LOW ENERGY CONFIGURATIONS OF TOPOLOGICAL SINGULARITIES IN TWO DIMENSIONS: A Γ-CONVERGENCE ANALYSIS OF DIPOLES

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ABSTRACT. This paper deals with the variational analysis of topological singularities in two dimensions. We consider two canonical zero-temperature models: the core radius approach and the Ginzburg-Landau energy. Denoting by ε the length scale parameter in such models, we focus on the $|\log \varepsilon|$ energy regime. It is well known that, for configurations whose energy is bounded by $c|\log \varepsilon|$, the vorticity measures can be decoupled into the sum of a finite number of Dirac masses, each one of them carrying $\pi|\log \varepsilon|$ energy, plus a measure supported on small zero-average sets. Loosely speaking, on such sets the vorticity measure is close, with respect to the flat norm, to zero-average clusters of positive and negative masses.

Here we perform a compactness and Γ-convergence analysis accounting also for the presence of such clusters of dipoles (on the range scale ε^s, for $0 < s < 1$), which vanish in the flat convergence and whose energy contribution has, so far, been neglected. Our results refine and contain as a particular case the classical Γ-convergence analysis for vortices, extending it also to low energy configurations consisting of just clusters of dipoles, and whose energy is of order $c|\log \varepsilon|$ with $c < \pi$.

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INTRODUCTION

Beyond its relevant applications in Physics and Materials Science, the analysis of topological singularities, as the length-scale parameter ε tends to zero, is a very fascinating problem in mathematical analysis. A celebrated model for the study of topological singularities is the so-called Ginzburg-Landau functional. We deal with its very basic version, i.e., without magnetic field. Let $\Omega \subset \mathbb{R}^2$ be open, bounded, with smooth boundary. For any $\varepsilon > 0$, the Ginzburg-Landau functionals $GL_\varepsilon : H^1(\Omega; \mathbb{R}^2) \to [0, +\infty]$ are defined as

$$(0.1) \qquad GL_\varepsilon(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2} W(|u|) \, \mathrm{d}x,$$

where $W \in C^0([0, +\infty))$ is such that $W(t) \geq 0$, $W^{-1}\{0\} = \{1\}$ and

$$\liminf_{t \to 1} \frac{W(t)}{(1-t)^2} > 0, \quad \liminf_{t \to \infty} W(t) > 0 \,.$$

In the monography [5], Bethuel, Brezis and Hélein collect the main results about the asymptotic behaviour (as $\varepsilon \to 0$) of the minimizers of GL_ε with a prescribed boundary datum $g : \partial\Omega \to \mathcal{S}^1$ having non-zero degree. Since then, much work has been devoted to understand the behaviour of sequences of functions $\{u_\varepsilon\}$ which are not necessarily minimizers but satisfy prescribed energy bounds; the natural language to face this problem is provided by the notion of Γ-convergence.

The starting point of such analysis has been the study of the regime $|\log \varepsilon|$, corresponding to a finite number of singularities in the limit. Sharp lower bounds for the energy $\frac{GL_\varepsilon}{|\log \varepsilon|}$ are given in [18, 13]. In [14] a Γ-convergence result in $W^{1,1}(\Omega; \mathbb{R}^2)$ is provided together with a compactness analysis of the vorticity measures, identified with the Jacobians Ju_ε of u_ε. Specifically, up to a subsequence, the Jacobians Ju_ε converge in the dual norm of Hölder continuous functions to a measure consisting of a finite sum of Dirac masses, representing the limit vortices. Self-contained

short proofs of the compactness of the Jacobians in the flat norm and of the Γ- convergence result are collected in [4]. The "first-order" Γ-convergence of the functional $GL_\varepsilon - M\pi|\log\varepsilon|$ (where M is the number of the singularities) has been largely studied (see, for instance, [20, 19, 4]). The Γ-limit is the so-called *renormalized energy*, depending on the position of the limit singularities and governing their dynamics [21, 22, 16, 10]. We end up with this list by recalling that in [1] the Γ-convergence analysis of GL_ε is developed in any dimension and codimension.

Another natural and perhaps simpler model for topological singularities, particularly popular in Materials Science, is the *core radius approach*. Here the order parameter takes values in $\mathcal{S}^1$ and has a finite number of singularities. The core radius approach consists in drilling the domain, by removing disks of radius ε around each singularity, cutting in this way the logarithmic tail of the energy. In the specific model we deal with, we consider convenient to enforce that the singularities have minimal mutual distance of order ε: The set of admissible configurations of topological singularities is defined as

$$X_\varepsilon(\Omega) := \{\mu = \sum_{i=1}^N z_i \delta_{x_i} \,:\, N \in \mathbb{N},\, z_i \in \mathbb{Z},\, x_i \in \Omega,\, \text{dist}\,(x_i, \partial\Omega) \geq 2\varepsilon,\, |x_i - x_j| \geq 4\varepsilon \quad \forall i \neq j\}.$$

The energy functional $\mathcal{E}_\varepsilon : X_\varepsilon(\Omega) \to \mathbb{R}$ induced by the distribution of singularities μ is given by

$$(0.2) \qquad\qquad \mathcal{E}_\varepsilon(\mu) := \frac{1}{2} \min_{v \in \mathcal{AS}_\varepsilon(\mu)} \int_{\Omega_\varepsilon(\mu)} |\nabla v|^2 \,\mathrm{d}x \,,$$

where

$$\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{x_i \in \text{supp}\,\mu} B_\varepsilon(x_i)$$

and the class of admissible order parameters associated to μ is given by

$$\mathcal{AS}_\varepsilon(\mu) := \left\{ v \in H^1(\Omega_\varepsilon(\mu); \mathcal{S}^1) \,:\, \deg(v, \partial B_\varepsilon(x_i)) = \mu(x_i) \text{ for all } x_i \in \text{supp}\,\mu \right\}.$$

The Γ-convergence analysis for the functionals $\mathcal{E}_\varepsilon$ gives the same outcome of the one developed for GL_ε; in particular, a sequence of measures $\{\mu_\varepsilon\}$ with $\sup_\varepsilon \frac{\mathcal{E}_\varepsilon(\mu_\varepsilon)}{|\log\varepsilon|} < +\infty$ converges, up to a subsequence, to a finite sum μ of Dirac deltas (see [17]).

In the results mentioned so far, the Γ-convergence analysis is done with respect to the flat norm of the vorticity measures, neglecting somehow the contribution of short (in terms of ε) clusters of dipoles of vortices. In this respect a natural question is how to describe these clusters, quantifying their energy contribution. A first result in this direction has been proven in [15] for the Ginzburg-Landau functional. In such a paper, the authors provide some fine estimates on the flat distance between Ju_ε and the class of measures which are sum of Dirac masses with integer coefficients. In view of this result we can look at Ju_ε as a superposition of zero-average clusters and isolated vortices. However, to the best of our knowledge, a quantitative analysis - both in topology and in energy - of such zero-average clusters is not available in the literature. In the present paper we (partially) accomplish this task by analyzing the behaviour and the energy contribution of zero-average clusters whose size can be expressed in terms of ε^s, with $0 < s < 1$.

To this end, we first consider the measures μ_ε (resp. Ju_ε) and their convolution μ_ε^s (resp. $J^s u_\varepsilon$) with a mollifier whose support is of order ε^s. It turns out that these measures average out all clusters of dipoles whose size is smaller than ε^s, while they provide a good description for the effective vorticity at mesoscopic scales of order ε^s. We show that, in the $|\log\varepsilon|$ regime, μ_ε^s (resp. $J^s u_\varepsilon$) weak star converge, up to a subsequence, to the flat limit μ of μ_ε (resp. Ju_ε). In this respect we obtain a new compactness property, namely in the weak star topology, for the vorticity measures.

In order to account for the short dipoles, we consider the measures $|\mu_\varepsilon^s|$ (resp. $|J^s u_\varepsilon|$), and we prove that they also enjoy suitable weak star compactness properties. The advantage of considering these total variation measures is that their limits keep track of all clusters of vortices of size larger than ε^s. In particular, the limit family of measures parametrized by s classifies them according to their length. Specifically, we prove that, up to a subsequence independent of s, the measures $|\mu_\varepsilon^s|$ (resp. $|J^s u_\varepsilon|$) converge to some limit measure ν^s, with $\nu^s = |\mu| + 2\xi_{\text{def}}^s$, being ξ_{def}^s a positive

sum of Dirac deltas. Moreover, ν^s is piecewise constant and non-decreasing with respect to s and it has a countable set $S = \{0 = s_0 < s_1 < \ldots\}$ of jumps. The measures ξ_{def}^s describe the limit density of zero-average clusters of vortices at all the scales parametrized in terms of all powers $s \in (0,1)$ of ε. Finally, we prove that the Γ-limit of the energy $\frac{\mathcal{E}_\varepsilon}{|\log \varepsilon|}$ (resp. $\frac{GL_\varepsilon}{|\log \varepsilon|}$) with respect to the convergence of μ_ε, μ_ε^s and $|\mu_\varepsilon^s|$ (resp. Ju_ε, $J^s u_\varepsilon$ and $|J^s u_\varepsilon|$) is given by

$$\pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega).$$

In particular, this energy is minimized for $\xi_{\mathrm{def}}^s \equiv 0$, corresponding to $\nu^s \equiv |\mu|$, and in this case it gives back the classical Γ-limit $\pi|\mu|(\Omega)$. Moreover, our Γ-convergence analysis provides a new non-trivial outcome whenever the energy bound is lower than $\pi|\log \varepsilon|$, that is, lower than the minimal energy of a single isolated vortex. More precisely, for configurations with energy of order $c|\log \varepsilon|$, with $c < \pi$, the macroscopic limit vorticity is zero, while ξ_{def}^s, and in turn ν^s can be different than zero for all $0 < s < 1$ large enough (depending on the prefactor c).

The proofs of our results exploit the ball construction by Sandier [18] and Jerrard [13]. It is a powerful machinery providing lower bounds and compactness for the vorticity measures μ_ε in the flat norm. Here we slightly revisit it in order to get also weak star compactness for the measures μ_ε^s and $|\mu_\varepsilon^s|$, as well as to quantify their energy contribution in the limit as $\varepsilon \to 0$.

To conclude, let us mention that clusters of vortices with zero-average are relevant in many physical systems. They first appear (and then remain together with isolated vortices) as the temperature, and more in general the free energy, increases. In the context of screw dislocations in crystals, they are somehow identified with the so-called statistically stored dislocations. Our analysis is a first attempt to describe these objects quantifying their energy contribution, within a purely variational approach at zero temperature, in the rigorous framework of Γ-convergence. We believe that the analysis we have developed here for GL_ε and $\mathcal{E}_\varepsilon$ can be extended with minor variations to the case of discrete vortices in the XY model and screw dislocations in crystal plasticity [3, 2, 17], whereas an extension to semi-discrete models for edge dislocations [11] appears less clear, and in our opinion deserves future investigations.

1. NOTATIONS AND PRELIMINARY RESULTS

In this section we introduce the notations that we will use throughout the paper. We start by fixing an open bounded subset Ω of $\mathbb{R}^2$ with Lipschitz continuous boundary.

1.1. **Weak star and flat convergence.** Let $C_c(\Omega)$ be the space of continuous functions compactly supported in Ω endowed with the L^∞ norm. A sequence $\{\mu_n\}$ of measures *weak star converges* in Ω to a measure μ if for any $\varphi \in C_c(\Omega)$

$$\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle \qquad \text{as } n \to +\infty.$$

In the following, wherever it is not specified, $\overset{\star}{\rightharpoonup}$ will denote the weak star convergence in Ω. Moreover, let $C^{0,1}(\Omega)$ be the space of Lipschitz continuous functions on Ω endowed with the norm

$$\|\psi\|_{C^{0,1}} := \sup_{x \in \Omega} |\psi(x)| + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\psi(x) - \psi(y)|}{|x - y|},$$

and let $C_c^{0,1}(\Omega)$ be its subspace of functions with compact support. The norm in the dual of $C_c^{0,1}(\Omega)$ will be denoted by $\|\cdot\|_{\mathrm{flat}}$ and referred to as *flat norm*, while $\overset{\mathrm{flat}}{\rightarrow}$ denotes the convergence with respect to the flat norm.

1.2. **Jacobian, current and degree.** Given $u \in H^1(\Omega; \mathbb{R}^2)$, the Jacobian Ju of u is the L^1 function defined by

$$Ju := \det \nabla u.$$

For every $u \in H^1(\Omega; \mathbb{R}^2)$, we can consider Ju as an element of the dual of $C^{0,1}(\Omega)$ by setting

$$\langle Ju, \psi \rangle := \int_\Omega Ju \, \psi \, \mathrm{d}x \qquad \text{for any } \psi \in C^{0,1}(\Omega).$$

Notice that Ju can be written in a divergence form as $Ju = \text{div}\ (u_1(u_2)_{x_2}, -u_1(u_2)_{x_1})$, i.e., for any $\psi \in C_c^{0,1}(\Omega)$,

$$(1.1) \qquad \langle Ju, \psi \rangle = -\int_\Omega u_1(u_2)_{x_2}\psi_{x_1} - u_1(u_2)_{x_1}\psi_{x_2}\,\mathrm{d}x.$$

Equivalently, we have $Ju = \text{curl}\ (u_1\nabla u_2)$ and $Ju = \frac{1}{2}\text{curl}\ j(u)$, where

$$j(u) := u_1\nabla u_2 - u_2\nabla u_1$$

is the so-called *current*.

Let $A \subset \Omega$ be open with Lipschitz boundary, and let $h \in H^{\frac{1}{2}}(\partial A; \mathbb{R}^2)$ with $|h| \geq \alpha > 0$. The *degree* of h is defined as follows

$$\deg(h, \partial A) := \frac{1}{2\pi}\int_{\partial A} \frac{h}{|h|} \cdot \frac{\partial}{\partial\tau}\Big(\frac{h_2}{|h|}, -\frac{h_1}{|h|}\Big)\,\mathrm{d}\mathcal{H}^1\,,$$

where τ is the tangent field to ∂A and the product in the above formula is understood in the sense of the duality between $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$; in [8, 9] it is proven that the definition above is well-posed, it is stable with respect to the strong convergence in $H^{\frac{1}{2}}(\partial A; \mathbb{R}^2 \setminus B_\alpha)$ and that $\deg(h, \partial A) \in \mathbb{Z}$ (here $B_\alpha = B_\alpha(0)$ stands for the ball of radius α centered at zero). Moreover, if $u \in H^1(A; \mathbb{R}^2 \setminus B_\alpha)$ for some $\alpha > 0$, then $\deg(u, \partial A) = 0$ (here and in the following we identify u with its trace). Finally, if $u \in H^1(A; \mathbb{R}^2)$ and $|u| = 1$ on ∂A, by Stokes theorem (and by approximating u with smooth functions) we deduce

$$(1.2) \qquad \int_A Ju\,\mathrm{d}x = \frac{1}{2}\int_A \text{curl}\ j(u)\,\mathrm{d}x := \frac{1}{2}\int_{\partial A} j(u)\cdot\tau\,\mathrm{d}\mathcal{H}^1 = \pi\deg(u, \partial A).$$

Notice that any $u \in H^1(A; \mathbb{R}^2 \setminus B_\alpha)$ can be written in polar coordinates as $u(x) = \rho(x)e^{i\theta(x)}$ on ∂A with $|\rho| \geq \alpha$, where θ is the so called *lifting* of u. By [6] (see also [7, Theorem 3] and [7, Remark 3]), if A is simply connected, then $\deg(u, \partial A) = 0$ and the lifting can be selected in $H^{\frac{1}{2}}(\partial A)$ with the map $u \mapsto \theta$ continuous (but the image of a bounded subset of $H^{\frac{1}{2}}(\partial A; \mathcal{S}^1)$ is not necessarily bounded in $H^{\frac{1}{2}}(\partial A)$). For A not necessarily simply connected, if Γ is a connected component of ∂A and the degree of u on Γ is equal to $d \in \mathbb{Z}$, then the lifting can be locally selected in $H^{\frac{1}{2}}(\Gamma)$ with a "jump" of order $2\pi d$.

Let us introduce a notion of modified Jacobian (a variant of the notion introduced in [1]), which we will use in our Γ-convergence results. Given $0 < \tau < 1$ and $u \in H^1(\Omega; \mathbb{R}^2)$, set

$$(1.3) \qquad u_\tau := T_\tau(|u|)\frac{u}{|u|}, \qquad J_\tau u := Ju_\tau, \qquad \text{where } T_\tau(\rho) = \min\Big\{\frac{\rho}{\tau}, 1\Big\}.$$

Notice that, for every $v := (v_1, v_2)$, $w := (w_1, w_2)$ belonging to $H^1(\Omega; \mathbb{R}^2)$ we have

$$(1.4) \qquad Jv - Jw = \frac{1}{2}\big(J(v_1 - w_1, v_2 + w_2) - J(v_2 - w_2, v_1 + w_1)\big).$$

By (1.1) and (1.4) we immediately deduce the following lemma.

Lemma 1.1. *There exists a universal constant $C > 0$ such that for any $v, w \in H^1(\Omega; \mathbb{R}^2)$, there holds*

$$\|Jv - Jw\|_{\text{flat}} \leq C\|v - w\|_{L^2}(\|\nabla v\|_2 + \|\nabla w\|_{L^2})\,.$$

By Lemma 1.1 we easily obtain the following proposition.

Proposition 1.2. *Let $\{u_n\}$ be a sequence in $H^1(\Omega; \mathbb{R}^2)$ such that $GL_{\varepsilon_n}(u_n) \leq C|\log\varepsilon_n|$, and let $\delta \in (0, \frac{1}{2})$. Then there exists $C_\delta > 0$ such that*

$$\sup_{\tau\in(\delta,1-\delta)}\|Ju_n - J_\tau u_n\|_{\text{flat}} \leq C_\delta\varepsilon_n|\log\varepsilon_n|,$$

$$\sup_{\tau\in(\delta,1-\delta)}|J_\tau u_n|(\Omega) \leq C_\delta|\log\varepsilon_n|\,.$$

1.3. **Mollifiers.** We denote by ρ a mollifier in $\mathbb{R}^2$, i.e., a positive, C^∞ and radially symmetric scalar function compactly supported in $B_1(0)$ with $\int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x = 1$. Moreover, for any $\eta > 0$, we define $\rho_\eta(\cdot) := \frac{1}{\eta^2}\rho(\frac{\cdot}{\eta})$. We recall that $\rho_\eta \in C^\infty$, $\operatorname{supp}\rho_\eta \subset B_\eta(0)$ and $\int_{\mathbb{R}^2} \rho_\eta(x)\,\mathrm{d}x = 1$. Finally, for any function $f \in L^1$, we define the mollification of f as

$$f * \rho_\eta(x) := \int_{\mathbb{R}^2} f(y)\rho_\eta(x - y)\,\mathrm{d}y\,;$$

analogously, the mollification of a Radon measure μ is defined via duality by

$$\langle \mu * \rho_\eta, \varphi \rangle := \langle \mu, \varphi * \rho_\eta \rangle, \qquad \text{for any } \varphi \in C_c.$$

By the standard properties of mollifiers, for any $\varphi \in C_c$, we have

$$\tag{1.5} \|\nabla(\varphi * \rho_\eta)\|_{L^\infty} \leq \|\varphi\|_{L^\infty}\|\nabla\rho\|_{L^1}\eta^{-1}\,.$$

2. Ball construction

We start this section by presenting the celebrated *ball construction* (introduced in [18, 12, 13]), which provides lower bounds of the Dirichlet energy in presence of topological singularities. We slightly revisit it, following the approach by Sandier [18], adopting the notations in [4].

Let $\mathcal{B} = \{B_{r_1}(x_1), \ldots, B_{r_N}(x_N)\}$ be a finite family of open balls in $\mathbb{R}^2$ with $\bar{B}_{r_i}(x_i) \cap \bar{B}_{r_j}(x_j) = \emptyset$ for $i \neq j$, and let $\mu = \sum_{i=1}^N z_i\delta_{x_i}$ with $z_i \in \mathbb{Z} \setminus \{0\}$.

Let moreover $F(\mathcal{B}, \mu, \cdot)$ be a function defined on open subsets of $\mathbb{R}^2$ satisfying the following properties:

(i) $F(\mathcal{B}, \mu, A \cup B) \geq F(\mathcal{B}, \mu, A) + F(\mathcal{B}, \mu, B)$ for all A, B open disjoint subsets of $\mathbb{R}^2$;

(ii) for any annulus $A_{r,R}(x) = B_R(x) \setminus \bar{B}_r(x)$ with $A_{r,R}(x) \cap \bigcup_i \bar{B}_{r_i}(x_i) = \emptyset$, there holds

$$\tag{2.1} F(\mathcal{B}, \mu, A_{r,R}(x)) \geq \pi|\mu(B_r(x))|\log\frac{R}{r}\,.$$

Remark 2.1. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bigcup_{B \in \mathcal{B}} B; \mathcal{S}^1)$ be such that $\mu = \sum_{B \in \mathcal{B}} \deg(u, \partial B)\delta_{x_B}$, where x_B is the center of B. Then, an explicit example of admissible functional $F(\mathcal{B}, \mu, \cdot)$ is given by

$$F(\mathcal{B}, \mu, A) := \frac{1}{2}\int_{A \setminus \bigcup_{B \in \mathcal{B}} B} |\nabla u|^2\,\mathrm{d}x,$$

for every open set $A \subset \mathbb{R}^2$. For further details see Remark 2.5.

Proposition 2.2. *There exists a one-parameter family of open balls $\mathcal{B}(t)$ with $t \geq 0$ such that, setting $U(t) := \bigcup_{B \in \mathcal{B}(t)} B$, the following properties hold true:*

(1) $\mathcal{B}(0) = \mathcal{B}$;

(2) $U(t_1) \subset U(t_2)$ *for any* $0 \leq t_1 < t_2$;

(3) *the balls in $\mathcal{B}(t)$ are pairwise disjoint;*

(4) *for any* $0 \leq t_1 < t_2$ *and for any open set* $U \subseteq \mathbb{R}^2$,

$$\tag{2.2} F(\mathcal{B}, \mu, U \cap (U(t_2) \setminus U(t_1))) \geq \pi \sum_{\substack{B \in \mathcal{B}(t_2) \\ B \subseteq U}} |\mu(B)|\log\frac{1 + t_2}{1 + t_1}\,;$$

(5) $\displaystyle\sum_{B \in \mathcal{B}(t)} r(B) \leq (1 + t)\sum_i r_i$, *where $r(B)$ denotes the radius of the ball B.*

Proof. In order to construct the family $\mathcal{B}(t)$, we closely follow the strategy of the ball construction due to Sandier and Jerrard. The ball construction consists in letting the balls alternatively expand and merge into each other as follows. The expansion phase consists in letting the balls expand, without changing their centers, in such a way that, at each (artificial) time t the radius $r_i(t)$ of the ball centered at x_i satisfies

$$\tag{2.3} \frac{r_i(t)}{r_i} = 1 + t \qquad \text{for all } i.$$

The first expansion phase stops at the first time T_1 when two balls bump into each other. Then the merging phase begins. It consists in identifying a suitable partition $\{S_j^1\}_{j=1,\ldots,N_n}$ of the family $\{B_{r_i(T_1)}(x_i)\}$, and, for each subclass S_j^1, in finding a ball $B_{r_j^1}(x_j^1)$ which contains all the balls in S_j^1 such that the following properties hold:

    i) $B_{r_j^1}(x_j^1) \cap B_{r_l^1}(x_l^1) = \emptyset$ for all $j \neq l$;

    ii) $r_j^1 \leq \sum_{B \in S_j^1} r(B)$.

After the merging, another expansion phase begins: we let the balls $\left\{B_{r_j^1}(x_j^1)\right\}$ expand in such a way that, for $t \geq T_1$, for every j we have

$$(2.4) \qquad\qquad \frac{r_j^1(t)}{r_j^1} = \frac{1+t}{1+T_1}.$$

Again note that $r_j^1(T_1) = r_j^1$. We iterate this procedure thus obtaining a set of merging times $\{T_1, \ldots, T_K\}$ with $K \leq N$ and a family $\mathcal{B}(t)$ for all $t \geq 0$; precisely, $\mathcal{B}(t)$ is given by $\{B_{r_j(t)}(x_j)\}_j$ for $t \in [0, T_1)$; for $t \in [T_k, T_{k+1})$, $\mathcal{B}(t)$ can be written as $\{B_{r_j^k(t)}(x_j^k)\}_j$ for all $k = 1, \ldots, K-1$, while it consists of a single expanding ball for $t \geq T_k$. By construction, we clearly have properties (1), (2) and (3). Moreover, (5) is an easy consequence of (2.3), (2.4) and property ii).

It remains to show property (4). We preliminarily note that, by (2),

$$(2.5) \qquad\qquad \sum_{\substack{B \in \mathcal{B}(\tau_1) \\ B \subseteq U}} |\mu(B)| \geq \sum_{\substack{B \in \mathcal{B}(\tau_2) \\ B \subseteq U}} |\mu(B)| \qquad \text{for any } 0 < \tau_1 < \tau_2.$$

Let $t_1 < \bar{t} < t_2$. In view of (2.5), if we show that (4) holds true for the pairs $(t_1, \bar{t})$ and $(\bar{t}, t_2)$, then (4) follows also for t_1 and t_2. Therefore, we can assume without loss of generality that $T_k \notin \,]t_1, t_2[$ for any $k = 1, \ldots, K$.

Let $t_1 < \tau < t_2$ and let $B \in \mathcal{B}(\tau)$. Then there exists a unique ball $B' \in \mathcal{B}(t_1)$ such that $B' \subset B$. By construction, $\mu(B) = \mu(B')$ and by (2.1) we have

$$F(\mathcal{B}, \mu, B \setminus B') \geq \pi |\mu(B)| \log \frac{1+\tau}{1+t_1},$$

which, summing up over all $B \in \mathcal{B}(\tau)$ with $B \subseteq U$, and using (2.5), yields

$$F(\mathcal{B}, \mu, U \cap (U(t_2) \setminus U(t_1))) \geq \pi \sum_{\substack{B \in \mathcal{B}(\tau) \\ B \subseteq U}} |\mu(B)| \log \frac{1+\tau}{1+t_1} \geq \pi \sum_{\substack{B \in \mathcal{B}(t_2) \\ B \subseteq U}} |\mu(B)| \log \frac{1+\tau}{1+t_1}.$$

Property (4) follows by letting $\tau \to t_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma collects some convergence results that will be used in the proofs of our main results. For any given $\psi : E \subset \mathbb{R}^2 \to \mathbb{R}$ we set $\mathrm{osc}_E(\psi) := \sup_E \psi - \inf_E \psi$. Moreover, for any family $\mathscr{B}$ of balls we define $\mathcal{R}ad(\mathscr{B}) := \sum_{B \in \mathscr{B}} r(B)$. Finally, we often denote by x_B the center of a ball B.

Lemma 2.3. *There exists a constant $C > 0$ such that the following holds true. Let $\mathscr{B}$ be a family of pairwise disjoint balls in $\mathbb{R}^2$ and let $\mathscr{C}$ be the family of balls in $\mathscr{B}$ which are contained in Ω. Let moreover α, β be two Radon measures supported in Ω with*

$$(2.6) \qquad \mathrm{supp}\,\alpha \subset \bigcup_{B \in \mathscr{C}} B, \qquad \mathrm{supp}\,\beta \subset \bigcup_{B \in \mathscr{B}} B \qquad \text{and} \quad \alpha(B) = \beta(B) \quad \text{for any } B \in \mathscr{C}.$$

Then, for any $\eta > 0$, there holds:

    (i) $\|\alpha - \beta\|_{\mathrm{flat}} \leq C \, \mathcal{R}ad(\mathscr{B})(|\alpha| + |\beta|)(\Omega)$;

    (ii) $\sum_{B \in \mathscr{C}} |((\alpha - \beta) \llcorner B) * \rho_\eta| \leq C \|\nabla \rho\|_{L^1} \eta^{-1} \mathcal{R}ad(\mathscr{B})(|\alpha| + |\beta|)(\Omega)$;

    (iii) *for any $\varphi \in C_c(\Omega)$*

$$\sum_{B \in \mathscr{C}} |\langle |\alpha(B)| \delta_{x_B} - |\alpha(B)| \delta_{x_B} * \rho_\eta, \varphi \rangle| \leq C \, |\alpha|(\Omega) \, \omega_\varphi(\eta),$$

where ω_φ denotes the modulus of continuity of φ.

Proof. We divide the proof in three steps corresponding to the three facts stated in the Lemma.
Step 1: Proof of (i). Set $\mathscr{D} := \mathscr{B} \setminus \mathscr{C}$. Let $\psi \in C_c^{0,1}(\Omega)$ with $\|\psi\|_{C^{0,1}} \le 1$. By (2.6) we have

$$\langle \alpha - \beta, \psi \rangle = \sum_{B \in \mathscr{C}} \int_B \psi \, \mathrm{d}(\alpha - \beta) - \sum_{B \in \mathscr{D}} \int_B \psi \, \mathrm{d}\beta$$

$$\le \sum_{B \in \mathscr{C}} \mathrm{osc}_B(\psi) \, (|\alpha| + |\beta|)(B) + \sum_{B \in \mathscr{D}} \max_B |\psi| \, |\beta|(B)$$

$$\le \sum_{B \in \mathscr{C}} \mathrm{diam}(B) \, (|\alpha| + |\beta|)(B) + \sum_{B \in \mathscr{D}} \mathrm{diam}(B) \, |\beta|(B)$$

$$\le \sum_{B \in \mathscr{B}} \mathrm{diam}(B) \, (|\alpha| + |\beta|)(\Omega) = 2 \, \mathcal{R}ad(\mathscr{B})(|\alpha| + |\beta|)(\Omega).$$

By taking the sup over all ψ we get (i).
Step 2: Proof of (ii). Let $B \in \mathscr{C}$ and let $\varphi \in C_c(B)$. By (2.6) and (1.5), we have

$$\langle ((\alpha - \beta) \llcorner B) * \rho_\eta, \varphi \rangle = \langle (\alpha - \beta) \llcorner B, \varphi * \rho_\eta \rangle$$

$$\le \mathrm{osc}_B(\varphi * \rho_\eta)(|\alpha| + |\beta|)(B) \le \|\nabla(\varphi * \rho_\eta)\|_{L^\infty} \mathrm{diam}(B) \, (|\alpha| + |\beta|)(\Omega)$$

$$\le \|\varphi\|_{L^\infty} \|\nabla \rho\|_{L^1} \eta^{-1} \mathrm{diam}(B) \, (|\alpha| + |\beta|)(\Omega) \,.$$

By taking the sup over all φ and summing over all $B \in \mathscr{C}$, we get (ii).
Step 3: Proof of (iii). Let $\varphi \in C_c(\Omega)$. Then

$$\sum_{B \in \mathscr{C}} |\langle |\alpha(B)| \delta_{x_B} - |\alpha(B)| \delta_{x_B} * \rho_\eta, \varphi \rangle| = \sum_{B \in \mathscr{C}} |\alpha(B)||\langle \delta_{x_B}, \varphi - \varphi * \rho_\eta \rangle|$$

$$\le \sum_{B \in \mathscr{C}} |\alpha(B)| \, \omega_\varphi(\eta) = |\alpha|(\Omega) \, \omega_\varphi(\eta) \,.$$

This concludes the proof of (iii) and of the lemma.

$\square$

We set

$$X(\Omega) := \{\mu = \sum_{i=1}^N z_i \delta_{x_i} \, : \, N \in \mathbb{N}, \, z_i \in \mathbb{Z}, \, x_i \in \Omega\}.$$

Moreover, for any countable set $S \subset \mathbb{R}$ we denote by $\sharp S$ the cardinality of S. If S is infinitely countable, with a little abuse of notations, we write "$k = 1, \dots, \sharp S$" in place of "$k \in \mathbb{N}$". Finally, here and throughout the whole paper, C denotes a positive universal constant which may change from line to line. We write C_a whenever we want to stress the dependence of C on some parameter a.

Theorem 2.4. *Let $\mathcal{B}_n = \{B_{r_{i,n}}(x_{i,n})\}$ be a sequence of finite families of disjoint balls in $\mathbb{R}^2$ with $\mathcal{R}_n := \mathcal{R}ad(\mathcal{B}_n) \to 0$ as $n \to +\infty$, and let $\mu_n := \sum_i z_{i,n} \delta_{x_{i,n}}$, with $z_{i,n} \in \mathbb{Z}$. Assume that*

$$(2.7) \qquad F(\mathcal{B}_n, \mu_n, \Omega) + |\mu_n|(\Omega) \le C|\log \mathcal{R}_n|,$$

for some constant $C > 0$ independent of n. Set $\mu_n^s := \mu_n * \rho_{\mathcal{R}_n^s}$ for any $0 < s < 1$.
There exist $\mu \in X(\Omega)$, a countable (finite or infinite) set $S := \{0 = s_0 < s_1 < s_2 < \dots\}$ with $\sup S = 1$, and a family $\{\nu^s\}_{s \in [0,1)} \subset X(\Omega)$ such that the following facts hold true.

(i) *Flat and weak-$\star$ compactness for vorticity measures: Up to a subsequence $\mu_n \overset{\mathrm{flat}}{\to} \mu$ and for any $s \in (0,1)$, $\mu_n^s \overset{\star}{\rightharpoonup} \mu$ up to a subsequence (independent of s).*
(ii) *Weak-$\star$ compactness for vorticity densities: For any $s \in (0,1) \setminus S$, $|\mu_n^s| \overset{\star}{\rightharpoonup} \nu^s$ up to a subsequence (independent of s).*

(iii) *Structure of vorticity densities: For all* $s \in [0, 1)$, $\nu^s = |\mu| + 2\xi^s_{\text{def}}$ *for some defect measure* $\xi^s_{\text{def}} \in X(\Omega)$ *with* $\xi^s_{\text{def}} \geq 0$. *Moreover,* ν^s *is constant in* $[s_{k-1}, s_k)$ *with* $\nu^{s_{k-1}} \leq \nu^{s_k}$ *for any* $k = 1, \dots, \sharp S$.

(iv) *Lower bound:* $\displaystyle \liminf_{n \to +\infty} \frac{F(\mathcal{B}_n, \mu_n, \Omega)}{|\log \mathcal{R}_n|} \geq \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega).$

Proof. Although the proof of the first part of (i) is well-known to experts (see for instance [4] and the references therein), we provide it together with all the needed estimates, since they will be used also in the proof of the other items.

Let $\mathcal{B}_n(t)$, for any $n \in \mathbb{N}$, be a time parametrized family of balls, starting from $\mathcal{B}_n$, as in Proposition 2.2. Set $\mathcal{C}_n(t) := \{B \in \mathcal{B}_n(t),\, B \subset \Omega\}$, $\mathcal{D}_n(t) := \mathcal{B}_n(t) \setminus \mathcal{C}_n(t)$, $U_n(t) := \bigcup_{B \in \mathcal{B}_n(t)} B$.

Moreover, for any $0 < s < 1$ set

$$t^s_n := \frac{1}{\mathcal{R}_n^{1-s}} - 1, \qquad \tilde{\mu}^s_n := \sum_{B_r(x) \in \mathcal{C}_n(t^s_n)} \mu_n(B_r(x)) \delta_x.$$

By the energy bound (2.7) and by applying (2.2) with $U = \Omega$, $t_1 = 0$ and $t_2 = t^s_n$, we have

$$C|\log \mathcal{R}_n| \geq F(\mathcal{B}_n, \mu_n, \Omega \cap (U_n(t^s_n) \setminus U_n(0)))$$
$$\geq \pi \sum_{B \in \mathcal{C}_n(t^s_n)} |\mu_n(B)|(1-s)|\log \mathcal{R}_n| = \pi(1-s)|\tilde{\mu}^s_n|(\Omega)|\log \mathcal{R}_n|$$

and hence

$$(2.8) \qquad\qquad\qquad |\tilde{\mu}^s_n|(\Omega) \leq \frac{C}{1-s}.$$

By (2.8), up to a subsequence, $\tilde{\mu}^s_n$ converges to some μ^s, both in the weak star and in the flat sense, for some $\mu^s \in X(\Omega)$. Let us show that in fact $\mu := \mu^s$ does not depend on s. To this purpose, it is enough to prove that for any $0 < \sigma_1 < \sigma_2 < 1$, $\|\tilde{\mu}^{\sigma_1}_n - \tilde{\mu}^{\sigma_2}_n\|_{\text{flat}} \to 0$ as $n \to +\infty$. By construction, $(\tilde{\mu}^{\sigma_1}_n - \tilde{\mu}^{\sigma_2}_n)(B) = 0$ for every $B \in \mathcal{C}_n(t^{\sigma_1}_n)$. Therefore, by applying Lemma 2.3 (i) with $\mathscr{B} = \mathcal{B}_n(t^{\sigma_1}_n)$, $\alpha = \tilde{\mu}^{\sigma_1}_n$ and $\beta = \tilde{\mu}^{\sigma_2}_n$, we get

$$(2.9) \quad \|\tilde{\mu}^{\sigma_1}_n - \tilde{\mu}^{\sigma_2}_n\|_{\text{flat}} \leq C\,\mathcal{R}ad(\mathcal{B}_n(t^{\sigma_1}_n))(|\tilde{\mu}^{\sigma_1}_n|(\Omega) + |\tilde{\mu}^{\sigma_2}_n|(\Omega)) \leq \frac{C}{1-\sigma_2}\mathcal{R}_n^{\sigma_1} \to 0 \quad \text{as } n \to +\infty,$$

where in the last inequality we have used (2.8) and Proposition 2.2 (5).

Moreover, by applying again Lemma 2.3 (i) and Proposition 2.2 (5), and using also the energy bound (2.7), we easily deduce that for every $0 < s < 1$

$$(2.10) \qquad\qquad \|\mu_n - \tilde{\mu}^s_n\|_{\text{flat}} \leq C|\log \mathcal{R}_n|\mathcal{R}_n^s \quad \to 0 \qquad \text{as } n \to +\infty.$$

Therefore, by (2.8), (2.9) and (2.10),

$$(2.11) \qquad\qquad \tilde{\mu}^s_n \overset{\star}{\rightharpoonup} \mu \qquad \text{for any } 0 < s < 1 \qquad \text{and} \qquad \mu_n \overset{\text{flat}}{\to} \mu$$

up to a subsequence independent of s.

We now prove the second part of (i). In view of (2.8), it is immediate to see that for any $0 < \sigma_1, \sigma_2 < 1$

$$\tilde{\mu}^{\sigma_2}_n * \rho_{\mathcal{R}_n^{\sigma_1}} - \tilde{\mu}^{\sigma_2}_n \overset{\star}{\rightharpoonup} 0 \qquad \text{as } n \to +\infty.$$

Therefore, in virtue of (2.11), the claim (i) is proven if we show that for any $0 < \sigma_1 < \sigma_2 < 1$

$$\mu^{\sigma_1}_n - (\tilde{\mu}^{\sigma_2}_n * \rho_{\mathcal{R}_n^{\sigma_1}}) \overset{\star}{\rightharpoonup} 0 \qquad \text{as } n \to +\infty.$$

Let $\varphi \in C_c(\Omega)$. By applying Lemma 2.3 (ii) with $\mathscr{B} = \mathcal{B}_n(t^{\sigma_2}_n)$, $\alpha = \tilde{\mu}^{\sigma_2}_n$, $\beta = \mu_n$ and $\eta = \mathcal{R}_n^{\sigma_1}$, for n large enough we get

$$\left|\langle \mu^{\sigma_1}_n - \tilde{\mu}^{\sigma_2}_n * \rho_{\mathcal{R}_n^{\sigma_1}}, \varphi \rangle\right| = \sum_{B \in \mathcal{C}_n(t^{\sigma_2}_n)} \left|\langle ((\mu_n - \tilde{\mu}^{\sigma_2}_n) \llcorner B) * \rho_{\mathcal{R}_n^{\sigma_1}}, \varphi \rangle\right|$$
$$\leq C\|\varphi\|_{L^\infty} \|\nabla \rho\|_{L^1} \mathcal{R}_n^{-\sigma_1} \mathcal{R}ad(\mathcal{B}_n(t^{\sigma_2}_n))(|\mu_n| + |\tilde{\mu}^{\sigma_2}_n|)(\Omega)$$
$$\leq C\|\varphi\|_{L^\infty} \|\nabla \rho\|_{L^1} \mathcal{R}_n^{\sigma_2 - \sigma_1} |\log \mathcal{R}_n| \to 0,$$

where in the last inequality we have used again Proposition 2.2 (5) and the bound (2.7). We pass to the proof of (ii) and (iii). First we recall that, in view of (2.8), the measures $|\tilde{\mu}_n^s|$ are pre-compact. Let now $0 < \sigma_1 < \sigma_2 < 1$ and assume that for $i = 1, 2$

$$|\tilde{\mu}_{n_h}^{\sigma_i}| \xrightarrow{\star} \tilde{\nu}^{\sigma_i} \qquad \text{as } n_h \to +\infty,$$

for some subsequence $n_h \to +\infty$ (independent of i) and for some measures $\tilde{\nu}^{\sigma_i}$, that by construction satisfy

$$\frac{1}{2}(\tilde{\nu}^{\sigma_i} - |\mu|) \in X(\Omega).$$

Then, it is easy to see that $\tilde{\nu}^{\sigma_1} \leq \tilde{\nu}^{\sigma_2}$ in the sense of measures. Since this happens for every pairs of limit measures $(\tilde{\nu}^{\sigma_1}, \tilde{\nu}^{\sigma_2})$, arguing as in the proof of the classical Helly's Theorem we deduce that there exists a non-decreasing family of measures $\tilde{\nu}^s$, with $s \in (0, 1)$, such that, up to a subsequence (not relabeled and independent of s)

(2.12) $$|\tilde{\mu}_n^s| \xrightarrow{\star} \tilde{\nu}^s \qquad \text{for all } 0 < s < 1.$$

By the monotonicity property, and recalling that $\tilde{\nu}^s$ are finite sum of Dirac masses with positive integer weights, we have that the map $s \mapsto \tilde{\nu}^s$ is piecewise constant.

Let $\psi_n : (0, 1) \to \mathbb{N} \cup \{0\}$ be the function that to any $s \in (0, 1)$ associates the number of balls B in $\mathcal{C}_n(t_n^s)$ with $\mu_n(B) \neq 0$, and let S_n be the union of the set $\{0, 1\}$ with set of discontinuity points of ψ_n. By (2.8) the cardinality of $S_n \cap (0, t)$ is uniformly bounded from above by a constant depending only on t. Therefore, up to a subsequence, S_n converge in the Hausdorff sense to some discrete set $S \subset [0, 1]$ with $0, 1 \in S$. Moreover, by construction S contains all the discontinuity points of the map $s \mapsto \tilde{\nu}^s$. Let ν^s be the right continuous extension to the whole interval $[0, 1)$ of $\tilde{\nu}_s$ restricted to $(0, 1) \setminus S$. By construction ν^s satisfies all the properties in (iii).

We pass to the proof of (ii). In virtue of (2.12), (ii) follows provided that for any $s < \sigma$ with $[s, \sigma] \subset (0, 1) \setminus S$ there holds

(2.13) $$|\mu_n^s| - |\tilde{\mu}_n^\sigma| \xrightarrow{\star} 0 \qquad \text{as } n \to +\infty.$$

Set

$$\hat{\mu}_{n,\neq 0}^\sigma := \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} \mu_n \llcorner B \quad \text{and} \quad \hat{\mu}_{n,=0}^\sigma := \mu_n - \hat{\mu}_{n,\neq 0}^\sigma.$$

Let $A \subset\subset \Omega$ be open. By applying Lemma 2.3 (ii) with $\mathscr{B} = \mathcal{B}_n(t_n^\sigma)$, $\alpha = 0$ and $\beta = \hat{\mu}_{n,=0}^\sigma$, and by using Proposition 2.2 (5) and the bound (2.7), for n large enough we obtain

$$|\hat{\mu}_{n,=0}^\sigma * \rho_{\mathcal{R}_n^s}|(A) \leq \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B)=0}} |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}| \leq C\|\nabla \rho\|_{L^1} |\log \mathcal{R}_n| \mathcal{R}_n^{\sigma-s} \to 0 \qquad \text{as } n \to +\infty.$$

As a consequence, (2.13) is equivalent to

(2.14) $$|\hat{\mu}_{n,\neq 0}^\sigma * \rho_{\mathcal{R}_n^s}| - |\tilde{\mu}_n^\sigma| \xrightarrow{\star} 0 \qquad \text{as } n \to +\infty.$$

Since $[s, \sigma] \subset (0, 1) \setminus S$, it is easy to see that, for n large enough, the supports of the measures $(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}$ (for $B \in \mathcal{C}_n(t_n^\sigma)$) are pairwise disjoint, whence

(2.15) $$|\hat{\mu}_{n,\neq 0}^\sigma * \rho_{\mathcal{R}_n^s}| = \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B)\neq 0}} |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}|.$$

Let $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \leq 1$. By (2.15) and by triangular inequality we have

$$|\langle |\hat{\mu}_{n,\neq 0}^\sigma * \rho_{\mathcal{R}_n^s}| - |\tilde{\mu}_n^\sigma|, \varphi\rangle| \leq \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |\langle |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}| - |\mu_n(B)|\delta_{x_B}, \varphi\rangle|$$

$$(2.16) \qquad\qquad \leq \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |\langle |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}| - |\mu_n(B)|\delta_{x_B} * \rho_{\mathcal{R}_n^s}, \varphi\rangle|$$

$$(2.17) \qquad\qquad + \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |\langle |\mu_n(B)|\delta_{x_B} * \rho_{\mathcal{R}_n^s} - |\mu_n(B)|\delta_{x_B}, \varphi\rangle|.$$

As for the addendum in (2.16), we can apply Lemma 2.3 (ii), Proposition 2.2 (5) and (2.7), thus obtaining

$$\sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |\langle |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}| - |\mu_n(B)|\delta_{x_B} * \rho_{\mathcal{R}_n^s}, \varphi\rangle| \leq \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} \langle |(\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} - \mu_n(B)\delta_{x_B} * \rho_{\mathcal{R}_n^s}|, |\varphi|\rangle$$

$$\leq \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |((\mu_n \llcorner B) - \mu_n(B)\delta_{x_B}) * \rho_{\mathcal{R}_n^s}| \leq C\|\nabla\rho\|_{L^1} |\log \mathcal{R}_n| \mathcal{R}_n^{\sigma-s} \to 0 \quad \text{as } n \to +\infty,$$

whereas for the term in (2.17), by applying Lemma 2.3 (iii) and recalling (2.8), we get

$$\sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B) \neq 0}} |\langle |\mu_n(B)|\delta_{x_B} * \rho_{\mathcal{R}_n^s} - |\mu_n(B)|\delta_{x_B}, \varphi\rangle| \leq C\omega_\varphi(\mathcal{R}_n^s) \to 0 \qquad \text{as } n \to +\infty.$$

Then, (2.14) follows and (ii) is proven.

Finally, we prove the lower bound (iv). Let $L \in \mathbb{N}$ with $L \leq \sharp S$ and let $\eta > 0$ (small enough). By (2.2) we have

$$F(\mathcal{B}_n, \mu_n, \Omega) \geq \sum_{l=1}^L F(\mathcal{B}_n, \mu_n, \Omega \cap (U_n(t_n^{s_{l-1}+\eta}) \setminus U_n(t_n^{s_l-\eta})))$$

$$\geq \pi |\log \mathcal{R}_n| \sum_{l=1}^L (s_l - s_{l-1} - 2\eta) \sum_{B \in \mathcal{C}_n(t_n^{s_{l-1}+\eta})} |\mu_n(B)| = \pi |\log \mathcal{R}_n| \sum_{l=1}^L (s_l - s_{l-1} - 2\eta) |\tilde{\mu}_n^{s_{l-1}+\eta}|(\Omega).$$

By (2.12) and recalling that the map $s \to \nu^s$ is the right-continuous extension of $s \to \tilde{\nu}^s$, we have

$$|\tilde{\mu}_n^{s_{l-1}+\eta}| \overset{\star}{\rightharpoonup} \tilde{\nu}^{s_{l-1}+\eta} = \nu^{s_{l-1}}.$$

By the lower semicontinuity property of the total variation with respect to the weak star convergence we get

$$\liminf_{n \to +\infty} \frac{F(\mathcal{B}_n, \mu_n, \Omega)}{|\log \mathcal{R}_n|} \geq \pi \sum_{l=1}^L (s_l - s_{l-1} - 2\eta)\nu^{s_{l-1}}(\Omega),$$

from which the lower bound (iv) follows by sending first $\eta \to 0$ and then $L \to \sharp S$. $\qquad\square$

For further use, we fix the "minimal" functional F satisfying the assumptions (i) and (ii) in Section 2. First, if $A_{r,R}(x) := B_R(x) \setminus B_r(x)$ is an annulus that does not intersect any $B_{r_i}(x_i)$, we set

$$G(\mathcal{B}, \mu, A_{r,R}(x)) := \pi |\mu(B_r(x))| \log\left(\frac{R}{r}\right).$$

Then, for every open set $A \subset \mathbb{R}^2$ we set

$$(2.18) \qquad\qquad F(\mathcal{B}, \mu, A) := \sup \sum_j G(\mathcal{B}, \mu, A_j),$$

where the sup is over all finite families of disjoint annuli $A_j \subset A$ that do not intersect any $B_{r_i}(x_i)$. Notice that, if A is an annulus that does not intersect any $B_{r_i}(x_i)$, then $F(\mathcal{B}, \mu, A) = G(\mathcal{B}, \mu, A)$.

Remark 2.5. The definition of F in (2.18) is justified by the following observation. Let $\tilde{\Omega} = \Omega \setminus \cup_{B \in \mathcal{B}} B$. Given $u \in H^1(\tilde{\Omega}; \mathcal{S}^1)$, let $\mu := \sum_{B \in \mathcal{C}} \deg(u, \partial B) \delta_{x_B}$, where $\mathcal{C}$ denotes the family of balls in $\mathcal{B}$ that are contained in Ω, and x_B is the center of B. Then, by Jensen inequality we easily deduce (see for instance [18]) that

$$F(\mathcal{B}, \mu, U) \leq \frac{1}{2} \int_{U \cap \tilde{\Omega}} |\nabla u|^2 \, dx,$$

for every open set $U \subset \Omega$.

3. Γ-CONVERGENCE OF THE CORE RADIUS APPROACH

In this section we exploit the results in Section 2 in order to develop a Γ-convergence analysis for the core radius approach (0.2).

Theorem 3.1. Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ as $n \to +\infty$. The following Γ-convergence result holds true.

    (**Compactness**) Let $\{\mu_n\}$ with $\mu_n \in X_{\varepsilon_n}(\Omega)$ for all $n \in \mathbb{N}$ be such that

(3.1) $$\mathcal{E}_{\varepsilon_n}(\mu_n) \leq C|\log \varepsilon_n| \qquad \text{for all } n \in \mathbb{N},$$

for some constant $C > 0$ independent of n. Set $\mu_n^s := \mu_n * \rho_{\varepsilon_n^s}$ for any $0 < s < 1$.

    Then, there exist a measure $\mu \in X(\Omega)$, a countable (finite or infinite) set $S := \{0 = s_0 < s_1 < s_2 < \ldots\}$ with $\sup S = 1$, and a family of positive Radon measures $\{\nu^s\}_{s \in [0,1)} \subset X(\Omega)$, such that the following compactness and structure properties hold true:

    (1) Flat and weak star compactness for vorticity measures: Up to a subsequence, $\mu_n \stackrel{\text{flat}}{\to} \mu$ and for any $s \in (0,1)$, $\mu_n^s \stackrel{\star}{\rightharpoonup} \mu$ up to a subsequence (independent of s).

    (2) Weak star compactness for vorticity densities: For any $s \in (0,1) \setminus S$, $|\mu_n^s| \stackrel{\star}{\rightharpoonup} \nu^s$ up to a subsequence (independent of s).

    (3) Structure of vorticity densities: For all $s \in [0,1)$, $\nu^s = |\mu| + 2\xi_{\text{def}}^s$ for some defect measure $\xi_{\text{def}}^s \in X(\Omega)$ with $\xi_{\text{def}}^s \geq 0$. Moreover, ν^s is constant in $[s_{k-1}, s_k)$ and $\nu^{s_{k-1}} \leq \nu^{s_k}$ for any $k = 1, \ldots, \sharp S$.

    (**Γ-liminf inequality**) Let $\{\mu_n\}$ be such that for all $s \in (0,1)$, $|\mu_n^s| \stackrel{\star}{\rightharpoonup} \nu^s$ for some family of measures $\{\nu^s\}_{s \in [0,1)}$ as in (3). Then,

$$\liminf_{n \to +\infty} \frac{\mathcal{E}_{\varepsilon_n}(\mu_n)}{|\log \varepsilon_n|} \geq \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega).$$

    (**Γ-limsup inequality**) For any $\mu \in X(\Omega)$ and for any family of measures $\{\nu^s\}_{s \in [0,1)}$ as in (3) with $\sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega) < +\infty$, there exists $\{\mu_n\}$ with $\mu_n \in X_{\varepsilon_n}(\Omega)$ for all $n \in \mathbb{N}$ such that $\mu_n \stackrel{\text{flat}}{\to} \mu$, $\mu_n^s \stackrel{\star}{\rightharpoonup} \mu$ for any $s \in (0,1)$, $|\mu_n^s| \stackrel{\star}{\rightharpoonup} \nu^s$ for any $s \in (0,1) \setminus S$, and

(3.2) $$\lim_{n \to +\infty} \frac{\mathcal{E}_{\varepsilon_n}(\mu_n)}{|\log \varepsilon_n|} = \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega).$$

Proof. For any $n \in \mathbb{N}$ we denote by $x_{i,n}$ the points in the support of μ_n and we set $\mathcal{B}_n := \{B_{\varepsilon_n}(x_{i,n})\}$ and $\mathcal{R}_n := \mathcal{R}ad(\mathcal{B}_n)$. Trivially,

(3.3) $$\mathcal{R}_n \leq \varepsilon_n |\mu_n|(\Omega).$$

Let $\mathcal{B}_n(t)$ be a time parametrized family of balls given by Proposition 2.2 starting from $\mathcal{B}_n =: \mathcal{B}_n(0)$; we denote by $\mathcal{C}_n(t)$ the family of balls in $\mathcal{B}_n(t)$ that are contained in Ω and by $U_n(t)$ the union of the balls in $\mathcal{B}_n(t)$. Moreover we set

$$\mu_n(t) := \sum_{B \in \mathcal{C}_n(t)} \deg(u_n, \partial B) \delta_{x_B}.$$

Let F be as defined in (2.18).

Step 1: Proof of compactness. By applying Proposition 2.2 (5) with $t_1 = 0$ and $t_2 = 1$, we have

$$(3.4) \qquad F(\mathcal{B}_n, \mu_n, \Omega) \geq \pi \log 2 \sum_{\substack{B \in \mathcal{B}_n(1) \\ B \subset \Omega}} |\mu_n|(B) = \pi \log 2 \, |\mu_n|(\Omega),$$

where the equality follows by the fact that $\mu_n \in X_{\varepsilon_n}(\Omega)$. Therefore, by (3.4), Remark 2.5 and the energy bound (3.1), we obtain

$$(3.5) \qquad F(\mathcal{B}_n, \mu_n, \Omega) + |\mu_n|(\Omega) \leq C \, F(\mathcal{B}_n, \mu_n, \Omega) \leq C \, \mathcal{E}_{\varepsilon_n}(\mu_n) \leq C |\log \varepsilon_n|,$$

which, in view of (3.3), immediately implies

$$F(\mathcal{B}_n, \mu_n, \Omega) + |\mu_n|(\Omega) \leq C |\log \mathcal{R}_n|.$$

By Theorem 2.4 there exist a measure $\mu \in X(\Omega)$, a countable set $S := \{0 < s_1 < s_2 < \ldots\}$ with $\sup S = 1$ and a family of measures $\{\nu^s\}_{s \in [0,1)} \subset X(\Omega)$ satisfying the structure properties in (3), such that up to a subsequence independent of s, there holds

$$
\begin{aligned}
& \mu_n \overset{\text{flat}}{\to} \mu, \\
(3.6) \qquad & \mu_n * \rho_{\mathcal{R}_n^s} \overset{\star}{\rightharpoonup} \mu \qquad \text{for any } s \in (0, 1), \\
& |\mu_n * \rho_{\mathcal{R}_n^s}| \overset{\star}{\rightharpoonup} \nu^s \qquad \text{for any } s \in (0, 1) \setminus S.
\end{aligned}
$$

Therefore, in order to conclude the proof of the compactness it is enough to show that

$$(3.7) \qquad \mu_n * \rho_{\varepsilon_n^s} \overset{\star}{\rightharpoonup} \mu \qquad \text{for any } s \in (0, 1)$$

$$(3.8) \qquad |\mu_n * \rho_{\mathcal{R}_n^s}| - |\mu_n * \rho_{\varepsilon_n^s}| \overset{\star}{\rightharpoonup} 0 \qquad \text{for any } s \in (0, 1) \setminus S.$$

We preliminarily notice that if $\mu_n \equiv 0$ for n large enough, then (3.7) and (3.8) are trivially satisfied, so that we can assume without loss of generality that $\mu_n \neq 0$ and hence

$$(3.9) \qquad \mathcal{R}_n \geq \varepsilon_n.$$

We start by proving (3.7). Let $\sigma > 0$ and set $t_n^\sigma := \frac{1}{\varepsilon_n^{1-\sigma}} - 1$. By construction

$$(\mu_n - \mu_n(t_n^\sigma))(B) = 0 \qquad \text{for any} \quad B \in \mathcal{C}_n(t_n^\sigma).$$

Therefore, by Lemma 2.3 (i), (3.5), Proposition 2.2 (5) and (3.3), we have

$$(3.10) \quad \|\mu_n - \mu_n(t_n^\sigma)\|_{\text{flat}} \leq C \, \mathcal{R}ad(\mathcal{B}_n(t_n^\sigma)) \left(|\mu_n|(\Omega) + |\mu_n(t_n^\sigma)|(\Omega)\right)$$

$$\leq C \, \mathcal{R}ad(\mathcal{B}_n(t_n^\sigma)) |\log \varepsilon_n| \leq C_l |\log \varepsilon_n|^{\sigma+1} \varepsilon_n^\sigma \to 0 \quad \text{as } n \to +\infty,$$

which in virtue of (3.6) yields that, up to subsequences,

$$(3.11) \qquad \mu_n(t_n^\sigma) \overset{\text{flat}}{\to} \mu.$$

Moreover, by (3.5) and by (2.2) in Proposition 2.2,

$$C |\log \varepsilon_n| \geq F(\mathcal{B}_n, \mu_n, \Omega \cap (U_n(t_n^\sigma) \setminus U_n(0))) \geq \pi |\mu_n(t_n^\sigma)|(1 - \sigma)|\log \varepsilon_n|,$$

which, together with (3.11), implies that, up to a subsequence independent of s, $\mu_n(t_n^\sigma) \overset{\star}{\rightharpoonup} \mu$ and

$$(3.12) \qquad \mu_n(t_n^\sigma) * \rho_{\varepsilon_n^s} \overset{\star}{\rightharpoonup} \mu.$$

Let now $\sigma > s$. By (3.10), for any $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \leq 1$, we have

$$|\langle \mu_n * \rho_{\varepsilon_n^s} - \mu_n(t_n^\sigma) * \rho_{\varepsilon_n^s}, \varphi \rangle| = |\langle \mu_n - \mu_n(t_n^\sigma), \varphi * \rho_{\varepsilon_n^s} \rangle|$$

$$\leq C \|\mu_n - \mu_n(t_n^\sigma)\|_{\text{flat}} \, \varepsilon_n^{-s} \leq C_l |\log \varepsilon_n|^{\sigma+1} \varepsilon_n^{\sigma-s} \to 0 \qquad \text{as } n \to +\infty.$$

This fact combined with (3.12) implies (3.7).

Finally we prove (3.8). Let $s \in (0, 1) \setminus S$ and let k be such that $s \in (s_{k-1}, s_k)$. Moreover, let $\sigma \in (s, s_k)$ and set $\hat{t}_n^\sigma := \frac{1}{\mathcal{R}_n^{1-\sigma}} - 1$. By construction, $(\mu_n - \mu_n(\hat{t}_n^\sigma))(B) = 0$ for any $B \in \mathcal{C}_n(\hat{t}_n^\sigma)$.

Therefore, by applying Lemma 2.3 (ii) with $\mathscr{B} = \mathcal{B}_n(\hat{t}_n^\sigma)$, $\alpha = 0$, $\beta = \mu_n \llcorner \cup_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B) = 0}}$, for any open set $A \subset\subset \Omega$ and for n large enough we obtain

$$(3.13) \quad \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)=0}} (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} + \sum_{B \in \mathcal{B}_n(\hat{t}_n^\sigma) \setminus \mathcal{C}_n(\hat{t}_n^\sigma)} (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} \right| (A)$$

$$= \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)=0}} (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} \right| (A) \leq \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)=0}} |(\mu_n \llcorner B) * \rho_{\varepsilon_n^s}| (A)$$

$$\leq \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)=0}} |(\mu_n \llcorner B) * \rho_{\varepsilon_n^s}| \leq C_l \|\nabla \rho\|_{L^1} |\log \varepsilon_n|^{1+\sigma} \varepsilon_n^{\sigma-s} \to 0 \qquad \text{as } n \to +\infty\,,$$

where the last inequality follows by the energy bound (3.5) and by (3.3). Similarly, one can prove that also

$$(3.14) \quad \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)=0}} (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} + \sum_{B \in \mathcal{B}_n(\hat{t}_n^\sigma) \setminus \mathcal{C}_n(\hat{t}_n^\sigma)} (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} \right| (A) \to 0\,.$$

Moreover, by arguing as in (2.15) and by using (3.9), it is easy to see that, for n large enough,

$$(3.15) \quad \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} \right| = \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} \left| (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} \right|,$$

$$\left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} \right| = \sum_{\substack{B \in \mathcal{C}_n(t_n^\sigma) \\ \mu_n(B)\neq 0}} \left| (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} \right|.$$

Let now $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \leq 1$. By (3.15) and by triangular inequality, we have

$$(3.16) \quad \left| \left\langle \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} (\mu_n \llcorner B) * \rho_{\varepsilon_n^s} \right| - \left| \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s} \right|, \varphi \right\rangle \right|$$

$$\leq \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} \left\langle |(\mu_n \llcorner B) * \rho_{\varepsilon_n^s} - (\mu_n \llcorner B) * \rho_{\mathcal{R}_n^s}|, |\varphi| \right\rangle$$

$$\leq \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} \left\langle |(\mu_n \llcorner B - (\mu_n)(B)\delta_{x_B}) * \rho_{\varepsilon_n^s}|, |\varphi| \right\rangle + \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} \left\langle |(\mu_n \llcorner B - \mu_n(B)\delta_{x_B}) * \rho_{\mathcal{R}_n^s}|, |\varphi| \right\rangle$$

$$+ \sum_{\substack{B \in \mathcal{C}_n(\hat{t}_n^\sigma) \\ \mu_n(B)\neq 0}} \left\langle |\mu_n(B)\delta_{x_B} * \rho_{\varepsilon_n^s} - \mu_n(B)\delta_{x_B} * \rho_{\mathcal{R}_n^s}|, |\varphi| \right\rangle \to 0 \qquad \text{as } n \to +\infty\,,$$

where the convergence to zero of the first two addenda can be proven by applying Lemma 2.3 (ii) whereas the convergence to zero of the last addendum follows by Lemma 2.3 (iii) and the triangular inequality.

Then (3.8) follows by (3.13), (3.14) and (3.16).

Step 2: Proof of the Γ-liminf inequality. We can assume without loss of generality that the upper bound (3.1) is satisfied, so that all the convergences in (3.6) hold true. Therefore, by

Remark 2.5, (3.3), (3.5) and Theorem 2.4 (iv) we have

$$\liminf_{n\to+\infty}\frac{\mathcal{E}_{\varepsilon_n}(\mu_n)}{|\log\varepsilon_n|}\geq\liminf_{n\to+\infty}\frac{F(\mathcal{B}_n,\mu_n,\Omega)}{|\log\mathcal{R}_n|}\geq\pi\sum_{k=1}^{\sharp S}(s_k-s_{k-1})\nu^{s_{k-1}}(\Omega)\,.$$

Step 3: Proof of the Γ-limsup inequality. Let μ, $\{\nu_s\}$ and S be as in the assumptions. By standard density arguments we can assume that $K:=\sharp S<+\infty$, so that $S=\{0=s_0<s_1<\ldots<s_K=1\}$. We set $\eta_{\mathrm{def}}^{s_0}:=\xi_{\mathrm{def}}^{s_0}$ and $\eta_{\mathrm{def}}^{s_k}:=\nu^{s_k}-\nu^{s_{k-1}}$ for any $k=1,\ldots,K$. It is easy to see that

$$(3.17)\qquad\sum_{k=1}^{K}(s_k-s_{k-1})\nu^{s_{k-1}}(\Omega)=|\mu|(\Omega)+\sum_{k=0}^{K}(1-s_k)\eta_{\mathrm{def}}^{s_k}(\Omega)\,.$$

Again by density arguments we can assume that $\mu=\sum_{i=1}^{N}z_i\delta_{x_i}$ with $|z_i|=1$ and $x_i\neq x_j$ for $i\neq j$, and that $\eta_{\mathrm{def}}^{s_k}=2\delta_{y^{s_k}}$ with all y^{s_k} different from each other and from all x_i's. For any $n\in\mathbb{N}$, we define

$$\mu_n:=\mu+\sum_{k=0}^{K}(\delta_{y_{n,+}^{s_k}}-\delta_{y_{n,-}^{s_k}})\,,$$

where $y_{n,+}^{s_k}$ and $y_{n,-}^{s_k}$ are two points in Ω such that $\mathrm{dist}\,(y_{n,+}^{s_k},y_{n,-}^{s_k})=2\,\mathrm{dist}\,(y_{n,+}^{s_k},y^{s_k})=2\,\varepsilon_n^{s_k}$ for any $k=1,\ldots,K$, while for $k=0$ we consider a dipole whose length tends to zero slower than any power of ε_n; for instance such that $\mathrm{dist}\,(y_{n,+}^{s_0},y_{n,-}^{s_0})=2\,\mathrm{dist}\,(y_{n,+}^{s_0},y^{s_0})=2\,\frac{1}{|\log\varepsilon_n|}$. It is immediate to see that, for n large enough $\mu_n\in X_{\varepsilon_n}(\Omega)$, $\mu_n\overset{\mathrm{flat}}{\to}\mu$, $\mu_n*\rho_{\varepsilon_n}^s\overset{\star}{\rightharpoonup}\mu$ for any $s\in(0,1)$ and $|\mu_n*\rho_{\varepsilon_n}^s|\overset{\star}{\rightharpoonup}\nu^s$ for any $s\in(0,1)\setminus S$.

For any $\xi\in\mathbb{R}^2$, consider the standard polar coordinates centered at ξ and let θ_ξ denote the phase, namely the angular coordinate. We set

$$(3.18)\qquad\vartheta_n(\cdot):=\sum_{i=1}^{M}z_i\theta_{x_i}(\cdot)+\sum_{k=0}^{K}(\theta_{y_{n,+}^{s_k}}(\cdot)-\theta_{y_{n,-}^{s_k}}(\cdot))$$

and $u_n:=e^{i\vartheta_n}$. Trivially, $u_n\in\mathcal{AS}_{\varepsilon_n}(\mu_n)$ and $\int_{\Omega_{\varepsilon_n}(\mu_n)}|\nabla u_n|^2\,\mathrm{d}x=\int_{\Omega_{\varepsilon_n}(\mu_n)}|\nabla\vartheta_n|^2\,\mathrm{d}x$.

Now we show that the pair (μ_n,u_n) is a recovery sequence for the Γ-limsup inequality (3.2). Recalling (3.17), we have to prove that

$$(3.19)\qquad\limsup_{n\to+\infty}\frac{1}{2|\log\varepsilon_n|}\int_{\Omega_{\varepsilon_n}(\mu_n)}|\nabla\vartheta_n|^2\,\mathrm{d}x\leq\pi|\mu|(\Omega)+\pi\sum_{k=0}^{K}(1-s_k)\eta_{\mathrm{def}}^{s_k}(\Omega)\,.$$

Fix $r>0$ such that the balls $B_r(x_i)$ and $B_r(y^{s_k})$ are pairwise disjoint and compactly contained in Ω. Then

$$(3.20)\quad\Omega_{\varepsilon_n}(\mu_n)=\Omega_r(\mu_n)\cup\bigcup_{i=1}^{N}(B_r(x_i)\setminus B_{\varepsilon_n}(x_i))\cup\bigcup_{k=0}^{K}\Big(B_r(y^{s_k})\setminus\big(B_{\varepsilon_n}(y_{n,+}^{s_k})\cup B_{\varepsilon_n}(y_{n,-}^{s_k})\big)\Big).$$

By construction

$$(3.21)\qquad\frac{1}{2}\int_{\Omega_r(\mu_n)}|\nabla u_n|^2\,\mathrm{d}x\leq C_r,$$

where C_r is a constant independent of n. Moreover, for any $i=1,\ldots,M$, by using Cauchy-Schwarz inequality together with the fact that $|\nabla\theta_\xi(\cdot)|=\frac{1}{|\cdot-\xi|}$, we deduce

$$(3.22)\qquad\frac{1}{2}\int_{B_r(x_i)\setminus B_{\varepsilon_n}(x_i)}|\nabla\vartheta_n|^2\,\mathrm{d}x\leq\frac{1}{2}\int_{B_r(x_i)\setminus B_{\varepsilon_n}(x_i)}|\nabla\theta_{x_i}|^2\,\mathrm{d}x+C=\pi\log\frac{r}{\varepsilon_n}+C\,,$$

for some positive constant $C > 0$. Arguing analogously, one can easily check that for any $k = 0, 1, \ldots, K$ there holds

$$(3.23) \quad \frac{1}{2} \int_{B_{\varepsilon_n^{s_k}}(y_{n,\pm}^{s_k}) \setminus B_{\varepsilon_n}(y_{n,\pm}^{s_k})} |\nabla \vartheta_n|^2 \, \mathrm{d}x \le \frac{1}{2} \int_{B_{\varepsilon_n^{s_k}}(y_{n,\pm}^{s_k}) \setminus B_{\varepsilon_n}(y_{n,\pm}^{s_k})} |\nabla \theta_{y_{n,\pm}^{s_k}}|^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \int_{B_{\varepsilon_n^{s_k}}(y_{n,\pm}^{s_k}) \setminus B_{\varepsilon_n}(y_{n,\pm}^{s_k})} |\nabla \theta_{y_{n,\mp}^{s_k}}|^2 \, \mathrm{d}x + \int_{B_{\varepsilon_n^{s_k}}(y_{n,\pm}^{s_k}) \setminus B_{\varepsilon_n}(y_{n,\pm}^{s_k})} |\nabla \theta_{y_+^{s_k}}| \, |\nabla \theta_{y_-^{s_k}}| \, \mathrm{d}x + C$$

$$\le \pi(1 - s_k)|\log \varepsilon_n| + C \, .$$

Furthermore, by straightforward computations, it follows that for any $k = 0, 1, \ldots, K$

$$(3.24) \quad \frac{1}{2} \int_{B_r(y^{s_k}) \setminus B_{8\varepsilon_n^{s_k}}(y^{s_k})} |\nabla \vartheta_n|^2 \, \mathrm{d}x \le \frac{1}{2} \int_{B_r(y^{s_k}) \setminus B_{8\varepsilon_n^{s_k}}(y^{s_k})} |\nabla (\theta_{y_{n,+}^{s_k}} - \theta_{y_{n,-}^{s_k}})|^2 \, \mathrm{d}x + C \le C.$$

Finally, by scaling arguments, it is easy to see that for any $k = 0, 1, \ldots, K$

$$(3.25) \quad \frac{1}{2} \int_{B_{8\varepsilon_n^{s_k}}(y^{s_k}) \setminus (B_{\varepsilon_n^{s_k}}(y_+^{s_k}) \cup B_{\varepsilon_n}(y_-^{s_k}))} |\nabla \vartheta_n|^2 \, \mathrm{d}x \le C \, .$$

By summing (3.21), (3.22), (3.23), (3.24), (3.25), and recalling (3.20), we obtain (3.19), i.e., the claim. $\qquad\square$

4. Γ-CONVERGENCE OF GL_ε

This section is devoted to the Γ-convergence analysis of the Ginzburg-Landau functionals (0.1).

Theorem 4.1. *Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ as $n \to +\infty$. The following Γ-convergence result holds true.*

    *(**Compactness**) Let $\{u_n\} \subset H^1(\Omega; \mathbb{R}^2)$ be such that*

$$(4.1) \qquad\qquad GL_{\varepsilon_n}(u_n) \le C|\log \varepsilon_n| \qquad \text{for all } n \in \mathbb{N},$$

for some constant $C > 0$ independent of n. Set $J^s u_n := J u_n * \rho_{\varepsilon_n^s}$ for any $n \in \mathbb{N}$ and for any $0 < s < 1$.

    Then, there exist a measure $\mu \in X(\Omega)$, a countable (finite or infinite) set $S := \{0 = s_0 < s_1 < s_2 < \ldots\}$ with $\sup S = 1$, and a family of positive Radon measures $\{\nu^s\}_{s \in [0,1)} \subset X(\Omega)$, such that the following compactness and structure properties hold true:

  (1) *Flat and weak star compactness for the Jacobians: Up to a subsequence $J u_n \overset{\text{flat}}{\to} \pi\mu$ and for any $s \in (0,1)$, $J^s u_n \overset{\star}{\rightharpoonup} \pi\mu$ up to a subsequence (independent of s).*

  (2) *Weak star compactness for vorticity densities: For any $s \in (0,1) \setminus S$, $|J^s u_n| \overset{\star}{\rightharpoonup} \pi\nu^s$ up to a subsequence (independent of s).*

  (3) *Structure of vorticity densities: For all $s \in [0,1)$, $\nu^s = |\mu| + 2\xi_{\text{def}}^s$ for some defect measure $\xi_{\text{def}}^s \in X(\Omega)$ with $\xi_{\text{def}}^s \ge 0$. Moreover, ν^s is constant in $[s_{k-1}, s_k)$ and $\nu^{s_{k-1}} \le \nu^{s_k}$ for any $k = 1, \ldots, \sharp S$.*

    *(**Γ-liminf inequality**) Let $\{u_n\} \subset H^1(\Omega; \mathbb{R}^2)$ be such that for all $s \in (0,1)$, $|J^s u_n| \overset{\star}{\rightharpoonup} \pi\nu^s$ for some family of measures $\{\nu_s\}_{s \in [0,1)}$ as in (3). Then,*

$$\liminf_{n \to +\infty} \frac{GL_{\varepsilon_n}(u_n)}{|\log \varepsilon_n|} \ge \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega) \, .$$

    *(**Γ-limsup inequality**) For any $\mu \in X(\Omega)$ and for any family of measures $\{\nu^s\}_{s \in [0,1)}$ as in (3) with $\sum_{k=1}^{\sharp S}(s_k - s_{k-1})\nu^{s_{k-1}}(\Omega) < +\infty$, there exists $\{u_n\} \subset H^1(\Omega; \mathbb{R}^2)$ such that $J u_n \overset{\text{flat}}{\to} \pi\mu$, $J^s u_n \overset{\star}{\rightharpoonup} \pi\mu$ for any $s \in (0,1)$, $|J^s u_n| \overset{\star}{\rightharpoonup} \pi\nu^s$ for any $s \in (0,1) \setminus S$, and*

$$(4.2) \qquad \lim_{n \to +\infty} \frac{GL_{\varepsilon_n}(u_n)}{|\log \varepsilon_n|} = \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega) \, .$$

Proof. First we will prove the compactness properties and the Γ-liminf inequality, following the approach in [18, 4]. By standard density arguments in Γ-convergence we may assume that the functions u_n are smooth.

Step 1: Energy estimates. Following the notations in [18, 4], for any $n \in \mathbb{N}$ and for any $\tau \in (0,1)$ we set

$$\Omega_{n,\tau} := \{|u_n| > \tau\}, \qquad \gamma_{n,\tau} := \partial\Omega_{n,\tau} \setminus \partial\Omega, \qquad K_{n,\tau} := \Omega \setminus \Omega_{n,\tau},$$

$$\Theta_n(\tau) := \frac{1}{2} \int_{\Omega_{n,\tau}} \left|\nabla \frac{u_n}{|u_n|}\right|^2 \mathrm{d}x, \qquad m_n(\tau) := \int_{\gamma_{n,\tau}} |\nabla|u_n|| \,\mathrm{d}\mathcal{H}^1.$$

By the Coarea Formula we have

$$(4.3) \qquad GL_{\varepsilon_n}(u_n) \geq \frac{1}{2} \int_0^\infty \left(m_n(\tau) + \frac{2W(\tau)}{\varepsilon_n^2} \int_{\gamma_{n,\tau}} \frac{1}{|\nabla|u_n||} \,\mathrm{d}\mathcal{H}^1 \right) \mathrm{d}\tau - \int_0^\infty \tau^2 \,\mathrm{d}\Theta_n'(\tau),$$

where $\Theta_n'(\tau)$ is the distributional derivative of the decreasing function $\Theta_n(\tau)$ and the inequality is due to the possible presence of flat regions $\{\nabla|u_n| = 0\}$ with positive measure.

Fix $l \in \mathbb{N}$ (large). For any $\tau \leq 1 - \frac{1}{l}$ and n large enough we have $|K_{n,\tau}| \leq C_l \varepsilon_n^2 |\log \varepsilon_n| \leq \frac{1}{2}|\Omega|$, so that, using also that Ω is Lipschitz we have

$$\mathcal{H}^1(\partial K_{n,\tau}) \leq C_l \mathcal{H}^1(\gamma_{n,\tau}).$$

Notice that, by definition of Hausdorff measure, since $\partial K_{n,\tau}$ is compact, it is always contained in a finite union of balls $B_{r_i}(y_i)$ such that $\sum_i r_i \leq \mathcal{H}^1(\partial K_{n,\tau})$. Moreover, after a merging procedure, we can always assume that such balls are disjoint. As a consequence, either $K_{n,\tau}$ or $\Omega \setminus K_{n,\tau} = \Omega_{n,\tau}$ is contained in the union of such balls. In the latter case, since $|\Omega \setminus K_{n,\tau}| \geq \frac{1}{2}|\Omega|$ we have $\sum_i r_i \geq C$, and we replace these balls by one single ball containing Ω. In both cases, we have a family of balls $\mathcal{B}_{n,\tau}$ whose union contains $K_{n,\tau}$, such that

$$(4.4) \qquad \mathcal{R}ad(\mathcal{B}_{n,\tau}) \leq C\mathcal{H}^1(\partial K_{n,\tau}) \leq C_l \mathcal{H}^1(\gamma_{n,\tau}).$$

Notice that we can always assume (just by enlarging an arbitrarily chosen ball in $\mathcal{B}_{n,\tau}$) that

$$\varepsilon_n \leq \mathcal{R}ad(\mathcal{B}_{n,\tau}) \leq C_l \mathcal{H}^1(\gamma_{n,\tau}) + \varepsilon_n.$$

Since $K_{n,\tau}$ is monotone in τ (with respect to inclusion), for any given $\eta > 0$ (possibly depending on n) we can always assume that the map $\tau \mapsto \mathcal{R}ad(\mathcal{B}_{n,\tau})$ is measurable and

$$(4.5) \qquad \varepsilon_n \leq \mathcal{R}ad(\mathcal{B}_{n,\tau_1}) \leq \mathcal{R}ad(\mathcal{B}_{n,\tau_2}) + \eta \qquad \text{for all } 0 < \tau_1 < \tau_2 < 1 - \frac{1}{l}.$$

By Hölder inequality we have

$$(4.6) \qquad \mathcal{H}^1(\gamma_{n,\tau})^2 \leq C \, m_n(\tau) \int_{\gamma_{n,\tau}} \frac{1}{|\nabla|u_n||} \,\mathrm{d}\mathcal{H}^1.$$

Fix $\bar{\tau} \in (\frac{2}{l}, 1 - \frac{2}{l})$. By using, in order of appearance, (4.3), (4.6), Young inequality, integration by parts of $\tau^2 \,\mathrm{d}\Theta_n'(\tau)$, and (4.4), we obtain

$$GL_{\varepsilon_n}(u_n) \geq \frac{1}{2} \int_0^\infty m_n(\tau) + \frac{\mathcal{H}^1(\gamma_{n,\tau})^2 W(\tau)}{C\varepsilon_n^2 m_n(\tau)} \,\mathrm{d}\tau - \int_0^\infty \tau^2 \,\mathrm{d}\Theta_n'(\tau)$$

$$\geq \int_0^\infty \frac{\sqrt{W(\tau)}}{C\varepsilon_n} \mathcal{H}^1(\gamma_{n,\tau}) + 2\tau\Theta_n(\tau) \,\mathrm{d}\tau$$

$$(4.7) \qquad \geq \int_0^{1-\frac{1}{l}} \frac{\sqrt{W(\tau)}}{C_l\varepsilon_n} (\mathcal{R}ad(\mathcal{B}_{n,\tau}) - \varepsilon_n) + 2\tau\Theta_n(\tau) \,\mathrm{d}\tau$$

$$(4.8) \qquad \geq \int_0^{\bar{\tau}} \frac{\sqrt{W(\tau)}}{C_l\varepsilon_n} (\mathcal{R}ad(\mathcal{B}_{n,\tau}) - \varepsilon_n) + 2\tau\Theta_n(\tau) \,\mathrm{d}\tau$$

$$(4.9) \qquad + \int_{\bar{\tau}}^{\bar{\tau}+\frac{1}{l}} \frac{\sqrt{W(\tau)}}{C_l\varepsilon_n} (\mathcal{R}ad(\mathcal{B}_{n,\tau}) - \varepsilon_n) + 2\tau\Theta_n(\tau) \,\mathrm{d}\tau.$$

By (4.5), (4.9) and the energy bound (4.1), it follows that, for η small enough (depending on n),

$$(4.10) \qquad \varepsilon_n \leq \mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}}) \leq C_l(\varepsilon_n|\log\varepsilon_n| + \varepsilon_n + \eta) \leq C_l\,\varepsilon_n|\log\varepsilon_n|\,.$$

Let $\mathcal{B}_{n,\bar{\tau}}(t)$ be a time parametrized family of balls given by Proposition 2.2 starting from $\mathcal{B}_{n,\bar{\tau}} =: \mathcal{B}_{n,\bar{\tau}}(0)$; we denote by $\mathcal{C}_{n,\bar{\tau}}(t)$ the family of balls in $\mathcal{B}_{n,\bar{\tau}}(t)$ that are contained in Ω and by $U_{n,\bar{\tau}}(t)$ the union of the balls in $\mathcal{B}_{n,\bar{\tau}}(t)$. Moreover we set

$$\mu_{n,\bar{\tau}}(t) := \sum_{B \in \mathcal{C}_{n,\bar{\tau}}(t)} \deg(u_n, \partial B)\delta_{x_B}\,.$$

Let F be as defined in (2.18). By (2.2) in Proposition 2.2, for any $t_n \geq 1$ we have

$$(4.11) \qquad F(\mathcal{B}_{n,\bar{\tau}}(0), \mu_{n,\bar{\tau}}(0), \Omega \cap (U_{n,\bar{\tau}}(t_n) \setminus U_{n,\bar{\tau}}(0))) \geq \pi|\mu_{n,\bar{\tau}}(t_n)|\log(1+t_n)\,.$$

Moreover, (4.8) implies in particular that

$$GL_{\varepsilon_n}(u_n) \geq \int_0^{\bar{\tau}} 2\tau\Theta_n(\tau)\,\mathrm{d}\tau \geq C_l\Theta_n(\bar{\tau})\,,$$

which together with (4.11), Remark 2.5, the energy bound and (4.10) yields

$$(4.12) \quad \log(1+t_n)|\mu_{n,\bar{\tau}}(t_n)| \leq CF(\mathcal{B}_{n,\bar{\tau}}(0), \mu_{n,\bar{\tau}}(0), \Omega) \leq C\Theta_n(\bar{\tau})$$
$$\leq C_l|\log\varepsilon_n| \leq C_l|\log\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}})|\,.$$

In particular, by applying (4.12) with $t_n = 1$, we have

$$(4.13) \qquad |\mu_{n,\bar{\tau}}(1)| \leq C_l\,F(\mathcal{B}_{n,\bar{\tau}}(0), \mu_{n,\bar{\tau}}(0), \Omega) \leq C_l|\log\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}})|\,.$$

Moreover, by construction (see also (2.18)),

$$F(\mathcal{B}_{n,\bar{\tau}}(1), \mu_{n,\bar{\tau}}(1), \Omega) \leq F(\mathcal{B}_{n,\bar{\tau}}(0), \mu_{n,\bar{\tau}}(0), \Omega) \leq C_l|\log\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}})|\,,$$

which, in virtue of (4.13) and Proposition (2.2) (5), implies

$$(4.14) \qquad F(\mathcal{B}_{n,\bar{\tau}}(1), \mu_{n,\bar{\tau}}(1), \Omega) + |\mu_{n,\bar{\tau}}(1)| \leq C_l|\log\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}})| \leq C_l|\log\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}}(1))|\,.$$

Step 2: Proof of compactness. Let $\bar{\tau} \in (\frac{2}{l}, 1 - \frac{1}{l})$. By (4.14) and by Theorem 2.4, there exist $\mu_{\bar{\tau}} \in X(\Omega)$ and $\{\nu_{\bar{\tau}}^s\}_{s\in[0,1)} \subset X(\Omega)$ such that, up to a subsequence independent of s,

$$(4.15) \qquad \mu_{n,\bar{\tau}}(1) \overset{\text{flat}}{\to} \mu_{\bar{\tau}}, \quad \mu_{n,\bar{\tau}}^s(1) \overset{\star}{\rightharpoonup} \mu_{\bar{\tau}} \qquad \text{for any } s \in (0,1)$$

and

$$|\mu_{n,\bar{\tau}}^s(1)| \overset{\star}{\rightharpoonup} \nu_{\bar{\tau}}^s \qquad \text{for any } s \in (0,1) \setminus S_{\bar{\tau}},$$

where $S_{\bar{\tau}}$ is the set constructed in the proof of Theorem 2.4. Moreover the measures $\nu_{\bar{\tau}}^s$ satisfy all the structure properties in (3).

Firstly, we show that, up to a subsequence, $Ju_n \overset{\text{flat}}{\to} \pi\mu_{\bar{\tau}}$ and that actually $\mu_{\bar{\tau}}$ does not depend on $\bar{\tau}$. By construction and by the very definition of $J_{\bar{\tau}}u_n$ (see (1.2) and (1.3)) we have that for any $t_n \geq 1$

$$(J_{\bar{\tau}}u_n - \pi\mu_{n,\bar{\tau}}(t_n))(B) = 0 \qquad \text{for any} \quad B \in \mathcal{C}_{n,\bar{\tau}}(t_n)\,.$$

Therefore, by triangular inequality, Proposition 1.2, Lemma 2.3 (i), Proposition 2.2 (5), (4.10) and (4.14), for any $t_n \geq 1$ and for n large enough we have

$$(4.16) \quad \|Ju_n - \pi\mu_{n,\bar{\tau}}(t_n)\|_{\text{flat}} \leq \|Ju_n - J_{\bar{\tau}}u_n\|_{\text{flat}} + \|J_{\bar{\tau}}u_n - \pi\mu_{n,\bar{\tau}}(t_n)\|_{\text{flat}}$$
$$\leq C_l\varepsilon_n|\log\varepsilon_n| + C\,\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}}(t_n))\,(|J_{\bar{\tau}}u_n|(\Omega) + \pi|\mu_{n,\bar{\tau}}(t_n)|(\Omega))$$
$$\leq C_l\varepsilon_n|\log\varepsilon_n| + C_l\,\mathcal{R}ad(\mathcal{B}_{n,\bar{\tau}}(t_n))(|\log\varepsilon_n| + |\mu_{n,\bar{\tau}}(1)|(\Omega))$$
$$\leq C_l|\log\varepsilon_n|^2(1+t_n)\,\varepsilon_n\,.$$

Therefore, by (4.15), applying (4.16) with $t_n = 1$ and setting $\mu := \mu_{\bar{\tau}}$, we get

$$(4.17) \qquad Ju_n \overset{\text{flat}}{\to} \pi\mu, \qquad \mu_{n,\bar{\tau}}(1) \overset{\text{flat}}{\to} \mu \qquad \text{for any } \bar{\tau} \in \left(\frac{2}{l}, 1 - \frac{2}{l}\right),$$

up to a subsequence independent of $\bar{\tau}$.

We now prove that $J^s u_n \equiv J u_n * \rho_{\varepsilon_n^s} \overset{\star}{\rightharpoonup} \pi\mu$ for any $s \in (0,1)$ up to a subsequence independent of s. To this end, fix $s \in (0,1)$ and let $\sigma > s$. By applying (4.16) with $t_n = t_n^\sigma := \frac{1}{\varepsilon_n^{1-\sigma}} - 1$, for any $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \le 1$, we get

$$(4.18) \quad |\langle J u_n * \rho_{\varepsilon_n^s} - \pi\mu_{n,\bar\tau}(t_n^\sigma) * \rho_{\varepsilon_n^s}, \varphi \rangle| = |\langle J u_n - \pi\mu_{n,\bar\tau}(t_n^\sigma), \varphi * \rho_{\varepsilon_n^s} \rangle|$$
$$\le C\|J u_n - \pi\mu_{n,\bar\tau}(t_n^\sigma)\|_{\text{flat}}\, \varepsilon_n^{-s} \le C_l |\log \varepsilon_n|^2 \varepsilon_n^{\sigma-s}\,.$$

By using (4.16) with $t_n = t_n^\sigma$ and by (4.17), $\mu_{n,\bar\tau}(t_n^\sigma) \overset{\text{flat}}{\rightarrow} \mu$; moreover, by (4.12), $|\mu_{n,\bar\tau}(t_n^\sigma)| \le \frac{C_l}{1-\sigma}$, so that $\mu_{n,\bar\tau}(t_n^\sigma) \overset{\star}{\rightharpoonup} \mu$ and $\mu_{n,\bar\tau}(t_n^\sigma) * \rho_{\varepsilon_n^s} \overset{\star}{\rightharpoonup} \mu$. This latter fact, combined with (4.18) implies that $J u_n * \rho_{\varepsilon_n^s} \overset{\star}{\rightharpoonup} \pi\mu$, as claimed above.

In order to conclude the proof of (i), we show that, up to a subsequence independent of s, $|J u_n * \rho_{\varepsilon_n^s}| \overset{\star}{\rightharpoonup} \pi\nu_{\bar\tau}^s$ for any $s \in (0,1) \setminus S_{\bar\tau}$ and that actually $\nu_{\bar\tau}^s$ do not depend on $\bar\tau$ and, in turn, $S_{\bar\tau}$ can be also chosen independent of $\bar\tau$. We denote by s_k the points in the set $S_{\bar\tau}$ with $s_0 = 0$ and $s_{k-1} < s_k$ for any $k = 1, \dots, \sharp S$. Set $\mathcal{R}_{n,\bar\tau} := \mathcal{R}ad(\mathcal{B}_{n,\bar\tau})$ and $t_{n,\bar\tau}^\sigma := \frac{1}{\mathcal{R}_{n,\bar\tau}^{1-\sigma}} - 1$ for any $\sigma \in (0,1)$. We recall that, by (2.13), if $s \in (s_{k-1}, s_k)$ for some k, then for any $\sigma \in (s, s_k)$

$$(4.19) \qquad\qquad |\mu_{n,\bar\tau}(t_{n,\bar\tau}^\sigma)| \overset{\star}{\rightharpoonup} \nu_{\bar\tau}^s \qquad \text{as } n \to +\infty\,.$$

By triangular inequality, to prove the claim it is enough to show that for any $s \in (0,1) \setminus S_{\bar\tau}$, (up to a subsequence independent of s) there holds

$$(4.20) \qquad\qquad |J u_n * \rho_{\varepsilon_n^s}| - |J_{\bar\tau} u_n * \rho_{\varepsilon_n^s}| \overset{\star}{\rightharpoonup} 0 \qquad \text{as } n \to +\infty\,,$$

$$(4.21) \qquad\qquad |J_{\bar\tau} u_n * \rho_{\varepsilon_n^s}| - |J_{\bar\tau} u_n * \rho_{\mathcal{R}_{n,\bar\tau}^s}| \overset{\star}{\rightharpoonup} 0 \qquad \text{as } n \to +\infty\,,$$

$$(4.22) \qquad\qquad |J_{\bar\tau} u_n * \rho_{\mathcal{R}_{n,\bar\tau}^s}| \overset{\star}{\rightharpoonup} \nu_{\bar\tau}^s \qquad \text{as } n \to +\infty\,.$$

As for the proof of (4.20), notice that, by Proposition 1.2 and the energy bound, for any $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \le 1$ we have

$$\left|\langle |J u_n * \rho_{\varepsilon_n^s}| - |J_{\bar\tau} u_n * \rho_{\varepsilon_n^s}|, \varphi \rangle\right| \le \langle |(J u_n - J_{\bar\tau} u_n) * \rho_{\varepsilon_n^s}|, |\varphi| \rangle \le \langle |J u_n - J_{\bar\tau} u_n| * \rho_{\varepsilon_n^s}, |\varphi| \rangle$$
$$\le \|J u_n - J_{\bar\tau} u_n\|_{\text{flat}} \|\nabla(|\varphi| * \rho_{\varepsilon_n^s})\|_{L^\infty} \le C_l \varepsilon_n^{1-s} |\log \varepsilon_n| \to 0 \qquad \text{as } n \to +\infty\,.$$

Let us pass to the proof of (4.21). Let $s \in (0,1) \setminus S_{\bar\tau}$, let k be such that $s \in (s_{k-1}, s_k)$, and let $\sigma \in (s, s_k)$. Let $A \subset\subset \Omega$ be open. By arguing as in (3.13) and (3.14) and replacing μ_n with $J_{\bar\tau} u_n$, we get

$$(4.23) \quad \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)=0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\varepsilon_{n,\bar\tau}^s} + \sum_{B \in \mathcal{B}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \setminus \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma)} (J_{\bar\tau} u_n \llcorner B) * \rho_{\varepsilon_{n,\bar\tau}^s} \right|(A) \to 0\,,$$

$$(4.24) \quad \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)=0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta_{n,\bar\tau}^s} + \sum_{B \in \mathcal{B}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \setminus \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma)} (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta_{n,\bar\tau}^s} \right|(A) \to 0\,.$$

Moreover, by arguing as in (3.15) and by using the first inequality in (4.10), it is easy to see that

$$(4.25) \quad \begin{aligned} \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)\neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\varepsilon_n^s} \right| &= \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)\neq 0}} \left| (J_{\bar\tau} u_n \llcorner B) * \rho_{\varepsilon_n^s} \right|, \\[2ex] \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)\neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\mathcal{R}_{n,\bar\tau}^s} \right| &= \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t_{n,\bar\tau}^\sigma) \\ J_{\bar\tau} u_n(B)\neq 0}} \left| (J_{\bar\tau} u_n \llcorner B) * \rho_{\mathcal{R}_{n,\bar\tau}^s} \right|, \end{aligned}$$

which, by arguing as in (3.16), implies that for any $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \leq 1$

$$(4.26) \quad \left| \left\langle \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\varepsilon^s_n} \right| - \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right|, \varphi \right\rangle \right| \to 0 \qquad \text{as } n \to +\infty,$$

Then (4.21) follows by (4.23), (4.24) and (4.26).

We end up with the proof of (i) by showing (4.22) which, in view of (4.19) and (4.24) is equivalent to

$$(4.27) \quad \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right| - \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) \right| \xrightarrow{\star} 0 \qquad \text{as } n \to +\infty .$$

Let $\varphi \in C_c(\Omega)$ with $\|\varphi\|_{L^\infty} \leq 1$. By (4.25) and by triangular inequality we have

$$\left| \left\langle \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right| - \left| \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) \right|, \varphi \right\rangle \right|$$

$$\leq \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} \left| \left\langle \left| (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right| - \left| (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) \right|, \varphi \right\rangle \right|$$

$$\leq \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} \left| \left\langle \left| (J_{\bar\tau} u_n \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right| - \left| (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right|, \varphi \right\rangle \right|$$

$$+ \sum_{\substack{B \in \mathcal{C}_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \\ J_{\bar\tau} u_n(B) \neq 0}} \left| \left\langle \left| (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) * \rho_{\delta^s_{n,\bar\tau}} \right| - \left| (\mu_{n,\bar\tau}(t^\sigma_{n,\bar\tau}) \llcorner B) \right|, \varphi \right\rangle \right| \to 0 \qquad \text{as } n \to +\infty ,$$

where the convergence to zero of the first addendum and of the second addendum follow by Lemma 2.3 (ii) and (iii) respectively.

This concludes the proof of (4.27), of (4.22) and, in turn, of all the compactness properties in (i).

Step 3: Proof of the Γ-liminf inequality. We can assume without loss of generality that the upper bound (4.1) holds true. Then, by (4.7), Fatou Lemma, Remark 2.5, (4.10), (4.14) and Theorem 2.4 (iv) we have

$$\liminf_{n \to +\infty} \frac{GL_{\varepsilon_n}(u_n)}{|\log \varepsilon_n|} \geq \liminf_{n \to +\infty} \frac{1}{|\log \varepsilon_n|} \int_{\frac{2}{l}}^{1-\frac{2}{l}} 2\tau \Theta_n(\tau) \, d\tau$$

$$\geq \int_{\frac{2}{l}}^{1-\frac{2}{l}} 2\tau \liminf_{n \to +\infty} \frac{\Theta_n(\tau)}{|\log \varepsilon_n|} \, d\tau \geq \int_{\frac{2}{l}}^{1-\frac{2}{l}} 2\tau \liminf_{n \to +\infty} \frac{F(\mathcal{B}_{n,\tau}(1), \mu_{n,\tau}(1), \Omega)}{|\log \mathcal{R}_{n,\tau}|} \, d\tau$$

$$\geq \left(1 - \frac{4}{l}\right) \pi \sum_{k=1}^{\sharp S} (s_k - s_{k-1}) \nu^{s_{k-1}}(\Omega),$$

whence the claim follows by sending $l \to \infty$.

Step 4: Proof of the Γ-limsup inequality. The construction of the recovery sequence closely resembles the one in the proof of Theorem 3.1. We briefly sketch it.

Let μ, $\{\nu^s\}$ and S be as in the assumption. By standard density arguments we can assume that $K := \sharp S < +\infty$ so that $S := \{0 = s_0 < s_1 < \ldots < s_K = 1\}$. We set $\eta^{s_0}_{\text{def}} := \xi^{s_0}_{\text{def}}$ and $\eta^{s_k}_{\text{def}} := \nu^{s_k} - \nu^{s_{k-1}}$ for any $k = 1, \ldots, K$. Again by density arguments we can assume that $\mu = \sum_{i=1}^N z_i \delta_{x_i}$ with $|z_i| = 1$ and $x_i \neq x_j$ for $i \neq j$, and that $\eta^{s_k}_{\text{def}} = 2\delta_{y^{s_k}}$ with all y^{s_k} different from each other and from all x_i's. Let moreover $y^{s_k}_{n,+}$ and $y^{s_k}_{n,-}$ be two points in Ω such that $\text{dist}\,(y^{s_k}_{n,+}, y^{s_k}_{n,-}) = 2\,\text{dist}\,(y^{s_k}_{n,+}, y^{s_k}) = 2\varepsilon^{s_k}_n$ for any $k = 1, \ldots, K$, while for $k = 0$ we enforce that

dist $(y_{n,+}^{s_0}, y_{n,-}^{s_0}) = 2$ dist $(y_{n,+}^{s_0}, y^{s_0}) = 2 \frac{1}{|\log \varepsilon_n|}$. For any $n \in \mathbb{N}$, let $g_n(r) := \min\{\frac{r}{\varepsilon_n}, 1\}$ and let ϑ_n be defined as in (3.18), for any $x \in \Omega$ we set

$$u_n(x) := \prod_{i=1}^{N} g_n(|\cdot - x_i|) \cdot \prod_{k=0}^{K} g_n(|\cdot - y_{n,+}^{s_k}|) \cdot g_n(|\cdot - y_{n,-}^{s_k}|) \cdot e^{i\vartheta_n(\cdot)}.$$

One can easily check that $u_n \in H^1(\Omega; \mathbb{R}^2)$ and that Ju_n satisfy all the desired convergence properties. By arguing as in the proof of Theorem 3.1, one can prove that

$$\limsup_{n \to +\infty} \frac{GL_{\varepsilon_n}(u_n)}{|\log \varepsilon_n|} \le \pi|\mu|(\Omega) + \pi \sum_{k=0}^{K}(1 - s_k)\eta_{\text{def}}^{s_k}(\Omega),$$

which, in view of (3.17), and sending eventually $K \to +\infty$, yields (4.2). $\qquad\square$

References

[1] Alberti, G.; Baldo, S.; Orlandi, G.: Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.* **54** (2005), no. 5, 1411–1472.

[2] Alicandro, R.; Cicalese, M.: Variational analysis of the asymptotics of the XY model. *Arch. Ration. Mech. Anal.* **192** (2009), no. 3, 501–536.

[3] Alicandro, R.; De Luca, L.; Garroni, A.; Ponsiglione M.: Metastability and dynamics of discrete topological singularities in two dimensions: a Γ-convergence approach. *Arch. Ration. Mech. Anal.* **214** (2014), no. 1, 269–330.

[4] Alicandro, R.; Ponsiglione, M.: Ginzburg-Landau functionals and renormalized energy: A revised Γ-convergence approach. *J. Funct. Anal.* **266** (2014), no. 8, 4890–4907.

[5] Bethuel, F.; Brezis, H.; Hélein, F.: *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications. vol. 13. Birkhäuser Boston, Boston, 1994.

[6] Bethuel, F; Zheng, X.: Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.* **80** (1988), no. 1, 60–75.

[7] Bourgain, J.; Brezis, H.; Mironescu, P.: Lifting in Sobolev spaces. *J. Anal. Math.* **80** (2000), no. 1, 37–86.

[8] Boutet de Monvel-Berthier, A.; Georgescu, V.; Purice R.: A boundary value problem related to the Ginzburg-Landau model. *Commun. Math. Phys.* **142** (1991), no.1, 1–23.

[9] Brezis, H.; Nirenberg, L.: Degree theory and BMO: Part i: compact manifolds without boundaries. *Selecta Math.* (N.S.) **1** (1995), no. 2, 197–263.

[10] Colliander, J. E.; Jerrard, R. L.: Ginzburg-Landau vortices: weak stability and Schrödinger equation dynamics. *J. Anal. Math.* **77** (1999), no. 1, 129–205.

[11] De Luca, L.; Garroni, A.; Ponsiglione, M.: Γ-convergence analysis of systems of edge dislocations: the self energy regime. *Arch. Ration. Mech. Anal.* **206** (2012), no. 3, 885–910.

[12] Jerrard, R.: Lower bounds for generalized Ginzburg-Landau functionals. Preprint (1995).

[13] Jerrard, R.: Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* **30** (1999), no. 4, 721–746.

[14] Jerrard, R.; Soner, H. M.: The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differ. Equ.* **14** (2002), no. 2, 151–191.

[15] Jerrard, R.; Spirn, D.: Refined Jacobian Estimates for Ginzburg-Landau functionals. *Indiana Univ. Math. J.* **56** (2007), no.1, 135–186.

[16] Lin, F.-H.; Xin, J.: On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation. *Commun. Math. Phys.* **200** (1999), no. 2, 249–274.

[17] Ponsiglione, M.: Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuous. *SIAM J. Math. Anal.* **39** (2007), no. 2, 449–469.

[18] Sandier, E.: Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998), no. 2, 379–403.

[19] Sandier, E.; Serfaty, S.: A product-estimate for Ginzburg-Landau and corollaries, *J. Funct. Anal.* **211** (2004), no. 1, 219–244.

[20] Sandier, E.; Serfaty, S.: *Vortices in the Magnetic Ginzburg-Landau Model.* Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston, Inc., Boston, MA, 2007.

[21] Sandier, E.; Serfaty, S.: Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.* **57** (2004), no. 12, 1627–1672.

[22] Serfaty, S.: Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow, part II: The dynamics. *J. Eur. Math. Soc.* **9** (2007), no. 3, 383–426.

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