Monopoly with differentiated final goods and heterogeneous markets

Andrea Caravaggio[*] and Mauro Sodini[*]

[*]Department of Economics and Management, University of Pisa

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Abstract

This work attempts to characterise the dynamic properties of a nonlinear model in which a monopolist produces a fixed amount of an intermediate good. The firm employs this good in producing two vertically differentiated final commodities, sold in two distinct markets. Consumers' preferences are described by quasi-linear and quadratic utility functions respectively yielding isoelastic and linear demand functions. In addition, we assume that the monopolist adopts a gradient adjustment rule based on profit variation to adjust its choice about the amount of intermediate good to employ in the production of each final good. Two alternative scenarios emerge: in the first, there exists coexistence of two markets; in the second, the monopolist specialises on the production of a single final good. The dynamics show that the elasticities of market demands play an ambiguous role in affecting the stability of the equilibrium, whereas the speed of adjustment unambiguously destabilises the system. Moreover, the article investigates the role of the productive capacity and the gap in marginal costs in defining the different scenarios. In particular, economic dynamics may be chaotic and/or multiple attractors may appear.

Keywords: Monopoly Dynamics; Intermediate Good; Differentiated Final Goods; Multiple Equilibria; Chaos.

1 Introduction

This article aims to characterise the dynamic properties of a nonlinear monopoly model in which the firm produces a fixed amount of an intermediate good that is then employed in the production of two vertically differentiated final commodities, placed on two heterogeneous markets. In order to emphasise this heterogeneity, we consider two markets characterised by different inverse demand functions deriving from different preferences.

Regarding the supply side, a common assumption in the literature on monopolistic and oligopolistic markets is that firms, due to the high cost of getting information, do not have complete knowledge regarding the consumers' preferences. For this reason, firms make their decisions on the basis of *heuristics* derived from market experiments, conjectures and local demand estimates. Clower (1959) and Baumol and Quandt (1964) started to investigate this issue by considering monopolistic markets where the firm has reduced rationality and adopts simple rules to adjust the level of production. By following this research line, Puu (1995) shows that such rules, in a monopolistic context, may generate nonlinear dynamics. In particular, the author assumes that the firm makes its decisions using a rule based on profit differentials obtained by varying the quantity produced.

Recently, other several works have investigated monopoly dynamics both in discrete time and continuous time with delays settings. With regard to the former set up, Naimzada and Ricchiuti (2008) reconsider the model proposed in Puu (1995) and analyse the dynamic properties of a monopoly with a cubic market demand without inflexion points, by taking into account a *gradient rule* through which the firm adjusts its choice according to the profit variation. Differently, Naimzada and Ricchiuti (2011) consider a monopolistic firm which does not know the entire demand and estimates a *subjective* inverse demand function, whose convexity/concavity may generate multiple steady states and induce a transition from stability to complex dynamics. More recently, Cavalli and Naimzada (2015) adopt the decisional mechanism proposed in Naimzada and Ricchiuti (2008) to investigate the emergence of complex dynamics in a monopoly model with an isoelastic demand.

In continuous time, Matsumoto and Szidarovszky (2012) provide a dynamic monopoly model with delays in which information lags on profits, linear market demand and non-constant speed of adjustment of the output are assumed. Matsumoto and Szidarovszky (2014) study the dynamics of a monopoly model with delays in which nonlinear demand and non-constant marginal costs are assumed. In addition, Matsumoto et al. (2013) and Matsumoto and Szidarovszky (2015) discuss two nonlinear monopoly models with linear demand in which the firm adopts the gradient mechanism and delays are continuously distributed. Finally, Gori et al. (2016) describe different long run dynamics in monopoly models in which the introduction of delays is associated with different assumptions related to the bounded rationality of the monopolist.

While the above-mentioned literature focuses on the decision-making mechanisms through which the firm decides the amount of the unique final good, our work takes a different context into consideration. In fact, we consider a monopolistic firm producing a fixed amount of an intermediate good that is then employed as an input for the production of two differentiated final goods. From an economic point of view, our model examines the dynamics of particular production sectors characterised by merceological and geographical peculiarities. First, our analysis focuses on sectors - such as agriculture or livestock farming - where it is reasonable to assume that the amount of intermediate good (used as an input for producing different final goods) is constant and at most affected by exogenous factors. In this regard, we can refer to the breeding of cows and goats for the production of milk and its derivatives or the cultivation of seasonal fruit for the production of jams and drinks. Second, we take into account specific geographical contexts, such as small tourist destinations, for which one can reasonably think that a small firm operates as a monopolist and differentiates its final production between goods for tourists, and goods purchased by local consumers.[1] In such a geographical and market context, the two different market demands for the differentiated final goods significantly affect the firm's choices concerning the allocation of the fixed amount of intermediate goods between the two different production lines.

Therefore, by considering a monopolistic firm which adopts the gradient adjustment rule to allocate the intermediate good, we study how (i) the heterogeneity in the demand side, (ii) the different costs incurred in the differentiation of final products and (iii) the productive capacity may affect the behaviour of the monopolist. Indeed, depending on the parameter configurations, we classify scenarios in which the two final productions coexist and other scenarios in which specialisation on a single production becomes preferable. In addition, we show the emergence of periodic cycles and chaotic regimes.

The rest of the paper is organised as follows: Section 2 shows the main features of the monopoly model and the definition of the piecewise map; Section 3 describes the properties of the map and the existence of steady states both in the general case and in a simplified case; Section 4 refers to the results on the local stability of equilibria; Section 5 shows several interesting global phenomena generated by the model and Section 6 concludes.

2 The model

We consider a monopolistic firm producing, at every discrete time period t, a fixed amount $P \in (0, +\infty)$ of an intermediate good. The firm employs the intermediate good in producing two vertically differentiated final goods. In particular, it allocates the share a_t of P, with $a_t \in [0, 1]$, to the production of good 1 and the remaining share $(1 - a_t)$ to the production of good 2. Then, we have that the amounts of the two final goods are respectively

$$q_{1,t} = a_t P, \tag{1}$$

$$q_{2,t} = (1 - a_t)P. \tag{2}$$

With regard to the demand side, we further assume that the two differentiated goods are produced to match the preferences of two different categories of

[1] An economic evidence of this heterogeneity can be found in the consumption of cheese on the island of Crete. In fact, in such island there is a strong heterogeneity between local consumers, who buy nearly exclusively traditional Cretan cheeses (i.e., Graviera and Myzithra), and tourists, who (also because of a lack of perfect information on the different varieties of cheese produced in the different Greek islands) buy largely the most famous Greek cheese, i.e. Feta, although this is not really a typical Cretan product (see the website https://www.rethymno.gr for details).

consumers, one that consumes only q_1 (tourists) and the other that consumes only q_2 (local consumers). In particular, we consider tourists' preferences described by the utility function $U(q_1, y_1) = \frac{\epsilon}{\epsilon-1} q_1^{\frac{\epsilon-1}{\epsilon}} + y_1$ where $\epsilon > 0$ (with $\epsilon \neq 1$) measures the demand elasticity in the market 1, whereas local consumers' preferences are described by $V(q_2, y_2) = \alpha\, q_2 - \frac{1}{2} q_2^2 + y_2$ where $\alpha > 0$ represents the dimension of the market 2. We assume y_i, $i = 1, 2$, as a numeraire (composite) good for tourists and local consumers, respectively, whose price is normalised to 1. Each type of agent maximises his utility function subject to the budget constraint $p_i q_i + y_i = M_i$, $i = 1, 2$ where $M_i > 0$ is the exogenous nominal income of the consumer. Solving these problems gives the following isoelastic inverse demand function for tourists

$$p_1(q_{1,t}) = q_{1,t}^{-\frac{1}{\epsilon}} \tag{3}$$

and the linear inverse demand function for local consumers

$$p_2(q_{2,t}) = \alpha - q_{2,t}. \tag{4}$$

In economic terms, we can notice that tourists perceive q_1 as an experiential good rather than a consumption good (see Hall et al., 2004), and consequently, for every price p_1, someone is always willing to buy it (the demand $q_1(p_1)$ is always positive). Differently, local consumers choose q_2 until its price becomes too high. Indeed, for an enough large price p_2, they replace q_2 with others consumption goods, identified by the numeraire y_2.
In what follows, we assume that the dimension of the second market is enough large, that is:

Assumption 1 $\alpha > P$.

We can notice that since the quantity placed in each of the two markets can vary in the interval $[0, P]$, the prices in the first and in the second market can vary in the interval $[P^{-\frac{1}{\epsilon}}, +\infty)$ and $[\alpha - P, \alpha]$, respectively.

By assuming constant positive marginal (and average) costs $c_1, c_2 > 0$ for markets 1 and 2, respectively, the profit function Π of the monopolist (where $\Pi : [0, P]^2 \longrightarrow \mathbb{R}$) is defined by

$$\Pi(q_{1,t}, q_{2,t}) = p_1(q_{1,t}) q_{1,t} + p_2(q_{2,t}) q_{2,t} - c_1 q_{1,t} - c_2 q_{2,t}. \tag{5}$$

Consequently, by substituting expressions (1)-(2)-(3)-(4) in (5), we can write the profits as a function of the variable a_t

$$\pi(a_t) = P\left[\left(\left(\frac{1}{a_t P}\right)^{\frac{1}{\epsilon}} + 2P + c_2 - c_1 - \alpha\right) a_t - P a_t^2 - (c_2 + P - \alpha)\right], \tag{6}$$

where $\pi : [0, 1] \to \mathbb{R}$. Hence, we get the marginal profit of the monopolistic firm

$$\pi'(a_t) = P\left[2(1-a_t)P - (\alpha + c_1 - c_2) + \frac{(\epsilon-1)}{\epsilon}\left(\frac{1}{a_t P}\right)^{\frac{1}{\epsilon}}\right]. \tag{7}$$

As in Naimzada and Ricchiuti (2008) and Cavalli and Naimzada (2015), we assume that the monopolist, due to its bounded knowledge of the markets, adapt the production level of the two goods q_1 and q_2 according to a "rule of thumb". In particular, we consider a *gradient-like* adjustment mechanism, in which the variation of the share a is proportional to the marginal profit, that is

$$a_{t+1} = f(a_t) := a_t + k\pi'(a_t), \tag{8}$$

where $k > 0$ represents the speed of adjustment of the share a at time $t+1$ with respect to the marginal profit at time t. This means that the share of the production factor P, used for one of the two productions, increases (respectively decreases) when the corresponding marginal profit is positive (respectively negative).

From the expression of $\pi'(a_t)$, it follows that f is defined in the range $(0, +\infty)$ and assumes values in $(-\infty, +\infty)$. Because an initial share $0 < a_0 < 1$ may generate an unfeasible trajectory with $a_t < 0$ or $a_t > 1$, in order to avoid this occurrence we assume that the monopolistic decision process is actually described by the following piecewise defined map $H : [0,1] \to [0,1]$

$$H : a_{t+1} = \begin{cases} 0 & if\ f(a_t) \leq 0 \\ f(a_t) & if\ f(a_t) \in (0,1) \\ 1 & if\ f(a_t) \geq 1. \end{cases} \tag{9}$$

We notice that if $f(a_t) \notin (0,1)$, the monopolist produces only one of the two final goods at $t+1$.

3 Steady state analysis

Before analysing the existence of steady states and then their local stability, we characterise the shape of the map.

3.1 The shape of the graph of f

By considering the behaviour of f at the boundaries of the interval $(0,1)$, the following two Lemmas hold:

Lemma 1 *Let f be defined in (8).*
(i) If $\epsilon > 1$, then $\lim_{a_t \to 0^+} f(a_t) = +\infty$;
(ii) if $\epsilon < 1$, then $\lim_{a_t \to 0^+} f(a_t) = -\infty$.

Proof. By calculating $\lim_{a_t \to 0^+} f(a_t)$, the results in *(i)* and *(ii)* are straightforward. $\square$

Lemma 2 *Let f be defined in (8) and $\widetilde{P} := \left[\frac{\epsilon-1}{\epsilon(\alpha+c_1-c_2)}\right]^{\epsilon}$.*
(i) Assume that $\epsilon > 1$ and $\alpha > c_2 - c_1$. If $P > \widetilde{P}$, then $f(1) < 1$. Otherwise, if $P \leq \widetilde{P}$, then $f(1) \geq 1$.
(ii) Assume $\epsilon < 1$ and $\alpha < c_2 - c_1$. If $P < \widetilde{P}$, then $f(1) < 1$. Otherwise, if $P \geq \widetilde{P}$, then $f(1) \geq 1$.
(iii) Assume that $\epsilon > 1$ and $\alpha < c_2 - c_1$. Then, $f(1) > 1$.
(iv) Assume that $\epsilon < 1$ and $\alpha > c_2 - c_1$. Then, $f(1) < 1$.

Proof. (i) Let us consider

$$f(1) = 1 + \frac{k}{\epsilon}\Big((\epsilon - 1)\,P^{\frac{\epsilon-1}{\epsilon}} - \epsilon\,(\alpha + c_1 - \, c_2)\,P\Big). \tag{10}$$

By solving the inequality $f(1) < 1$ in terms of P, the results follow. $\square$

By considering the first derivative of f, we obtain the following result:

Proposition 1 *Let $\widetilde{\widetilde{k}} := \frac{\epsilon^2}{(\epsilon-1)P^{\frac{\epsilon-1}{\epsilon}}+2P^2\epsilon^2}$ and $\overline{P} := \left(\frac{1-\epsilon}{2\epsilon^2}\right)^{\frac{\epsilon}{\epsilon+1}}$.*
The function f admits at most one critical point, whose coordinate is $\widetilde{a} := P^{\frac{\epsilon-1}{1+\epsilon}}\left(\frac{\left(1-2\,P^2 k\right)\epsilon^2}{k(\epsilon-1)}\right)^{-\frac{\epsilon}{1+\epsilon}}$. It belongs to the interval $(0,1)$ if one of the following conditions is satisfied:
(i) $\epsilon > 1$ and $k < \widetilde{\widetilde{k}}$ (in this case it is the global minimum of f);
or
(ii) $\epsilon < 1$, $P > \overline{P}$ and $k > \widetilde{\widetilde{k}}$ (in this case it is the global maximum of f).

Proof. Let us consider the first derivative of f:

$$f'(a_t) = 1 - 2\,P^2 k - \frac{k\,(\epsilon - 1)}{\epsilon^2\,a_t}\,{a_t}^{-\frac{1}{\epsilon}}\,P^{\frac{\epsilon-1}{\epsilon}}.$$

We have that $\widetilde{a}$ solves $f'(a_t) = 0$ if $\begin{cases}\epsilon > 1 \\ k \leq \frac{1}{2P^2}\end{cases}$ or $\begin{cases}\epsilon < 1 \\ k \geq \frac{1}{2P^2}\end{cases}$. The results follow by studying the inequalities $0 < \widetilde{a} < 1$. $\square$

Remark 1 *We notice that in the cases in which the parametric configuration does not satisfy the conditions in (i) or (ii) of Proposition 1, the function f is monotone.*

By studying the sign of the second derivative of f, we state the following Proposition:

Proposition 2 *Let f be defined in (8).*
(i) If $\epsilon > 1$, then f is convex for every $a_t \in (0,1)$;
(ii) if $\epsilon < 1$, then f is concave for every $a_t \in (0,1)$.

By means of Proposition 2, it follows that when $\widetilde{a} \in (0,1)$ (see Proposition 1), then $\widetilde{a}$ is the global minimum for f on $(0,1)$ if $\epsilon > 1$, while it is the global maximum for f on $(0,1)$ if $\epsilon < 1$.

3.2 Boundary stationary points

In the light of the study performed in the previous subsection, we are now able to analyse the existence and stability of the boundary fixed points 0 and 1 for the map H. In this regard, the following Propositions are stated:

Proposition 3 *Let H be the map defined in (9). Then, 0 is a fixed point for H if and only if $\epsilon < 1$.*

Proof. By means of the result in Lemma 1, in the case $\epsilon < 1$ we have $\lim_{a_t \to 0^+} f(a_t) = -\infty$ and then the first piece of H in (9) applies. Therefore, 0 is a fixed point for H. □

Proposition 4 *Let H be the map defined in (9) and $\widetilde{P} = \left[\frac{\epsilon-1}{\epsilon(\alpha+c_1-c_2)}\right]^{\epsilon}$. The following cases arise:*
(i) if $\epsilon > 1$ and $\alpha > c_2 - c_1$, then 1 is a fixed point for H if and only if $P \leq \widetilde{P}$;
(ii) if $\epsilon < 1$ and $\alpha < c_2 - c_1$, then 1 is a fixed point for H if and only if $P \geq \widetilde{P}$.

Proof. (i) By means of the result in Lemma 2, in the case $\epsilon > 1$ and $\alpha > c_2 - c_1$ we have that $f(1) \geq 1$ for $P \leq \widetilde{P}$. Then, the third piece of H in (9) applies and it follows that 1 is a fixed point for H. In a similar manner, we get the result in *(ii)*. □

We notice that when the fixed point 0 exists, the graph of the map H, in a right neighbourhood of 0, is defined by a horizontal line and it follows that 0 is locally superstable.[2] The same result applies when the fixed point 1 exists and $P \neq \widetilde{P}$. In this case, the map H, in a left neighbourhood of 1, is defined by a horizontal line and it follows that 1 is locally superstable. Differently, if $P = \widetilde{P}$, 1 is a fixed point but the map H, in a left neighbourhood of 1, is defined by f. In the latter case, 1 can be a stable or unstable fixed point, depending on the parametric configuration.

[2]A fixed point is said to be locally superstable if starting from a sufficiently close initial condition, it is reached through only one iteration.

3.3 Different stationary scenarios

By moving on the analysis of the existence of interior fixed points for H, we can state the following Propositions:

Proposition 5 *Let $\epsilon > 1$, the following cases arise:*
(I) Let $k < \widetilde{\widetilde{k}}$:
(a) if $f(\widetilde{a}) > \widetilde{a}$, then no interior fixed points exist;
(b) if $f(\widetilde{a}) \leq \widetilde{a}$ and $f(1) \leq 1$, then there exists a unique interior fixed point;
(c) if $f(\widetilde{a}) \leq \widetilde{a}$ and $f(1) > 1$, then there exist two interior fixed points.
(II) Let $k \geq \widetilde{\widetilde{k}}$:
(a) if $f(1) < 1$, then there exists a unique interior fixed point;
(b) if $f(1) \geq 1$, then no interior fixed points exist.

Proof. By means of the results in Lemma 1 and Proposition 2, $\lim_{a_t \to 0^+} f(a_t) = +\infty$ and f is convex in the interval $(0, 1)$. Let us now consider the following exhaustive cases:
(I) for $k < \widetilde{\widetilde{k}}$, $\widetilde{a}$ exists and belongs to the interval $(0, 1)$. (I.a) The assumption implies that the point $(\widetilde{a}, f(\widetilde{a}))$ lies above the main diagonal and then the result follows; (I.b) the conditions imply that the points $(\widetilde{a}, f(\widetilde{a}))$ and $(1, f(1))$ lie below or on the main diagonal. Then, the convexity of f on $(0, 1)$ guarantees the result; (I.c) in the same manner, because $(1, f(1))$ stays above the 45-degree line and $f(\widetilde{a}) \leq \widetilde{a}$ we have that, due to its convexity, the graph of f crosses the main diagonal twice on $(0, 1)$ and then the result follows.

(II) For $k \geq \widetilde{\widetilde{k}}$, $\widetilde{a}$ does not exist and f is monotonically decreasing. (II.a) Because $f(1) < 1$, the graph of f crosses the 45-degree line once and then the result follows; (II.b) in the same manner, because $f(1) \geq 1$ the graph of f does not cross the main diagonal on $(0, 1)$ and then the result follows. □

Proposition 6 *Let $\epsilon < 1$ and $\widetilde{k} = \frac{1}{2P^2}$, the following cases arise:*
(I) Let $P > \overline{P}$ and $k > \widetilde{\widetilde{k}}$:
(a) if $f(\widetilde{a}) \geq \widetilde{a}$ and $f(1) < 1$, then two interior fixed points exist;
(b) if $f(\widetilde{a}) > \widetilde{a}$ and $f(1) \geq 1$, then there exists a unique interior fixed point;
(c) if $f(\widetilde{a}) < \widetilde{a}$, then zero or two interior fixed points exist.
(II) Let $P < \overline{P}$ and $k > \widetilde{k}$ or let $P > \overline{P}$ and $k \in (\widetilde{k}, \widetilde{\widetilde{k}})$:
(a) if $f(\widetilde{a}) \geq \widetilde{a}$ and $f(1) \leq 1$, then no interior fixed points exist;
(b) if $f(\widetilde{a}) \geq \widetilde{a}$ and $f(1) > 1$, then there exists a unique interior fixed point;
(c) if $f(\widetilde{a}) < \widetilde{a}$ and $f(1) < 1$, then zero or two interior fixed points exist;
(d) if $f(\widetilde{a}) < \widetilde{a}$ and $f(1) > 1$, then there exists a unique interior fixed point;
(e) if $f(\widetilde{a}) < \widetilde{a}$ and $f(1) = 1$, then zero or one interior fixed point exists.
(III) Otherwise:
(a) if $f(1) > 1$, then there exists a unique interior fixed point;
(b) if $f(1) < 1$, then zero or two interior fixed points exist;
(c) if $f(1) = 1$, then zero or one interior fixed points exists.

Proof. By means of the results in Lemma 1 and Proposition 2, $\lim_{a_t \to 0^+} f(a_t) = -\infty$ and f is concave in the interval $(0,1)$. Let us now consider the following exhaustive cases:
(I) For $P > \overline{P}$ and $k > \widetilde{\widetilde{k}}$, $\widetilde{a}$ exists and belongs to $(0,1)$. (I.a) The assumptions imply that the point $(\widetilde{a}, f(\widetilde{a}))$ lies above or on the main diagonal while the point $(1, f(1))$ stays below it. Then, the graph of f crosses the 45-degree line twice in the interval $(0,1)$ and the result follows; (I.b) because $f(\widetilde{a}) > \widetilde{a}$ and $f(1) \geq 1$, the graph of f crosses the main diagonal once and then a unique interior fixed point exists; (I.c) because $(\widetilde{a}, f(\widetilde{a}))$ lies below the 45-degree line, the concavity of f in $(0,1)$ guarantees that (i) the graph of f is defined by points which stay all below the main diagonal or (ii) it crosses the 45-degree line twice. Then, the result follows.
(II) For $P < \overline{P}$ and $k > \widetilde{k}$ or $P > \overline{P}$ and $k \in (\widetilde{k}, \widetilde{\widetilde{k}})$, $\widetilde{a}$ exists and belongs to the interval $[1, +\infty)$ and f is monotonically increasing in the interval $(0,1)$. (II.a) Because $f(\widetilde{a}) \geq \widetilde{a}$ and $f(1) \leq 1$, the graph of f does not cross the main diagonal and the result follows; (II.b) because $f(\widetilde{a}) \geq \widetilde{a}$ and $f(1) > 1$, the graph of f crosses the 45-degree line once in the interval $(0,1)$, and the result follows; (II.c) the assumptions imply that the points $(\widetilde{a}, f(\widetilde{a}))$ and $(1, f(1))$ stay below the main diagonal. Then, it follows that (i) the graph of f is defined on $(0,1)$ by points which stay all below the main diagonal, or (ii) the graph of f crosses the 45-degree line twice. Then, the result follows; (II.d) because $f(\widetilde{a}) < \widetilde{a}$ and $f(1) > 1$, the graph of f crosses the main diagonal once. Then, the result follows; (II.e) because $f(\widetilde{a}) < \widetilde{a}$ and $f(1) = 1$, the graph of f (i) lies below the main diagonal for every $a_t \in (0,1)$ or (ii) crosses the latter once. Then, the result follows.
(III) Otherwise, $\widetilde{a}$ does not exist and f is monotonically increasing. (III.a) Because $f(1) > 1$, the graph of f crosses the 45-degree line once on $(0,1)$ and then the result follows; (III.b) because $f(1) < 1$, the graph of f (i) lies below the main diagonal for every $a_t \in (0,1)$ or (ii) it crosses the latter twice. Then, the result follows; (III.c) because $f(1) = 1$, the graph of f (i) lies below the main diagonal for every $a_t \in (0,1)$ or (ii) crosses the latter once. Then, the result follows. □

Remark 2 *We notice that the inequalities introduced in the Propositions 5-6 (i.e., $f(\widetilde{a}) \gtreqless \widetilde{a}$ and $f(1) \gtreqless 1$) can be defined in terms of α. Since the expressions we obtain are rather cumbersome and not easy to read from an economic point of view, we have preferred to furnish a description based on the graph of the map H. Moreover, although not all the cases discussed in Propositions 5-6 will be treated in what follows, they can be all generated by appropriate parametric configurations.*

In addition, the following result on the relationship between fixed points of the map and stationary states for the firm's profit can be stated (see Tuinstra, 2004; Bischi et al., 2007):

Proposition 7 *If a^* is an interior steady state of the map H defined in (9), then it is also an admissible solution of $\pi'(a_t) = 0$, and viceversa.*

Proof. Let a^* be an interior steady state of H. By means of simple calculations, we get $\pi'(a^*) = 0$. Let us now assume that $\overline{a}$ is such that $\pi'(\overline{a}) = 0$. It follows that $a_{t+1} = a_t = \overline{a}$ and therefore $\overline{a}$ is a steady state for f. □

4 Stability analysis

4.1 The simplified case

Because of the specification of the model, in general, it is not possible to obtain stationary solutions in a closed form. One way to get explicit solutions is to introduce the following technical assumption:

Assumption 2

$$c_2 - c_1 = \alpha - 2P. \tag{11}$$

Proposition 8 *Let the Assumption 2 holds.*
(i) If $\epsilon > 1$ and $P > \left(\frac{\epsilon-1}{2\epsilon}\right)^{\frac{\epsilon}{\epsilon+1}}$, then the map (9) admits a unique interior steady state

$$a^* = \left(\frac{1}{P}\right)\left(\frac{\epsilon-1}{2\epsilon}\right)^{\frac{\epsilon}{\epsilon+1}}, \tag{12}$$

in which the profit function (5) achieves its global maximum;
(ii) if (a) $\epsilon > 1$ and $P \leq \left(\frac{\epsilon-1}{2\epsilon}\right)^{\frac{\epsilon}{\epsilon+1}}$ or (b) $\epsilon \leq 1$, then no interior fixed points exist.

Proof. The fixed point a^* in (12) is obtained by solving $f(a) = a$. Since

$$\pi''(a_t) = -\left(\frac{(\epsilon-1)P^{\frac{\epsilon-1}{\epsilon}}}{\epsilon^2 a_t^{\frac{\epsilon+1}{\epsilon}}} + 2P^2\right) < 0, \qquad \forall\, a_t \in (0, +\infty)$$

then π is concave on $(0, 1]$ and a^* is the global maximum for π. □

In order to investigate the local stability of the interior steady state, we consider the first derivative of f, evaluated at a^*:

$$f'(a^*) = \left[1 - 2kP^2\left(\frac{\epsilon+1}{\epsilon}\right)\right]. \tag{13}$$

The following proposition holds:

Proposition 9 *Let* $\overline{k} = \frac{\epsilon}{P^2(1+\epsilon)}$ *and* $P > \left(\frac{\epsilon-1}{2\epsilon}\right)^{\frac{\epsilon}{\epsilon+1}}$. *Under the Assumption 2 and* $\epsilon > 1$,
(i) if $k \in (0, \overline{k})$*, then* a^* *is locally asymptotically stable;*
(ii) if $k > \overline{k}$*, then* a^* *is unstable.*

Proof. The steady state is locally stable if $-1 < f'(a^*) < 1$, where $f'(a^*)$ is defined in (13). Since the inequality $f'(a^*) < 1$ is always fulfilled, we have only to analyse the inequality

$$\left[1 - 2kP^2\left(\frac{\epsilon+1}{\epsilon}\right)\right] > -1,$$

which is satisfied for

$$k < \frac{\epsilon}{P^2(1+\epsilon)}.$$

The results *(i)* and *(ii)* then follow. □

Remark 3 *Proposition 9 allows to deduce that* a^* *loses its stability for* $k = \overline{k} = \frac{\epsilon}{P^2(1+\epsilon)}$ *through a flip bifurcation.*

From an economic point of view, the result in Proposition 9 shows that, in the long run, the coexistence between the two markets may be achieved only for sufficiently low levels of the reactivity parameter k and starting from initial values a_0 sufficiently close to the interior equilibrium a^*.

In the case $\epsilon < 1$, the results on the behaviour of f at the boundaries of the interval $(0, 1)$ and its concavity allow to state the following Proposition:

Proposition 10 *Under Assumption 2, if* $\epsilon < 1$ *then for every* $a_0 \in [0, 1)$ *the trajectories converge to the boundary equilibrium* 0.

Proof. First notice that the Assumption 2 verifies the hypotheses of Lemma 1 and Proposition 2. Hence, in the case $\epsilon < 1$ the graph of f may not cross the main diagonal or cross it twice. In the light of the result in Proposition 8, the result follows and the map H admits only the boundary fixed point 0. □

The result in Proposition 10 shows that, under Assumption 2, if the elasticity of demand for market 1 is sufficiently low (that is, $\epsilon < 1$), in any case it is not possible to achieve long term scenarios in which the two markets coexist and, in particular, only the market 2 survives.

4.2 The general case

By moving back to the general specification of the map H, the local stability analysis becomes more complicated. Nevertheless, it is possible to deduce some results in analytical form. In particular, by considering the case $\epsilon < 1$, we get the following result:

Proposition 11 *Assume $\epsilon < 1$, $P < \overline{P}$ and $k > \widetilde{k}$. If the map admits two interior fixed points a_1^* and a_2^* with $a_2^* > a_1^*$, then a_2^* is locally asymptotically stable while a_1^* is always unstable.*

Proof. We are in the case (II.c) of Proposition 6, in which we have that $\lim_{a_t \to 0^+} f(a_t) = -\infty$ and f is monotonically increasing in the interval $(0, 1)$. Because the graph of f crosses the main diagonal in a_1^* and a_2^*, we have that a_1^* is intersected from below while a_2^* is intersected from above. Then, the result follows. $\square$

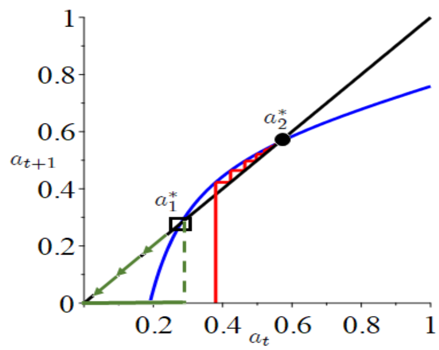


Figure 1: Parameter set: $c_1 = 0.01, c_2 = 1.94, k = 0.08, P = 2.05, \alpha = 3.2, \epsilon = 0.6$. In red it is depicted a trajectory starting from the initial value $a_0 = 0.38$ and converging to the stable equilibrium a_2^*, while the green horizontal segment defines the basin of attraction of the superstable boundary equilibrium 0.

Figure 1 shows a numerical example of the result in Proposition 11. In particular, the graph allows to notice that the map admits two interior fixed points, a_1^* and a_2^*, and the boundary fixed point 0. By starting from every initial value $a_0 < a_1^*$ (described by the green horizontal segment) all trajectories converge to the *superstable*[3] boundary fixed point 0, while by starting from an initial value $a_0 > a_1^*$ the trajectory converges to the locally asymptotically stable a_2^* (see red lines). Hence, a_1^* is always unstable and divides the basins of attraction of the two stable equilibria. In economic terms, this example shows that when $\epsilon < 1$, different initial conditions on the allocation of the intermediate good may lead the system in two really different final scenarios. Specifically, when the trajectories converge to a_2^* we observe a final state in which the two markets coexist; otherwise, when the trajectories go to 0 the firm chooses, in the long run, to allocate all the intermediate good in producing the final good 2.

When the map admits a fixed point a^*, although it is not possible to calculate it in closed form, we can still study how a^* and its local stability properties may change as parameters vary. In what follows, we focus on the role of marginal

[3]See Bischi et al. (2016) for details.

costs, and in particular on their difference $\Delta c = c_2 - c_1$. To this end, it is worth considering the explicit dependence of f on Δc and we refer with $f(a; \Delta c)$ the function f defined in (8).

Then, we can now provide the following results:

Lemma 3 *Let f be the function defined in (8) and a^* be an interior fixed point for H, given the value Δc^*, that is $f(a^*; \Delta c^*) = a^*$.*
(a) If $\frac{\partial f}{\partial a}(a^*; \Delta c^*) \neq 1$, then there exists a C^1 function $\hat{a} = \hat{a}(\Delta c)$, defined in a neighbourhood $I_{\Delta c^*}$ of Δc^* such that $\hat{a}(\Delta c^*) = a^*$ and $f(\hat{a}(\Delta c); \Delta c) = \hat{a}(\Delta c)$ for all $\Delta c \in I_{\Delta c^*}$.
(b) Consequently, the following cases arise:
(i) If $\frac{\partial f}{\partial a}(a^*; \Delta c^*) < 1$, then $\frac{d\ \hat{a}}{d\ \Delta c}(\Delta c^*) > 0$;
(ii) if $\frac{\partial f}{\partial a}(a^*; \Delta c^*) > 1$, then $\frac{d\ \hat{a}}{d\ \Delta c}(\Delta c^*) < 0$.

Proof.

Let us consider the function

$$F(a; \Delta c) = a - f(a; \Delta c).$$

Since $F(a^*; \Delta c^*) = 0$, the result (a) follows by the Implicit Function Theorem.
Therefore, we obtain

$$\frac{d\ \hat{a}}{d\ \Delta c}(\Delta c^*) = -\frac{\frac{\partial F}{\partial \Delta c}(a^*; \Delta c^*)}{\frac{\partial F}{\partial a}(a^*; \Delta c^*)} = \frac{Pk}{1 - \frac{\partial f}{\partial a}(a^*; \Delta c^*)}. \tag{14}$$

Being $Pk > 0$ for every P and k, the results *(i)* and *(ii)* in (b) are straightforward. $\square$

Proposition 12 *Let f and $\hat{a}(\Delta c)$ be the functions defined in (8) and Lemma 3, respectively, and a^* be an interior fixed point for the map H, given the value Δc^*. Then, the following cases arise:*
(i) Let $\epsilon > 1$:

(a) if $\frac{\partial f}{\partial a}(a^*; \Delta c^*) < 1$, then $\frac{d}{d\Delta c}\left[\frac{\partial f(\hat{a}(\Delta c); \Delta c)}{\partial a}\right](\hat{a}(\Delta c); \Delta c) > 0$;

(b) if $\frac{\partial f}{\partial a}(a^*; \Delta c^*) > 1$, then $\frac{d}{d\Delta c}\left[\frac{\partial f(\hat{a}(\Delta c); \Delta c)}{\partial a}\right](\hat{a}(\Delta c); \Delta c) < 0$.

(ii) Let $\epsilon < 1$:

(c) if $\frac{\partial f}{\partial a}(a^*; \Delta c^*) < 1$, then $\frac{d}{d\Delta c}\left[\frac{\partial f(\hat{a}(\Delta c); \Delta c)}{\partial a}\right](\hat{a}(\Delta c); \Delta c) < 0$;

(d) if $\frac{\partial f}{\partial a}(a^*; \Delta c^*) > 1$, then $\frac{d}{d\Delta c}\left[\frac{\partial f(\hat{a}(\Delta c); \Delta c)}{\partial a}\right](\hat{a}(\Delta c); \Delta c) > 0$.

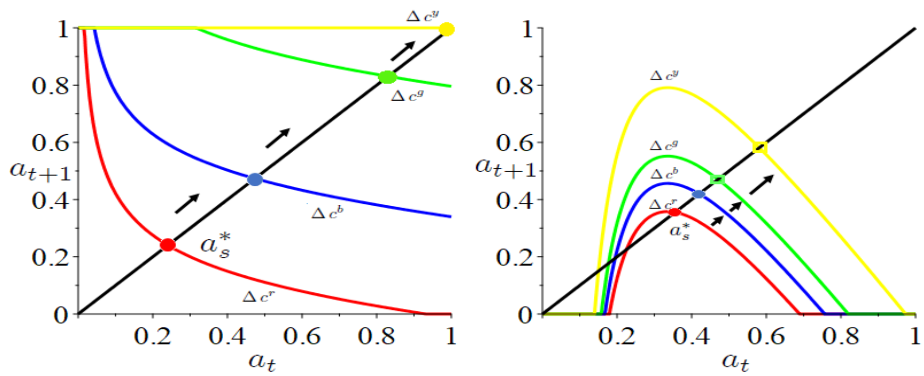
Proof. We compute via the Chain Rule that, in $I_{\Delta c^*}$,

$$\frac{d}{d\Delta c}\left[\frac{\partial f(\hat{a}(\Delta c);\Delta c)}{\partial a}\right](\hat{a}(\Delta c);\Delta c)=\frac{\partial^2 f(\hat{a}(\Delta c);\Delta c)}{\partial a^2}\frac{d\hat{a}(\Delta c)}{d\Delta c}+\frac{\partial^2 f(\hat{a}(\Delta c);\Delta c)}{\partial \Delta c\ \partial a}. \tag{15}$$

Since f is additively separable in a and Δc, we have that $\frac{\partial^2 f(\hat{a}(\Delta c);\Delta c)}{\partial \Delta c\ \partial a}=0$ for every $\Delta c \in I_{\Delta c^*}$. By evaluating the RHS of (15) in Δc^*, the results follow by means of Proposition 2 and Lemma 3. □

Considering the scenarios described in Propositions 5-6 and the results illustrated in Proposition 12, we can notice that in the case $\epsilon > 1$, as Δc increases, an unstable interior fixed point of H may become a locally asymptotically stable interior fixed point; instead, a stable interior fixed point of H, as long as it exists, remains stable as Δc increases. Differently, if we consider the case $\epsilon < 1$, we can notice that as Δc increases, an unstable interior fixed point of H remains unstable; instead, a stable interior fixed point may become unstable.

Figure 2 furnishes a graphical example of the phenomenon described above. In particular, Panel (a) in Figure 2 shows that by starting from a parametric configuration in which the interior fixed point is stable, an increase in Δc enlarges the value of a_s^* (which approaches the value 1) but not its stability. The yellow curve in Panel (a) of Figure 2 suggests the emergence of a first global phenomenon. Specifically, a significant distance in the marginal costs may induce the monopolist to exit one of the two markets and therefore to specialise only in a unique production. Differently, Panel (b) in Figure 2 shows a numerical exercise in which, for $\epsilon < 1$, the parametric configuration we observe the existence of two interior fixed points, one unstable and the other, a_s^*, stable. Specifically, the different curves depicted in the graph reveal that an increase in Δc, on the one hand, reduces the amplitude of the basin of attraction for the boundary fixed point 0 and on the other hand induces an increase in the value assumed by the stable interior fixed point a_s^*. Furthermore, we can notice that an increase in Δc towards high values may lead to the loss of stability in a_s^* and, consequently, to the occurrence of (attracting) cycles.



(a)

(b)

Figure 2: (a) Parameter set: $k = 0.38, P = 1.2, \alpha = 3.8, \epsilon = 2.8$. Changes in the map for the different values of Δc: $\Delta c^r = 0.975, \Delta c^b = 1.75, \Delta c^g = 2.75, \Delta c^y = 3.55$. The curve depicted in yellow describes a scenario in which the graph of f crosses the 45-degree line to the right of 1. Then, for every initial conditions the system leads, in one iterate, to the exit of the monopolist from the market 2 (1 is a superstable equilibrium); (b) Parameter set: $P = 2.85, \alpha = 3.2, k = 0.165, \epsilon = 0.6, \Delta c^r = 0.329, \Delta c^b = 0.538, \Delta c^g = 0.591, \Delta c^y = 1.25$. For $\Delta c < \Delta c^g$, a^* is stable while for $\Delta c > \Delta c^g$, a^* is unstable.
In both figures, the superscripts in Δc refer to the different colours of the curves (that is, r=red, b=black, g=green and y=yellow). The full (empty) dot identifies a stable (unstable) fixed point.

With regard to the connections between the monopolist's profit function and the steady states of the system defined by the map in (9), the following result holds:

Proposition 13 *Let π be the profit function in (6), f the function defined in (8) and a^* a generic interior fixed point for H.*
(i) If $f'(a^*) > 1$, then a^* is a local minimum for π;
(ii) if $f'(a^*) < 1$, then a^* is a local maximum for π.

Proof. By means of the definition of f in (8), we obtain

$$f'(a_t) = 1 + k \frac{d^2\ \pi(a_t)}{d\ a_t^2}. \tag{16}$$

By evaluating such equation at a^*, we have

$$\frac{d^2\ \pi(a^*)}{d\ a^2} = \frac{f'(a^*) - 1}{k}, \tag{17}$$

and then, for $f'(a^*) > 1$ the RHS of (17) is positive while, for $f'(a^*) < 1$, $\frac{d^2 \pi(a^*)}{da^2}$ is negative. It follows that for $f'(a^*) > 1$, the profit function is convex in a neighbourhood of a^* whereas for $f'(a^*) < 1$, the profit function is concave in a neighbourhood of a^*. Therefore, the results in *(i)* and *(ii)* follow. $\square$

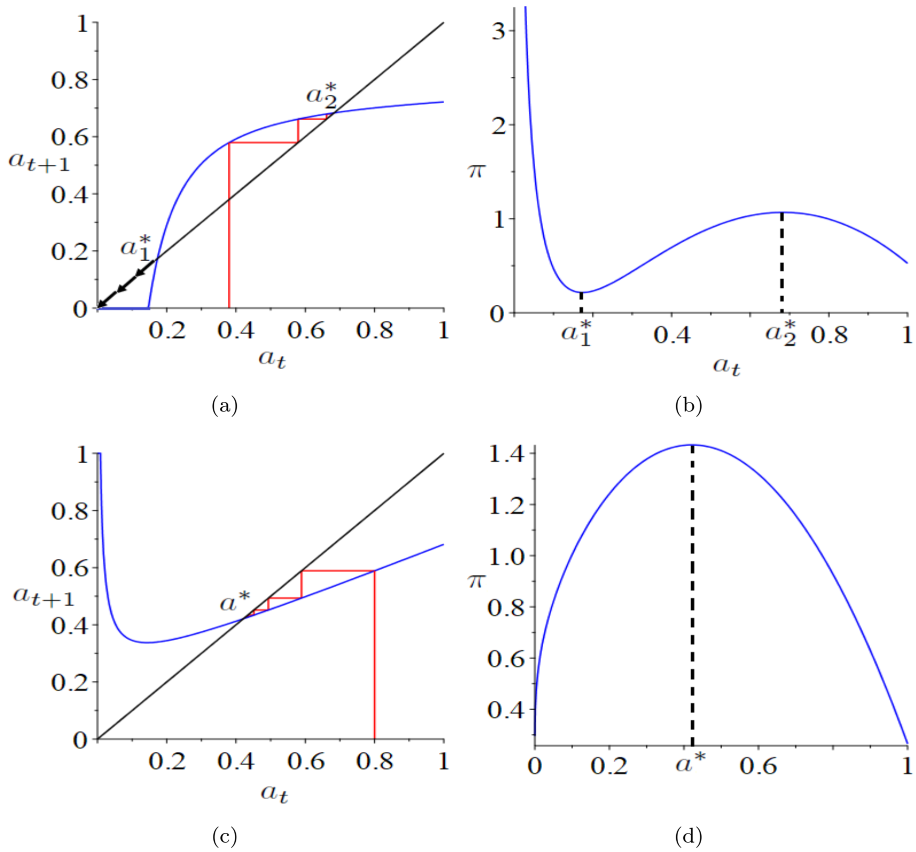


Figure 3: (a) Parameter set: $c_1 = 0.01, c_2 = 1.94, k = 0.08, P = 2.45, \alpha = 3.2, \epsilon = 0.6$. The map H admits two interior fixed points, $a_1^* \simeq 0.1732$ (unstable) and $a_2^* \simeq 0.6832$ (stable) and the boundary fixed point 0. In red it is depicted a trajectory starting from $a_0 \simeq 0.38$ and converging to a_2^* whereas the black arrows describe the convergence to 0 of the trajectories starting from initial values less than a_1^*. (b) The graph of the profit function $\pi(a_t)$ corresponding to the parameter set in Panel (a). (c) Parameter set: $c_1 = 0.48, c_2 = 1.34, k = 0.08, P = 1.8, \alpha = 3.2, \epsilon = 1.26$. The map H admits a unique interior fixed point, $a^* \simeq 0.4214$, which is locally asymptotically stable. In red it is depicted a trajectory starting from $a_0 \simeq 0.8$ and converging to a^*. (d) The graph of the profit function $\pi(a_t)$ corresponding to the parameter set in Panel (c).

The results in Proposition 13 allow to deduce an important relationship between the steady states of the map and the corresponding levels of the firm's profit: an unstable steady state minimises the profits of the monopolistic firm, while a local asymptotically stable steady state corresponds to a local, but not always global, maximum for the profit function.
Numerical evidences of the occurrence described above is provided in Figure 3. In particular, Panel (a) and (b) show that, for $\epsilon < 1$, a_1^* (unstable fixed point for H) and a_2^* (stable fixed point for H) are local minimum and local maximum, respectively, for the profit function π. Similarly, Panel (c) and (d) show that, for $\epsilon > 1$, the locally asymptotically stable fixed point for H is a local and global maximum for the profit function.

5 Global Analysis

This section is devoted to investigating some economically relevant dynamic phenomena, related to the results proposed in Proposition 5 and Proposition 6. More specifically, we describe some scenarios in which the elasticity of demand ϵ plays a key role and we consider separately the two cases $\epsilon > 1$ and $\epsilon < 1$, having observed, in previous sections, that these two cases represent a reason of widely different economic dynamics.

Figure 4 shows a numerical example in which, considering $\epsilon < 1$, several global phenomena occur as ϵ increases. In particular, the bifurcation diagram in Panel (a) shows that for very low values of ϵ, the unique attractor of the system is the boundary equilibrium 0. From an economic point of view, this means that when the elasticity of the demand in the market 1 is really low, the monopolist maximises its profits by specialising in the production of good 2. At $\epsilon \simeq 0.3102$, a switch to the coexistence of the two markets occurs until ϵ reaches the value $\epsilon_{flip} \simeq 0.452$, at which the interior attractor undergoes a flip bifurcation that generates a stable 2-cycle. As ϵ further increases, a sequence of period doubling bifurcations occurs, leading to the emergence of chaos for $\epsilon \simeq 0.6063$. When ϵ exceeds the value 0.6619, the chaotic regime closes and the boundary equilibrium 0 goes back to being the global attractor of the map. For $\epsilon \in (0.793, 0.86)$, we observe the existence of cycles, with periodicity depending on ϵ, whose coordinates are $(1, f(1), f^2(1), ..., f^n(1))$ with $f^{n+1}(1) = 1$ and $n > 1$. Finally, for $\epsilon \in (0.86, 1)$, the system converges to a stable 2-cycle whose coordinates are $a_1^c = 1$ and $a_2^c \in (0, 1)$ (see Panel (d) for details). In addition, the graph in Panel (a) allows to observe a phase of coexistence of attractors for $\epsilon \in (0.31, 0.66)$ when two different initial conditions $a_0^1 = 0.4$ and $a_0^2 = 0.99$ are considered. This phenomenon is described in further detail in Panel (b) where, for $\epsilon = 0.648$, two attractors coexist. In particular, we can notice that starting from an initial value a_0^1, the trajectory (depicted in red) falls into a chaotic regime; differently, by starting from an initial value a_0^2, the trajectory (depicted in green) converges towards the equilibrium 0 and then the monopolist chooses the specialisation on the market 2. Moreover, the unimodality of f induces the birth of a non-connected basin of attraction for the boundary equilibrium 0 (grey lines on the x-axis highlight the basin of attraction). In Panel (c) of Figure 4, we show one of the high order cycles obtained for a value of ϵ in the interval $(0.793, 0.86)$. Specifically, the graph illustrates the appearance of a stable 8-cycle which involves the second and third piece of the map H. Finally, Panel (d) displays the scenario in which the system converges to a stable 2-cycle where a continuous switching between $a_1^c = 1$ (that is, the specialisation on market 1) and $a_2^c \simeq 0.06$ (that is, the coexistence of the two markets) occurs.

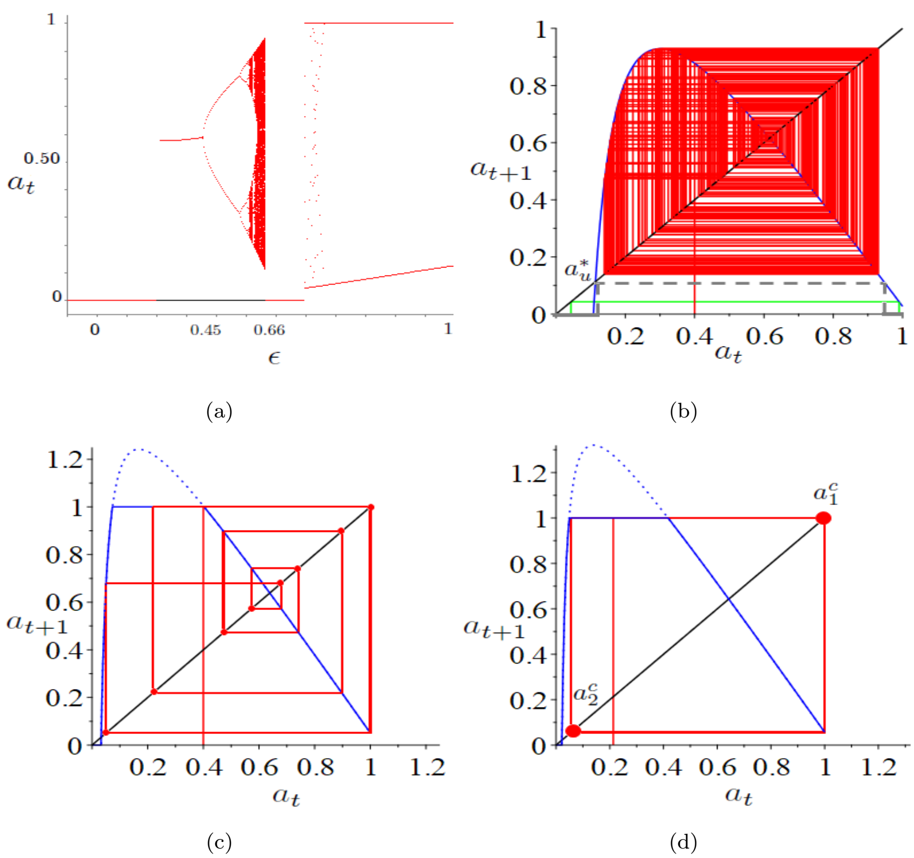


Figure 4: Parameter set: $c_1 = 0.15, c_2 = 1.389, k = 0.165, P = 2.85, \alpha = 3.2$. (a) The bifurcation diagram with respect to ϵ in which it is also underlined a coexistence of attractors, respectively depicted in red and black, for $\epsilon \in (0.31, 0.66)$; (b) coexistence of a chaotic regime and the boundary equilibrium 0 for two different initial conditions $a_0^1 = 0.4$ (red lines) and $a_0^2 = 0.99$ (green lines) when $\epsilon = 0.648$; (c) Convergence to a 8-cycle for $\epsilon = 0.83$; (d) the final convergence to a stable 2-cycle for $\epsilon = 0.87$. The full dots in Panels (c) and (d) identify the stable 8-cycle and the stable 2-cycle, respectively.

With regard to the case $\epsilon > 1$, Figure 5 illustrates an interesting scenario in which, as ϵ increases, several dynamic phenomena appear. Specifically, the bifurcation diagram in Panel (a) of Figure 5 shows that for $\epsilon \in (1, 2.5)$, the firm, at every period, continuously switches between specialising on the market 1 and the market 2. Therefore, at every period, the firm chooses to allocate the entire amount of intermediate good on only one of the two productions (see also Panel (b) which shows a time series depicted for $\epsilon = 2.24$). By increasing ϵ, first the system converges to a stable 2-cycle whose coordinates are $a_1^c = 1$ and $a_2^c \in (0, 1)$, then the map undergoes a sequence of period doubling bifurcations leading to chaos. As ϵ continues to increase, a series of period-halving bifurcations occurs until the value $\epsilon = 4.63$, at which a new stable 2-cycle, whose coordinates belong to the interval $(0, 1)$, appears. As ϵ assumes higher values, then the map undergoes a new sequence of period doubling bifurcations leading to chaos. The graph in Panel (c) highlights the emergence of a chaotic regime when $\epsilon = 4.2$ (we assume the initial condition $a_0 = 0.32$). Finally, Panel (d) depicts, for $\epsilon = 5.25$, the time series of a_t converging to a stable 2-cycle.

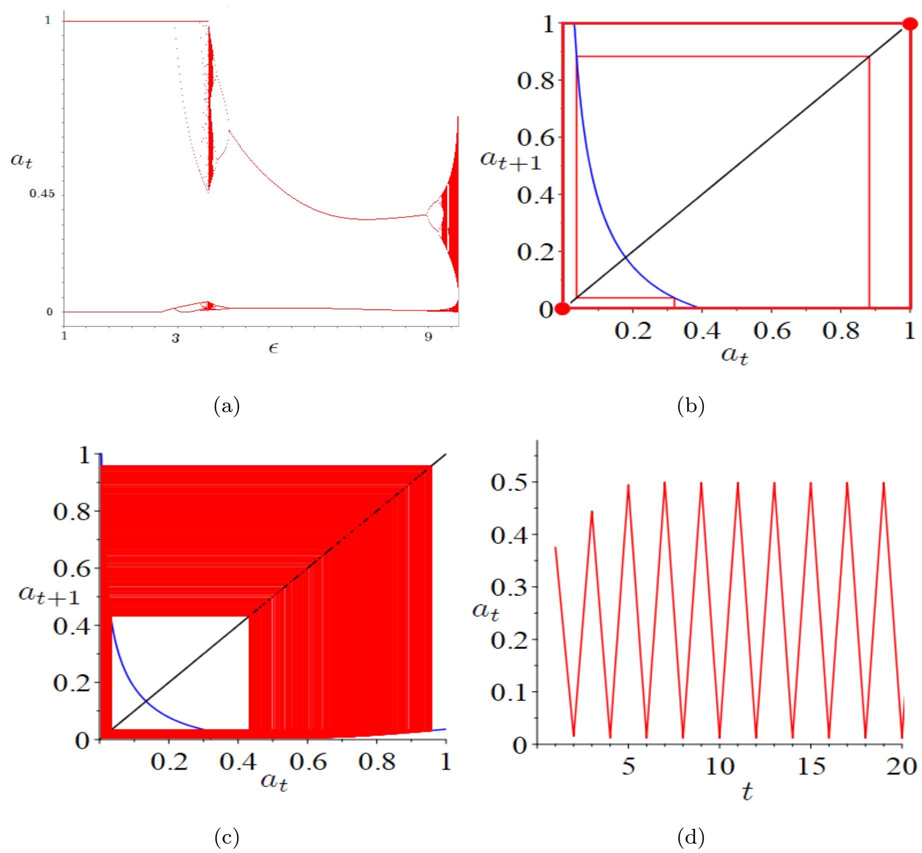


Figure 5: Parameter set: $c_1 = 0.5, c_2 = 1.8, k = 0.88, P = 0.66, \alpha = 3.8$. (a) The bifurcation diagram with respect ϵ in the case $\epsilon > 1$; (b) a trajectory converging to a continuous switching between 0 and 1 for $\epsilon = 2.44$ ($a_0 = 0.38$); (c) chaotic regime when $\epsilon = 4.2$ ($a_0 = 0.32$); (d) the time series of a_t converging to a stable 2-cycle, whose coordinates are 0.011 and 0.498, for $\epsilon = 5.25$.

In the case of $\epsilon > 1$, by using an appropriate parametric configuration, we may observe an interesting phenomenon of multistability. In this regard, Figure 6 shows a numerical exercise in which a stable 2-cycle (whose coordinates are 1 and its first order image a_c^c) and a stable interior fixed point a^* coexist. In particular, the left panel in Figure 6 shows the attractors of the map as well as the boundaries of the basins of attraction defined by the fixed points $\overline{x}_1$ and $\overline{x}_2$ of the second iterate f^2 (depicted in black). The right panel in figure 6 displays the time series related to the two attractors where the first trajectory (defined by green points)[4] starts inside the basin of attraction of the 2-cycle $(a_c^c, 1)$,

[4]For the sake of clarity, in the second trajectory (depicted in green) we have not traced the segments that connect two successive points of the time series.

while the second one (depicted in red) starts inside the basin of attraction of a^*. From an economic point of view, the graph shows how both the elasticity of the demand ϵ (which affects the shape of the map H) and the initial share a_0 of the intermediate good have a significant influence on the long run dynamics that the monopolist faces. In fact, different initial values determine (i) the convergence to an allocative stationary value within the interval $(0, 1)$ and therefore the survival of the two markets, or (ii) a cyclical behaviour of the firm. In the latter case we observe, in alternate periods, the exit from the market 2 and the specialisation on the market 1.

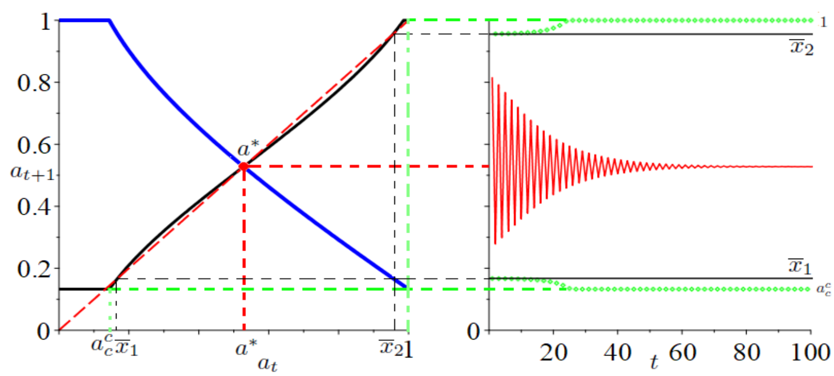


Figure 6: Parameter set: $c_1 = 0.5, c_2 = 1.3, k = 0.83, P = 0.996, \alpha = 2.65, \epsilon = 5$. Coexistence between a 2-cycle (involving the specialisation in the market 1) and the interior fixed point a^*. The time series of the map for the two different initial values $a_0^1 = 0.167$ and $a_0^2 = 0.84$, where a_0^1 is outside and a_0^2 belongs to the interval ($\overline{x}_1 = 0.167355$, $\overline{x}_2 = 0.955493$), displays the two paths. Indeed, the red line shows the convergence to the interior stable equilibrium while green points highlight the convergence to the stable 2-cycle.

6 Conclusions

In this article, we have analysed the emergence of nonlinear dynamics in a monopoly where (i) the firm produces a fixed amount of an intermediate good and then (ii) it employs this intermediate good as an input in the production of two vertically differentiated final goods. The monopolist adopts a gradient mechanism to adjust the share of intermediate good to be allocated between the two productions. Moreover, consumers of the two markets are heterogeneous.

It is interesting to notice that, in the literature on microeconomic dynamics, this kind of allocation problem has not been analysed. As a matter of fact, the majority of contributions has focused on characterising the production when capacity constraints are not considered. Among the few exceptions, it is worth mentioning Puu and Norin (2003), in which capacity constraints play an important role in defining the market dynamics.

The analysis of the dynamic properties of the model allows observing two alternative scenarios: one implying the coexistence of the two markets and another where there is specialisation on the production of a single final good. We have observed that both the elasticity of market demand and marginal costs play an ambiguous role on the stability of the long-term equilibrium. Moreover, by investigating how the production level of intermediate good and the speed of adjustment may affect the dynamic properties of the model, we have noticed the emergence of cases in which dynamics may be chaotic and others in which multiple attractors may appear. In this regard, we have proposed several numerical simulations showing the main features of the model and peculiar dynamic scenarios, as coexistence of attractors and cyclical dynamics. More specifically, we have shown that coexistence between a boundary and a chaotic attractor or between a cycle and an interior fixed point may emerge. Finally, numerical examples highlight the existence of parametric configurations whereby the dynamics of the monopolist are trapped in a continuous switching between periods in which it specialises in one production and others in which it specialises in the other.

With regard to possible future research scenarios, we can notice that the analysis of this model is based on assuming a *rigidity* in the quantity of intermediate good that is produced, i.e. we assume that it is totally allocated among the final productions considered. A natural extension of this work is then represented by considering the possibility that part of the intermediate good is discarded and therefore not employed in the final productions (for example, in order to increase the price of final goods). In addition, it will be interesting to include the presence of stochastic terms in order to consider variations in the amount of the intermediate good that are triggered by exogenous factors (for example, natural phenomena).

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References

W. J. Baumol and R. E. Quandt. Rules of thumb and optimally imperfect decisions. *The American economic review*, 54(2):23–46, 1964.

G. I. Bischi, A. K. Naimzada, and L. Sbragia. Oligopoly games with local monopolistic approximation. *Journal of economic behavior & organization*, 62(3):371–388, 2007.

G. I. Bischi, A. Panchuk, and D. Radi. Qualitative theory of dynamical systems. tools and applications for economic modelling. In *Lectures Given at the cost training school on new economic complex geography at Urbino, Italy, 17–19 September 2015.* Springer, 2016.

F. Cavalli and A. Naimzada. Effect of price elasticity of demand in monopolies with gradient adjustment. *Chaos, Solitons & Fractals*, 76:47–55, 2015.

R. W. Clower. Some theory of an ignorant monopolist. *The Economic Journal*, 69(276):705–716, 1959.

L. Gori, L. Guerrini, and M. Sodini. Different modelling approaches for time lags in a monopoly. In *Essays in Economic Dynamics*, pages 81–98. Springer, 2016.

C. M. Hall, L. Sharples, R. Mitchell, N. Macionis, and B. Cambourne. *Food tourism around the world*. Routledge, 2004.

A. Matsumoto and F. Szidarovszky. Nonlinear delay monopoly with bounded rationality. *Chaos, Solitons & Fractals*, 45(4):507–519, 2012.

A. Matsumoto and F. Szidarovszky. Complex dynamics of monopolies with gradient adjustment. *Economic Modelling*, 42:220–229, 2014.

A. Matsumoto and F. Szidarovszky. Dynamic monopoly with multiple continuously distributed time delays. *Mathematics and Computers in Simulation*, 108:99–118, 2015.

A. Matsumoto, C. Chiarella, and F. Szidarovszky. Dynamic monopoly with bounded continuously distributed delay. *Chaos, Solitons & Fractals*, 47:66–72, 2013.

A. K. Naimzada and G. Ricchiuti. Complex dynamics in a monopoly with a rule of thumb. *Applied Mathematics and Computation*, 203(2):921–925, 2008.

A. K. Naimzada and G. Ricchiuti. Monopoly with local knowledge of demand function. *Economic Modelling*, 28(1-2):299–307, 2011.

T. Puu. The chaotic monopolist. *Chaos, Solitons & Fractals*, 5(1):35–44, 1995.

T. Puu and A. Norin. Cournot duopoly when the competitors operate under capacity constraints. *Chaos, Solitons & Fractals*, 18(3):577–592, 2003.

J. Tuinstra. A price adjustment process in a model of monopolistic competition. *International Game Theory Review*, 6(03):417–442, 2004.