

Some new properties of a suitable weak solution to the Navier-Stokes equations

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Abstract - The paper is concerned with the IBVP of the Navier-Stokes equations. The goal is the construction of a weak solution enjoying some new properties. Of course, we look for properties which are global in time. The results hold assuming an initial data $v_0 \in J^2(\Omega)$.

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1 Introduction

This note concerns the 3D-Navier-Stokes initial boundary value problem:

$$\begin{aligned} v_t + v \cdot \nabla v + \nabla \pi_v &= \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, \\ v &= 0 \text{ on } (0, T) \times \partial\Omega, \quad v(0, x) = v_0(x) \text{ on } \{0\} \times \Omega. \end{aligned} \tag{1}$$

In system (1) $\Omega \subseteq \mathbb{R}^3$ is assumed bounded or exterior, and its boundary is smooth. The symbol v denotes the kinetic field, π_v is the pressure field, $v_t := \frac{\partial}{\partial t} v$ and $v \cdot \nabla v := v_k \frac{\partial}{\partial x_k} v$. In several papers, related to the Navier-Stokes initial boundary value problem, the

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authors give results concerning the partial regularity of a suitable weak solution (see Definition 2 below). This is made in order to highlight the properties of a weak solution, corresponding to a data $v_0 \in L^2(\Omega)$, divergence free, that can be suitable to state the well posedness of the equations, see e.g. [19, 18, 4, 10, 33, 5, 8, 7, 21, 9]¹. We believe that, in connection with the non-well posedness of the Navier-Stokes Cauchy or IBVP problem, this kind of investigation achieves a further interest. Actually, in the recent paper [3], it is considered the possibility of non uniqueness of a weak solution to the Navier-Stokes equations. This is proved for *very weak solutions*, that is solutions satisfying a variational formulation of the Navier-Stokes equations and simply belonging to $C([0, T]; L^2(\Omega))$. As a consequence of the weakness of the solutions, the result of non uniqueness fails to hold for regular solutions, but *a priori* it also does not work for a suitable weak solution, that is a solution verifying an energy inequality. So that, in order to better delimit the validity of a possible counterexample to the uniqueness in the set of weak solutions corresponding to an initial data in $L^2(\Omega)$, it seems of a certain interest to support the energy inequality, or its variants, by means of a wide set of global properties of the weak solutions not necessarily only consequences of the energy inequality, but of the coupling of other *a priori* estimates.

The aim of this note is to prove some new properties of a weak solution. We investigate two questions. One is related to a sort of energy equality for a suitable weak solution. It is easy to understand that the possible validity of the energy equality achieves a *mechanical* interest that goes beyond the above question concerning the well posedness. Actually, we construct a weak solution (v, π_v) to the Navier-Stokes initial boundary value problem such that the “energy equalities” of the kind

$$\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau - \|v(s)\|_2^2 = -H(t, s), \text{ a.e. in } t \geq s > 0 \text{ and for } s = 0 \quad (2)$$

and

$$2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau = F(t, s)(\|v(s)\|_2^2 - \|v(t)\|_2^2), \text{ a.e. in } t \geq s > 0 \text{ and for } s = 0 \quad (3)$$

are fulfilled. The functions $H(t, s)$ and $F(t, s)$ have suitable expressions, see formula (6) and formula (7). If $H(t, s) \leq 0$, then the energy equality holds (that is *a fortiori* $H(t, s) = 0$). If $F(t, s) \geq 1$, then the energy equality holds (that is *a fortiori* $F(t, s) =$

¹ There is a wide literature concerning extension to the IBVP of results proved for the Cauchy problem. One of the most interesting of these kinds of extensions is sure the energy inequality in strong form, see e.g. [15, 27]

1). These results are a consequence of the fact that we are able to prove that an approximating sequence $\{(v^m, \pi_{v^m})\}$ is strongly converging in $L^r(0, T; W^{1,2}(\Omega))$ for all $r \in [1, 2)$. The strong convergence, in turn, is a consequence of the property: $P\Delta v^m \in L^{\frac{2}{3}}(0, T; L^2(\Omega))$ for all $m \in \mathbb{N}$ and $T > 0$. Unfortunately we are not able to put $r = 2$, that should give the energy equality. For 2D-Navier-Stokes equations one proves that $H(t, s) = 0$. It is important to stress that the term $H(t, s)$ is equal to zero in 2D-case thanks to our approximating approach, without appealing to the regularity of the limit. Another result proves that $v \in L^{\mu(p)}(0, T; L^p(\Omega))$, with $\mu(p) := \frac{p}{p-2}$ and $p \in (6, \infty]$. This result is not new in literature. A first contribution in this sense is proved in [12] for a particular geometry and it is reconsidered in [10]. The proof given in [10] for exterior domains is not completely clear to the present authors. However our proof is alternative with respect to the ones of the quoted papers.

In order to better state our result we recall the following definitions. We denote by $J^2(\Omega)$ and $J^{1,2}(\Omega)$ the completion of $\mathcal{C}_0(\Omega)$ in $L^2(\Omega)$ and in $W^{1,2}(\Omega)$ respectively, where $\mathcal{C}_0(\Omega)$ is the set of smooth divergence free functions. Moreover (\cdot, \cdot) represents the scalar product in $L^2(\Omega)$.

Definition 1. Let $v_0 \in J^2(\Omega)$. A pair (v, π_v) , such that $v : (0, \infty) \times \Omega \rightarrow \mathbb{R}^3$ and $\pi_v : (0, \infty) \times \Omega \rightarrow \mathbb{R}$, is said a weak solution to problem (1) if

- i) for all $T > 0$, $v \in L^2(0, T; J^{1,2}(\Omega))$ and, for some q, r , $\pi_v \in L^r_{loc}([0, T]; L^q_{loc}(\overline{\Omega}))$,
- ii) $\lim_{t \rightarrow 0} \|v(t) - v_0\|_2 = 0$,
- iii) for all $t, s \in (0, T)$, the pair (v, π_v) satisfies the equation:

$$\int_s^t \left[(v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) + (\pi_v, \nabla \cdot \varphi) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),$$

for all $\varphi \in C_0^1([0, T] \times \Omega)$.

In [4] and in [30], in order to investigate the regularity of a weak solution, it is introduced an energy inequality having a local character:

Definition 2. A pair (v, π_v) is said a suitable weak solution if it is a weak solution in the sense of the Definition 1 and, moreover,

$$\begin{aligned} \int_{\Omega} |v(t)|^2 \phi(t) dx + 2 \int_s^t \int_{\Omega} |\nabla v|^2 \phi dx d\tau &\leq \int_{\Omega} |v(s)|^2 \phi(s) dx \\ &+ \int_s^t \int_{\Omega} |v|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_s^t \int_{\Omega} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dx d\tau, \end{aligned} \tag{4}$$

for all $t > s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \overline{\Omega})$. We denote by $\Sigma \subseteq [0, \infty)$ the set of the instants s for which inequality (4) holds.

Thanks to the properties of the pressure field furnished by the existence theorem, from inequality (4) one deduces the classical one:

$$\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2, \text{ for all } t > s \text{ and } s \in \Sigma. \quad (5)$$

We are going to prove the following result.

Theorem 1. *For all $v_0 \in J^2(\Omega)$ there exists a suitable weak solution (v, π_v) to problem (1) that is the weak limit in $L^2(0, T; J^{1,2}(\Omega)) \times L_{loc}^q([0, T]; L_{loc}^2(\overline{\Omega}))$, $q \in (1, \frac{12}{11})$, of a sequence $\{(v^m, \pi_{v^m})\}$ of solutions to (16). The sequence $\{v^m\}$ converges strongly to v in $L^p(0, T; W^{1,2}(\Omega))$ for all $p \in [1, 2)$. Further, for any $q \in (6, \infty]$, $v \in L^{\mu(q)}(0, T; L^q(\Omega))$ with $\mu(q) := \frac{q}{q-2}$, and v satisfies relation (2) with*

$$H(t, s) := \begin{cases} \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau, & \text{for } s > 0, \\ \lim_{s \rightarrow 0} \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau, & \text{for } s = 0, \end{cases} \quad (6)$$

for any arbitrary constant $K > 0$, as well as v satisfies relation (3) with

$$F(t, s) := \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \frac{1}{(K_1 + \|\nabla v^m(t_{\alpha, m})\|_2)^{\alpha}} \quad (7)$$

for any arbitrary constant $K_1 \geq 0$. Finally, the following inclusion holds: $\mathcal{G}_1 := \{t, s \text{ such that (2) is true}\} \subseteq \mathcal{G}_2 := \{t, s \text{ such that (3) is true}\}$.

REMARK 1. Since the proprieties of the pressure field π_v are not our main interest in this paper, we limit ourselves to point out the one that allows us to state that (v, π_v) is a suitable weak solution. Actually, in our construction the pressure π_v enjoys the properties that one can deduce by means of Lemma 9. For more exhaustive properties relative to the pressure field of a suitable weak solution (that is with an initial data only in $J^2(\Omega)$) a possible reference is [26].

We note that the quantity $H(t, s)$ is independent of the constant K . This fact is intriguing and somehow leads to conjecture that $H(t, s) = 0$.

If $v_0 \in J^2(\Omega) \setminus J^{1,2}(\Omega)$, almost everywhere in $t > 0$, following the proof idea we also get

$$\|v(t)\|_2^2 + 2 \int_0^t \|\nabla v(\tau)\|_2^2 d\tau = - \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \int_0^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau.$$

One proves that there exists an instant $\theta > 0$ such that $v^m(t, x) \in C([\theta, \infty); J^{1,2})$, in particular there exists a M such that $\|\nabla v^m(t)\|_2 \leq M$ for all $t \geq \theta$ and $m \in \mathbb{N}$ (one proves this result by repeating the arguments employed for the structure theorem by Leray). Hence, via estimate (21) and taking into account the energy inequality, we can deduce

$$\text{for all } t > \theta, \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_{\theta}^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau = 0.$$

Hence we get that function $H(t, s) = H(\theta, s)$ for all $t > \theta$, and H becomes a constant function for $t > \theta$.

Concerning the function F we remark that its values are independent of $K_1 \geq 0$. The fact that K_1 can be chosen equal to zero makes a difference with K in function H , as well as $\mathcal{G}_1 \subseteq \mathcal{G}_2$ is another difference.

In the introduction we remarked that if $F(t, s) \geq 1$, then the energy equality holds, that is $F(t, s) = 1$. Since we are not in a position to prove $F(t, s) \geq 1$, *a priori* we have to consider that $F(t, s) \leq 1$. However, we can claim that almost everywhere in $t > 0$, $F(t, 0) > 0$ holds. Actually, more in general, assume that $s \in \Sigma$ and $\|v(s)\|_2 \neq 0$ and that exists a sequence $\{t_p\}$ converging to s such that $F(t_p, s) = 0$. Then, from formula (3), we deduce that $\|v(\tau)\|_2 = 0$ a.e. in $\tau \in (s, t_p)$ holds for all $p \in \mathbb{N}$. Hence we can select a new sequence $\{t'_p\} \subset (s, t_p)$ such that $\|v(t'_p)\|_2 = 0$. By virtue of the right- L^2 -continuity in s , we get $\lim_{t'_p \rightarrow s} \|v(t'_p) - v(s)\|_2 = 0$, which is a contradiction with $\|v(s)\|_2 \neq 0$.

From formulas (2)-(3), a.e. in $t \in \mathcal{G}_1$ and for $s = 0$, we easily deduce that

$$H(t, 0) = (1 - F(t, 0))(\|v(0)\|_2^2 - \|v(t)\|_2^2).$$

Therefore, via (2) we deduce

$$H(t, 0) = \left(\frac{1}{F(t, 0)} - 1\right) \int_0^t \|\nabla v(\tau)\|_2^2 d\tau. \quad (8)$$

Recalling that, for $t \geq \theta$, $H(t, 0) = H(\theta, 0)$, then from (8) we deduce that $F(t, 0)$ is a continuous function for $t \geq \theta$.

REMARK 2. In paper [28] a new energy inequality is proposed:

$$\|v(t)\|_2^2 + N(t) + 2 \int_0^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v_0\|_2^2, \text{ for all } t > 0.$$

Function $N(t) := \limsup_{\delta \rightarrow 0} \int_{\delta}^t \left\| \frac{u(\tau) - u(\tau - \delta)}{\delta^{\frac{1}{2}}} \right\|_2^2 d\tau \geq 0$ can be interpreted as $\frac{1}{2}$ -time derivative. It is not known if $N(t) > 0$ holds. In the two dimensional case one proves that $N(t) = 0$. Of course, we are not able to compare the solution furnished in [28] and the one of Theorem 1.

In paper [21] the compatibility between an energy equality and an initial data $v_0 \in J^2(\Omega)$ is proved. This supports the idea that $H(t, s)$ can be equal to zero.

REMARK 3. We point out that by a proof completely similar to the one of Theorem 1, one can prove the validity of the following generalized energy equality

$$\begin{aligned} \int_{\Omega} |v(t)|^2 \phi(t) dx + 2 \int_s^t \int_{\Omega} |\nabla v|^2 \phi dx d\tau &= \int_{\Omega} |v(s)|^2 \phi(s) dx \\ &+ \int_s^t \int_{\Omega} |v|^2 (\phi_{\tau} + \Delta \phi) dx d\tau + \int_s^t \int_{\Omega} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dx d\tau - \tilde{H}(t, s), \end{aligned} \quad (9)$$

a.e. in $t \geq s > 0$ and for $s = 0$, where

$$\tilde{H}(t, s) := \begin{cases} \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|\phi^{\frac{1}{2}}(\tau) v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau, & \text{for } s > 0, \\ \lim_{s \rightarrow 0} \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|\phi^{\frac{1}{2}}(\tau) v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau, & \text{for } s = 0, \end{cases}$$

for any arbitrary constant $K > 0$ and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \overline{\Omega})$.

The plan of the paper is the following. In Section 2 we give some preliminaries and auxiliary lemmas. In Section 3 we give the proof of the theorem. In the appendix we recall some known properties of the pressure field that are employed in Section 2.

2 Some preliminary results

For $p \in (1, \infty)$ we set $J^p(\Omega) :=$ completion of $\mathcal{C}_0(\Omega)$ in $L^p(\Omega)$. By P_p we denote the projector from $L^p(\Omega)$ onto $J^p(\Omega)$. In the case of $p = 2$ we write $P_2 \equiv P$. For any $R > 0$ we set $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$.

We start with the following a priori estimate:

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in W^{2,2}(\Omega) \cap J^{1,2}(\Omega)$. Then there exists a constant c independent of u such that*

$$\|u\|_r \leq c \|P\Delta u\|_2^a \|u\|_q^{1-a}, \quad a\left(\frac{1}{2} - \frac{2}{n}\right) + (1-a)\frac{1}{q} = \frac{1}{r}, \quad (10)$$

provided that $a \in [0, 1)$.

Proof. The result of the lemma is a special case of a general one proved in [20, 22]. \square

Lemma 2 (Friedrichs's lemma). *Let Ω be bounded. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|u\|_2^2 \leq (1 + \varepsilon) \sum_{j=1}^N (u, a^j)^2 + \varepsilon \|\nabla u\|_2^2, \quad \text{for any } u \in W^{1,2}(\Omega), \quad (11)$$

where $\{a^j\}$ is an orthonormal basis of $L^2(\Omega)$.

Corollary 1. *Assume that $\{u^k(t, x)\}$ is a sequence with*

$$\int_0^T \|u^k(t)\|_{W^{1,2}(\Omega)}^2 dt + \operatorname{ess\,sup}_{(0,T)} \|u^k(t)\|_2 \leq M < \infty, \quad \text{for all } k \in \mathbb{N}, \quad (12)$$

and

$$\|u^k(t)\|_{L^2(|x|>R)}^2 \leq \|u_0^k\|_{L^2(|x|>\frac{R}{2})}^2 + c(t)\psi(R), \quad \text{for all } k \in \mathbb{N}, \quad (13)$$

with $c(t) \in L^\infty((0, T))$, and $\lim_{R \rightarrow \infty} \psi(R) = 0$.

Also, assume that

$$u_0^k \rightarrow u_0 \text{ strongly in } L^2(\Omega) \text{ and, a.e. in } t \in (0, T), \quad u^k(t) \rightarrow u(t) \text{ weakly in } L^2(\Omega). \quad (14)$$

Then there exists a subsequence of $\{u^k\}$ strongly converging to u in $L^2(0, T; L^2(\Omega))$.

Proof. The result for Ω bounded is well known, a proof is given in [18]. In the case of Ω exterior, a proof is due to Leray in [19]. For the sake of the completeness we furnish the following proof.

Let u be the weak limit of $\{u^k\}$ in $L^2(0, T; L^2(\Omega))$. By virtue of (13), for any $R > 0$ we have

$$\begin{aligned} \int_0^T \int_{|x|>R} |u^k - u|^2 dx dt &\leq \int_0^T \int_{|x|>R} |u^k|^2 + |u|^2 dx dt \\ &\leq \|u_0^k\|_{L^2(|x|>\frac{R}{2})}^2 + \psi(R) \int_0^T c(t) dt + \int_0^T \int_{|x|>R} |u|^2 dx dt \\ &\leq 2[\|u_0^k - u_0\|_{L^2(|x|>\frac{R}{2})} + \|u_0\|_{L^2(|x|>\frac{R}{2})}] + \psi(R) \int_0^T c(t) dt + \int_0^T \int_{|x|>R} |u|^2 dx dt. \end{aligned}$$

By (13) for $\psi(R)$ and (14) for u_0^k , and by the absolute continuity of the integral, we get that, for any $\varepsilon > 0$, there exist \bar{R} and \bar{k} , such that

$$\int_0^T \int_{|x|>R} |u^k - u|^2 dx dt < \varepsilon \text{ for all } R > \bar{R} \text{ and } k > \bar{k}.$$

In the bounded set $\Omega \cap B_R$ we apply Lemma 2 and we use estimate (12), obtaining, for any $k \in \mathbb{N}$,

$$\int_0^T \int_{\Omega \cap B_R} |u^k - u|^2 dx dt \leq (1 + \varepsilon) \sum_{j=1}^N \int_0^T \left[\int_{\Omega \cap B_R} (u^k - u) a^j dx \right]^2 dt + 2M\varepsilon. \quad (15)$$

By the uniform bound (12) we have that

$$\left[\int_{\Omega \cap B_R} (u^k - u) a^j dx \right]^2 \leq 2M^2 \|a^j\|_2^2,$$

and we use the dominated convergence theorem to pass to the limit as $k \rightarrow \infty$ in (15). The property (14) allows us to complete the proof. \square

We recall also two basic results.

Lemma 3. *Let Ω be a measurable subset of \mathbb{R}^n and let $v \in L^q(\Omega)$ for any $q \geq \bar{q} \geq 1$. If $\liminf_{q \rightarrow \infty} \|v\|_q = l$ then $v \in L^\infty(\Omega)$ and $\|v\|_\infty = l$.*

Proof. There exists an increasing sequence $\{q_h\}$ such that $q_h \rightarrow \infty$ and $\lim_{h \rightarrow \infty} \|v\|_{q_h} = l$. Hence, for any $\varepsilon > 0$ we can find \bar{h} such that $\|v\|_{q_h} \leq l + \varepsilon$ for any $h \geq \bar{h}$. Moreover, if $q > q_{\bar{h}}$ we can find $h \geq \bar{h}$ such that $q_h \leq q < q_{h+1}$. By interpolation, there exists $\theta_h \in [0, 1]$ such that

$$\|v\|_q \leq \|v\|_{q_h}^{\theta_h} \|v\|_{q_{h+1}}^{1-\theta_h} \leq l + \varepsilon.$$

It follows that $\|v\|_q \leq l + \varepsilon$ for any $q > q_{\bar{h}}$. Hence $v \in L^\infty(\Omega)$ and

$$l = \liminf_{q \rightarrow \infty} \|v\|_q \leq \limsup_{q \rightarrow \infty} \|v\|_q \leq l + \varepsilon$$

for any $\varepsilon > 0$. It follows that

$$\|v\|_\infty = \lim_{q \rightarrow \infty} \|v\|_q = l.$$

□

Lemma 4. *Let $\{g^k\}$ and g be summable functions such that $g^k \rightarrow g$ almost everywhere and*

$$\lim_{k \rightarrow \infty} \int g^k dx = \int g dx.$$

If $\{f^k\}$ and f are measurable functions such that $|f^k| \leq g^k$ almost everywhere and $f^k \rightarrow f$ almost everywhere, then

$$\lim_{k \rightarrow \infty} \int |f^k - f| dx = 0.$$

Proof. The result of lemma is contained in Theorem 1.20 of [11]. □

It is well known that in [4] and in [30] it is furnished an existence theorem of suitable weak solutions to the Navier-Stokes Cauchy problem. Here, in order to achieve the same result in the case of problem (1), that is, in the case of the initial boundary value problem in bounded or exterior domains Ω , we give the chief steps of the proof in Lemma 5 and in the Appendix. For this goal we consider a mollified Navier-Stokes system. Hence problem (1) becomes

$$\begin{aligned} v_t^m + J_m[v^m] \cdot \nabla v^m + \nabla \pi_{v^m} &= \Delta v^m, \quad \nabla \cdot v^m = 0, \quad \text{in } (0, T) \times \Omega, \\ v^m &= 0 \text{ on } (0, T) \times \partial\Omega, \quad v^m(0, x) = v_0^m(x) \text{ on } \{0\} \times \Omega, \end{aligned} \tag{16}$$

where $J_m[\cdot]$ is a mollifier and $\{v_0^m\} \subset J^{1,2}(\Omega)$ converges to v_0 in $J^2(\Omega)$. The result of existence is established proving that the sequence of solutions $\{(v^m, \pi_{v^m})\}$ to problem (16) converges with respect to the metric stated in Definition 1, as well as proving that the limit satisfies the energy inequality (4). All this is a consequence of the following

Lemma 5. *There exists a sequence of solutions $\{(v^m, \pi_{v^m})\}$ such that, for all $m \in \mathbb{N}$ and $T > 0$, $v^m \in C([0, T]; J^{1,2}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega))$. Moreover, for Ω exterior domain, for \bar{R} sufficiently large, we get*

$$\|v^m(t)\|_{L^2(|x|>R)}^2 \leq \|v_0^m\|_{L^2(|x|>\frac{R}{2})}^2 + c(t)\psi(R) \text{ for any } t > 0, R > 2\bar{R} \text{ and } m \in \mathbb{N}, \quad (17)$$

with $c(t) \in L^\infty(0, T)$ and $\psi(R) = o(1)$.

Proof. The above result is well known. The existence and uniqueness of the solutions and related properties of regularity can be proved as in Theorem 3 of [16] (see also [6]). Concerning estimate (17), in the case of the Cauchy problem it was due to Leray in [19]. Subsequently the result is extended to the initial boundary value problem in exterior domains by several authors, in different contexts. Actually, the technique employed by the authors is essentially the same. In this connection, without the aim of being exhaustive, we refer to [15, 27]. In Appendix we give the details of the proof of (17). \square

Lemma 6. *For all $T > 0$ the sequence of solutions to problem (16) furnished by Lemma 5, uniformly in $m \in \mathbb{N}$, satisfies the estimate*

$$\left(\int_0^T (\|P\Delta v^m(t)\|_2^2 + \|v_t^m(t)\|_2^2)^{\frac{1}{3}} dt \right)^3 \leq c \left(\frac{1}{1 + \|\nabla v^m(T)\|_2^2} + \|v_0\|_2^6 \right). \quad (18)$$

Proof. By virtue of the regularity of (v^m, π_{v^m}) stated in Lemma 5, we multiply equation (16)₁ by $P\Delta v^m - v_t^m$. Integrating by parts on Ω , and applying the Hölder inequality, we get

$$\|P\Delta v^m - v_t^m\|_2 \leq \|v^m \cdot \nabla v^m\|_2, \text{ a.e. in } t > 0. \quad (19)$$

Applying inequality (10) with $r = \infty$ and $q = 6$, by virtue of the Sobolev inequality, we obtain

$$\|v^m \cdot \nabla v^m\|_2 \leq \|v^m\|_\infty \|\nabla v^m\|_2 \leq c \|P\Delta v^m\|_2^{\frac{1}{2}} \|\nabla v^m\|_2^{\frac{3}{2}}. \quad (20)$$

By inequalities (19) and (20), we get

$$\begin{aligned} \frac{d}{dt} \|\nabla v^m\|_2^2 + \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 &= \|P\Delta v^m - v_t^m\|_2^2 \leq c \|P\Delta v^m\|_2 \|\nabla v^m\|_2^3 \\ &\leq \frac{1}{2} \|P\Delta v^m\|_2^2 + c \|\nabla v^m\|_2^6, \end{aligned} \quad (21)$$

for all $m \in \mathbb{N}$ and a.e. in $t > 0$. We can divide by $(1 + \|\nabla v^m(t)\|_2^2)^2$, and the following holds

$$\frac{\frac{d}{dt} \|\nabla v^m\|_2^2}{(1 + \|\nabla v^m\|_2^2)^2} + \frac{\frac{1}{2} \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1 + \|\nabla v^m\|_2^2)^2} \leq c \|\nabla v^m\|_2^2.$$

Integrating on $(0, T)$, we have

$$\frac{1}{1 + \|\nabla v_0^m\|_2^2} - \frac{1}{1 + \|\nabla v^m(T)\|_2^2} + \int_0^T \frac{\frac{1}{2}\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1 + \|\nabla v^m\|_2^2)^2} dt \leq c \int_0^T \|\nabla v^m\|_2^2 dt.$$

Employing the reverse Hölder inequality (see [1, Theorem 2.12]) with exponents $\frac{1}{3}$ and $-\frac{1}{2}$, we get

$$\int_0^T \frac{\frac{1}{2}\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1 + \|\nabla v^m\|_2^2)^2} dt \geq \left[\int_0^T \left[\frac{1}{2}\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 \right]^{\frac{1}{3}} dt \right]^3 \left[\int_0^T (1 + \|\nabla v^m\|_2^2) dt \right]^{-2}.$$

Coupling the above inequalities with the energy inequality (5), estimate (18) follows. \square

3 Proof of Theorem 1

The idea of the proof is the following. We consider the sequence of solutions to problem (16) furnished by Lemma 5. It is well known that there exists a subsequence $\{(v^m, \pi_{v^m})\}$ whose weak limit (v, π_v) in $L^2(0, T; J^{1,2}(\Omega))$ is a weak solution in the sense of Definition 2. All this is contained in [19] or, for example, also in [4]. Now, our aim is to prove further estimates on the extract $\{(v^m, \pi_{v^m})\}$ that ensure the thesis of Theorem 1.

3.1 The strong convergence in $L^p(0, T; L^2(\Omega))$ for all $p \in [1, 2)$

We start by proving that the sequence $\{v^m\}$ strongly converges in $L^p(0, T; W^{1,2}(\Omega))$, for $p \in [1, 2)$ and for all $T > 0$. We recall that

$$\|\nabla u\|_2 \leq \|P\Delta u\|_2^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}}, \text{ for all } u \in W^{2,2}(\Omega) \cap J^{1,2}(\Omega).$$

Hence, integrating on $(0, T)$ and applying the Hölder inequality, we get

$$\int_0^T \|\nabla v^k(t) - \nabla v^m(t)\|_2 dt \leq \left[\int_0^T \|P\Delta v^k(t) - P\Delta v^m(t)\|_2^{\frac{2}{3}} dt \right]^{\frac{3}{4}} \left[\int_0^T \|v^k(t) - v^m(t)\|_2^2 dt \right]^{\frac{1}{4}}.$$

By virtue of Lemma 6, we get the existence of a $M(T)$ such that

$$\int_0^T \|\nabla v^k(t) - \nabla v^m(t)\|_2 dt \leq (2M(T))^{\frac{3}{4}} \left[\int_0^T \|v^k(t) - v^m(t)\|_2^2 dt \right]^{\frac{1}{4}}, \text{ for all } k, m \in \mathbb{N},$$

and, via (17), we can apply Corollary 1 to deduce the strong convergence of the sequence $\{v^m\}$ in $L^1(0, T; W^{1,2}(\Omega))$. Since the energy inequality holds uniformly with respect to $m \in \mathbb{N}$, by interpolation we arrive at the strong convergence of $\{v^m\}$ in $L^p(0, T; W^{1,2}(\Omega))$, for any $p \in [1, 2)$. In order to identify the limit point, we remark that $\{v^m\}$ weakly converges to v in $L^2(0, T; W^{1,2}(\Omega))$, hence v has to coincide with the strong limit in each space $L^p(0, T; W^{1,2}(\Omega))$. Thus for all $T > 0$ we deduce that

$$\begin{aligned} \int_s^t \|\nabla v(\tau)\|_2^2 d\tau &= \lim_{p \rightarrow 2^-} \int_s^t \|\nabla v(\tau)\|_2^p d\tau \\ &= \lim_{p \rightarrow 2^-} \lim_{m \rightarrow \infty} \int_s^t \|\nabla v^m(\tau)\|_2^p d\tau, \text{ for all } t, s \in (0, T). \end{aligned} \quad (22)$$

3.2 Proof of formula (2)

By the strong convergence in $L^1(0, T; W^{1,2}(\Omega))$, there exists a negligible set (for the Lebesgue measure) $\mathcal{I} \subset (0, T)$, such that for any $t \in \mathcal{G}_1 := (0, T) - \mathcal{I}$ the following limits are finite

$$\lim_{m \rightarrow \infty} \|v^m(t)\|_2 = \|v(t)\|_2 \quad \text{and} \quad \lim_{m \rightarrow \infty} \|\nabla v^m(t)\|_2 = \|\nabla v(t)\|_2. \quad (23)$$

From the energy equality for the approximating solutions $\{v^m\}$ we obtain, for any $t \in \mathcal{G}_1$ and any $\alpha, K > 0$,

$$\frac{1}{(K + \|\nabla v^m(t)\|_2^2)^\alpha} \frac{d}{dt} \|v^m(t)\|_2^2 + \frac{2\|\nabla v^m(t)\|_2^2}{(K + \|\nabla v^m(t)\|_2^2)^\alpha} = 0. \quad (24)$$

Integrating by parts we get

$$\begin{aligned} \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau + 2 \int_s^t \frac{\|\nabla v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^\alpha} d\tau \\ = \frac{\|v^m(s)\|_2^2}{(K + \|\nabla v^m(s)\|_2^2)^\alpha} - \frac{\|v^m(t)\|_2^2}{(K + \|\nabla v^m(t)\|_2^2)^\alpha}. \end{aligned}$$

We remark that, for almost every $\tau \in (0, T)$, by (23),

$$\frac{\|\nabla v(\tau)\|_2^2}{(K + \|\nabla v(\tau)\|_2^2)^\alpha} \leftarrow \frac{\|\nabla v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^\alpha} \leq \|\nabla v^m(\tau)\|_2^{2-2\alpha} \rightarrow \|\nabla v(\tau)\|_2^{2-2\alpha}$$

and that, for $\alpha \in (0, \frac{1}{2}]$, by virtue of the strong convergence in $L^{2-2\alpha}(0, T; W^{1,2}(\Omega))$,

$$\lim_{m \rightarrow \infty} \int_s^t \|\nabla v^m(\tau)\|_2^{2-2\alpha} d\tau = \int_s^t \|\nabla v(\tau)\|_2^{2-2\alpha} d\tau.$$

Hence we can apply Lemma 4 to obtain that, for any $t, s \in \mathcal{G}_1$,

$$\begin{aligned} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau + 2 \int_s^t \frac{\|\nabla v(\tau)\|_2^2}{(K + \|\nabla v(\tau)\|_2^2)^\alpha} d\tau \\ = \frac{\|v(s)\|_2^2}{(K + \|\nabla v(s)\|_2^2)^\alpha} - \frac{\|v(t)\|_2^2}{(K + \|\nabla v(t)\|_2^2)^\alpha}. \end{aligned} \quad (25)$$

Applying once again Lemma 4, we get

$$\lim_{\alpha \rightarrow 0} \int_s^t \frac{\|\nabla v(\tau)\|_2^2}{(K + \|\nabla v(\tau)\|_2^2)^\alpha} d\tau = \int_s^t \|\nabla v(\tau)\|_2^2 d\tau.$$

Then, letting $\alpha \rightarrow 0$ in (25), we deduce (2) with

$$H(t, s) := \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau. \quad (26)$$

3.3 Proof of formula (3)

We denote by \mathcal{G}_2 the set of $t \geq 0$ such that the extract $\{v^m\}$ is strongly convergent in $L^2(\Omega)$. Recalling the definition of \mathcal{G}_1 , we have $\mathcal{G}_1 \subseteq \mathcal{G}_2$. By virtue of Lemma 11 we claim that $\|\nabla v^m(t)\|_2 \neq 0$ for all $t > 0$ and $m \in \mathbb{N}$. Hence we consider formula (24) that rewrite with K_1

$$\frac{1}{(K_1 + \|\nabla v^m(t)\|_2^2)^\alpha} \frac{d}{dt} \|v^m(t)\|_2^2 + \frac{2\|\nabla v^m(t)\|_2^2}{(K_1 + \|\nabla v^m(t)\|_2^2)^\alpha} = 0,$$

where, by the above claim, we can consider $K_1 \geq 0$. Integrating on (s, t) , for $s, t \in \mathcal{G}_2$, and applying the mean value theorem for the integrals, we get

$$\frac{1}{(K_1 + \|\nabla v(t_{\alpha, m})\|_2^2)^\alpha} = (\|v^m(s)\|_2^2 - \|v^m(t)\|_2^2)^{-1} 2 \int_s^t \frac{\|\nabla v^m(\tau)\|_2^2}{(K_1 + \|\nabla v^m(\tau)\|_2^2)^\alpha} d\tau.$$

Since the right hand side admits limit as $m \rightarrow \infty$ and as $\alpha \rightarrow 0$, the limit $F(t, s) := \lim_{\alpha \rightarrow 0} \lim_{m \rightarrow \infty} \frac{1}{(K_1 + \|\nabla v^m(t_{\alpha, m})\|_2^2)^\alpha}$ is well posed and (3) is proved.

3.4 The $L^{\mu(q)}(0, T; L^q(\Omega))$ property

By virtue of estimate (10) we get

$$\|v^m(t)\|_\infty \leq c \|P\Delta v^m\|_2^{\frac{1}{2}} \|\nabla v^m\|_2^{\frac{1}{2}}.$$

Employing the energy relation (5) and estimate (18), applying Hölder's inequality, for all $T > 0$, we deduce that, for any $m \in \mathbb{N}$,

$$\int_0^T \|v^m(\tau)\|_\infty d\tau \leq c \left[\int_0^T \|P\Delta v^m(\tau)\|_2^{\frac{2}{3}} d\tau \right]^{\frac{3}{4}} \left[\int_0^T \|\nabla v^m(\tau)\|_2^2 d\tau \right]^{\frac{1}{4}} \leq C(v_0),$$

where, here and in the following, $C(v_0)$ are constants depending only on $\|v_0\|_2$. Therefore, by L^p -interpolation and recalling that $v^m \in L^\infty(0, T; J^2(\Omega))$, uniformly in $m \in \mathbb{N}$ and $T > 0$, we arrive at

$$\int_0^T \|v^m(\tau)\|_q^{\frac{q}{q-2}} d\tau \leq \sup_{(0,T)} \|v^m(\tau)\|_2^{\frac{2}{q-2}} \int_0^T \|v^m(\tau)\|_\infty d\tau \leq C(v_0).$$

This allows us to claim that, for all $T > 0$, the weak solution v to problem (1), limit of the sequence $\{v^m\}$, belongs to $L^{\frac{q}{q-2}}(0, T; L^q(\Omega))$, for all $q \in (6, \infty)$, with

$$\int_0^T \|v(\tau)\|_q^{\frac{q}{q-2}} d\tau \leq C(v_0).$$

This limit property and Fatou's lemma ensure that, for all $T > 0$ and for any sequence $q_h \rightarrow \infty$, the following estimate holds true

$$\int_0^T \liminf_{h \rightarrow \infty} \|v(\tau)\|_{q_h} \leq \lim_{h \rightarrow \infty} T^{\frac{2}{q_h}} C(\|v_0\|_2)^{\frac{q_h-2}{q_h}} = C(v_0). \quad (27)$$

The thesis of the theorem in the case $q = \infty$ follows straightforward by Lemma 3. \square

REMARK 4. We verify that $H(t, s) = 0$ in the case of 2D-Navier-Stokes equations. It is important to realize the result in the framework of the construction given in the above proof, that is, not relying on the regularity of the limit solution v . We start remarking that estimate (20), for $\Omega \subset \mathbb{R}^2$, via (10), becomes

$$\|v^m \cdot \nabla v^m\|_2 \leq c \|v^m\|_2^{\frac{1}{2}} \|P\Delta v^m\|_2^{\frac{1}{2}} \|\nabla v^m\|_2.$$

Hence, in place of (21), we deduce the differential inequality

$$\frac{d}{dt} \|\nabla v^m\|_2^2 + \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 \leq c \|v^m\|_2^2 \|\nabla v^m\|_2^4 \leq c \|v_0^m\|_2^2 \|\nabla v^m\|_2^4. \quad (28)$$

Hence we achieve the result of Lemma 6 also for $\Omega \subset \mathbb{R}^2$, with the only difference that on the right hand side of estimate (18) we have $\frac{1}{(1+\|\nabla v^m(T)\|_2^2)} + c\|v_0\|_2^2$. By the same arguments of the three-dimensional case, we obtain that $\{v^m\}$ strongly converges in $L^p(0, T; W^{1,2}(\Omega))$, for all $p \in [1, 2)$, that is the key ingredient to arrive at the identity (2).

Now we prove that $H(t, s) \leq 0$, which implies, by virtue of the energy inequality (5), that $H(t, s) = 0$. By (28), (5) and the Hölder inequality, we have

$$\begin{aligned} & \alpha \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\alpha+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau \\ & \leq \alpha c \|v_0^m\|_2^2 \int_s^t \|v^m(\tau)\|_2^2 \frac{\|\nabla v^m\|_2^4}{(K + \|\nabla v^m\|_2^2)^{\alpha+1}} d\tau \leq \alpha c \|v_0^m\|_2^4 \int_s^t \|\nabla v^m(\tau)\|_2^{2-2\alpha} d\tau \\ & \leq \alpha c \|v_0^m\|_2^4 (t-s)^{\frac{1}{\alpha}} \left(\int_s^t \|\nabla v^m(\tau)\|_2^2 d\tau \right)^{1-\alpha} \leq \alpha c \|v_0^m\|_2^{6-2\alpha} (t-s)^{\frac{1}{\alpha}}. \end{aligned}$$

Passing to the limit for $m \rightarrow \infty$ and then for $\alpha \rightarrow 0$, we get that $H(t, s) \leq 0$.

4 Appendix

4.1 Some results related to the construction of the weak solution

In this section we recall some results which are fundamental in order to construct a suitable weak solution. These results essentially concern estimates of the pressure field π_{v^m} which appears in (16). Of course we look for estimates that are uniform with respect to $m \in \mathbb{N}$. Our aim is to justify estimate (17).

We start by recalling that the energy relation holds uniformly in $m \in \mathbb{N}$:

$$\|v^m(t)\|_2^2 + 2 \int_0^t \|\nabla v^m(\tau)\|_2^2 d\tau = \|v_0^m\|_2^2 \leq \|v_0\|_2^2 \text{ for all } t > 0.$$

We introduce the following functionals:

$$\lambda \in (0, 1), \quad q \in (1, \infty), \quad \langle a \rangle_q^\lambda := \left[\int_{\partial\Omega} \int_{\partial\Omega} \frac{|a(x) - a(y)|^q}{|x - y|^{2+\lambda q}} d\sigma_y d\sigma_x \right]^{\frac{1}{q}},$$

$$\|a\|_{W^{1-\frac{1}{q}, q}(\partial\Omega)} := \|a\|_{L^q(\partial\Omega)} + \langle a \rangle_q^{1-\frac{1}{q}}.$$

We consider the following Neumann problem:

$$\Delta\pi = 0, \quad \pi \rightarrow 0 \text{ for } |x| \rightarrow \infty, \quad \frac{d\pi}{d\nu} = \nu \cdot \nabla \times a \text{ on } \partial\Omega. \quad (29)$$

Lemma 7. *In (29) assume $a \in W^{1-\frac{1}{q}, q}(\partial\Omega)$. Then for all $\lambda \in (0, 1 - \frac{1}{q}]$ and R_0 sufficiently large there exists a constant c independent of a such that*

$$\|\pi\|_{L^q(\Omega \cap B_{R_0})} \leq c \langle a \rangle_q^\lambda. \quad (30)$$

The lemma is due to Solonnikov in [31, 32]. A recent proof of the same result, by similar techniques, can be found, for example, in [23].

Applying the Hölder inequality and the Gagliardo trace theorem, one gets

$$\begin{aligned} \|\pi\|_{L^q(\Omega \cap B_{R_0})} &\leq c \langle a \rangle_q^\lambda \leq c \|a\|_{L^q(\partial\Omega)}^\beta \left[\langle a \rangle_q^{1-\frac{1}{q}} \right]^{1-\beta} \\ &\leq c \left[\|a\|_{L^q(\Omega \cap B_{R_0})} + \|a\|_{L^q(\Omega)}^{\frac{1}{q'}} \|\nabla a\|_{L^q(\Omega)}^{\frac{1}{q}} \right]^\beta \|\nabla a\|_q^{1-\beta} \end{aligned} \quad (31)$$

with $\beta := \frac{q(1-\lambda)-1}{1+q}$. Now, we consider (U, π) as a solution to the Stokes problem

$$\begin{aligned} U_t + \nabla\pi &= \Delta U, \quad \nabla \cdot U = 0, \quad \text{in } (0, T) \times \Omega, \\ U &= 0 \text{ on } (0, T) \times \partial\Omega, \quad U = v_0 \text{ on } \{0\} \times \Omega. \end{aligned} \quad (32)$$

We estimate π by means of (31). That is, we set $a := \text{curl}U$, we assume $v_0 \in J^2(\Omega)$, and, via the semigroup properties of U (see, e.g., [24]), for $q = 2$, for all $T > 0$, we get

$$\|\pi(t)\|_{L^2(\Omega \cap B_{R_0})} \leq c(T) \|v_0\|_2 \left[t^{-1+\frac{\beta}{2}} + t^{-1+\frac{\beta}{4}} \right], \quad \text{for all } t \in (0, T), \quad (33)$$

with $\beta = \frac{1-2\lambda}{3}$. Keeping this in hand, we can also deduce an estimate in the exterior of B_{R_0} . Actually, by means of a cut of the equation (29) in B_{R_0} , we get

$$\Delta(\pi h_{R_0}) = \pi \Delta h_{R_0} + 2\nabla\pi \cdot \nabla h_{R_0},$$

with h_{R_0} smooth function such that $h_{R_0}(x) = 1$ for $|x| > R_0$, $h_{R_0}(x) = 0$ for $|x| < \frac{R_0}{2}$. Then, by the representation formula of the solution, we obtain

$$\pi(t, x) = - \int_{\mathbb{R}^3} \mathcal{E}(x - y) \pi \Delta h_{R_0} dy - 2 \int_{\mathbb{R}^3} \nabla \mathcal{E}(x - y) \nabla h_{R_0} \pi dy,$$

with \mathcal{E} fundamental solution. So that, for $\bar{r} > 3$ and for $|x| > 2R_0$, we easily get

$$\begin{aligned} \|\pi(t)\|_{L^{\bar{r}}(|x| > R_0)} &\leq c \|\pi(t)\|_{L^2(\Omega \cap B_{R_0})} \\ &\leq c(T) \|v_0\|_2 \left[t^{-1 + \frac{\beta}{2}} + t^{-1 + \frac{\beta}{4}} \right], \text{ for all } t \in (0, T). \end{aligned} \quad (34)$$

Consider the following initial boundary value problem for the Stokes system:

$$\begin{aligned} W_t - \Delta W + \nabla \pi_W &= F, \quad \nabla \cdot W = 0, \quad \text{on } (0, T) \times \Omega, \\ W &= 0 \text{ on } (0, T) \times \Omega, \quad W = 0 \text{ on } \{0\} \times \Omega. \end{aligned} \quad (35)$$

Lemma 8. *In problem (35) assume $F \in L^r(0, T; L^s(\Omega))$, $\frac{3}{s} + \frac{2}{r} = 4$, $s \in (1, \frac{3}{2})$. Then there exists a unique solution to problem (35) such that*

$$\int_0^T \left[\|D^2 W(\tau)\|_s^r + \|\nabla \pi_W(\tau)\|_s^r + \|W_\tau(\tau)\|_s^r \right] d\tau \leq c \int_0^T \|F(\tau)\|_s^r d\tau, \quad (36)$$

with c independent of F and T .

Proof. This result is well known, a proof can be found in [24, 25]. \square

Lemma 9. *Let $\{(v^m, \pi_{v^m})\}$ be the sequence of solutions to problem (16) furnished by Lemma 5. Then there exist functions $\pi_{v^m}^1, \pi_{v^m}^2$ such that $\pi_{v^m} = \pi_{v^m}^1 + \pi_{v^m}^2$, and, for all $\bar{r} > 3$, $R_0 > 0$ and $\lambda \in (0, \frac{1}{2})$, we also obtain*

$$\begin{aligned} \|\pi_{v^m}^1(t)\|_{L^2(\Omega \cap B_{R_0})} + \|\pi_{v^m}^1(t)\|_{L^{\bar{r}}(|x| > R_0)} &\leq c(T) \|v_0\|_2 t^{-1 + \frac{\beta}{4}}, \text{ with } \beta := \frac{1-2\lambda}{3}, \\ \text{and} \\ \int_0^T \|\nabla \pi_{v^m}^2(\tau)\|_s^r d\tau &\leq c \|v_0\|_2^{2r}, \quad \frac{3}{s} + \frac{2}{r} = 4. \end{aligned} \quad (37)$$

Proof. The result of the lemma is an immediate consequence of the following decomposition:

$$\text{for all } m \in \mathbb{N}, v^m = U^m + W^m \quad \text{and} \quad \pi_{v^m} = \pi_{U^m} + \pi_{W^m},$$

with (U_m, π_{U^m}) solution to problem (32) with initial data $U^m = v_0^m$, and (W^m, π_{W^m}) solution to problem (35) with $F^m = J_m(v^m) \cdot \nabla v^m$. Since $v^m \in L^2(0, T; J^{1,2}(\Omega))$, one easily deduces that $F^m \in L^r(0, T; L^s(\Omega))$ provided that $\frac{3}{s} + \frac{2}{r} = 4$. \square

From estimate (37)₂, via the Sobolev embedding theorem (cf. Lemma 5.2 of [13]), there exists a function $\pi_{v^m}^0(\tau)$ such that

$$\int_0^T \|\pi_{v^m}^2(\tau) - \pi_{v^m}^0(\tau)\|_{\frac{3s}{3-s}}^r d\tau \leq c \int_0^T \|\nabla \pi_{v^m}^2(\tau)\|_s^r d\tau \leq c \|v_0\|_2^{2r}, \quad (38)$$

for $\frac{3}{s} + \frac{2}{r} = 4$. In the sequel we assume that $\pi_{v^m} := \pi_{v^m}^1(t, x) + \pi_{v^m}^2(t, x) - \pi_{v^m}^0(t)$.

Now, we are in a position to prove estimate (17).

We consider $R > 0$ such that $R > 2R_0$. We denote by h_R a smooth function such that $h_R(x) = 1$ for $|x| > R$, $h_R = 0$ for $|x| < \frac{R}{2}$ with $|D^2 h_R| + |\nabla h_R| \leq \frac{c}{R}$. We multiply equation (16)₁ by $v^m(t, x)h_R(x)$. Integrating by parts on $(0, T) \times \Omega$, we get

$$\begin{aligned} \|v^m(t)\|_{L^2(|x|>R)}^2 &\leq \|v_0^m\|_{L^2(|x|>\frac{R}{2})}^2 + cR^{-1} \int_0^t \left[\|v^m\|_2^2 + \|v^m\|_3^3 + \|\pi_{v^m}|v^m\|_{L^1(R<|x|<2R)} \right] d\tau \\ &=: \|v_0^m\|_{L^2(|x|>\frac{R}{2})}^2 + cR^{-1} \int_0^t [I_1(\tau) + I_2(\tau) + I_3(\tau)] d\tau. \end{aligned}$$

Applying the Gagliardo-Nirenberg inequality and the energy relation, we deduce

$$I_1(\tau) + I_2(\tau) \leq \|v_0^m\|_2^2 + c\|v_0^m\|_2^{\frac{3}{2}} \|\nabla v^m(\tau)\|_2^{\frac{3}{2}}.$$

By virtue of Lemma 9, assuming $\bar{r} \in (3, 6)$ and $\frac{3}{s} + \frac{2}{r} = 4$, by applying the Hölder inequality, for I_3 we get

$$\begin{aligned} \|\pi_{v^m}|v^m\|_{L^1(R<|x|<2R)} &\leq \left[cR^{3\frac{\bar{r}-2}{2\bar{r}}} \|\pi_{v^m}^1(\tau)\|_{L^{\bar{r}}(|x|>\frac{R}{2})} + cR^{\frac{5s-6}{2s}} \|\pi_{v^m}^2 - \pi_{v^m}^0\|_{\frac{3s}{3-s}} \right] \|v^m\|_2 \\ &\leq c \left[R^{3\frac{\bar{r}-2}{2\bar{r}}} \|\pi_{v^m}^1(\tau)\|_{L^{\bar{r}}(|x|>\frac{R}{2})} + R^{\frac{5s-6}{2s}} \|\nabla \pi_{v^m}^2\|_s \right], \end{aligned}$$

for all $R > 2R_0$. Increasing the terms I_i , $i = 1, 2, 3$, by means of the above estimates, we arrive at

$$\begin{aligned} \|v^m(t)\|_{L^2(|x|>R)}^2 &\leq \|v_0^m\|_{L^2(|x|>\frac{R}{2})}^2 + cR^{-1} \int_0^t \left[\|v_0^m\|_2^2 + c\|v_0^m\|_2^{\frac{3}{2}} \|\nabla v^m(\tau)\|_2^{\frac{3}{2}} \right. \\ &\quad \left. + R^{3\frac{\bar{r}-2}{2\bar{r}}} \|\pi_{v^m}^1(\tau)\|_{L^{\bar{r}}(|x|>\frac{R}{2})} + R^{\frac{5s-6}{2s}} \|\nabla \pi_{v^m}^2\|_s \right] d\tau. \end{aligned}$$

Via estimates (37), applying the Hölder inequality and the energy relation we prove that

$$\|v^m(t)\|_{L^2(|x|>R)}^2 \leq \|v_0^m\|_{L^2(|x|>\frac{R}{2})}^2 + cR^{-1} \left[t + t^{\frac{1}{4}} \|v_0^m\|_2 + t^{\frac{\beta}{4}} R^{3\frac{\bar{r}-2}{2\bar{r}}} + t^{1-\frac{1}{r}} \|v_0^m\|_2 R^{\frac{5s-6}{2s}} \right] \|v_0^m\|_2^2,$$

which furnishes (17).

4.2 Uniqueness backward in time for the sequence $\{v^m\}$

For the sake of the completeness we prove a result concerning the uniqueness backward in time for solutions (v^m, π^m) to the IBVP (16), whose existence is furnished by Lemma 5. A wide literature on the topic can be found in [29] and [2]. Here we employ the logarithmic convexity method developed in [17, 14]. In order to prove the result we premise a result

Lemma 10. *Let $v_0^m \in \mathcal{C}_0(\Omega)$ and (v^m, π_{v^m}) the solution to problem (16). Then, for all $T > 0$, there exists a constant A_m such that*

$$\|v^m(t)\|_\infty \leq A_m \text{ for all } t \in [0, T]. \quad (39)$$

Proof. The result of the lemma is classical in the case of a solution to problem (1), provided that one considers $[0, T]$ as a subset of the local interval of existence of the solution. In the case of a solution to problem (16), by employing the properties of the mollifier, one can prove property (39) on $[0, T]$ for all $T > 0$ with a bound depending on m . Actually, for all $T > 0$, one proves Ladyzhenskaya's estimate (see [18] or [16]), that is $\|v^m(t)\|_{2,2} \leq A_m$ for any $t \in [0, T]$. These considerations allow us to omit further details related to estimate (39). \square

We are going to prove

Lemma 11. *If $v_0^m \neq 0$, then the solution (v^m, π^m) to problem (16) enjoys the property $\|\nabla v^m(t)\|_2 > 0$ for all $t > 0$.*

Proof. We start from (19), that furnishes:

$$\frac{d}{dt} \|\nabla v^m\|_2^2 + \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 \leq \|v\|_\infty^2 \|\nabla v^m\|_2^2, \text{ a.e. in } t > 0. \quad (40)$$

We recall that the following estimates hold:

$$\|\nabla v^m\|_2^2 \leq \|P\Delta v^m\|_2 \|v^m\|_2 \quad \text{and} \quad \|\nabla v^m\|_2^2 \leq \|v_t^m\|_2 \|v^m\|_2. \quad (41)$$

By virtue of the energy equation for (v^m, π^m) , we get

$$\dot{E}(t) = -2\|\nabla v^m\|_2^2, \quad \ddot{E}(t) = -2\frac{d}{dt} \|\nabla v^m\|_2^2, \quad (42)$$

where we set $E(t) := \|v^m\|_2^2$. Therefore, by (39) and (40)-(42) we arrive at

$$-\ddot{E} + \frac{\dot{E}^2}{E} \leq 2A_m^2 \|\nabla v^m\|_2^2. \quad (43)$$

We prove the result of the lemma claiming that if $\bar{t} > 0$ is the first instant such that $\|\nabla v^m(\bar{t})\|_2 = 0$, then $\|v^m(t)\|_2 = 0$ for all $t \in [0, \bar{t}]$, that is a contradiction. Since, for all $T > 0$, $v^m \in C([0, T]; J^{1,2}(\Omega))$, if there exists $\bar{t} > 0$ such that $\|v^m(\bar{t})\|_2 = 0$, then there exists $\delta > 0$ such that $\|v^m(t)\|_2 \leq 1$ for all $t \in [\bar{t} - \delta, \bar{t}]$. So that, for a suitable $h > 0$, the inequality (43) can be written as

$$\ddot{E} - \frac{\dot{E}^2}{E} \geq h\dot{E} \Rightarrow \frac{\ddot{E}E - \dot{E}^2}{E^2} \geq h\frac{\dot{E}}{E} \Leftrightarrow \frac{d}{dt} \left[e^{-ht} \frac{\dot{E}}{E} \right] \geq 0, \text{ for all } t \in [\bar{t} - \delta, \bar{t}].$$

Set $\sigma = e^{-ht}$, we deduce

$$\frac{d}{d\sigma} \left[\frac{1}{E} \frac{d}{d\sigma} E \right] \geq 0.$$

This last implies that $\log E$ is a convex function. That is, for all $h \in [0, 1]$,

$$\log E(ht + (1-h)t_0) \leq h \log E(t) + (1-h) \log E(\bar{t} - \delta). \quad (44)$$

Since in $\bar{t} > 0$ it is $\|v^m(\bar{t})\|_2 = 0$, then we arrive at $\|v^m(t)\|_2 = 0$ for all $t < \bar{t}$, which is a contradiction with the hypothesis $v_0^m \neq 0$. \square

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