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Authors:

Name: Anna Baccaglini-Frank ORCID: 0000-0002-8116-5370
Email: anna.baccaglinifrank@unipi.it
Address: Dipartimento di Matematica, University of Pisa, Largo Bruno Pontecorvo 5, 56126, Pisa, Italy
Name: Gemma Carotenuto
Email: gemmacarotenuto@gmail.com
Affiliation: Dipartimento di Matematica e Applicazioni, University of Naples "Federico II", Italy
Name: Nathalie Sinclair ORCID: 0000-0003-0600-7062
Email: nathalie_sinclair@sfu.ca
Affiliation: Faculty of Education, Simon Fraser University, Canada

Title: **Eliciting preschoolers' number abilities using open, multi-touch environments**

Abstract:

Research has highlighted the potential of digital technology to support the development of children's number sense abilities. However, the main focus of such research has been on apps affording directed interactions, where only one solution strategy is available, and it has targeted mostly cardinality. Little is known, in these terms, about task design and implementation in more open environments where several different solution strategies are available. To explore this direction, we chose to study *TouchCounts*, an open environment that combines multi-touch affordances with aural, visual and symbolic ones as well. Using tasks that were designed to address different number sense abilities, we experimented with 4-year-old preschoolers. In this paper we present two tasks, their expected potential with respect to strengthening number sense abilities, and analyses of data collected during the preschoolers' interactions with *TouchCounts*. The analyses reveal that the children used different strategies in response to the two tasks, and that a broad range of abilities related to number sense were elicited, including both cardinality and ordinality. An important contribution of this study is also a theoretical framework capable of identifying children's learning in a multi-touch environment.

Keywords:

App design, Cardinality, Multitouch app, Number Sense, Ordinality, Scheme, TouchCounts

Eliciting preschoolers' number abilities using open, multi-touch environments

1. Children's number abilities and digital touch environments for their development

More and more apps are being developed and advertised as effective in "developing children's arithmetic competence" or in "strengthening children's number sense". These claims apply especially to apps that capture multi-touch gestures and provide feedback that is somehow coherent with these input gestures. However, what is actually meant by these claims is not always clear, especially when the children are very young (e.g., 3-4 years old). In this study we wish to explore whether and how an appropriately designed multitouch app, *TouchCounts*, and tasks within it can promote number sense in very young children. In particular, we hypothesise that appropriate use of fingers can plant important seeds that can eventually blossom into managing ordinality and cardinality, and, therefore, contribute to developing children's number sense, especially at young ages. This is why this study focuses particularly on children's utterances and ways of using fingers to operate with numbers in a multitouch environment and attempts to put these visible actions in relationship with specific number sense abilities.

In order to do this, we will first provide a critical review of the literature in the intricate terrain of "number sense", investigating the abilities it involves and its relationships to arithmetic competence; in particular we will present the somewhat controversial issues of cardinality and ordinality. Since in this study fingers and gestures play an important role, the literature review includes studies addressing the importance of specific ways of using fingers when learning about numbers and discuss, specifically, how these can contribute to the discussion on ordinality and cardinality. The critical literature review concludes with an examination of current research on apps designed to exploit the multi-touch potential with respect to number sense, as described in Baccaglini-Frank and Maracci (2015), who examine the potential of multi-touch technology "to foster important aspects of children's development of number-sense, including the ability to use fingers to represent numbers in an analogical format" (p. 12).

1.1 The multiple faces of number sense

The notion of number sense still lacks a universal interpretation across the communities of cognitive scientists and of mathematics educators, and even within the community of mathematics educators (e.g., Baccaglini-Frank & Maracci, 2015). However, its development is unanimously seen as a necessary condition for gaining arithmetic competence at the early elementary level (Mulligan et al., 2018; Elia, Mulligan, Anderson, Baccaglini-Frank, & Benz, 2018). Moreover, literature from the fields of neuroscience, developmental psychology, and mathematics education agree that a variety of abilities are fundamental for the development of number sense (e.g., Butterworth, 2005; Coles, 2014). These include: subitising, one-to-one correspondence (between number words and objects, but also between fingers and objects), approximate estimation, managing ordinality and managing cardinality; but also abilities that involve particular ways of using fingers, such as finger tapping and finger gnosis, that are not even explicitly related to numbers (Butterworth, 2005; Penner-Wilger et al., 2007; Gracia-Bafalluy & Noël, 2008). We will return to the importance of fingers in the next section.

The most complex, and perhaps controversial, components of number sense discussed in the literature are those included in the list of abilities above as "managing ordinality and managing cardinality". Though research concurs in highlighting the need for these two aspects to be woven together, there is an ongoing discussion as to whether one or the other should be favoured in the early years of schooling (e.g.: Coles, 2014; Coles & Sinclair, 2018a; Nesher, 2018). Indeed, in most Western school practices cardinality is favoured over ordinality (Coles & Sinclair, 2018a). However, recent evidence points out the importance of ordinality in predicting mathematical success (Lyons & Beilock, 2011), and recent educational research suggests promoting ordinal awareness earlier and to a greater extent than what is done traditionally in Western schools (Coles, 2014). This suggestion is based on the hypotheses that a great source of children's difficulties with numbers is related to linking symbols to other symbols (as opposed to symbols to sets of objects) and that ordinality is extremely relevant to place value (Coles & Sinclair, 2018a, 2018b).

To better understand these positions, it is necessary to clarify what is meant by "ordinality" and "cardinality". Typical definitions of cardinality refer to the "cardinality principle", that is, understanding that the last number spoken in a counting sequence names the quantity for that set, whereas "ordinality" is knowing the number words in the correct sequence, but also knowing what number comes after (or before) a given one. These definitions are intrinsically related to the task of counting, typically in response to the question "How many?". In this line of reasoning, according to researchers such as Fuson and Hall (1983) and Gelman and Gallistel (1978), and as noted by Nesher (2018), ordinality is considered to precede cardinality in children's development of number sense. Indeed, "as soon as the child succeeds in tagging the sets correctly (i.e. mastering one-to-one correspondence between the ordered words and the objects), he is already in the mode of ordinality, which is embedded in the notion of cardinality" (Nesher, 2018, p. 173).

However, from a mathematical point of view, researchers have argued that cardinality also involves an aspect that is independent from counting: cardinality is based on Fraenkel's (1942) basic concept of ‘one-to-one correspondence' between sets, and establishing such a correspondence between sets may not involve counting at all. Indeed, knowing that two sets are equal in number does not necessarily involve specifying *the number* of items in such sets. We agree that establishing such one-to-one correspondences between elements of sets (one of which could be of fingers) is key in acquiring number sense, and we see it as an aspect of cardinality.

1.2 Numbers on fingers bridging ordinality and cardinality

Cognitive science research now shows that using fingers for counting and representing numbers seems to be beneficial in formal and informal settings prior to entering school at 5 or 6 years of age (e.g.: Lafay, Thevenot, Castel & Fayol, 2013). Some researchers have even stressed the importance for developing number sense by considering quantities independently of numbers, and of using fingers to operate on them (Margolinas & Wosniak, 2012). For example, comparing the quantities of two collections of objects or representing a certain quantity with fingers can be done without using numbers explicitly, but by establishing one-to-one correspondences between objects or between objects and fingers. These processes are intertwined with the development of the so-called "finger symbol sets" (Brissiaud, 1992), which is the representation of numbers and numbers operations and relations through finger gestures.

However, the importance of using fingers in early arithmetic has not always been understood, especially in mathematics education: for decades children were pushed to avoid using fingers for mathematics (e.g., Moeller, Martignon, Wessolowski, Engel & Nuer, 2011).

We hypothesise that this is why definitions of ordinality and cardinality such as the ones introduced above have not given deep enough consideration to finger representations of numbers. Indeed, if a child raises four fingers of one hand and says "four" what can we infer about the number sense abilities she has used at that precise moment? She has not counted, but she did correctly associate a number word to a set with that number of elements. Does she *know* that if she had counted her fingers, the last number word said would have been "four", thus having mastered the cardinality principle? Does she *know* that "four" comes after "three" and that her set of fingers contains "one more" than a set of “three” objects, thus having mastered ordinality? Or does she *know* the name of a configuration of fingers, giving a name to a gesture—in which case, did she develop this through cardinal or ordinal knowing, or something entirely different?

We believe that through the use of fingers for dealing with quantities, children can start developing the needed mastery of number sense, including the counting principles. In particular, we hypothesise that fingers can play a bridging role in developing both ordinality and cardinality.

In order to contextualize this study and its results with respect to other studies on number sense abilities that can be developed through multitouch interactions with apps, we will refer to "number sense abilities" as the following set of abilities that the literature has highlighted. We assign to each the following working definitions (Table 1):

Table 1: Number sense abilities considered and the definitions used in this study

number sense ability	**definition**
finger gnosis and control	knowing finger positions in one's body and controlling the precision of gestures with fingers[1]
subitising	recognizing small numerosities (up to ****) without counting
representing numbers with fingers (constructive or immediate)	knowing the number name or symbol corresponding to certain configurations of fingers - the configurations can be obtained by counting or immediately
ordinality	associating number symbols to number words; knowing the number symbols sequence; knowing the number words sequence; knowing what number comes before or after a certain one
cardinality	establishing one-to-one correspondence between elements of two sets (of number words, of material or virtual objects or of fingers); understanding that the last number spoken in a counting sequence names the quantity for that set; assigning the correct symbol or number word to a set

1.3 How can number sense abilities be elicited through apps designed to exploit the multi-touch potential?

[1] We note that a child who taps with a single finger (usually the index) is showing a very different level of this kind of knowledge with respect to a child who can precisely carry out a multiple finger tap with, say, four fingers.

Though some potentials and pitfalls of apps with respect to children's development of number sense abilities have emerged in recent literature (e.g., Rothschild & Williams, 2015; Baccaglini-Frank & Maracci, 2015; Holgersson, Barendregt, Emanuelsson, Ottosson, Rietz, & Lindström, 2016; Tucker, Moyer-Packenham, Westenskow, & Jordan, 2016), research on apps designed to exploit the multi-touch potential with respect to number sense (Baccaglini-Frank & Maracci, 2015) with preschoolers' is still in its infancy (Moyer-Packenham et al., 2015). Indeed, from an analysis of recent literature (see the upcoming literature review by ICME-14 Survey Team 2 "Early childhood mathematics education (up to age 7)") on studies published in research journals or conference proceedings in mathematics education, cognitive psychology and pedagogy, we found less than twenty relevant publications.

The main findings are the following.

- Different children attend to different affordances, and scaffolding is key (Bullock, Shumway, Watts, & Moyer-Packenham, 2017).
- While representing the majority of apps, instructive apps are less beneficial than manipulative and constructive ones (e.g., Goodwin & Highfield, 2013).
- The multi-touch app Fingu has the potential of enhancing children’s speed and accuracy in subitising and their ability to identify part-whole relationships (Barendregt et al., 2012; Holgersson et al., 2016; Broda et al., 2018).
- Certain multi touch apps can support one-to-one correspondence between fingers and virtual objects, finger gnosia, and counting – both of virtual objects and of fingers (Baccaglini-Frank & Maracci, 2015; Baccaglini-Frank, 2018).

These studies have mostly focused on apps affording interactions that are quite constrained or directed, and that focus particularly on cardinality. A few more recent studies (we only found 6 but they constitute over a third of the existing literature (16 publications) of the past 7 years on the development of number sense abilities in preschoolers between 3 and 5 when interacting with digital technology) focus on the app TouchCounts (TC) (Jackiw & Sinclair, 2014), which affords an open and unconstrained environment (tasks are not given explicitly to the child by the app, nor is the feedback designed to constrain children's interactions) that combines multi-touch affordances with aural, visual and symbolic ones as well. A handful of studies have explored how TC can be used to foster the development of number sense abilities in children between the ages of 3 and 5, such as cardinality (Sinclair & Sedaghatjou, 2013; Sedaghatjou & Campbell, 2017), ordinality (Sinclair & Heyd-Metzuyanim) and finger gnosis (Sinclair & Pimm, 2015).

In this study we are interested in extending the body of research that explores relationships between children's strategies used when interacting with multi-touch technology and their development of number sense abilities. In particular, we seek to explore such relationships in the context of open-ended tasks such as those that can be posed in TC. This is a novel and important direction of research because, to our knowledge, these relationships have been studied only in the context of constrained or directed interactions, while the literature suggests that open-ended tasks offer a higher potential for young children's learning.

Research Questions

Specifically, we ask:

RQ1. What is the potential of certain tasks in TC to mobilise specific number sense abilities in 3 and 4 year old children?

RQ2. What number sense abilities can actually be mobilised in 3 and 4 year old children in TC? Specifically, what "seeds" of ordinality and cardinality are planted during the interactions with TC analysed?

Answers to these questions fuel a broader discussion on children's development of number sense fostered by multi-touch digital interactions, and on task design within open environments, which at the moment constitute important challenges in mathematics education in early childhood.

2. Theoretical Framework: intertwining the notions of *semiotic potential* of an artefact and *scheme*

A major challenge in this study was to construct an appropriate theoretical framework. Such a framework should allow us to design tasks with the potential of mobilising the number sense abilities in focus, and to describe the potential of such a design. Moreover, the framework should also allow us to make inferences on the number sense abilities that children are using. This can be done only by making inferences on what is visible, that is their utterances and gestures as they interact with virtual objects in a digital environment. We elaborated an analytical toolbox for this purpose, using the notion of *semiotic potential of an artefact* for the task design and a priori analysis, and Vergnaud's notion of *scheme* for the children's responses to the designed task. In this section we explain our framework more in detail.

The Theory of Semiotic Mediation (TSM) is a Vygotskian lens introduced by Bartolini Bussi and Mariotti (2008) and designed to study links between artefacts, tasks, the mathematical knowledge to be *mediated* in the classroom. For this study we specifically use the TSM for designing tasks within different environments in TC, so that they potentially elicit certain number sense abilities[2]. Indeed, the TSM, among other things, makes explicit the relationships between an artefact (here specific environments in TC), a task (the ones designed), specific mathematical knowledge (here number sense abilities), and the signs generated by people solving the tasks (here gestures and utterances of the interviewed children).

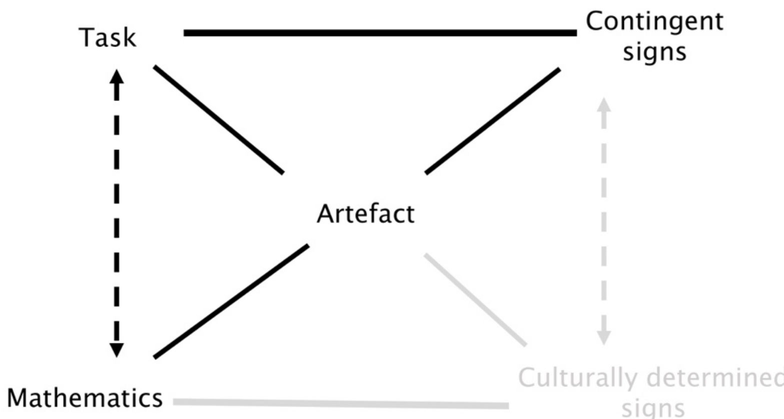

Figure 1: Diagram of main processes in the TSM (adapted from Bartolini Bussi & Mariotti, 2008, p. 753); part of the diagram is greyed out because we do not focus on the development of culturally determined signs in this paper

The TSM explicitly introduces the notion of *semiotic potential of an artefact*, describing it as a "double semiotic relationship" (Bartolini Bussi & Mariotti, 2008, p. 754), on the one hand, between personal meanings related to the use of the artefact with the goal of accomplishing a specific task, and, on the other hand, between the use of the artefact and mathematics (see the left-hand triangle in Figure 1). In section 3 we will introduce the specific tasks designed and in section 4 we will describe the potential of certain tasks in TC to mobilise specific number sense abilities in 3 and 4 year old children, thus answering RQ1.

According to the *instrumental approach* (Béguin & Rabardel, 2000) a subject, engaged in a goal-directed activity, can build *schemes of instrumented action* for an artefact. Thanks to the visible contingent signs elaborated by the solver (e.g., words, gestures), we can make inferences about the schemes she is developing for our tasks in TC. An instrumented action scheme describes the organization of activity with a tool, in order to realize a given task; it is oriented towards carrying out a specific task (Bèguin & Rabardel, 2000). Such a notion draws on Vergnaud's *scheme*:

[2] We see such abilities not as innate but as a cultural product, in accordance with the TSM.

"The function of schemes, in the present theory, is both to describe ordinary ways of doing, for situations already mastered, and give hints on how to tackle new situations. Schemes are adaptable resources: they assimilate new situations by accommodating to them. Therefore the definition of schemes must contain ready-made rules, tricks and procedures that have been shaped by already mastered situations; but these components should also offer the possibility to adapt to new situations. On the one hand, a scheme is the invariant organisation of activity for a certain class of situations; on the other hand, its analytic definition must contain open concepts and possibilities of inference." (Vergnaud, 2009, p. 88).

Vergnaud proceeds to describe how schemes comprise of several aspects; these include: an intentional aspect involving "a goal or several goals that can be developed in sub-goals and anticipations" (goals and anticipations), a generative aspect involving "rules to generate activity, namely the sequences of actions" (rules of action), an epistemic aspect of involving operational invariants whose "main function is to pick up and select the relevant information and infer from it goals and rules" (Vergnaud, 2009, p. 88). Seeing a scheme as a sort of iceberg, some of the aspects listed above can be considered its "visible tip". Specifically, the rules of action can be seen, as well as the student's goals and anticipations which sometimes are made explicit (and in any case they are usually easy to infer).

We fit together the theoretical constructs presented as shown in the diagram in Figure 2. The <u>number sense abilities</u> constitute the "mathematics" (lower left-hand corner), and the <u>rules of action</u> and visible <u>anticipations</u> are the contingent signs (upper right-hand corner). The artefacts are the specific environments within TC in which each task is proposed (see section 3.2 for their description).

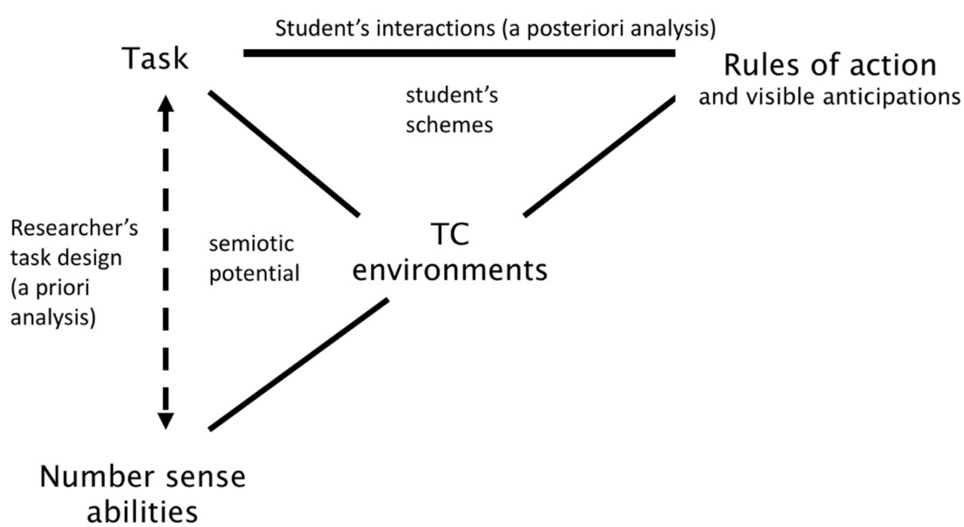

Figure 2: Adaptation of the diagram in Fig. 1 to this study

This theoretical framework will allow us to analyse the relationships between tasks in TC, children's schemes, and number sense abilities. Using this lens a priori to analyse the tasks designed in TC, and a posteriori to analyse children's interactions, we will be able to answer RQ1 and RQ2, respectively.

3. Methodology

3.1 Participants and data collection

The study involves a sequence of TC-based tasks that were used by the second author in a private preschool in Southern Italy. The children worked in turns over a time period of two months, from December 2018 to January 2019. The children involved were a total of 19, between the ages of 3-and-a-half and 4-and-a-half years of age.

One-on-one initial interviews were used to select eight children who we expected would be able to individually interact with the app and with the interviewer during the sessions, without being too shy, and with sufficient experience with numbers to be able to understand the tasks. The initial interview questions asked children to count, to show numbers with their fingers, and read a written numeral, and to say what precedes and follows six. 8 children were selected: they answered at least four of the 13 questions of the initial interview correctly and were between the ages of 42 and 52 months. Specifically, 3 children answered four of the initial interview questions correctly, 1 answered six questions, 1 answered eight questions, and 3 children answered ten questions.

All the meetings of the researcher with each child were video recorded. After the initial interviews, the researcher worked with each child individually for two sessions with the app.

The 8 children selected for the activities with TC interacted with the app in two different moments. During the first session, the children were first invited to explore both TC environments (see section 3.2 for a description of the TC and the tasks proposed). Immediately after, they were asked Task 1 and Task 2. During the second session the children were given tasks that addressed number representations with fingers (as “Make four/seven”) and the creation of one-to-one correspondence between dots in a picture and fingers on the screen.

The length of the sessions varied, depending on the child’s involvement and success in the task: if a child was unsuccessful in accomplishing a task, the researcher would interrupt it when the child stopped interacting or began to risk becoming frustrated. The length of the first session varied from child to child between a minimum of 8’30’’ to a maximum of 17’; the second session from a minimum of 6’40’’ to a maximum of 23’20’’.

We first explored all children's schemes of both sessions, individually, to find out how the potential of the activities was actually exploited. However, here we focus on the children's interactions with the tasks of the first section only. We make this choice for two reasons: the schemes developed by the children in response to Task 1 and Task 2 included many of the rules of action implemented for the tasks of the second session; moreover, these two tasks are possible (and make sense) only within the TC, because of its design, and therefore they are different from the tasks generally reported in the literature on early development of number sense.

3.2 The app *TouchCounts* (TC) and the designed tasks

TC consists of two environments, named *Enumerating World* and *Operating World*[3]; the researcher proposed tasks in both. For the benefit of the reader, we describe the two environments together with the two tasks that we analyse, *Task 1* and *Task 2*, and provide anticipations of interactions that could be used to accomplish them.

3.2.1 *Task 1: "Can you put only this on the shelf?"*

In the Enumerating World, the learner sees a horizontal line—called a "shelf"—near the top of the screen. If the learner taps anywhere on the screen with one or more fingers, she produces numbered yellow discs. As each single tap produces a new numbered disc, TC audibly speaks the number word for its number (in English: "one", "two", …). As long as the learner's finger remains pressed to the screen, the disc holds its position beneath her fingertip. But as soon as she 'let's go' (by lifting that finger), the disc falls down, as under the effect of gravity, and then disappears "off" the bottom of the screen. If the learner releases her disc above the shelf, it falls only to the shelf, and comes to rest there, visibly and permanently on screen, rather than vanishing out of sight 'below' (see Figure 3).

Task 1 is performed in the Enumerating World and the child is asked to put only the disc with 4 on the shelf. The researcher says "Can you put only this on the shelf?", while showing a paper card with the symbol "4". (Use this link to watch a video of children putting 10 on the shelf: https://www.youtube.com/watch?v=7xD-pqnsce0).

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[3] The names of the two environments are connected with this difference: in the Enumerating World children create the counting numbers in sequence, advancing with each touch; while in the Operating World children can explore basic number operation concepts, by manipulating previously created herds.

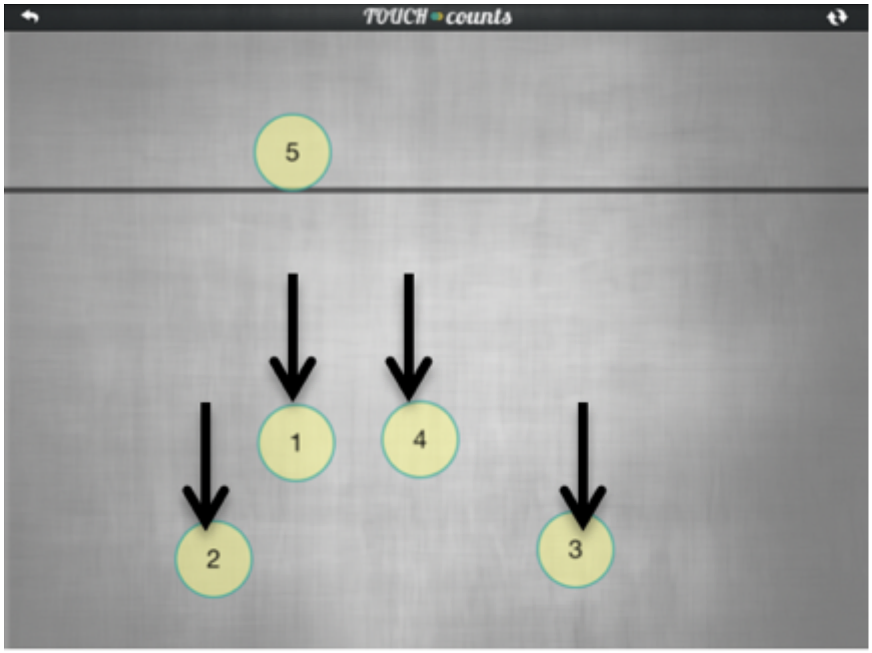

Figure 3: Effect on the screen of two consecutive gestures: a four-finger all-at-once tap below the shelf, followed by a single tap above (the arrows indicate the fall of the discs)

3.2.2 *Task 2: Make Nadia count*

Whilst tapping on the screen in the Enumerating World with three fingers in a three-finger all-at-once tap creates three independent discs labelled ‘1’, ‘2’ and ‘3’, the same gesture in the Operating World produces a single herd labelled ‘3’ containing three smaller unlabelled discs (Figure 4a-4b). When the three fingers are lifted off the screen (Figure 3b), “three” is spoken aloud. Unlike in the Enumerating World, herds can be interactively touched, for example to move them around with single finger tap, without producing new herds.

In Task 2, the researcher asks the child to “make Nadia count” in the Operating World (Nadia is the name we gave to "the voice" in TC). To make Nadia count, a learner needs to make a one-finger all-at-once tap followed by a two-finger all-at-once tap, followed by a three-finger all-at-once tap, and so on.

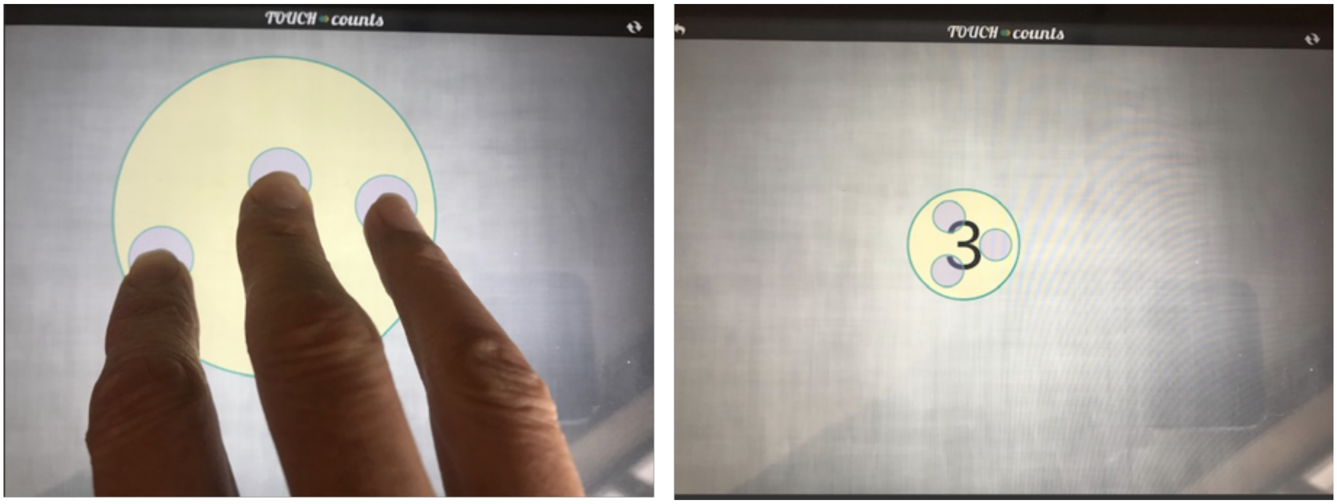

Figure 4: (a) Placing three fingers all-at-once on the screen in the Operating World; (b) the herd of 3 that is formed after the fingers are lifted

4. A priori analysis of the tasks: addressing RQ1

We now use our theoretical framework to describe the potential of certain tasks in TC to mobilise specific number sense abilities in 3 and 4 year old children (RQ1). In the analyses below we identify sets of abilities corresponding to the hypothetical schemes we describe, however it is possible the abilities actually elicited by each task will depend on the scheme used by the solver. Some of the schemes actually used by the solvers are analysed in section 5.

Let us consider Task 1, "Can you put only this on the shelf?" (see section 3.2.1). In this case the artefact is the Enumerating World, and a scheme developed by an expert solver could be: a three-finger all-at-once tap below the shelf followed by a single finger tap above the shelf, and be satisfied with the outcome. We could infer her scheme and, in particular her number sense abilities as follows:

- anticipations: a disc with "4" sitting on the shelf, nothing else on the screen;
- rules of action: prepare three fingers immediately on one hand, use a three finger all-at-once touch on the screen below the shelf, then tap once above the shelf;
- number sense abilities mobilised: finger gnosis and control (three fingers are lifted and used to touch the screen all-at-once, a gesture that requires precise and coordinate movements as well as awareness of the fingers raised and lowered), ordinality (the symbol "4" is associated to what comes after "three" or "3"), representing numbers with fingers ("3" or "three", depending on the solver's use of ordinality, corresponds to a gesture with three fingers raised, a known gesture to the expert solver).

For this solver cardinality seems to be elicited only to the extent that she expects to hear "three" pronounced by TC as feedback for the three-finger all-at-once tap and three discs are shown falling off the screen. Also, subitising would not seem to be directly mobilised by this scheme, although we could argue that subitising could be strengthened by the appearance of the three discs produced by the three-fingers all-at-once tap. Other schemes could mobilise to a greater extent the abilities of cardinality and subitising.

Now let us consider Task 2, "Make Nadia count" (see section 3.2.2), in which the artefact is the Operating World. An expert solver would expect to hear TC pronounce "one, two, three, four, ..." in response to each gesture, and remember that in this world TC pronounces the number word corresponding to the number of fingers placed on the screen simultaneously before lifting them off all together (and thus corresponding to the number of discs created on the screen and floating within a larger disc). So the solver could tap the screen with one finger, then with two fingers, then with three fingers simultaneously, and so on. Depending on the expertise of the solver she could make Nadia count up to a higher number. We could analyse her actions and infer the number sense abilities mobilised as follows:

- anticipations: TC pronouncing "one, two, three, four, ..." and the screen ending up with floating discs containing, respectively, one, two, three, four, ... smaller discs and the corresponding number symbols in the middle;
- rules of action: tap the screen with one finger, prepare two fingers immediately on one hand and tap the screen with those two fingers simultaneously, prepare three fingers immediately on one hand, use a three finger all-at-once touch on the screen, prepare four fingers immediately on one hand, use a four finger all-at-once touch on the screen, and so on;
- number sense abilities mobilized: finger gnosis and control (one, then two, then three, then four fingers – and so on - are lifted and used to touch the screen all-at-once, gestures that requires precise and coordinate movements as well as awareness of the fingers raised and lowered), ordinality (knowing the verbal number sequence and knowing that, for example, either “three” is followed by “four” or “3” is followed by “4”, or both), representing numbers with fingers (gestures for each number can be known by the solver who creates them quickly, or they can be created gradually by counting up on one's fingers).

In this task, cardinality and subitising can be elicited, for example, if the solver loses count, or makes a mistake in one of her tap gestures, and needs to recognize the numbers that Nadia/TC has already pronounced by observing the smaller discs contained in the numbered ones.

5. Two children's insightful interactions with the tasks: answers to RQ2

In the following two sections we report on excerpts from two children's interactions: one for Task 1 and one for Task 2. We chose these excerpts for two reasons, which concern both the particular children involved and for their duration. Indeed, the children that appear in the excerpts, Fred (Task 1) and Maria (Task 2) were not selected depending on their initial interviews[4], but because they were the more engaged and open to communicate with the interviewer, and they were able to perform relatively "clean" taps on tap screen, showing a sufficient level of finger gnosis and control to be able to interact almost always with the iPad according to what seemed to be the strategies they planned. Moreover, the excerpts show changes in the schemes used over a particularly short time span, and so they can be described accurately and almost entirely included in a limited space.

In the excerpts, we refer to the interviewer by her first name, Gemma.

5.1 Qualitative analysis for Task 1: schemes and number sense abilities mobilised

Fred seems to be positively engaged during each meeting with the researcher. He is so enthusiastic over the three attempts he made to accomplish Task 1 that, after having been successful, he invents a new—more difficult—task for himself: this consists in putting only "Cristiano's [a soccer player] number" (number "7") on the shelf. In this section, we describe Fred's interaction with TC and with Gemma in response to Task 1. During Fred's interaction there were two changes of schemes, and different rules of action came into play. We divide the interaction into three phases: for each of these, we include a description of the video excerpt, which contains the rules of action, and we comment it, highlighting the changes in the schemes and the number sense abilities mobilised.

We provide a link to the video of Fred's interaction with TC, Gemma and Task 1, as supplementary material (see supplementary material "Fred and task 1.m4v").

Phase 1: First scheme (and free exploration)

The first phase of Fred's interaction with Task 1 lasts 35'' and begins with his first attempt to accomplish the task, which we show in the following; it continues for a few seconds during which Fred freely explores the Enumerating World, of which we omit the description.

[4] Fred answered 4 of the 13 initial interview questions correctly, i.e. the minimum number to be selected for the tasks; while Maria answered 10 questions correctly, i.e. the maximum number reached by the involved children.

Fred: *Taps index finger on the screen, producing 1 on the shelf.*

Gemma: Ah, it stayed up!

Fred: *With a fluid gesture he taps above the shelf, making a 2 and, without lifting his finger off the screen, he drags it down off the shelf. He also tries to drag the 1, but in doing this he makes and then drags down a 3; again, tapping on the disc with the 1, he makes and drags down a 4.* Ohhh! (*disappointed*).

Gemma: Oh, it fell, let's start over! (*She refreshes the screen*).

Fred: (*He smiles, amused*).

From the beginning, Fred's anticipation seems aligned with the task as it was designed: to put only the disc with "4" on the shelf. Even before making the 4 he tries to remove the 1 he had placed above the shelf and, as soon as he accidentally let 4 fall, he stopped, disappointed. He did not seem to correctly anticipate how the Enumerating World works (which is that previously made discs cannot be moved because this would break the process of creating the counting numbers in sequence, advancing with each touch), and this seemed to inhibit his initial success in the task.

The number sense abilities mobilised during the implementation of this first scheme belong mainly to ordinality (associating number symbols to number words; knowing the number symbols sequence; knowing the number words sequence). We note that finger control, and in particular controlling the precision of gestures with finger, is only marginally mobilised, since Fred only used a simple single-finger tap gesture.

Phase 2: Second scheme

After the first phase, Gemma calls Fred's attention back to the task and reminds him of the experience with the numbers that fell down when released under the shelf. Then Fred makes 1, 2, 3 and 4 sequentially, all under the shelf and stops. The second phase lasts 20''.

Gemma: Do something for me: try to leave on the shelf (*pointing to the shelf*) only the number four.

Fred: Four, four (*waving his open hand above the iPad without touching it*).

Gemma: Do you remember that before the numbers were falling (*pointing to the area of the screen under the shelf*)? Why were they falling before?

Fred: *He makes 1 under the shelf and lets it fall.*

Gemma: Good job!

Fred: *He makes 2 under the shelf and lets it fall; he does the same with 3 and with 4.* Ahhhh (*disappointed*).

Gemma: It fell down.

The researcher's intervention prompted the use of a new scheme, in which all four single finger taps are done below the shelf. The implementation of this scheme appears to be useful to Fred for two reasons. First, it allowed him to rediscover the functioning of the Enumerating World and its shelf. Second, the fact that this time the four discs are all created with the same gesture, making the four numbers appear (and immediately fall) one after the other, sequentially, might have led Fred to a deeper awareness of the sequence of the first four numbers (and so about ordinality). This hypothesis is supported by the third and last scheme that Fred developed to accomplish Task 1.

Phase 3: Third (successful) scheme

Fred decides to try again, and this time he is successful: he taps three times under the shelf and then once above it.

Gemma: Shall we try again?

Fred: *Smiles*.

Gemma: Let's try again! (*She refreshes the screen*).

Fred: *He quickly makes 1 under the shelf and lets it fall; he does the same for 2; he is about to tap again but stops and looks at Gemma, then hesitates pointing to the screen under the shelf, and finally makes 3 and lets it fall;*

Gemma: Three.

Fred: *Immediately, he taps above the shelf and makes 4.*

Gemma: Very good! Four on the shelf: that's exactly what we wanted! (*very enthusiastic*)

Fred: Yes.

A careful analysis of Fred's gestures and actions in this third phase, and in particular of the duration of the pauses between one gesture and thenext, seems to suggest that Fred is on the verge of developing new knowledge about what numbers follow and precede each (ordinality) in the succession of the first four numbers. Indeed, the immediacy with which he created the first two discs, his hesitation in making 3 below the shelf, as well as the rapidity with which he made the fourth disc above the shelf, can be seen as representative of *a new emerging awareness: that 4 is the successor of 3, and not the successor of 2.*

5.2 Qualitative analysis for Task 2: schemes and number sense abilities mobilised

In this section, we present excerpts from the interactions of Maria with Task 2. Maria successfully makes Nadia count up to eight. The excerpt shows different schemes for the production of herds of different numbers; the schemes seem to depend on the position of the number to make in the number sequence, or Maria's immediately previous experiences. The entire interaction between Maria, Gemma, TC and the task is quite long (it lasts almost three minutes) and contains several moments in which Maria correctly represents on her fingers the number that she wanted on the iPad, but she fails in producing a "clean" (and so successful) multi-touch on the screen for the herds 7 and 9. We provide a link to the video of Maria's entire interaction with TC, Gemma and Task 2, as supplementary material (see supplementary material "Maria and task 2.m4v").

Phase 1 (Maria): Producing herds of 1, 2, 3 and 4

Encouraged by the interviewer, Maria produces the first four numbers along a horizontal line, from right to left, in a short time (14''), without any hesitation.

> Gemma: Let's see!
>
> Maria: *Taps with her right index finger,* one *(pronounced simultaneously with the vocal feedback of TC), then taps with her right index and middle fingers,* two *(also pronounced simultaneously with TC).*
>
> Gemma: Two, bravissima. Now three…
>
> Maria: *Taps with her right thumb, index and middle fingers,* three (anticipating TC's vocal feedback)
>
> Gemma: Bravissima. Now?
>
> Maria: *Prepares four fingers, extending her right thumb, index, middle finger and ring finger all-at-once, and taps,* four (anticipating TC's vocal feedback again).
>
> Gemma: Brava!

We can identify two slightly different types of rules of action: in order to produce the first three herds, Maria immediately taps on the screen with the correct number of fingers; instead, before tapping to make the fourth herd, she first prepares her fingers. So, to produce the first three numbers, her scheme is supported by knowledge of the number sequence, by knowledge of configurations of fingers immediately associated to each number and by well-developed finger control (that she is aware of). For four, her scheme also seemed to be supported by knowledge of the number sequence to four, by knowledge of a configuration of fingers immediately associated to four, and by slightly less well-developed finger control: she seems to be aware of the fact that accurately tapping with the four finger she has prepared needs careful control, hence her slight hesitation.

Phase 2 (Maria): Producing a herd of 5

After producing the first four numbers, Maria says that she does not remember what follows, but she is helped by Gemma, who counts up from one to four verbally; Maria says “five” and then produces a herd of 5, after a brief hesitation. The second phase lasts 19’’.

Gemma:	And then?
Maria:	and then…
Gemma:	What comes next?
Maria:	I don’t remember.
Gemma:	One, two, three, four…
Maria:	Five, *with her five fingers stretched out, palm down, about to touch, hesitating.*
Gemma:	Five! How is five?
Maria:	Like this? (*hesitating while looks at Gemma with her five fingers stretched out, palm down, about to touch)*.
Gemma:	Try, try!
Maria:	*Places her fingers on the screen and makes 5*.
Gemma:	Brava! *(clapping her hands).* What a big number!

First, Maria is helped in this task by listening to the number sequence from one to four, which enables her to say the number after four. Once she had remembered that "five" comes after "four", she hesitates when it comes to representing it with her fingers; only after being encouraged by Gemma does she produce the five-fingers gesture touch.

Phase 3 (Maria): Producing a herd of 6

In order to go on after the herd of 5, Maria counts on her fingers up from one to six and then she taps on the screen; it takes her 16’’ to do this.

Gemma: And what comes next?

Maria: *Counting on her fingers, left hand palm up*, one, two, three, four, five, *then brings out her right hand and extends her thumb,* six (*pause*), six (*excitedly).*

Gemma: Do you want to try to do six? With two hands.

Maria: Yes, *and taps with the prepared fingers on the screen, with a clean gesture.*

Gemma: Bravissima. You're getting these number right away?!

Maria: Yes.

The scheme used to produce a herd of 6 is different from the previous ones: Maria uses both her voice and her fingers to find the correct number word and finger configuration, and thus prepares the required gesture. The number sense abilities that seem to be most elicited in this phase are: knowing what number comes after five—this knowledge is not immediate, but Maria does seem to be able to reconstruct it—which belongs to ordinality; and, representing a certain number on her fingers (in this case, six), which she also does not recall immediately, but is able to reconstruct.

Phase 4 (Maria): Two different schemes for a herd of 7

Gemma continues to encourage Maria to make the next number. Maria proceeds as she did for the herd of 6, but this time she struggles to find the number word that comes after six and to make a clean gesture on the screen. The fourth phase lasts 1'. In the excerpt below, we include all Maria's gestures and utterances other than those related to her realizing a clean gesture through appropriate finger control, since they are difficult to describe in writing.

Gemma: And what comes next?

Maria: *Counting quickly on fingers, from the left to the right hand, up to extending all of them* one, two, three, four, five, six, seven, eight, nine, (*pause*) nine! (*She fails in associating the raising of a new finger with the pronunciation of a new number word*).

Gemma: What comes after six?

Maria: *Counting more slowly on fingers of both hands* one, two, three, four, five, six (*slowly*), seven *and folds the two fingers on her right hand and counts them again*, six seven (*pause*) seven! (*excitedly*, *placing her correctly prepared fingers on the screen but making 4*).

[Over the course of 37’’, Maria makes two new attempts, in which she makes 6 twice, due to imprecise touches].

Maria: *Extends seven fingers all-at-once, then places them on the screen and makes 7.*

The scheme used to produce the herd of 7 is similar to the one used for the herd of 6, so it elicits analogous number sense abilities (see Phase 3). However, Maria faces some difficulties, and this gives her the opportunity to develop a new aspect of number sense. At first, she does not associate the gesture of raising a new finger with the pronunciation of a new number word. Gemma’s feedback suggests that she count more carefully, which lead to success. Maria seems to be developing the ability to establish one-to-one correspondence (cardinality). We notice that at her third and successful attempt, she extends all her fingers at once, showing a possible new awareness of a finger representation of 7, which she did not have at the beginning of the phase.

Task 2 is made up of various subtasks that depend on the number that the child wants to make Nadia say. Therefore, a skilled solver like Maria will use different schemes, that will surely vary based on the specific anticipation for the subtask (e.g., to produce a heard of 5 or of 6). In this excerpt we also noticed that the variation depends on the size of the number to generate. While for the first three numbers Maria produced the herds immediately, for the fourth she needed to prepare her fingers and look at them before tapping, even though she extended them all at once. Later she made use of the verbal sequence of number words, with the help of Gemma, to understand that she needed to make a herd of 5. For six Maria changed her scheme again. She seemed to need to prepare the correct gesture raising her fingers one at a time as she counted up from one. For the herd of 7, something quite interesting happened: Maria initially proceeded as she had for 6, but after having failed repeatedly, she prepared seven fingers all-together and successfully produces the herd with a clean touch. This kind of immediate production of the multiple fingers, as opposed to counting them out, has been described in previous research as *inverse gestural subitising*[5] (Sinclair & Pimm, 2015; Sinclair & Zazkis, 2017). *It seems that at the end of Task 2 Maria had attained the inverse gestural subitising for 7.*

[5] Sinclair and Zazkis (2017) describe how the request of “making four all-at-once” in TC “requires that the children produce an action (quickly lifting up their fingers and pressing them on the screen, instead of pressing fingers one by one) based on an oral prompt” (p. 185).

6. Discussion and conclusions

The initial aim of our research was to identify the numerical abilities elicited in 3-to-4 year-old children through their interactions with a digital multi-touch app. Since most of the literature has focused on apps that invite quite constrained or directed interactions, and that focus particularly on cardinality, we chose to use the more open environment TC. Its open nature comes with the requirement of task design, and in our study we were particularly interested in pursuing a task design that explicitly addressed ordinality. This, in turn, prompted us to take up the definition of ordinality as it relates to number sense. A critical analysis of existing studies led to Table 1, a schematic organisation of number sense ability that centrally features ordinality. Of course, given the interconnectedness of the number sense abilities listed in the table, there may exist other ways of organising them. However, we are convinced that an organisation of this kind—which takes into account some of the most basic ordinal and cardinal aspects of the number, as well as finger gnosis and finger control, subitising and representing numbers with fingers—is necessary for careful analysis. This is the first generalisable aspect of our study.

A major challenge in this study was to construct an appropriate theoretical framework for our study that aimed at designing tasks with the potential of mobilising the number sense abilities in focus, describing the potential of such a design (a priori), and making inferences on the number sense abilities that children were using (a posteriori).

The theoretical framework we developed indeed made it possible to identify the specific number sense abilities that could be mobilised through our tasks in TC, and then to identify the ones actually mobilised in the interviews that were conducted. So, a second important generalisable contribution of the study is having developed a theoretical framework that allow us to *see children's learning*, which is a notoriously difficult endeavour because young children have developed fewer linguistical means and more inferences need to be made by the researchers. The information that this theoretical framework allowed us to obtain can be very useful in the design of new tasks and in the analysis of the (even very young) children's responses to them, enabling us to pursue specific educational objectives related to the development of number sense.

Moreover, we wish to discuss what we consider to be a very interesting finding from an educational point of view that emerged from the analyses in section 5. In an open environment, and in the context of tasks that seem developmentally appropriate—some might say in the "zone of proximal development", in the sense of Vygotsky (1986)–*the children were free to experiment with different schemes and demonstrate new numerical abilities in the context of the TC activity, even during relatively brief interactions*. Consider both the progress of Fabio, who by the end of the extract, seemed to know that 4 comes after 3 (ordinality) and that of Maria, who succeeded in immediately producing the finger configuration 7 (representing numbers with fingers). Indubitably, the children's changing of schemes was also facilitated by the interventions of the interviewing researcher. These included encouraging the children to try again when they received unexpected feedback, showing the children how to make a multi-finger tap, as well as asking probing questions such as "how can you do it with your fingers?" or "and what number would come next?".

Finally, we remark that unlike the apps discussed by Baccaglini-Frank and Maracci (2015), which are designed to be used without teacher/adult intervention, the effectiveness of TC can heavily depend on teacher/adult mediation. For this reason, studying the researcher's interventions and the effect they produced on the children's responses, can be useful for adapting these and other tasks so they can be used explicitly for teaching. In this respect, an important difference between open environments like those proposed in TC and other closed environments, that are much more common, emerges clearly: TC is designed to satisfy educational aims such as exploration and open-ended tasks, where several different solution strategies are available and can be negotiated and discussed to deepen children's learning.

References

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-Touch Technology and Preschoolers' Development of Number-Sense. *Digital Experiences in Mathematics Education, 1*(1), 7–27. DOI:10.1007/s40751-015-0002-4

Baccaglini-Frank, A. (2018). What schemes do preschoolers develop when using multi-touch applications to foster number sense (and why)? In I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education* (pp. 223–243), ICME-13 Monographs. Springer.

Barendregt, W., Lindstrom, B., Rietz-Leppanen, E., Holgersson, I., & Ottosson, T. (2012). Development and evaluation of Fingu: A mathematics iPad game using multi-touch interaction. *IDC 2012*, June 12–15, 2012, Bremen, Germany, 1–4.

Bartolini Bussi, M. G. & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English & al. (Eds.), *Handbook of International Research in Math. Educ.*, second edition (pp. 746–783). New York: Routledge.

Bèguin, P., & Rabardel, P. (2000). Designing for instrument-mediated activity. *Scandinavian Journal of Information Systems*, *12*, 173–190.

Broda, M., Tucker, S., Ekholm, E., Johnson, T. N., & Liang, Q. (2018): Small fingers, big data: Preschoolers' subitizing speed and accuracy during interactions with multitouch technology. *The Journal of Educational Research*, online first, DOI: 10.1080/00220671.2018.1486281

Bullock, E.P., Shumway, J.F., Watts, C.M., & Moyer-Packenham, P.S. (2017). Affordance Access Matters: Preschool Children's Learning Progressions While Interacting with Touch-Screen Mathematics Apps. *Technology Knowledge and Learning, 22*, 485–511.

Butterworth, B. (2005). The development of arithmetical abilities. *Journal of Child Psychology and Psychiatry, 46*, 3–18.

Coles, A. (2014). Ordinality, neuroscience and the early learning of number. In P. Liljedahl, Nicol, C., Oesterle, S., & Allan, D. (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 2* (pp. 329–336). Vancouver, Canada: PME.

Coles, A., & Sinclair, N. (2018a). Re-Thinking 'Normal' Development in the Early Learning of Number. *Journal of Numerical Cognition, 4*(1), 136–158.

Coles, A., & Sinclair, N. (2018b). Re-thinking 'concrete to abstract': towards the use of symbolically structured environments. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 275–282). Umeå, Sweden: PME.

Elia, I., Mulligan, J., Anderson, A., Baccaglini-Frank, A., & Benz, C. (2018). *Contemporary Research and Perspectives on Early Childhood Mathematics Education*. ICME-13 Monographs. Springer.

Fraenkel, A. (1942). *Introduction to mathematics*. Tel-Aviv: Masada.

Fuson, K. C., & Hall, J. W. (1983). The acquisition of early number word meanings. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 49–107). New York: Academic.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge: Harvard University Press.

Gracia-Bafalluy, M. G., & Noël, M. P. (2008). Does finger training increase young children's numerical performance? *Cortex, 44*, 368–375.

Holgersson, I., Barendregt, W., Emanuelsson, J., Ottosson, T., Rietz, E., & Lindström, B. (2016). Fingu—A Game to Support Children's Development of Arithmetic Competence: Theory, Design and Empirical Research. In Moyer-Packenham, P.S. (Ed.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives* (pp. 123–146). Mathematics Education in the Digital Era. Springer.

Jackiw, N., & Sinclair, N. (2014). *Touchcounts* [computer software]. Tangible mathematics project, Simon Fraser University.

Lafay, A., Thevenot, C., Castel, C., & Fayol, M. (2013). The role of fingers in number processing in young children. *Frontiers in Psychology, 4*, 488. doi:10.3389/fpsyg.2013.00488.

Lyons, I., & Beilock, S. (2011). Numerical ordering ability mediates the relation between number-sense and arithmetic competence. *Cognition*, *121*(2), 256–261.

Margolinas, C., & Wosniak, F. (2012). *Le nombre à l'école maternelle. Une approche didactique*. Bruxelles: De Boeck.

Moeller, K., Martignon, L., Wessolowski, S., Engel, J., & Nuerk, H.-C. (2011). Effects of Finger Counting on Numerical Development? The Opposing Views of Neurocognition and Mathematics Education. *Frontiers in Psychology, 2*, 1–5. http://doi.org/10.3389/fpsyg.2011.00328

Moyer-Packenham, P., Shumway, J.F., Bullock, E., Tucker, S.I., Anderson-Pence, K.L., Westenskow, A., ..., Jordan, K. (2015). Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps. *J. Comput. Math. Sci. Teach. 34*(1), 41–69.

Mulligan, J., Verschaffel, L., Baccaglini-Frank, A., Coles, A., Gould, P., He, S., ..., Yang, D.-C. (2018). Whole number thinking, learning and development: neuro-cognitive, cognitive and developmental approaches. In M.G. Bartolini Bussi, X.H. Sun (eds.), *Building the Foundation: Whole Numbers in the Primary Grades* (pp. 137–167), New ICMI Study Series, Springer.

Nesher, P. (2018). Whole Number Thinking, Learning and Development: A Commentary on Chapter 7. In M.G. Bartolini Bussi, X.H. Sun (eds.), Building the Foundation: Whole Numbers in the Primary Grades, New ICMI Study Series (pp. 169–179). Spinger.

Penner-Wilger, M., Fast, L., LeFevre, J. A., Smith-Chant, B. L, Skwarchuk, S.L, Kamawar, D., & Bisanz, J. (2007). The Foundations of Numeracy: Subitizing, Finger Gnosia, and Fine Motor Ability. In D. S. McNamara & J. G. Trafton (Eds.), Proceedings of the 29th annual conference of the cognitive science society. Austin, TX: Cognitive Science Society. pp. 1385–1390.

Pimm, D., & Sinclair, N. (2015). "How do you make numbers?": Rhythm and turn-taking when coordinating ear, eye and hand. In K. Krainer & N. Vondrová (Eds.), Proceedings of the 9th Congress of the European Society for Research in Mathematics Education (pp. 1961–1968). Prague, Czech Republic: ERME.

Rothschild, M and Caroline C. Williams. (2015). Apples and Coconuts: Young Children 'Kinecting' with Mathematics and Sesame Street. In T. Lowrie, R. Jorgensen (Eds.), *Digital games and mathematical learning Potential, promises and pitfalls* (pp. 123–141). Springer

Sedaghat, J. M., & Campbell, S. (2017). Exploring cardinality in the era of touchscreen-based technology. *International Journal of Mathematical Education in Science and Technology*, *48*(8), 1225–1239.

Sinclair, N. & SedaghatJou, M. (2013). Finger Counting and Adding with Touch Counts. In Ubuz, Haser, Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (pp. 2198–2207). Antalya, Turkey.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning Number with TouchCounts: The Role of Emotions and the Body in Mathematical Communication. *Tech Know Learn*, 19, 81–99.

Sinclair, N., & Pimm, D. (2015). Mathematics using multiple sense: Developing finger gnosis with three- and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education*, 9(3), 99–109.

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 662 – 686). Taylor and Francis.

Sinclair, N., & Zazkis, R. (2017). Everybody counts: designing tasks for TouchCounts. In Leung, A. & Baccaglini-Frank, A. (Eds.), *Digital Technologies in Designing Mathematics Education Tasks, Mathematics Education in the Digital Era 8* (pp. 175 – 191). Springer.

Tucker, S.I., Moyer-Packenham, P.S., Westenskow, A. & Jordan, K.E. (2016). The Complexity of the Affordance–Ability Relationship When Second-Grade Children Interact with Mathematics Virtual Manipulative Apps. *Technology Knowledge and Learning, 21*,341–360.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94.

Vygotsky, L. (1986). *Thought and Language*. Translated and edited by Alex Kozulin. Cambridge, MA: MIT Press.