i is a Lipschitz simple curve starting at p_i and ending at p_{i+1};

(b) if $m > 2$ then $\Sigma_i \cap \Sigma_{i+1} = \{p_{i+1}\}$ for $i = 1, \dots, m-2$;

(c) $\Sigma_i \cap \Sigma_l = \emptyset$ for any $i, l = 1, \dots, m-1$ such that $|l - i| > 1$.

Moreover we denote by $\mathbf{Cuts}(S)$ the set of all pairs $\mathbf{\Sigma} := (\Sigma, \Sigma')$ such that

(i) $\Sigma, \Sigma' \in \mathrm{Cuts}(S)$ and $\Sigma \cap \Sigma' = S$;

(ii) if $m > 2$, for every $i = 2, \dots, m-1$ let $C_i(p_i, \varepsilon)$ be a circle centered at p_i with radius ε such that $C_i(p_i, \varepsilon) \cap \Sigma_{i-1} \neq \emptyset$ and $C_i(p_i, \varepsilon) \cap \Sigma_i \neq \emptyset$. Denote by x_i (resp y_i) the intersection of C_i with Σ_{i-1} (resp. with Σ_i). Then there exists an arc of C_i connecting x_i and y_i and not intersecting Σ'.

Fix $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(S)$ and define

$$D := \mathbb{R}^2 \setminus \Sigma\,, \qquad D' := \mathbb{R}^2 \setminus \Sigma'$$

and

$$X = \bigcup_{j=1}^{m} (D, j) \;\cup\; \bigcup_{j'=m+1}^{2m} (D', j')$$

that is the space made of m disjoint copies of D and of D'.

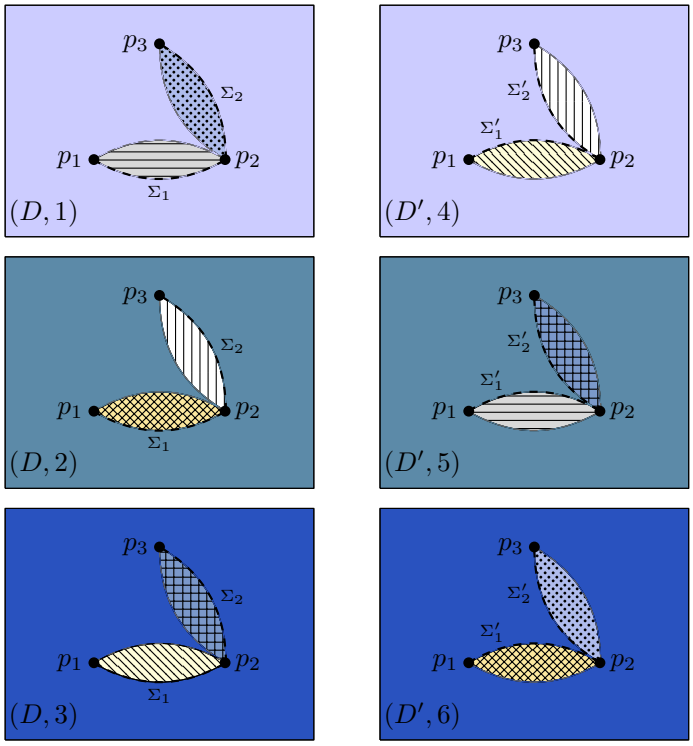


Figure 1: An explicit construction of X when $m = 3$ and $S = \{p_1, p_2, p_3\}$. The regions identified by the equivalence relation $\sim$ are represented with the same pattern and color.

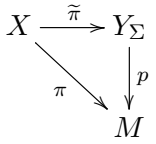
Let I_i be the open, bounded set enclosed by Σ_i and Σ_i' and $O = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m-1} \overline{I_i}$. Given $(x, j) \in (D, j)$ with $j \in \{1, \dots, m\}$ and $(x', j') \in (D', j')$ with $j' \in \{m+1, \dots, 2m\}$, we define the equivalence relation $\sim$ in X as $(x, j) \sim (x', j')$ if and only if one of the following conditions holds:

$$\begin{cases} j \equiv j' \pmod m, & x = x' \in O\,, \\ j \equiv j' - i \pmod m, & x = x' \in I_i,\ i = 1, \dots, m-1\,. \end{cases} \tag{2.1}$$

Definition 2.2. We define Y_Σ to be the topological quotient space induced by $\sim$, i.e.

$$Y_\Sigma := X \Big/ \sim \ .$$

Finally we denote by $\tilde{\pi} : X \to Y_\Sigma$ the projection induced by the equivalence relation, by π the projection from X to the space M and by $p : Y_\Sigma \to M$ the map that makes the following diagram commutative:



Proposition 2.3. *The map $p : Y_\Sigma \to M$ is well-defined and the pair (Y_Σ, p) is a covering space of M.*

Remark 2.4. The covering (Y_Σ, p) of M inherits the local structure of M, hence, in our case, it is a Riemannian manifold.

In order to be able to work on Euclidean spaces we define the following natural local parametrizations of Y_Σ.

Definition 2.5 (Local parametrizations)**.** For every $j = 1, \dots m$ we define the local parametrizations $\psi_j : D \to \widetilde{\pi}\left((D, j)\right)$ as

$$\psi_j(x) := \widetilde{\pi}\left((x, j)\right) \quad \text{for every } x \in D\,,$$

or equivalently as

$$\psi_j(x) := \left(p\big|_{\widetilde{\pi}((D,j))}\right)^{-1}(x) \quad \text{for every } x \in D\,.$$

The local parametrizations $\psi_{j'} : D' \to \widetilde{\pi}\left((D', j')\right)$, for $j' = m+1, \dots 2m$ are analogously defined.

2.2 Sets of finite perimeter on the covering space

It is natural to endow the space Y_Σ with a measure μ defined as the sum on every sheet of the covering space of the push-forward by the local parametrization introduced in Definition 2.5 of the Lebesgue measure $\mathscr{L}^2$. Given a Borel set $E \subset Y_\Sigma$ we define

$$\mu(E) := \sum_{j=1}^{m} \psi_{j\#}\mathscr{L}^2(E \cap \tilde{\pi}((D, j)))\,.$$

We define the pullback of a function f by the local parametrizations ψ_j and $\psi_{j'}$ in the following way:

Definition 2.6. Consider a function $f : Y_\Sigma \to \mathbb{R}^n$. For $j = 1, \dots, m$ and $j' = m+1, \dots 2m$ we let $f^j : D \to \mathbb{R}^n$, $f^{j'} : D' \to \mathbb{R}^n$ be the maps defined by

$$f^j := f \circ \psi_j \qquad \text{and} \qquad f^{j'} := f \circ \psi_{j'}\,.$$

By construction, thanks to the identifications given by $\sim$, we have

$$\begin{cases} f^j = f^{j'} & \text{in } O \quad \text{if } j \equiv j' \pmod m\,, \\ f^j = f^{j'} & \text{in } I_i \quad \text{if } j \equiv j' - i \pmod m,\ i = 1, \dots, m-1\,. \end{cases} \tag{2.2}$$

From now on by χ_E we mean the characteristic function of the set E.

Definition 2.7. Given a μ–measurable set $E \subset Y_\Sigma$ we define

$$E^j := \{x \in D \,:\, \chi_E^j(x) = 1\} = p(E \cap \tilde{\pi}((D, j)))$$

for $j \in \{1, \dots, m\}$ and analogously

$$E^{j'} := \{x \in D' \,:\, \chi_E^{j'}(x) = 1\} = p(E \cap \tilde{\pi}((D', j')))$$

for $j' \in \{m+1, \dots, 2m\}$.

Remark 2.8. It is useful sometimes to define functions f (resp. sets E) on the covering space Y_Σ prescribing first the parametrizations $f^j : D \to \mathbb{R}^n$ (resp. sets E^j) for every $j = 1 \dots, m$ and then deducing $f^{j'}$ in $D' \setminus \Sigma$ (resp. sets $E^{j'}$) for every $j' = m+1, \dots, 2m$, according to (2.2).

We set $L^1(Y_\Sigma) \coloneqq L^1(Y_\Sigma; \mathbb{R}; \mu)$ and analogously, $L^1_{loc}(Y_\Sigma) \coloneqq L^1_{loc}(Y_\Sigma; \mathbb{R}; \mu)$. We also define the distributional gradient of a function $u \in L^1(Y_\Sigma)$ as the linear map

$$Du(\psi) = -\int_{Y_\Sigma} u \operatorname{div} \psi \, d\mu$$

for $\psi \in C^1_c(Y_\Sigma, \mathbb{R}^2)$, where the space $C^1_c(Y_\Sigma, \mathbb{R}^2)$ is defined in the natural way by the local parametrizations.

Definition 2.9. Given $u \in L^1(Y_\Sigma)$ we say that $u \in BV(Y_\Sigma)$ if Du is represented by a Radon measure with bounded total variation.

Definition 2.10. Given $E \subset Y_\Sigma$ a μ-measurable set and $\Lambda \subset Y_\Sigma$ open, we define

$$P(E, \Lambda) := |D\chi_E|(\Lambda) = \sup \left\{ \int_E \operatorname{div} \psi \, d\mu : \psi \in C^1_c(\Lambda, \mathbb{R}^2), \|\psi\|_\infty \le 1 \right\}. \tag{2.3}$$

We say that E is a set of finite perimeter in Λ if $P(E, \Lambda) < \infty$. In the case E is of finite perimeter in Y_Σ, the definition of P can be extended to all Borel sets $\Lambda \subset Y_\Sigma$ and $\Lambda \to P(E, \Lambda)$ is a Borel measure in Y_Σ.

For every $t \in [0, 1]$ define the set $E^t \subset Y_\Sigma$ as

$$E^t = \left\{ x \in Y_\Sigma : \lim_{r \to 0} \frac{\mu(E \cap B_r(x))}{\mu(B_r(x))} = t \right\}.$$

We denote by $\partial^* E$ the essential boundary of E defined as $\partial^* E = Y_\Sigma \setminus (E^0 \cup E^1)$ (see [6, page 158] for the definition of the essential boundary in the Euclidean setting).

Lemma 2.11 (Representation formula for the perimeter and for $D\chi_E$)**.** *Let Λ be a Borel set in Y_Σ and E a set of finite perimeter in Y_Σ. Then defining $\Lambda^{j'}_\Sigma := p(\Lambda \cap \tilde{\pi}((\Sigma \setminus S, j')))$ we have*

$$P(E,\Lambda) = \sum_{j=1}^{m} P(E^j, \Lambda^j) + \sum_{j'=m+1}^{2m} P(E^{j'}, \Lambda^{j'}_\Sigma) \tag{2.4}$$

and

$$D\chi_E(\Lambda) = \sum_{j=1}^{m} D\chi_{E^j}(\Lambda^j) + \sum_{j'=m+1}^{2m} D\chi_{E^{j'}}(\Lambda^{j'}_\Sigma)\,. \tag{2.5}$$

Proof. We notice that a Borel set $\Lambda \subset Y_\Sigma$ can be decomposed in the union of the disjoint sets:

$$\Lambda \cap \tilde{\pi}((D,j)), \quad j = 1, \dots m\,, \qquad \Lambda \cap \tilde{\pi}((\Sigma \setminus S, j')), \quad j' = m+1, \dots, 2m\,. \tag{2.6}$$

Hence it is enough to prove the statements for a $\Lambda \subset \tilde{\pi}((D,j))$ for a fixed j or a $\Lambda \subset \tilde{\pi}((\Sigma \setminus S, j'))$ for a fixed j'. Let us assume without loss of generality that $\Lambda \subset \tilde{\pi}((D,j))$ is open. Consider $\eta \in C^1(\Lambda)$ and compactly supported in Λ. Then noticing that as $p\big|_\Lambda$ is bijective we have

$$D\chi_E(\eta) = \int_\Lambda \chi_E \operatorname{div} \eta \, d\mu = \int_{p(\Lambda \cap E)} \operatorname{div}(\eta \circ p^{-1})\, d\mathscr{L}^2 = D\chi_{p(E\cap\Lambda)}(\eta \circ p^{-1})\,.$$

Moreover as $p\big|_\Lambda$ is an homeomorphism it is easy to verify that $\eta \in C^1_c(\Lambda)$ if and only if $\eta \circ p^{-1} \in C^1_c(p(\Lambda))$. Therefore taking the supremum on η we have that

$$P(E,\Lambda) = P(p(E \cap \Lambda)) = P(p(E \cap \Lambda \cap \tilde{\pi}((D,j)))) = P(E^j, \Lambda^j).$$

If $\Lambda \subset \tilde{\pi}((\Sigma \setminus S, j'))$ the decomposition (2.5) follows performing the same computation in (D', j') and then using the outer regularity of the measure $\Lambda \to P(E,\Lambda)$. Formula (2.5) can be proven similarly. □

Remark 2.12. From the computations of the previous lemma, one can easily see that if E is a set of finite perimeter in Y_Σ, then E^j is a set of finite perimeter in D for $j = 1 \dots m$ (and, respectively, in a similar way one can show that $E^{j'}$ is a set of finite perimeter in D' for $j' = m+1 \dots 2m$).

2.3 The constrained minimum problem

We define our simplified version of the minimization problem introduced by Amato, Bellettini and Paolini in [2], where we use sets of finite perimeter instead of vector valued BV functions.

Definition 2.13 (Constrained sets)**.** We denote by $\mathscr{P}_{constr}(Y_\Sigma)$ the space of the sets E of finite perimeter in Y_Σ such that

i) $\sum_{p(y)=x} \chi_E(y) = 1$ for almost every $x \in M = \mathbb{R}^2 \setminus S$,

ii) $\chi_{E^1}(x) = 1$ for every $x \in \mathbb{R}^2 \setminus \Omega$.

In other words a set E of finite perimeter in Y_Σ belongs to $\mathscr{P}_{constr}(Y_\Sigma)$ if for almost every x in the base space there exists exactly one point y of E such that $p(y) = x$.

Remark 2.14. Notice that it is possible to produce different sets in Y_Σ (by a permutation of the sheets) satisfying condition i) in Definition 2.13 that have the same projection on to the base space. In order to avoid this unpleasant effect we decide to add the condition ii) in Definition 2.13.

We state the constrained minimization problem as follows:

$$\mathscr{A}_{constr}(S) = \inf \{P(E) : E \in \mathscr{P}_{constr}(Y_\Sigma)\} \, .$$

Remark 2.15. It can be proved as in [2] (see also [7, 8]) that given $\mathbf{\Sigma}, \hat{\mathbf{\Sigma}} \in \mathbf{Cuts}(S)$ and $E \in \mathscr{P}_{constr}(Y_\Sigma)$ there exists $\hat{E} \in \mathscr{P}_{constr}(Y_{\hat{\Sigma}})$ such that $p(\partial^* E) = p(\partial^* \hat{E})$. This implies that the quantity $\mathscr{A}_{constr}(S)$ is independent on the choice of the cuts $\mathbf{\Sigma}$.

Lemma 2.16 (Compactness)**.** *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathscr{P}_{constr}(Y_\Sigma)$ such that*

$$\sup_{n\in\mathbb{N}} P(E_n) < +\infty \, .$$

Then there exists $E \in \mathscr{P}_{constr}(Y_\Sigma)$ and a subsequence $(E_n)_{n\in\mathbb{N}}$ (not relabelled) converging to E in $L^1(Y_\Sigma)$ as $n \to +\infty$.

Proof. Notice that, thanks to (2.4), there holds $P(E_n^j, D) \leq P(E_n) < c < \infty$. Therefore, up to subsequences, $E_n^j \to E^j$ in $L^1(D)$ for every $j = 1, \dots, m$ (without relabelling the subsequence). The set E is determined according to Remark 2.8. As for every $x \in D$ we have that $p^{-1}(x) = \tilde{\pi} \circ \pi^{-1}(x) = \bigcup_{j=1}^m \psi_j(x)$, the property in Definition 2.13 can be rephrased using the local parametrization in the following way:

$$\sum_{j=1}^m \chi_{E_n^j}(x) = 1 \, .$$

for almost every $x \in M$. Hence letting $n \to +\infty$ we get that $\sum_{i=1}^m \chi_{E^j}(x) = 1$ for almost every $x \in M$. Moreover, as $\chi_{E_n^1} = 1$ in $\mathbb{R}^2 \setminus \Omega$ for every n, we have $\chi_{E^1} = 1$ in $\mathbb{R}^2 \setminus \Omega$ and therefore $E \in \mathscr{P}_{constr}(Y_\Sigma)$. □

Theorem 2.17 (Existence of minimizers)**.** *There exists a minimizer for Problem (1.2).*

Proof. The proof follows by the application of the direct method thanks to Lemma 2.16 and the lower semicontinuity of the perimeter in $L^1_{loc}(Y_\Sigma; \mathbb{R}; \mu)$, that holds because the perimeter defined in (2.3) is supremum of continuous functional. □

2.4 Some properties of the projection of the essential boundary

Proposition 2.18. *Given a set E in $\mathscr{P}_{constr}(Y_\Sigma)$, we have*

$$P(E) = 2\mathcal{H}^1(p(\partial^* E)) \, .$$

Proof. Consider a set E in $\mathscr{P}_{constr}(Y_\Sigma)$, then thanks to Lemma 2.11 we get

$$P(E) = \sum_{j=1}^m P(E^j, D) + \sum_{j'=m+1}^{2m} P(E^{j'}, \Sigma \setminus S) \, . \tag{2.7}$$

Notice firstly that $\partial^* E^j$ and $\partial^* E^{j'}$ are rectifiable sets in D and D' respectively, therefore they admit a generalized unit normal that we denote by ν_j and $\nu_{j'}$. Define, for $h,k=1,\dots,m$, the set

$$A_{h,k}=\left\{x\in\partial^* E^h\cap\partial^* E^k : \nu_h(x),\nu_k(x) \text{ exist and } \nu_h(x)=-\nu_k(x)\right\},$$

and, for $h',k'=m+1,\dots,2m$

$$A_{h',k'}=\left\{x\in\partial^* E^{h'}\cap\partial^* E^{k'} : \nu_{h'}(x),\nu_{k'}(x) \text{ exist and } \nu_{h'}(x)=-\nu_{k'}(x)\right\}.$$

Suppose that $A_{h,k}\subset D$ for every $h,k=1,\dots,m$.
As $E\in\mathscr{P}_{constr}(Y_\Sigma)$ the sets $A_{h,k}$ satisfy the following properties:

- the sets $\{A_{h,k} : k=1,\dots,m\}$ are pairwise disjoint for every $h=1,\dots,m$;
- $\mathcal{H}^1\left(p(\partial^* E)\setminus\bigcup_{h<k}A_{h,k}\right)=0$;
- $\mathcal{H}^1(A_{h,k}\cap\partial^* E^j)=0$ for every $h,k\neq j$.

Hence using (2.7) and the previous properties we have

$$\begin{aligned}P(E) &= \sum_{h=1}^m P(E^h,\bigcup_{k=1}^m A_{h,k})=\sum_{h,k=1}^m P(E^h,A_{h,k})\\ &= \sum_{h<k}P(E^h,A_{h,k})+\sum_{h>k}P(E^h,A_{h,k})\\ &= 2\mathcal{H}^1(p(\partial^* E))\end{aligned}$$

as we wanted to prove.
If $A_{h,k}\cap\Sigma\neq\emptyset$ for some h,k, then one can repeat the previous argument decomposing $p(\partial^* E)$ with the sets $A_{h,k}$ in D and with the sets in $A_{h',k'}$ in $\Sigma\setminus S$ and then use (2.7). □

Remark 2.19. With a similar proof it is possible to prove that given a μ–measurable set $A\subset\mathbb{R}^2$ and $E\in\mathscr{P}_{constr}(Y_\Sigma)$ we have

$$P(E,p^{-1}(A))=2\mathcal{H}^1(A\cap p(\partial^* E)).$$

Lemma 2.20 (Non-constancy)**.** *Let $A\subset\mathbb{R}^2$ be a nonempty, open set such that $p^{-1}(A\setminus S)$ is connected. Then, for every $E\in\mathscr{P}_{constr}(Y_\Sigma)$*

$$\mathcal{H}^1(A\cap p(\partial^* E))>0.$$

Proof. By contradiction there exists $E\in\mathscr{P}_{constr}(Y_\Sigma)$ such that $\mathcal{H}^1(A\cap p(\partial^* E))=0$. Then by Remark 2.19 we obtain $P(E,p^{-1}(A\setminus S))=2\mathcal{H}^1(A\cap p(\partial^* E))=0$. By Lemma 2.11 there holds $P(E^j,(p^{-1}(A\setminus S))^j)\le P(E,p^{-1}(A\setminus S))=0$, for every $j\in\{1,\dots,m\}$ and $P(E^{j'},(p^{-1}(A\setminus S))^{j'})\le P(E,p^{-1}(A\setminus S))=0$, for every $j'\in\{m+1,\dots,2m\}$.
Hence $D\chi_{E^j}=0$ in $(p^{-1}(A\setminus S))^j$, that implies that the function χ_{E^j} is constant in $(p^{-1}(A\setminus S))^j$ for every $j\in\{1,\dots,m\}$ (similarly $\chi_{E^{j'}}$ is constant in $(p^{-1}(A\setminus S))^{j'}$). Moreover writing the set $p^{-1}(A\setminus S)$ as

$$p^{-1}(A\setminus S)=\bigcup_{j=1}^m\left[p^{-1}(A\setminus S)\cap\tilde{\pi}(D,j)\right]\bigcup_{j'=m+1}^{2m}\left[p^{-1}(A\setminus S)\cap\tilde{\pi}(D',j')\right]$$

and applying p to both sides we obtain that

$$A \setminus S = \bigcup_{j=1}^{m} (p^{-1}(A \setminus S))^j \bigcup_{j'=m+1}^{2m} (p^{-1}(A \setminus S))^{j'}.$$

Hence $A \setminus S$ is a connected set, union of open sets in which χ_{E^j} and $\chi_{E^{j'}}$ are constant. This implies that the value of all χ_{E^j} and $\chi_{E^{j'}}$ is the same for every j, j' and, as a consequence, χ_E is constant in $p^{-1}(A \setminus S)$. This contradicts the validity of the constraint i) of Definition 2.13. □

The previous lemma implies that the projection via p of the essential boundary of $E \in \mathscr{P}_{constr}(Y_\Sigma)$ touches S.

Corollary 2.21. *Let $E \in \mathscr{P}_{constr}(Y_\Sigma)$. Then*

$$S \subset \overline{p(\partial^* E)}\,.$$

Proof. Suppose by contradiction that there exists $p_i \in S \setminus \overline{p(\partial^* E)}$. Then there exists a ball B with center in p_i and such that $B \cap \overline{p(\partial^* E)} = \emptyset$. Notice that $p^{-1}(B \setminus \{p_i\})$ is path connected in Y_Σ. Indeed given two points q_1 and q_2 in $p^{-1}(B \setminus \{p_i\})$ it is possible to construct a connected path in Y_Σ joining q_1 and q_2 following the identifications given by $\sim$ and crossing the cuts in the right order (see Definition 2.1 and the identifications defined in (2.1)); so we can apply Lemma 2.20 to deduce that

$$\mathcal{H}^1(B \cap p(\partial^* E)) > 0\,,$$

that is a contradiction. □

Proposition 2.22. *Consider $E \in \mathscr{P}_{constr}(Y_\Sigma)$ such that $\mathcal{H}^1(\overline{p(\partial^* E)}) < +\infty$. If at least one point of S is contained in a connected component $\mathcal{C}$ of $\overline{p(\partial^* E)}$, then the whole S is contained in $\mathcal{C}$.*

Proof. For any $p_j \in S$ let $\mathcal{C}_j$ be the connected component of $\overline{p(\partial^* E)}$ containing p_j. Suppose by contradiction that there exists $k_1 \neq k_2$ such that $\mathcal{C}_{k_1} \neq \mathcal{C}_{k_2}$. As a consequence $\overline{p(\partial^* E)}$ is not connected. Hence there exist two non-empty disjoint sets $A, B \subset \overline{p(\partial^* E)}$, relatively closed in $\overline{p(\partial^* E)}$ with $A \cup B = \overline{p(\partial^* E)}$. Moreover it is possible to choose A and B satifying the properties above and such that there exists $p_A, p_B \in S$ with $p_A \in A$ and $p_B \in B$. Notice that, thanks to condition ii) in Definition 2.13, the set $\overline{p(\partial^* E)}$ is bounded. Hence A and B are compact in $\mathbb{R}^2$. Let $\varepsilon > 0$ be such that

$$A_\varepsilon \cap B = \emptyset\,,$$

where we have denoted by A_ε the open ε-neighbourhood of A. Notice that ∂A_ε is a Lipschitz manifold for ε small enough [19]. Hence ∂A_ε has a finite number of connected components and in particular it is a finite union of simple loops (see for example [5, Corollary 1]) that we denote by $\{\gamma_i\}_{i=1,\dots,q}$. As $p_A \in A_\varepsilon$,

$$1 = \mathrm{Ind}(\gamma_1 \circ \gamma_2 \dots \circ \gamma_q, p_A) = \sum_{i=1}^{q} \mathrm{Ind}(\gamma_i, p_A)\,, \tag{2.8}$$

where we have denoted by $\mathrm{Ind}(\gamma, p)$ the index of the loop γ with respect to the point p and by $\gamma \circ \sigma$ the concatenation of two curves γ and σ. From (2.8) we infer that there exists a loop

γ_Q such that $\mathrm{Ind}(\gamma_Q, p_A) = 1$ and, as p_B belongs to the unbounded connected component of $\mathbb{R}^2$ described by γ_Q, we have also that $\mathrm{Ind}(\gamma_Q, p_B) = 0$.
Hence the loop γ_Q is such that $\gamma_Q \cap A = \emptyset$, $\gamma_Q \cap B = \emptyset$ and it has index one with respect to at most $m-1$ points of S and index zero with respect to at least one point of S. This implies that γ_Q is crossing the cuts Σ and Σ' at least once and therefore $p^{-1}(\gamma_Q)$ is a closed loop in the covering space Y_Σ. Indeed tracking the path $p^{-1}(\gamma_Q)$ in Y_Σ one can see that for every crossing of the cuts Σ and Σ' in the covering space the path is continuing to the next sheet (following the identification given in (2.2)). In doing so, $p^{-1}(\gamma_Q)$ is visiting all the sheets of Y_Σ closing back at the starting point.
Additionally there exists an open ε-tubular neighborhood of γ_Q such that $(\gamma_Q)_\varepsilon \cap A = \emptyset$ and $(\gamma_Q)_\varepsilon \cap B = \emptyset$. We infer that $p^{-1}((\gamma_Q)_\varepsilon)$ is path connected. Indeed given two points $q_1, q_2 \in p^{-1}((\gamma_Q)_\varepsilon)$ it is enough to connect them to $p^{-1}(\gamma_Q)$ with a path that do not intersect the cuts and use that $p^{-1}(\gamma_Q)$ is a closed loop in Y_Σ. This is a contradiction with Lemma 2.20.

□

The next theorem is a regularity result for $p(\partial^* E)$ when $E \in \mathscr{P}_{constr}(Y_\Sigma)$ is a minimizer of Problem (1.2). To prove this theorem our strategy is to establish *locally* an equivalence between Problem (1.2) and the partition problem [3] and then use the known regularity results for it (see, for instance, [26, 28]).

Definition 2.23 (Local minimizer)**.** Given A an open subset of Y_Σ, we say that $E \in \mathscr{P}_{constr}(Y_\Sigma)$ is a local minimizer for Problem (1.2) in A if for every $F \in \mathscr{P}_{constr}(Y_\Sigma)$ such that $E \Delta F \subset\subset A$, there holds

$$P(E, A) \leq P(F, A)\,.$$

Consider the m–regular simplex in $\mathbb{R}^{m+1}$ centred in the origin and call $\{\alpha_1 \dots \alpha_m\}$ the vertices of the simplex and call $BV(A, \{\alpha_1 \dots \alpha_m\})$ the space of BV–functions with values in $\{\alpha_1 \dots \alpha_m\}$.

Definition 2.24 (Local minimizer for the partition problem)**.** Given A an open bounded subset of $\mathbb{R}^2$, we say that $u \in BV(A, \{\alpha_1 \dots \alpha_m\})$ is a local minimizer for the partition problem in A if for every $w \in BV(A, \{\alpha_1 \dots \alpha_m\})$ such that $\{u \neq w\} \subset\subset A$, there holds

$$|Du|\,(A) \leq |Dw|\,(A)\ .$$

Given a set $E \in \mathscr{P}_{constr}(Y_\Sigma)$, consider $x \in \overline{p(\partial^* E)}$ such that $x \notin \Sigma \cup \Sigma'$ and $r > 0$ small enough such that $B_r(x) \cap (\Sigma \cup \Sigma') = \emptyset$. The associated vector valued function u^α in $BV(B_r(x), \{\alpha_1 \dots \alpha_m\})$ is canonically defined as

$$u^\alpha(x) \coloneqq \alpha_j \quad \text{if } x \in E^j \cap B_r(x)\,,$$

for $j = 1, \dots, m$. Notice that by construction there holds

$$p(\partial^* E) \cap B_r(x) = J_{u^\alpha}\,, \tag{2.9}$$

where J_{u^α} is the jump set of u^α.

Lemma 2.25. *Suppose that $E_{\min} \in \mathscr{P}_{constr}(Y_\Sigma)$ is a local minimizer for Problem (1.2) in $p^{-1}(B_r(x))$ with $x \in \overline{p(\partial^* E)}$ and $r > 0$ such that $B_r(x) \cap (\Sigma \cup \Sigma') = \emptyset$. Then the associated vector valued function $u^\alpha_{\min}$ in $BV(B_r(x), \{\alpha_1 \dots \alpha_m\})$ is a local minimizer for the partition problem in $p^{-1}(B_r(x))$.*

Proof. Consider any $w \in BV(B_r(x), \{\alpha_1 \dots \alpha_m\})$ such that $\{u^\alpha_{\min} \neq w\} \subset\subset B_r(x)$. We associate to w a set F in $\mathscr{P}_{constr}(Y_\Sigma)$ defining its characteristic function as

$$\chi_{F^j}(x) := \begin{cases} 1 & \text{if } w(x) = \alpha_j\,, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \dots, m$ (see Remark 2.8). By construction we have that $J_w = p(\partial^* F) \cap B_r(x)$. Then applying Remark 2.19 and (2.9) we obtain

$$\begin{aligned} &2|Du^\alpha_{\min}|(B_r(x)) = 2\mathcal{H}^1\left(J_{u^\alpha_{\min}}\right) \\ &= 2\mathcal{H}^1\left(p(\partial^* E_{\min}) \cap B_r(x)\right) = P(E_{\min}, p^{-1}(B_r(x))) \\ &\leq P(F, p^{-1}(B_r(x))) = 2\mathcal{H}^1\left(p(\partial^* F) \cap B_r(x)\right) \\ &= 2\mathcal{H}^1\left(J_w\right) = 2|Dw|(B_r(x))\,, \end{aligned}$$

hence

$$|Du^\alpha_{\min}|(B_r(x)) \leq |Dw|(B_r(x))\,.$$

□

Remark 2.26. We have exhibited a way to pass *locally* from our minimization problem in the covering space to a problem of minimal partition in $\mathbb{R}^2$ for (1.2). This is enough for our aim, that is obtaining the regularity of the minimizer E, but clearly it is possible, with a similar procedure, to show that given a local minimizer for the minimal partition problem, then the associated set $E \in \mathscr{P}_{constr}(Y_\Sigma)$ is a local minimizer for (1.2). We underline that in general the equivalence between the partition problem and Problem (1.2) does not hold *globally*.

Theorem 2.27 (Regularity)**.** *Given $E_{min} \in \mathscr{P}_{constr}(Y_\Sigma)$ a minimizer of Problem (1.2), then there holds*

$$\mathcal{H}^1(\overline{p(\partial^* E_{\min})} \setminus p(\partial^* E_{\min})) = 0\,. \tag{2.10}$$

Moreover $\overline{p(\partial^* E_{\min})}$ is a finite union of segments meeting at triple junctions with angles of 120 degrees.

Proof. Consider $x \in \overline{p(\partial^* E)}$ and suppose without loss of generality (thanks to Remark 2.15) that $x \notin \Sigma \cup \Sigma'$; then there exists $r > 0$ such that $B_r(x) \cap (\Sigma \cup \Sigma') = \emptyset$ and $E_{\min} \in \mathscr{P}_{constr}(Y_\Sigma)$ is a local minimizer for (1.2) in $p^{-1}(B_r(x))$. Then by Lemma 2.25 the associated function $u^\alpha_{\min}$ is a local minimizer for the partition problem. Thanks to the regularity results for the local partition problem (see [28, Theorem 4.7]) we have that $\mathcal{H}^1(\overline{p(\partial^* E_{\min})} \setminus p(\partial^* E_{\min})) = 0$. Moreover $\overline{p(\partial^* E_{\min})}$ inherits all the regularity properties of the minimum of the partition problem, namely the set $\overline{p(\partial^* E_{\min})}$ is finite union of segments meeting at triple junctions with angles of 120 degree. □

2.5 Proof of the equivalence

In this section we prove that the minimization problem (1.2) is equivalent to the Steiner problem in $\mathbb{R}^2$.

First of all we need to prove that, given a solution of the Steiner problem for S, we can find a set $E \in \mathscr{P}_{constr}(Y_\Sigma)$ such that $\overline{p(\partial^* E)}$ is the Steiner network. We prove this statement for a smaller class of network, namely for the connected networks without loops. This result will be used again in Section 4.

Definition 2.28. A connected network is a finite union of C^0 injective curves that intersect each other only at their end points. We say that a connected network $\mathscr{S}$ connects the points of S if $S \subset \mathscr{S}$ and the end points of the curves of $\mathscr{S}$ either have order one and are points of $\mathscr{S}$ or have order greater or equal than one. In the first case the end points are called leaves, in the latter case they are called multipoint. We call $L \subset \mathscr{S}$ the set of all leaves of $\mathscr{S}$

Proposition 2.29. *Consider $S = \{p_1, \dots, p_m\}$ and $\mathscr{S}$ a connected network without loops that connects the m points of S. Then, for an appropriate relabeling of the points of S there exists an admissible pair of cuts $\hat{\mathbf{\Sigma}} \in \boldsymbol{Cuts}(S)$ and a set $E_{\mathscr{S}} \in \mathscr{P}_{constr}(Y_{\hat{\Sigma}})$ such that $\overline{p(\partial^* E_{\mathscr{S}})} = \mathscr{S}$.*

Proof. First of all we prove that, up to a permutation of the labelling of the point of S, there exists an admissible pair of cuts $\hat{\mathbf{\Sigma}} \in \mathbf{Cuts}(S)$ such that $\hat{\mathbf{\Sigma}} \cap \mathscr{S} = S$.
We notice that in order to prove the previous claim it is sufficient to find $\hat{\Sigma} \in Cuts(S)$ such that $\hat{\Sigma} \cap \mathscr{S} = S$. Then by a continuous deformation it is immediate to construct $\hat{\mathbf{\Sigma}} \in \mathbf{Cuts}(S)$ with $\hat{\mathbf{\Sigma}} \cap \mathscr{S} = S$. We build $\hat{\Sigma}$ in a constructive way. We remind that from Definition 2.28 follows that the set S can be written as $L \cup \mathcal{M} := \{\ell_1, \dots, \ell_h\} \cup \{m_1, \dots, m_k\}$ where L is the set of leaves and $\mathcal{M}$ is a subset (possibly empty) of the set of all the multipoints of $\mathscr{S}$ and $m = h + k$. The first step of our construction is the following: fix $\ell_1 \in L$ and follow the network $\mathscr{S}$ with the rule that at every multipoint we proceed along the closest curve with respect to the clockwise rotation. As the network is without loops, this procedure ends in a leaf $\ell_2 \neq \ell_1$. We call N_{ℓ_1} the subnetwork described by the just defined procedure, N_{ℓ_1} contains ℓ_1, ℓ_2 and possibly some points of $\mathcal{M}$, let us say m_i with $i \in \{1, \dots, j\}, j \leq k$. Then there exists a Lipschitz curve Σ_1 that connects ℓ_1 to m_1, for $i \in \{1, \dots, j-1\}$ there exist Lipschitz curves Σ_{i+1} that connects m_i to m_{i+1} and Σ_{j+1} a Lipschitz curve from m_j to ℓ_2. Moreover we can choose all the Lipschitz curves in such a way that they do not intersect $\mathscr{S} \setminus S$ (for instance they can be obtained by continuous deformation of the subnetwork N_{ℓ_1}). Step 2 to step h of the procedure are nothing else than an iteration of the procedure of the first step, starting from ℓ_i with $i \in \{2, \dots, h-1\}$ with the extra requirement that at the $n-th$ step one ones does not connect with Lipschitz curves the points m_i, already visited in the steps 1 to $n-1$ (see Figure 2). This will produce a family of Lipschitz curves $\{\Sigma_i\}_{i=1,\dots,m-1}$ connecting the points of S and not intersecting $\mathscr{S} \setminus S$. Then $\hat{\Sigma} = \cup_{i=1}^{m-1} \Sigma_i$ is the desired set of cuts.
Now we describe how to associate to $\mathscr{S}$ a set $E_{\mathscr{S}}$ in the covering space $Y_{\hat{\Sigma}}$. For $j = 1, \dots, m-1$ the set $E_{\mathscr{S}}^{m+1-j}$ is defined as the open set such that its boundary is composed by Σ_j and the part of $\mathscr{S}$ connecting $\tilde{p}_j$ and $\tilde{p}_{j+1}$ and $E_{\mathscr{S}}^1 = \mathbb{R}^2 \setminus \cup_{j=1}^{m-1} E_{\mathscr{S}}^{m+1-j}$. Thanks to Remark 2.8 the set $E_{\mathscr{S}}$ is well defined. By construction it is trivial that $E_{\mathscr{S}}$ satisfies the constraints of Definition 2.13 and that $\overline{p(\partial^* E_{\mathscr{S}})} = \mathscr{S}$. □

Remark 2.30. The choice of E^j in the previous construction is not arbitrary: if one chooses differently the sets E^j, one obtains a different set with perimeter greater than the perimeter of $E_{\mathscr{S}}$.

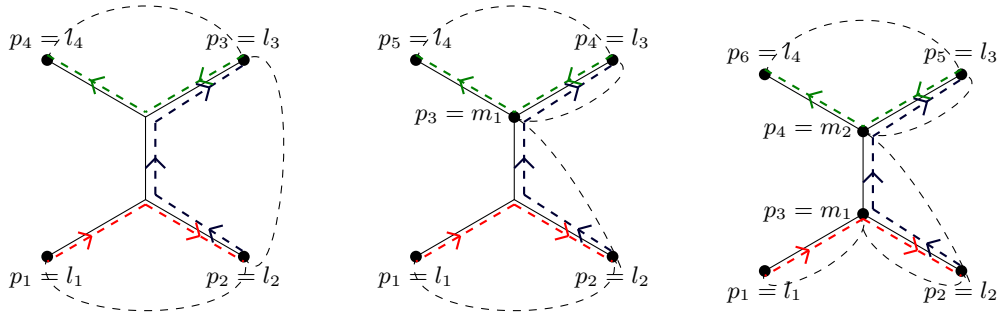


Figure 2: Examples of three possible situations occouring in Proposition 2.29

Theorem 2.31. *The Steiner problem and Problem (1.2) are equivalent. In particular if $E \in \mathscr{P}_{constr}(Y_\Sigma)$ is a minimizer of Problem (1.2), then $\overline{p(\partial^* E)}$ is a solution of the Steiner problem and if $\mathscr{S}$ is a minimizer for the Steiner problem, then its associated set $E_{\mathscr{S}}$ (constructed as in Proposition 2.29) is a minimizer for Problem (1.2).*

Proof. Consider E a minimizer for Problem (1.2) in Y_Σ and $\mathscr{S}$ a minimizer for the Steiner problem. Thanks to Proposition 2.29 there exists an admissible pair of cuts $\hat{\mathbf{\Sigma}} \in \mathbf{Cuts}(S)$ and a set $E_{\mathscr{S}} \in \mathscr{P}_{constr}(Y_{\hat{\Sigma}})$ such that $\overline{p(\partial^* E_{\mathscr{S}})} = \mathscr{S}$. Thanks to Remark 2.15 Problem (1.2) is independent on the choice of the cuts. Therefore

$$P(E, Y_\Sigma) \leq P(E_{\mathscr{S}}, Y_{\hat{\Sigma}}) \,. \tag{2.11}$$

Using Proposition 2.18 and inequality (2.11) we have

$$\mathcal{H}^1(p(\partial^* E)) = \frac{1}{2} P(E, Y_\Sigma) \leq \frac{1}{2} P(E_{\mathscr{S}}, Y_{\hat{\Sigma}}) = \mathcal{H}^1(p(\partial^* E_{\mathscr{S}})) \leq \mathcal{H}^1(\mathscr{S}) \,. \tag{2.12}$$

Thanks to Theorem 2.27 we know that

$$\mathcal{H}^1(\overline{p(\partial^* E)}) = \mathcal{H}^1(p(\partial^* E)) \,. \tag{2.13}$$

Thanks to Corollary 2.21 we have $S \subset \overline{p(\partial^* E)}$, and by Proposition 2.22 there exists a connected component $\mathcal{C}_E$ of $\overline{p(\partial^* E)}$ that contains S. Hence $\mathcal{C}_E$ is a competitor for the Steiner problem for S in $\mathbb{R}^2$, therefore from the minimality of $\mathscr{S}$ we get

$$\mathcal{H}^1(\mathscr{S}) \leq \mathcal{H}^1(\mathcal{C}_E) \leq \mathcal{H}^1(\overline{p(\partial^* E)}) \,. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) we have

$$\mathcal{H}^1(\overline{p(\partial^* E)}) = \mathcal{H}^1(\mathscr{S}) \,.$$

We obtain as well that $\mathcal{H}^1(\mathcal{C}_E) = \mathcal{H}^1(\overline{p(\partial^* E)})$; therefore using again Theorem 2.27 we infer that $\overline{p(\partial^* E)}$ is connected. The set $\overline{p(\partial^* E)}$ is a connected set that joints the point of S and (by the minimality of $\mathscr{S}$) such that, for every connected set $\mathscr{T} \subset \mathbb{R}^2$ that connects the point of S

$$\mathcal{H}^1(\overline{p(\partial^* E)}) = \mathcal{H}^1(\mathscr{S}) \leq \mathcal{H}^1(\mathscr{T}) \,.$$

Hence $\overline{p(\partial^* E)}$ is a minimizer for the Steiner problem. On the other hand the constrained set $E_{\mathscr{S}}$ has the same perimeter of E, hence is a solution of Problem (1.2). □

3 Calibrations

In this section we introduce the notion of calibration for the minimum problem (1.2) and we show some explicit examples. In doing so, it is often convenient to consider vector fields that are not continuous and for which a divergence theorem still holds. For this reason we employ the notion of *approximately regular vector field* (in a slightly stronger version than in [1]) and then we generalize it to the covering space setting.

Definition 3.1 (Approximately regular vector fields on $\mathbb{R}^n$)**.** Given $A \subset \mathbb{R}^n$, a Borel vector field $\Phi : A \to \mathbb{R}^n$ is approximately regular if it is bounded and for every Lipschitz hypersurface $\mathscr{M}$ in $\mathbb{R}^n$, Φ admits traces on $\mathscr{M}$ on the two sides of $\mathscr{M}$ (denoted by Φ^+ and Φ^-) and

$$\Phi^+(x) \cdot \nu_{\mathscr{M}}(x) = \Phi^-(x) \cdot \nu_{\mathscr{M}}(x) = \Phi(x) \cdot \nu_{\mathscr{M}}(x), \tag{3.1}$$

for $\mathcal{H}^{n-1}$–a.e. $x \in \mathscr{M} \cap A$.

Definition 3.2 (Approximately regular vector fields on Y_Σ)**.** Given $\Phi : Y_\Sigma \to \mathbb{R}^2$, we say that it is *approximately regular* in Y_Σ if Φ^j and $\Phi^{j'}$ (see Definition 2.6) are *approximately regular* for every $j = 1, \dots, m$ and $j' = m+1, \dots, 2m$.

Definition 3.3 (Calibration on coverings)**.** Given $E \in \mathscr{P}_{constr}(Y_\Sigma)$, a calibration for E (with respect to the minimum problem (1.2)) is an *approximately regular* vector field $\Phi : Y_\Sigma \to \mathbb{R}^2$ such that:

(**1**) $\operatorname{div} \Phi = 0$ (in the sense of the distributions);

(**2**) $|\Phi^i(x) - \Phi^j(x)| \leq 2$ for every $i, j = 1, \dots m$ and for every $x \in D$;

(**3**) $\int_{Y_\Sigma} \Phi \cdot D\chi_E = P(E)$.

Remark 3.4. At first sight the size condition (**2**) may sounds different in comparison with the classical notion of paired calibration. This difference is only apparent. Indeed we choose to minimize $P(E)$ that thanks to Proposition 2.18 is equal to the double of the length of the minimal network on to the base space.

Proposition 3.5 (Divergence theorem on coverings)**.** *Consider $E, F \in \mathscr{P}_{constr}(Y_\Sigma)$ and let $\Phi : Y_\Sigma \to \mathbb{R}^2$ be an approximately regular vector field such that* $\operatorname{div} \Phi = 0$ *(in the sense of the distributions) in Y_Σ. Then*

$$\int_{Y_\Sigma} \Phi \cdot D\chi_E = \int_{Y_\Sigma} \Phi \cdot D\chi_F. \tag{3.2}$$

Proof. See Appendix 4.1. □

In the following theorem we prove that our notion of calibration is indeed meaningful, in the sense that the existence of a calibration for a given $E \in \mathscr{P}_{constr}(Y_\Sigma)$ implies the minimality of E for Problem (1.2).

Theorem 3.6. *If $\Phi : Y_\Sigma \to \mathbb{R}^2$ is a calibration for E, then E is a minimizer of Problem* (1.2).

Proof. Let $\Phi : Y_\Sigma \to \mathbb{R}^2$ be a calibration for $E \in \mathscr{P}_{constr}(Y_\Sigma)$ and let $F \in \mathscr{P}_{constr}(Y_\Sigma)$ a competitor. By Proposition 3.5 and (**1**) of Definition 3.3 we have

$$\int_{Y_\Sigma} \Phi \cdot D\chi_E = \int_{Y_\Sigma} \Phi \cdot D\chi_F \tag{3.3}$$

and thanks to property (**3**) of Definition 3.3

$$P(E) = \int_{Y_\Sigma} \Phi \cdot D\chi_E \,. \tag{3.4}$$

Moreover, using (2.5) we have

$$\begin{aligned}
\int_{Y_\Sigma} \Phi \cdot D\chi_F &= \sum_{j=1}^{m} \int_D \Phi^j \cdot D\chi_{F^j} + \sum_{j'=m+1}^{2m} \int_\Sigma \Phi^{j'} \cdot D\chi_{F^{j'}} \\
&= \sum_{j=1}^{m} \int_{\partial^* F^j \cap D} \Phi^j \cdot \nu_{F^j} \, d\mathcal{H}^1 + \sum_{j'=m+1}^{2m} \int_{\partial^* F^{j'} \cap \Sigma} \Phi^{j'} \cdot \nu_{F^{j'}} \, d\mathcal{H}^1 \\
&= \int_{p(\partial^* F)\cap D} \sum_{j=1}^{m} \Phi^j \cdot \nu_{F^j} \chi_{\partial^* F^j} \, d\mathcal{H}^1 + \int_{p(\partial^* F)\cap \Sigma} \sum_{j'=m+1}^{2m} \Phi^{j'} \cdot \nu_{F^{j'}} \chi_{\partial^* F^{j'}} \, d\mathcal{H}^1 \\
&\leq \int_{p(\partial^* F)\cap D} \Big| \sum_{j=1}^{m} \Phi^j \cdot \nu_{F^j} \chi_{\partial^* F^j} \Big| \, d\mathcal{H}^1 + \int_{p(\partial^* F)\cap \Sigma} \Big| \sum_{j'=m+1}^{2m} \Phi^{j'} \cdot \nu_{F^{j'}} \chi_{\partial^* F^{j'}} \Big| \, d\mathcal{H}^1 .
\end{aligned}$$

As $F \in \mathscr{P}_{constr}(Y_\Sigma)$, for $\mathcal{H}^1$–a.e. $x \in p(\partial^* F) \cap D$ there exist exactly two distinct indices $j_1, j_2 \in \{1, \dots, m\}$ such that $x \in \partial^* F^{j_1} \cap \partial^* F^{j_2}$ and $\nu_{F^{j_1}} = -\nu_{F^{j_2}}$. Therefore using condition (**2**) of Definition 3.3 and the usual identifications given by $\sim$ we get that

$$\int_{Y_\Sigma} \Phi \cdot D\chi_F \leq 2\mathcal{H}^1(p(\partial^* F)) = P(F), \tag{3.5}$$

where the last equality follows from Proposition 2.18.
Combining Equations (3.3), (3.4) and (3.5) one obtains

$$P(E) = \int_{Y_\Sigma} \Phi \cdot D\chi_E = \int_{Y_\Sigma} \Phi \cdot D\chi_F \leq P(F) \,.$$

□

Remark 3.7. Given $\Phi : Y_\Sigma \to \mathbb{R}^2$ a calibration for $E \in \mathscr{P}_{constr}(Y_\Sigma)$, then for every $c \in \mathbb{R}^2$ we have that $\Phi + c$ is a calibration for E. Indeed if Φ is a calibration for $E \in \mathscr{P}_{constr}(Y_\Sigma)$ then it is easy to see that properties (**1**) and (**2**) hold for $\Phi + c$ as well. It remains to show that if $\int_{Y_\Sigma} \Phi \cdot D\chi_E = P(E)$, then

$$\int_{Y_\Sigma} (\Phi + c) \cdot D\chi_E = P(E) \,,$$

that is that for every $c \in \mathbb{R}^2$ we have $\int_{Y_\Sigma} c \cdot D\chi_E = 0$. Following the computation in the proof of Theorem 3.5 we have that

$$\int_{Y_\Sigma} c \cdot D\chi_E = \int_{\partial\Omega} c \cdot \nu_{\partial\Omega} \, d\mathcal{H}^1 = 0 \,.$$

3.1 Examples of calibrations

We present here several examples of calibrations for Steiner configurations in the covering space setting. In the figures below the vector field is implicitly defined as the constant $\Phi = (0,0)$, where no arrows are drawn. Notice that, thanks to Definition 3.2, a calibration can admit discontinuities in the domain of definition provided that (3.1) is fulfilled.

Example 3.8 (Calibration for the segment).

In order to introduce the reader to the calibration method in our setting we start with the trivial example of the minimality of the segment. In particular we show that the set $E \in \mathscr{P}_{constr}(Y_\Sigma)$ defined in such a way that the closure of its essential boundary is the segment connecting p_1 and p_2 is the minimizer of Problem (1.2). We recall that in this case the number of sheets of the covering is two. We define E^1 (resp. E^2) as the coloured subset of $(D,1)$ (resp. $(D,2)$) in Figure 3 and the set E is obtained as explained in Remark 2.8.

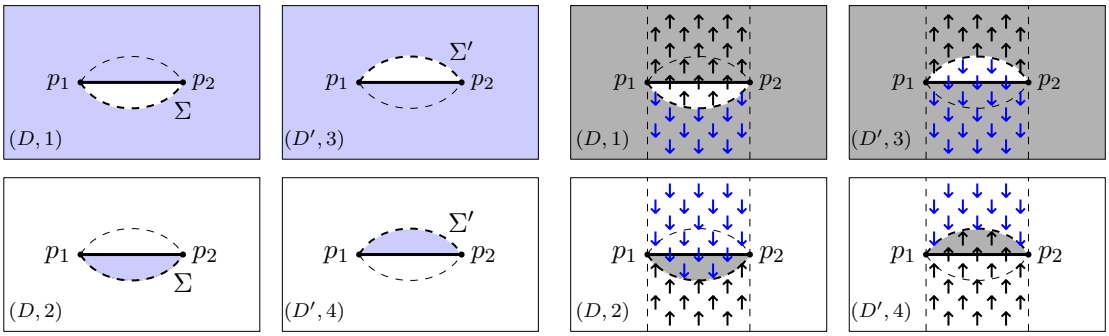


Figure 3: The candidate minimizer E and the vector field Φ

Let us denote by Q the dashed stripe in Figure 3, by A_1 the set enclosed by Q and the cut Σ and by A_2 the complement of A_1 with respect to Q. We define a vector field $\Phi : Y_\Sigma \to \mathbb{R}$ prescribing its parametrization on the sheets $(D,1)$ and $(D,2)$:

$$\Phi^1(x) = \begin{cases} (0,1) & x \in A_1 \\ (0,-1) & x \in A_2 \\ 0 & \text{otherwise} \end{cases} \qquad \Phi^2(x) = \begin{cases} (0,-1) & x \in A_1 \\ (0,1) & x \in A_2 \\ 0 & \text{otherwise} \end{cases}.$$

We verify that the unit vector field $\Phi : Y_\Sigma \to \mathbb{R}$ defined as in Figure 3 is a calibration of E. First notice that Φ is an approximately regular divergence free vector field in Y_Σ. Indeed, as a consequence of the identifications in the construction of the covering space, Φ is constant in $p^{-1}(Q)$. Since Φ is a piecewise constant vector field satisfying (3.1), its distributional divergence of is zero. Condition (**2**) in Definition 3.3 is trivially satisfied. Finally

$$\begin{aligned} \int_{Y_\Sigma} \Phi \cdot D\chi_E &= \sum_{j=1}^{2} \int_D \Phi^j \cdot D\chi_{E^j} \\ &= \int_{p(\partial^* E)} \Phi^1 \cdot \nu_{E^1} d\mathcal{H}^1 + \int_{p(\partial^* E)} \Phi^2 \cdot \nu_{E^2} d\mathcal{H}^1 \\ &= 2\mathcal{H}^1(p(\partial^* E)) \\ &= P(E) \end{aligned}$$

where we have used Lemma 2.11 and Proposition 2.18.
This shows, thanks to Theorem 3.6, that E is a minimizer for Problem (1.2).

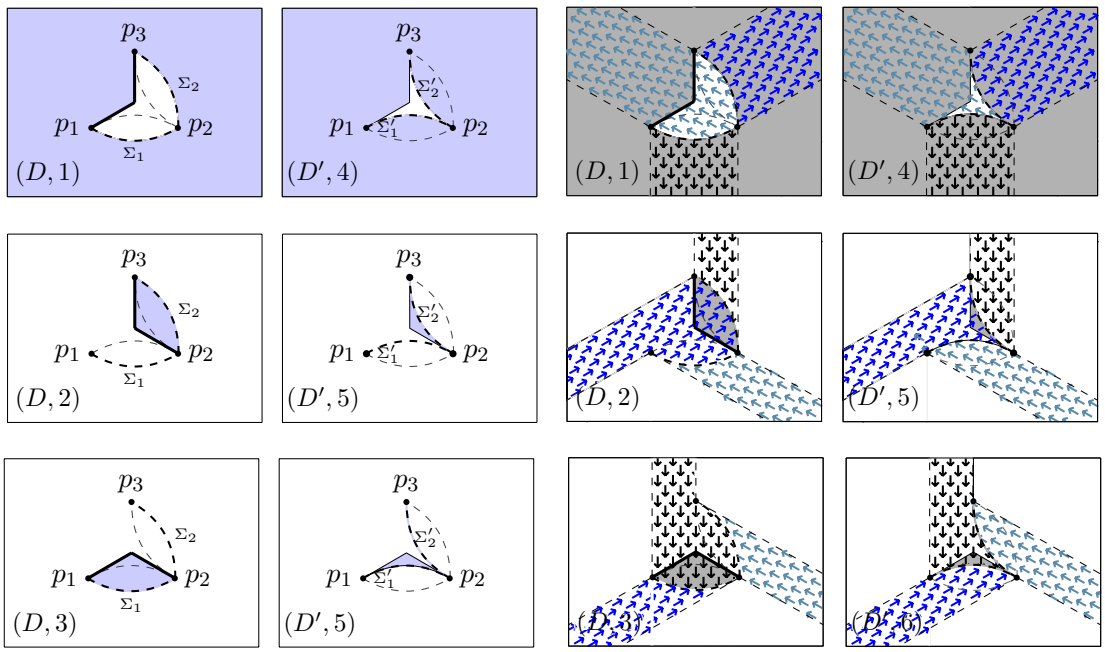


Figure 4: Minimizer and calibration for three points, located at the vertices of an equilater triangle

Example 3.9 (Calibration for three points).

Let us consider the case where S consists of three points p_1, p_2, p_3. First of all we focus our attention on the case in which the three points are the vertices of an equilateral triangle. We set without loss of generality $p_1 = (-\sqrt{3}/2, -1/2)$, $p_2 = (\sqrt{3}/2, -1/2)$ and $p_3 = (0, 1)$.
The set $E_{min} \in \mathscr{P}_{constr}(Y_\Sigma)$ (constructed from the minimal triple junction connecting p_1, p_2, p_3 following the procedure of Proposition 2.29) is colored in Figure 4. We define Φ as in Figure 4. The vectors represented by the arrows are the following:

$$\Phi^1 = (-1, 1/\sqrt{3}), \quad \Phi^2 = (1, 1/\sqrt{3}), \quad \Phi^3 = (0, -2/\sqrt{3})\,. \tag{3.6}$$

It is easy to check that the conditions in Definition 3.3 are satisfied.

Remark 3.10. Notice that the calibration for three points p_1, p_2 and p_3 which are the vertices of a triangle with all angles of amplitude less or equal than 120 degrees is the same (up to a rotation and minor modifications of the extension outside the cuts and the convex envelope of the points) of the calibration for the equilateral triangle that we have just explicitly shown. Indeed in this case the minimal Steiner network is again the union of three segments (possibly with different lenghts) meeting in a triple junction with angles of 120 degrees.

Hence, it remains to consider the cases in which the three points of S form a triangle with one angle greater or equal than 120 degrees. For simplicity let $d(p_1, p_2) = d(p_2, p_3)$ and α be the angle between the segment $\overline{p_1p_2}$ (respectively $\overline{p_2p_3}$) and the horizontal line (as in Figure 5).

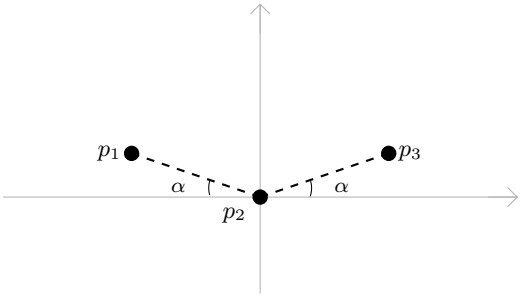


Figure 5: The set $S = \{p_1, p_2, p_3\}$ when the three points are the vertices of a triangle with one angle greater that 120 degrees.

It is well known that in this case the Steiner configuration that connects the points p_1, p_2, p_3 reduces to the two segments $\overline{p_1p_2}$ and $\overline{p_2p_3}$. Again we construct the set $E_{min} \in \mathscr{P}_{constr}(Y_\Sigma)$ from the minimal Steiner configuration following the procedure of Proposition 2.29.

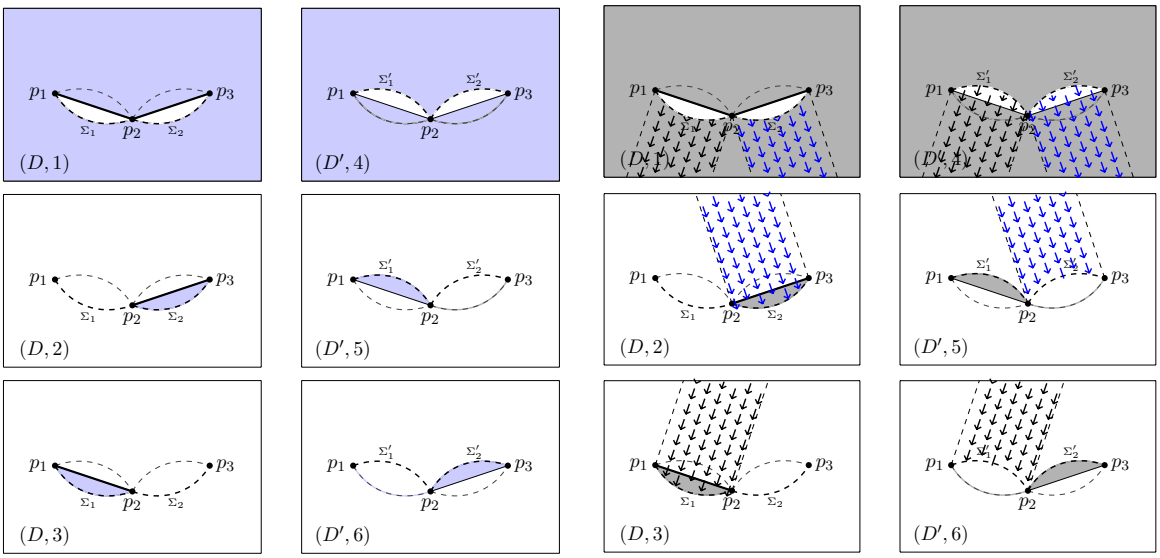


Figure 6: Minimizer and calibration for three points

The calibration, depending on the fixed angle $\alpha \in (0, \pi/6)$, is the following (see Figure 6):

$$\Phi^1 = (0, 0), \quad \Phi^2 = (2\sin\alpha, -2\cos\alpha), \quad \Phi^3 = (-2\sin\alpha, -2\cos\alpha)\,. \tag{3.7}$$

Remark 3.11. For $\alpha = \pi/6$ the calibration (3.7) coincides, up to a rotation and a translation (see Remark 3.7), to the one for the triangle (3.6).

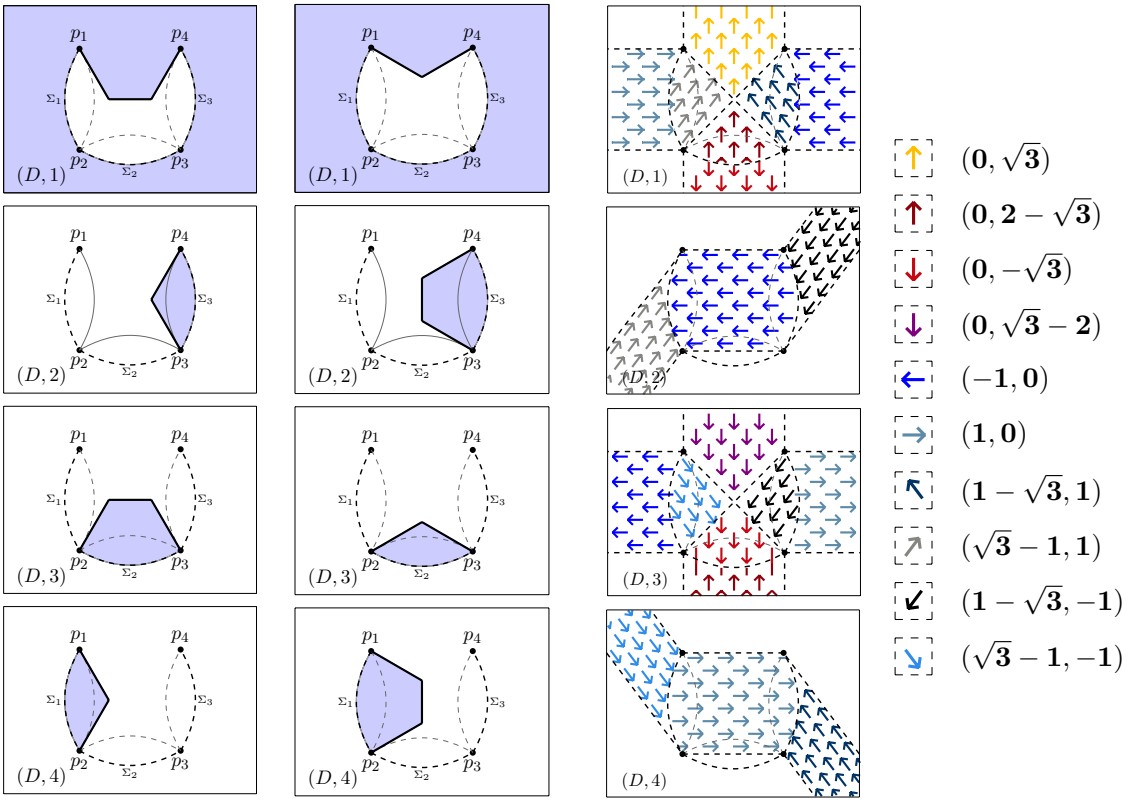


Figure 7: Minimizers and calibration for the vertices of a square

Example 3.12 (Calibration for the vertices of a square).
Given $S = \{p_1, p_2, p_3, p_4\}$ located at the vertices of a square, the Steiner problem does not admit a unique solution. Therefore a calibration (if it exists) must calibrate all the minimizers. The two candidates minimizers $E_{\min,1}, E_{\min,2}$ for Problem (1.2) are shown in Figure 7 on the left (we draw only the sheets (D, i) for $i = 1, \dots, 4$). The calibration for $E_{min,1}$ and $E_{min,2}$ is defined as in Figure 7 on the right. Again, we draw the vector field only in the sheets (D, i) for $i = 1, \dots, 4$ and we employ the usual convention that where the vector field is not written it is equal to $(0, 0)$. The reader can easily verify that the vector field defined in this way is a calibration on Y_Σ for both the minimizers $E_{min,1}$ and $E_{min,2}$.

4 Calibrations in families

The aim of this section is to introduce a weaker definition of calibration. The idea is finding a way to divide the set of competitors in different families defining an appropriate notion of calibration in each family with a weaker condition (**2**). Then we calibrate the minimizers of each family separately and in conclusion we compare the energy of the minimizers to find the explicit solutions of Problem (1.2).

Definition 4.1. Let $\mathcal{J} \subset \{1, \dots, m\} \times \{1, \dots, m\}$ be a subset of the Cartesian product of the indices. Given $E \in \mathscr{P}_{constr}(Y_\Sigma)$ we define

$$E^{i,j} := \partial^* E^i \cap \partial^* E^j$$

and

$$\mathcal{F}(\mathcal{J}) := \{E \in \mathscr{P}_{constr}(Y_\Sigma) : \mathcal{H}^1(E^{i,j}) = 0 \text{ for every } (i,j) \in \mathcal{J}\}. \tag{4.1}$$

Definition 4.2 (Calibration in the family $\mathcal{F}(\mathcal{J})$)**.** Given $E \in \mathscr{P}_{constr}(Y_\Sigma)$, a calibration for E in $\mathcal{F}(\mathcal{J})$ is an *approximately regular* vector field $\Phi : Y_\Sigma \to \mathbb{R}^2$ such that

1. $\operatorname{div} \Phi = 0$ (in the sense of the distributions);
2. $|\Phi^i(x) - \Phi^j(x)| \leq 2$ for every $i, j = 1, \dots m$ such that $(i,j) \notin \mathcal{J}$ and for every $x \in D$;
3. $\displaystyle\int_{Y_\Sigma} \Phi \cdot D\chi_E = P(E)$.

Proposition 4.3. *Given $\mathcal{J}$ as above and $E \in \mathcal{F}(\mathcal{J})$, if $\Phi : Y_\Sigma \to \mathbb{R}^2$ is a calibration for E in the family $\mathcal{F}(\mathcal{J})$, then*

$$P(E) \leq P(F)$$

for every $F \in \mathcal{F}(\mathcal{J})$. In particular E minimizes the perimeter in the class $\mathcal{F}(\mathcal{J})$.

Proof. The proof is similar to that of Theorem 3.6, and it is omitted. □

We want to use Definition 4.2 to validate the minimality of a candidate minimizer for the Steiner problem. To this aim we need to assign any competitor in $\mathscr{P}_{constr}(Y_\Sigma)$ to at least one family. Notice that there exist sets $E \in \mathscr{P}_{constr}(Y_\Sigma)$ such that $\mathcal{H}^1(E^{i,j}) > 0$ for every couple of indices $(i,j) \in \{1, \dots, m\} \times \{1, \dots, m\}$, hence it is not possible to cover $\mathscr{P}_{constr}(Y_\Sigma)$ with non-trivial families $\mathcal{F}(\mathcal{J}_i)$. For this reason we restrict to $\mathscr{P}^T_{constr}(Y_\Sigma)$.

Definition 4.4. We call T the set of all connected networks without loops (see Definition 2.28). Moreover we call $\mathscr{P}^T_{constr}(Y_\Sigma)$ the set of all $E \in \mathscr{P}_{constr}(Y_\Sigma)$ such that $\overline{p(\partial^* E)}$ is an element of T.

Consider the following:

Problem 4.5. *Given S a finite sets of points in $\mathbb{R}^2$ we look for a network in T with minimal length that connects the points of S.*

It is well known that Problem 4.5 is equivalent to the Steiner problem defined in (1.1) (see, for instance [27]). Therefore if we define $\mathscr{A}^T_{constr}(S) = \inf\left\{P(E) : E \in \mathscr{P}^T_{constr}(Y_\Sigma)\right\}$ we infer, thanks to Theorem 2.31, that $\mathscr{A}^T_{constr}(S) = \mathscr{A}_{constr}(S)$.

Proposition 4.6. *Suppose that there exists $\mathcal{J}_1, \dots, \mathcal{J}_N \subset \{1, \dots, m\} \times \{1, \dots, m\}$ such that*

$$\mathscr{P}^T_{constr}(Y_\Sigma) \subseteq \bigcup_{i=1}^{N} \mathcal{F}(\mathcal{J}_i). \tag{4.2}$$

If for every $i = 1, \dots, N$ there exists a calibration Φ_i for E_i in $\mathcal{F}(\mathcal{J}_i)$, then

$$\mathscr{A}_{\text{constr}}(S) = \min\{P(E_i) : i = 1, \dots, N\}. \tag{4.3}$$

In other words the set E_i with less perimeter is the absolute minimizer of Problem (1.2).

Proof. Fix $F \in \mathscr{P}^T_{constr}(Y_\Sigma)$. Thanks to (4.2) there exists at least one $i \in \{1,\dots,N\}$ such that $F \in \mathcal{F}(\mathcal{J}_i)$. Proposition 4.3 implies that $P(E_i) \le P(F)$. Thus

$$\min\{P(E_i) : i = 1,\dots,N\} \le P(F)\,,$$

that is (4.3). □

Remark 4.7. In Subsection 4.1 we will see that it is relevant *how coarse* is the decomposition of $\mathscr{P}^T_{constr}(Y_\Sigma)$ in families $\mathcal{F}(\mathcal{J}_i)$. Indeed if the cardinality of $\mathcal{J}_i$ is small, then one needs less families to cover $\mathscr{P}^T_{constr}(Y_\Sigma)$, but the task of finding an explicit calibration in $\mathcal{F}(\mathcal{J}_i)$ results more challenging.

4.1 Examples

Suppose that the finite set $S = \{p_1,\dots,p_m\}$ consists of m points located on the boundary of a smooth, open, convex set $A \subset \mathbb{R}^2$. For simplicity we label the points on the boundary of A in a anticlockwise order and from now on we consider the indices $i = 1,\dots,m$ cyclically identified modulus m.
It is not restrictive to suppose that each competitor $\Gamma \in T$ is contained in A, so that T induces a partition of $A \setminus \Gamma$ in m connected sets $\{A^1_\Gamma,\dots,A^m_\Gamma\}$ labelled in such a way that $\{p_i, p_{i+1}\} \subset \partial A^{m+1-i}$. Calling $A^{i,j}_\Gamma := \partial A^i_\Gamma \cap \partial A^j_\Gamma$ for every $i,j = 1,\dots,m$, the Steiner problem can be rephrased as

$$\min\left\{\sum_{i<j} \mathcal{H}^1(A^{i,j}_\Gamma) : \Gamma \in T\right\}. \tag{4.4}$$

We now suggest a general and explicit way to cover $\mathscr{P}^T_{constr}(Y_\Sigma)$ with families $\mathcal{F}(\mathcal{J}_i)$ in order to use the notion of calibration in $\mathcal{F}(\mathcal{J}_i)$ and Proposition 4.6 to show explicit solution of Problem (1.2).

Lemma 4.8. *Consider $\Gamma \in T$ inducing the partition $\{A^1_\Gamma,\dots,A^m_\Gamma\}$ and suppose that there exists $i \neq j$ such that $A^{i,j}_\Gamma \neq \emptyset$. Then for every $0 \le k_1 < i < k_2 < j \le m$ (or $0 \le j < k_1 < i < k_2 \le m$) we have that*

$$\mathcal{H}^1(A^{k_1,k_2}_\Gamma) = 0\,.$$

Proof. By contradiction it is enough to notice that if $\mathcal{H}^1(A^{k_1,k_2}_\Gamma) > 0$, then the interior of $\overline{A^{k_1}_\Gamma \cup A^{k_2}_\Gamma}$ is an open connected set that separates A^i_Γ and A^j_Γ. Hence we infer $A^{i,j}_\Gamma = \emptyset$. □

We construct the covering space Y_Σ choosing an admissible pair of cuts in the following way: the cut Σ' coincides with ∂A and the cut Σ lies outside A. Then, thanks to Proposition 2.29, it is possible to associate to the network Γ, and hence to the partition $\{A^1_\Gamma,\dots,A^m_\Gamma\}$, a set E_Γ in the covering space Y_Σ (simply setting $E^j_\Gamma = A^j_\Gamma$) such that

$$p(\overline{\partial^* E_\Gamma}) = \Gamma = \bigcup_{i,j} A^{i,j}_\Gamma \quad \text{and} \quad E^{i,j}_\Gamma = A^{i,j}_\Gamma. \tag{4.5}$$

Thanks to (4.5) Lemma 4.8 is trivially true replacing $A^{i,j}_\Gamma$ with $E^{i,j}_\Gamma$.

We define

$$\mathcal{F}_{i,j} = \{\Gamma \in T : A_\Gamma^{i,j} \neq \emptyset\}\,.$$

It is easy to see that T can be covered in the following way:

$$T = \bigcup_{k=2}^{\lfloor \frac{m}{2} \rfloor} \bigcup_{|i-j|=k} \mathcal{F}_{i,j}\,. \tag{4.6}$$

This covering induces automatically a covering of $\mathscr{P}_{constr}(Y_\Sigma)$. Consider for instance a family $\mathcal{F}_{i,i+\lfloor \frac{m}{2} \rfloor}$ for a fixed i in $\{1,\dots,m\}$. Thanks to Lemma 4.8 we have that

$$\mathcal{H}^1(E_\Gamma^{k,l}) = 0$$

for all $(k,l) \in \{i+1,\dots,i+\lfloor \frac{m}{2} \rfloor - 1\} \times \{i+\lfloor \frac{m}{2} \rfloor + 1, i-1\}$. This property defines a family $\mathcal{F}(\mathcal{J})$ according to (4.1) with $\mathcal{J} = \{i+1,\dots,i+\lfloor \frac{m}{2} \rfloor - 1\} \times \{i+\lfloor \frac{m}{2} \rfloor + 1, i-1\}$.

Example 4.9 (Calibration for the vertices of the regular pentagon).

Consider $S = \{p_1,p_2,p_3,p_4,p_5\}$ located at the vertices of a regular pentagon. First we divide the elements of $\mathscr{P}^T_{constr}(Y_\Sigma)$ in families. According to (4.6), we cover T with five families $\mathcal{F}_{i,j}$ as follows:

$$T = \bigcup_{|i-j|=2} \mathcal{F}_{i,j}\,.$$

Then we split again each family $\mathcal{F}_{i,j}$ in two subfamilies $\mathcal{F}^1_{i,j}$ and $\mathcal{F}^2_{i,j}$ defined as

$$\mathcal{F}^1_{i,j} = \mathcal{F}_{i,j} \cap \mathcal{F}_{i,j+1} \quad \text{and} \quad \mathcal{F}^2_{i,j} = \mathcal{F}_{i,j} \cap \mathcal{F}_{i-1,j}\,.$$

This produces in principle 10 families, but it is easy to see that $\mathcal{F}^k_{i,j} = \mathcal{F}^{k'}_{i',j'}$ for some $i,j,i',j' \in \{1,2,3,4,5\}$ and $k,k' \in \{1,2\}$. Hence, we obtain

$$\mathscr{P}^T_{constr}(Y_\Sigma) = \bigcup_{i=1}^{5} \mathcal{F}(\mathcal{J}_i)\,,$$

with

$$\mathcal{J}_1 = \{(1,3),(1,4),(2,4)\}, \quad \mathcal{J}_2 = \{(1,3),(1,4),(3,5)\}, \quad \mathcal{J}_3 = \{(1,3),(2,5),(3,5)\},$$
$$\mathcal{J}_4 = \{(1,4),(2,4),(2,5)\}, \quad \mathcal{J}_5 = \{(2,4),(2,5),(3,5)\}\,.$$

It is known that the Steiner problem for S has 5 minimizers $\mathscr{S}_i$ for $i = 1,\dots,5$ (obtained by rotation one from the other). Denoted by $E_{min,i} \in \mathscr{P}^T_{constr}(Y_\Sigma)$ for $i = 1,\dots,5$ the sets associated with the minimizers of the Steiner problem, it is easy to see that that $E_{min,i} \in \mathcal{F}(\mathcal{J}_i)$ for $i = 1,\dots,5$. Our aim is to prove that $E_{min,i}$ is a minimizer in $\mathcal{F}(\mathcal{J}_i)$ constructing a vector field Φ_i that is a calibration for $E_{min,i}$ in $\mathcal{F}(\mathcal{J}_i)$.
On the left of Figure 8 is shown the set $E_{min,5} \in \mathscr{P}^T_{constr}(Y_\Sigma)$, on the right a calibration for $E_{min,5}$ in $\mathcal{F}(\mathcal{J}_5)$. The vector field represented by the arrows is the following:

$$\Phi^1 = (0,0), \quad \Phi^2 = (2,0), \quad \Phi^3 = (1,-\sqrt{3}), \quad \Phi^4(-1,-\sqrt{3}), \quad \Phi^5 = (-2,0)$$

and it is easy to verify that it is indeed a calibration for $E_{min,5}$ in $\mathcal{F}(\mathcal{J}_5)$.

As the minimizers $\mathscr{S}_i$ with $i = 1, \dots, 5$ for the Steiner problem are obtained by rotation one from the other, it is easy to construct for $E_{min,i}$ (with $i = 1, 2, 3, 4$) a calibration in $\mathcal{F}(\mathcal{J}_i)$ similar to the one for $E_{min,5}$ in $\mathcal{F}(\mathcal{J}_5)$.
To summarize we have split the set $\mathscr{P}^T_{constr}(Y_\Sigma)$ and we have exhibited a calibration in each family for the corresponding $E_{min,i}$. As $\mathcal{H}^1(\mathscr{S}_i) = \mathcal{H}^1(\mathscr{S}_j)$ for $i, j \in \{1, \dots, 5\}$, thanks to Proposition 2.18 we have also that $P(E_{min,i}) = P(E_{min,j})$ for every $i, j \in \{1, \dots, 5\}$. Thus applying Proposition 4.6 we infer that $E_{min,i}$ are minimizers of Problem (1.2) for every $i = 1 \dots, 5$, as we wanted to prove.

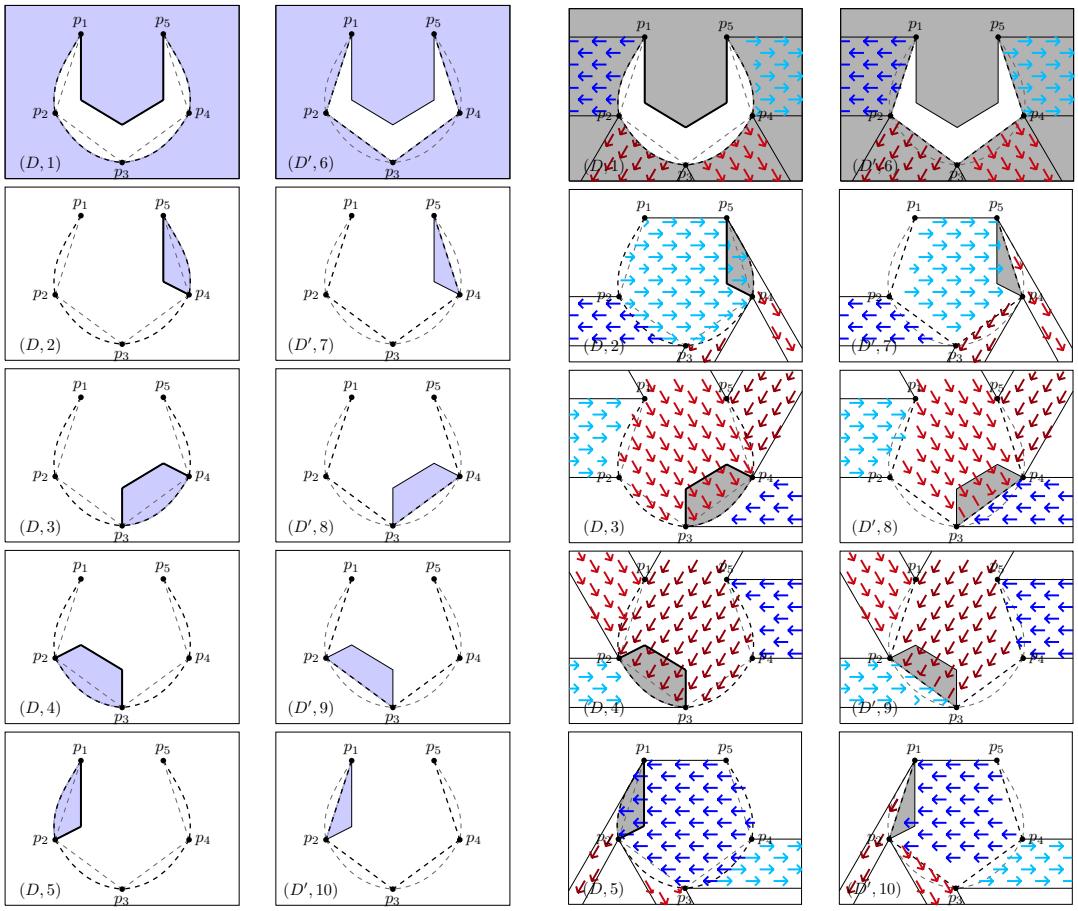


Figure 8: The minimizer $u_{min,5}$ for five points at the vertices of a regular pentagon and a calibration for the family $\mathcal{F}(\mathcal{J}_5)$

Remark 4.10. In [15] we prove that if $\Phi : Y \to \mathbb{R}^2$ is a calibration for $E \in \mathcal{P}_{constr}$, then E is a minimizer not only among all (constrained) finite perimeter sets, but also in the larger class of finite linear combinations of characteristic functions of (constrained) finite perimeter sets. Then if there exists an element of this larger class with strictly less energy of the minimizer of Problem (1.2), a calibration for such a minimizer cannot exist. This is the case when $S = \{p_1, \dots, p_5\}$ with p_i the vertices of a regular pentagon. This counterexample can be constructed adapting the example by Bonafini [10] in the framework of rank one tensor valued measures to our setting. Hence the tool of the calibration in families is necessary, in this case, to prove the minimality of the candidate by a calibration argument.

Example 4.11 (Calibration for the vertices of the regular hexagon).

We fix the points of S as the vertices of a regular hexagon in the following way: $p_1 = (-1/2, \sqrt{3}/2)$, $p_2 = (-1, 0)$, $p_3 = (-1/2, -\sqrt{3}/2)$, $p_4 = (1/2, -\sqrt{3}/2)$, $p_5 = (1, 0)$, $p_6 = (1/2, \sqrt{3}/2)$.
As in Example 4.9 we start covering $\mathscr{P}^T_{constr}(Y_\Sigma)$ with explicit families $\mathcal{F}(\mathcal{J})$ of competitors. From (4.6) we get

$$T = \left(\bigcup_{|i-j|=3} \mathcal{F}_{i,j} \right) \cup \left(\bigcup_{|i-j|=2} \mathcal{F}_{i,j} \right) .$$

For given $i, j \in \{1, \dots, 6\}$ such that $|i-j| = 3$ we further split $\mathcal{F}_{i,j}$ in four classes $(\mathcal{F}^k_{i,j})_{k=1,\dots,4}$ as follows:

$$\mathcal{F}^1_{i,j} = \mathcal{F}_{i,j} \cap (\mathcal{F}_{i,j-1} \cup \mathcal{F}_{i,j+1}), \qquad \mathcal{F}^2_{i,j} = \mathcal{F}_{i,j} \cap (\mathcal{F}_{i+1,j} \cup \mathcal{F}_{i-1,j}) ,$$

$$\mathcal{F}^3_{i,j} = \mathcal{F}_{i,j} \cap (\mathcal{F}_{i,j-1} \cup \mathcal{F}_{i+1,j}), \qquad \mathcal{F}^4_{i,j} = \mathcal{F}_{i,j} \cap (\mathcal{F}_{i-1,j} \cup \mathcal{F}_{i,j+1}) .$$

As in the previous example, thanks to Lemma 4.8 applied to the families $\mathcal{F}^k_{i,j}$ with $i, j \in \{1, \dots, 6\}$ and $k \in \{1, \dots, 4\}$, we can associate the respectively families $\mathcal{F}(\mathcal{J}_i)$ with J_i defined as follows:

$$\mathcal{J}_1 = \{(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\}, \quad \mathcal{J}_2 = \{(1,3),(1,4),(1,5),(3,5),(3,6),(4,6)\},$$
$$\mathcal{J}_3 = \{(1,4),(1,5),(2,4),(2,5),(2,6),(4,6)\}, \quad \mathcal{J}_4 = \{(1,3),(1,5),(2,5),(2,6),(3,5),(3,6)\},$$
$$\mathcal{J}_5 = \{(1,3),(1,4),(2,4),(2,6),(3,6),(4,6)\}, \quad \mathcal{J}_6 = \{(1,3),(1,4),(1,5),(2,4),(2,5),(3,5)\},$$
$$\mathcal{J}_7 = \{(1,5),(2,4),(2,5),(2,6),(3,5),(3,6)\}, \quad \mathcal{J}_8 = \{(1,3),(1,4),(2,6),(3,5),(3,6),(4,6)\},$$
$$\mathcal{J}_9 = \{(1,3),(1,4),(1,5),(2,4),(2,5),(4,6)\}, \quad \mathcal{J}_{10} = \{(1,3),(2,5),(2,6),(3,5),(3,6),(4,6)\},$$
$$\mathcal{J}_{11} = \{(1,3),(1,4),(1,5),(2,4),(3,6),(4,6)\}, \quad \mathcal{J}_{12} = \{(1,4),(1,5),(2,4),(2,5),(2,6),(3,5)\}.$$

We notice that the families $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{1, \dots, 6\}$ can be obtained from the first one by a cyclic permutation of the indices. Also the families $\mathcal{F}(\mathcal{J}_8)$ and $\mathcal{F}(\mathcal{J}_9)$ can be obtained by a cyclic permutation of the indices from $\mathcal{F}(\mathcal{J}_7)$, and the same holds for the families $\mathcal{F}(\mathcal{J}_{10})$, $\mathcal{F}(\mathcal{J}_{11})$, and $\mathcal{F}(\mathcal{J}_{12})$ (see Figure 9).
Consider now the case in which $|i-j| = 2$ for given $i, j \in \{1, \dots, 6\}$. Here the situation is easier, as we find two families $\mathcal{F}_{i,j}$ and it is not necessary to consider a further refinement of the classes. In terms of $\mathcal{F}(\mathcal{J}_i)$ we get

$$\mathcal{J}_{13} = \{(1,4),(2,4),(2,5),(2,6),(3,6),(4,6)\}, \ \mathcal{J}_{14} = \{(1,3),(1,4),(1,5),(2,5),(3,5),(3,6)\},$$

where again $\mathcal{F}(\mathcal{J}_{14})$ is obtained by a cyclic permutation of the indices from $\mathcal{F}(\mathcal{J}_{13})$. In conclusion the subdivision in families is as follows:

$$\mathscr{P}^T_{constr}(Y_\Sigma) = \bigcup_{i=1}^{14} \mathcal{F}(\mathcal{J}_i) .$$

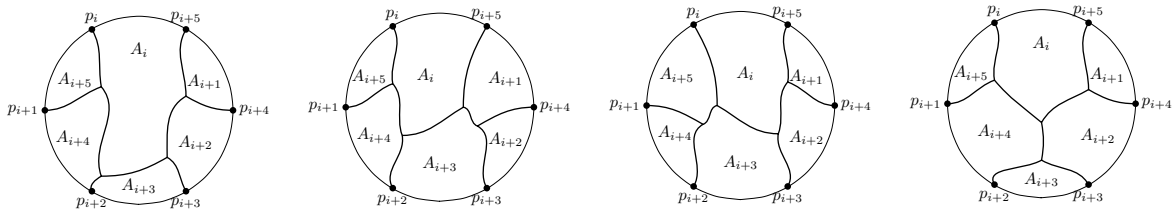


Figure 9: On the left we represent the projection onto the base set of the essential boundary of an element of the family $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{1, \dots, 6\}$, then of the family $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{7, 8, 9\}$ and $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{10, 11, 12\}$, on the right $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{13, 14\}$.

It is well know that if the points of S lies at the vertices of a regular hexagon, then there are six minimizers $\mathscr{S}_i$ for the Steiner problem. Calling $E_{\min,i}$ with $i \in \{1, \dots, 6\}$ the sets in $\mathscr{P}_{constr}(Y_\Sigma)$ associated to $\mathscr{S}_i$, we have that $E_{\min,i} \in \mathcal{F}(\mathcal{J}_i)$ for $i \in \{1, \dots, 6\}$.
In order to use Proposition 4.6 we have to find a calibration Φ^i for an explicit set $E_i \in \mathcal{F}(\mathcal{J}_i)$ for every $i \in \{1, \dots, 14\}$. The global minimizers $E_{\min,i}$ are clearly minimizers in their families, so they are the natural candidate minimizers for the families $\mathcal{F}(\mathcal{J}_i)$ for $i \in \{1, \dots, 6\}$; it is more challenging to propose a minimizer for the other families. Our candidate minimizers are shown in Figure 10.

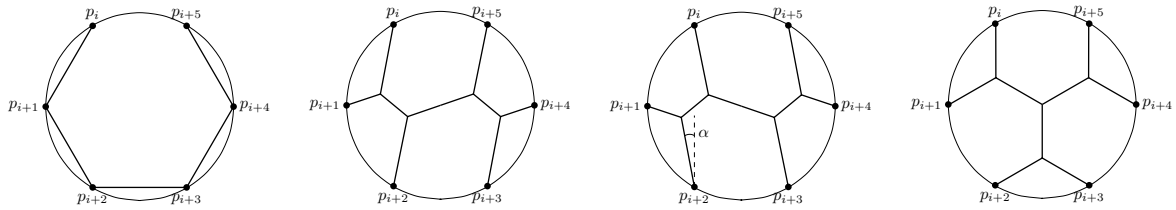


Figure 10: From left to right: the projection onto the base set of the essential boundary of candidate minimizers in the family $\mathcal{F}(\mathcal{J}_i)$ with $i \in \{1, \dots, 6\}$, $i \in \{7, 8, 9\}$ $i \in \{10, 11, 12\}$ and $i \in \{13, 14\}$.

We write explicitly the calibration Φ_i (written, as usual, by means of the pullbacks on every sheet Φ_i^j) for $E_{\min,i} \in \mathcal{F}(\mathcal{J}_i)$ with $i = 1, 10, 13$. The other calibrations are easy variants of the previous ones. Moreover we give the expression of the vector field only inside $p^{-1}(\mathrm{Conv}(S))$, as the divergence free extension in Y_Σ can be easily achieved. We get

$$\begin{aligned}
&\Phi_1^1 = (0,0)\,, && \Phi_1^2 = (\sqrt{3},1)\,, && \Phi_1^3 = (\sqrt{3},-1)\,,\\
&\Phi_1^4 = (0,-2)\,, && \Phi_1^5 = (-\sqrt{3},-1)\,, && \Phi_1^6 = (-\sqrt{3},1)\,,
\end{aligned}$$

$$\begin{aligned}
&\Phi_{10}^1 = (0,0)\,, && \Phi_{10}^2 = \left(\tfrac{3\sqrt{3}}{\sqrt{7}}, \tfrac{1}{\sqrt{7}}\right)\,, && \Phi_{10}^3 = \left(\tfrac{2\sqrt{3}}{\sqrt{7}}, -\tfrac{4}{\sqrt{7}}\right)\,,\\
&\Phi_{10}^4 = \left(-\tfrac{\sqrt{3}}{\sqrt{7}}, -\tfrac{5}{\sqrt{7}}\right)\,, && \Phi_{10}^5 = \left(-\tfrac{4\sqrt{3}}{\sqrt{7}}, -\tfrac{6}{\sqrt{7}}\right)\,, && \Phi_{10}^6 = \left(-\tfrac{3\sqrt{3}}{\sqrt{7}}, -\tfrac{1}{\sqrt{7}}\right)\,,
\end{aligned}$$

$$\begin{aligned}
&\Phi_{13}^1 = (0,0)\,, && \Phi_{13}^2 = (2,0)\,, && \Phi_{13}^3 = (1,-\sqrt{3})\,,\\
&\Phi_{13}^4 = (0,-2\sqrt{3})\,, && \Phi_{13}^5 = (-1,-\sqrt{3})\,, && \Phi_{13}^6 = (-2,0)\,.
\end{aligned}$$

Remark 4.12. The vector field Φ_{10} is a calibration for the set $E \in \mathscr{P}^T_{constr}(Y_\Sigma)$ associated to the third network represented in Figure 10 for $i = 1$. It can be computed easily as follows:

firstly we rotate the points of S and the network by a rotation of angle $\alpha = -\arctan\left(\frac{1}{3\sqrt{3}}\right)$ with the following rotation matrix:

$$R(\alpha) = \begin{bmatrix} \frac{3\sqrt{3}}{2\sqrt{7}} & \frac{1}{2\sqrt{7}} \\ -\frac{1}{2\sqrt{7}} & \frac{3\sqrt{3}}{2\sqrt{7}} \end{bmatrix}.$$

The network obtained in this way can be calibrated with the following vector field:

$$\begin{array}{lll} \widetilde{\Phi}^1_{10} = (0,0)\,, & \widetilde{\Phi}^2_{10} = (2,0)\,, & \widetilde{\Phi}^3_{10} = (1,-\sqrt{3})\,, \\ \widetilde{\Phi}^4_{10} = (-1,-\sqrt{3})\,, & \widetilde{\Phi}^5_{10} = (-3,-\sqrt{3})\,, & \widetilde{\Phi}^6_{10} = (-2,0)\,. \end{array}$$

Finally we rotate back with the matrix $R(-\alpha)$ to get Φ_{10}.

An easy computation shows that $P(E_{min,i}) < P(E_{min,j})$ for $i \in \{1,\dots,6\}$ and $j \in \{7,\dots,14\}$. Thus applying Proposition 4.6 we infer that $E_{min,i}$ are minimizers of $\mathscr{A}_{constr}(S)$ for every $i = 1,\dots,6$, as we wanted to prove.

Remark 4.13. The above division in families is the finest possible one. It has the advantage that we obtain constant calibrations. However from the numerical point of view this choice is not convenient. Indeed the complexity is simply shifted from solving the non–convex original problem to finding all the families. It would be better to consider a more coarse division in families in which one can in any case find a calibration.

Appendix: Divergence theorem on Y_Σ

Proposition 4.14 (Divergence theorem on coverings)**.** *Consider $E, F \in \mathscr{P}_{constr}(Y_\Sigma)$ and let $\Phi : Y_\Sigma \to \mathbb{R}^2$ be an approximately regular vector field such that* $\operatorname{div}\Phi = 0$ *(in the sense of distributions) in Y_Σ. Then*

$$\int_{Y_\Sigma} \Phi \cdot D\chi_E = \int_{Y_\Sigma} \Phi \cdot D\chi_F. \tag{4.7}$$

Proof. From (2.5), one gets:

$$D\chi_E = \sum_{j=1}^{m} D\chi_{E^j} \llcorner D + \sum_{j'=m+1}^{2m} D\chi_{E^{j'}} \llcorner (\Sigma \setminus S)\,.$$

Fix $\varepsilon > 0$ and call Ω_ε the ε–tubular neighbourhood of Ω. Using the divergence theorem for approximately regular vector fields in $D \subset \mathbb{R}^2$ (see [1]) and the definition of $D\chi_{E^{j'}}$ we have

$$\begin{aligned} &\int_{Y_\Sigma} \Phi \cdot D\chi_E = \sum_{j=1}^{m} \int_D \Phi^j \cdot D\chi_{E^j} + \sum_{j'=m+1}^{2m} \int_\Sigma \Phi^{j'} \cdot D\chi_{E^{j'}} \\ &= -\sum_{j=1}^{m} \int_\Sigma \left[(\Phi^j)^+ \chi^+_{E^j} - (\Phi^j)^- \chi^-_{E^j}\right] \cdot \nu_\Sigma \, d\mathcal{H}^1 - \sum_{j=1}^{m} \int_{\partial\Omega_\varepsilon} \chi_{E^j} \Phi^j \cdot \nu_{\partial\Omega_\varepsilon} \, d\mathcal{H}^1 \\ &+ \sum_{j'=m+1}^{2m} \int_\Sigma \left[(\chi_{E^{j'}})^+ - (\chi_{E^{j'}})^-\right] \Phi^{j'} \cdot \nu_\Sigma \, d\mathcal{H}^1\,, \end{aligned} \tag{4.8}$$

where $\nu_{\partial\Omega_\varepsilon}$ is the inner unit normal to $\partial\Omega_\varepsilon$. From now on we will call

$$a_h^+ := (\Phi^h)^+ \cdot \nu_\Sigma, \quad a_h^- := (\Phi^h)^- \cdot \nu_\Sigma \quad \text{for } h = 1, \dots, 2m\,.$$

As Φ is approximately regular (see Definition 3.1 and Definition 3.2) one has that

$$a_{j'}^+(x) = a_{j'}^-(x) = (\Phi^{j'} \cdot \nu_\Sigma)(x) \quad \text{for } \mathcal{H}^1 - a.e\ x \in \Sigma \tag{4.9}$$

for every $j' = m+1, \dots, 2m$. Moreover as $E \subset Y_\Sigma$, one can verify that

$$\sum_{j=1}^{m} a_j^+(x)(\chi_{E^j})^+(x) = \sum_{j'=m+1}^{2m} a_{j'}^+(x)(\chi_{E^{j'}})^+(x) \tag{4.10}$$

and

$$\sum_{j=1}^{m} a_j^-(x)(\chi_{E^j})^-(x) = \sum_{j'=m+1}^{2m} a_{j'}^-(x)(\chi_{E^{j'}})^-(x) \tag{4.11}$$

for $\mathcal{H}^1$- a.e $x \in \Sigma$.
Using (4.9), (4.10), (4.11) on Formula (4.8) it is easy to see that

$$\int_{Y_\Sigma} \Phi \cdot D\chi_E = -\sum_{j=1}^{m} \int_{\partial\Omega_\varepsilon} \Phi^j \chi_{E^j} \cdot \nu_{\partial\Omega_\varepsilon}\, d\mathcal{H}^1$$

and analogously

$$\int_{Y_\Sigma} \Phi \cdot D\chi_F = -\sum_{j=1}^{m} \int_{\partial\Omega_\varepsilon} \Phi^j \chi_{F^j} \cdot \nu_{\partial\Omega_\varepsilon}\, d\mathcal{H}^1\,.$$

As $E, F \in \mathscr{P}_{constr}(Y_\Sigma)$, the functions χ_{E^j} and χ_{F^j} have same boundary conditions on $\partial\Omega_\varepsilon$ for every j. This gives equation (3.2) as we wanted to prove. □

References

[1] G. Alberti, G. Bouchitté, and G. Dal Maso. The calibration method for the Mumford-Shah functional and free-discontinuity problems. *Calc. Var. Partial Differential Equations*, 16(3):299–333, 2003.

[2] S. Amato, G. Bellettini, and M. Paolini. Constrained BV functions on covering spaces for minimal networks and Plateau's type problems. *Adv. Calc. Var.*, 10(1):25–47, 2017.

[3] L. Ambrosio and A. Braides. Functionals defined on partitions in sets of finite perimeter. I. Integral representation and Γ-convergence. *J. Math. Pures Appl. (9)*, 69(3):285–305, 1990.

[4] L. Ambrosio and A. Braides. Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization. *J. Math. Pures Appl. (9)*, 69(3):307–333, 1990.

[5] L. Ambrosio, V. Caselles, S. Masnou, and J.-M. Morel. Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc. (JEMS)*, 3(1):39–92, 2001.

[6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[7] G. Bellettini, M. Paolini, and F. Pasquarelli. Triple covers and a non-simply connected surface spanning an elongated tetrahedron and beating the cone. *Interfaces Free Bound.*, 20(3):407–436, 2018.

[8] G. Bellettini, M. Paolini, F. Pasquarelli, and G. Scianna. Covers, soap films and BV functions. *Geom. Flows*, 3:57–75, 2018.

[9] G. Bellettini, M. Paolini, and C. Verdi. Numerical minimization of geometrical type problems related to calculus of variations. *Calcolo*, 27(3-4):251–278 (1991), 1990.

[10] M. Bonafini. Convex relaxation and variational approximation of the steiner problem: theory and numerics. *Geom. Flows*, 3:19–27, 2018.

[11] M. Bonafini, G. Orlandi, and E. Oudet. Variational Approximation of Functionals Defined on 1-dimensional Connected Sets: The Planar Case. *SIAM J. Math. Anal.*, 50(6):6307–6332, 2018.

[12] M. Bonnivard, A. Lemenant, and F. Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015.

[13] K. A. Brakke. Numerical solution of soap film dual problems. *Experiment. Math.*, 4(4):269–287, 1995.

[14] K. A. Brakke. Soap films and covering spaces. *J. Geom. Anal.*, 5(4):445–514, 1995.

[15] M. Carioni and A. Pluda. On different notions of calibrations for minimal partitions and minimal networks in $\mathbb{R}^2$. Accepted in Adv. Calc. Var., Arxiv preprint: https://arxiv.org/abs/1805.11397, 2018.

[16] A. Chambolle, B. Merlet, and L. Ferrari. A simple phase-field approximation of the steiner problem in dimension two. Accepted paper: Adv. Calc. Var., 2016.

[17] R. Courant and H. Robbins. *What Is Mathematics?* Oxford University Press, New York, 1941.

[18] D. Z. Du, F. K. Hwang, and J. F. Weng. Steiner minimal trees for regular polygons. *Discrete Comput. Geom.*, 2(1):65–84, 1987.

[19] J. H. G. Fu. Tubular neighborhoods in Euclidean spaces. *Duke Math. J.*, 52(4):1025–1046, 1985.

[20] A. O. Ivanov and A. A. Tuzhilin. *Minimal networks. The Steiner problem and its generalizations*. CRC Press, Boca Raton, FL, 1994.

[21] V. Jarník and M. Kössler. On minimal graphs containing n given points. *Časopis Pěst. Mat.*, 63:223–235, 1934.

[22] G. Lawlor and F. Morgan. Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms. *Pacific J. Math.*, 166(1):55–83, 1994.

[23] A. Lemenant and F. Santambrogio. A Modica-Mortola approximation for the Steiner problem. *C. R. Math. Acad. Sci. Paris*, 352(5):451–454, 2014.

[24] A. Marchese and A. Massaccesi. The Steiner tree problem revisited through rectifiable G-currents. *Adv. Calc. Var.*, 9(1):19–39, 2016.

[25] A. Massaccesi, E. Oudet, and B. Velichkov. Numerical calibration of steiner trees. *Applied Mathematics & Optimization*, 79(1):69–86, 2019.

[26] J.-M. Morel and S. Solimini. *Variational methods in image segmentation. With seven image processing experiments*, volume 14 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1995.

[27] E. Paolini and E. Stepanov. Existence and regularity results for the Steiner problem. *Calc. Var. Partial Differential Equations*, 46(3-4):837–860, 2013.

[28] I. Tamanini and G. Congedo. Density theorems for local minimizers of area-type functionals. *Rend. Sem. Mat. Univ. Padova*, 85:217–248, 1991.